Unspanned Risks, Negative Local Time Risk Premiums, and Empirical Consistency of Models of Interest-Rate Claims

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Abstract

We formalize the notion of *local time* risk premium in the context of a theory in which the pricing kernel is a general diffusion process with spanned and unspanned components. We derive results on the expected excess return of options on bond futures. These results are organized around our finding that the average returns of out-of-the-money puts and calls on Treasury bond futures are both negative. Our theoretical reconciliation warrants negative local time risk premiums, and our treatment considers models with market incompleteness and sources of volatility uncertainty.

Keywords: Expected return of options on Treasury bond futures, unspanned components of pricing kernel, interest-rate models, Tanaka's formula, local time risk premiums

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Contents

1 Introduction

The cornerstone of interest-rate theory is that prices of Treasury bonds, futures on bonds, and options on Treasury bond futures can be characterized by postulating the pricing kernel process and the evolution of the spot interest-rate. Despite progress on modeling bond dynamics, little priority has been given to depicting and reconciling the return pattern of state-contingent claims on interest-rate movements. This seems surprising since options on Treasury bond futures are actively sought markets. We develop a set of results regarding the expected excess return of claims on the downside and the upside, and then present the pattern of expected excess returns of options on bond futures that is generated by a host of model classes (we feature seven).

In light of our theoretical approach, we construct the time-series of option returns on Treasury bond futures. Our new empirical result is that the average returns of holding out-of-the-money (OTM) puts and calls on bond futures are both negative, implying that both up and down interest-rate movements can be disconcerting to a segment of investors. This finding can be used as a guardrail to improve the modeling of interest-rates, granted that all interest-rate claims are priced consistently under the same equivalent martingale measure (i.e., the risk-neutral measure).

We develop theoretical characterizations that synthesize the expected excess return of puts and calls on bond futures under general diffusion processes for the pricing kernel (with *spanned* and *unspanned* components) and spot interest-rate. We show that many (but not all) interest-rate models and macro-finance models imply that the expected excess return to holding a call option (respectively, put option) has the same sign (opposite sign) as the risk premium on bond futures. Thus, these models manifest the property that the expected excess return to holding puts and calls on bond futures cannot *both be negative*. Therefore, these models are not capable of matching our featured empirical findings. In essence, we offer an empirical dimension to anchor interest-rate models as well as a way for refining and differentiating among prospective macro-finance models.

Our tool of analysis is Tanaka's formula for continuous semimartingales, which decomposes any option-like payoff into three parts: First, the option's intrinsic value; second, the gain (or loss) process of a dynamic trading strategy; and, third, *local time* (e.g., Revuz and Yor (1991)). Suited for our analytical treatment, local time is a construct that embodies the (integrated) variance of, for example, the bond futures price at a *particular level*, and is a stochastic process, related to, but conceptually distinct from, quadratic variation.

Our development is to characterize the expectation of Tanaka's formula for the option payoff under the physical probability measure (i.e., the real-world measure) and the discounted option payoff under the risk-neutral measure. Quantifying the sign and magnitude of the disparity in the expectations leads us to study the *risk premium on local time,* which, in turn, yields concise, and testable, implications that many (but not all) interest-rate models fail to cohere with.

The quantitative effects can be summarized in three findings. First, we show that the average return of long positions in OTM options on the 10-year Treasury bond futures are negative. Second, our bootstrap-based tests identify an increasing (decreasing) pattern in average returns across the strike prices for puts (calls). Third, the results remain robust for options written on the 30-year Treasury bond futures.

Our theoretical results, valid without imposing any parametric specification, distill new economic insights into how investors' dislike both rising and falling interest-rates and into risk compensations tied to state-contingent movements in the Treasury bond markets.

Themes indispensable to our theoretical approach in this paper are market incompleteness (distinguished by the relevance of unspanned risks in the pricing kernel) and sources of volatility uncertainty. The role of some of these features is also studied by Collin-Dufresne and Goldstein (2002), Heidari and Wu (2003), Li and Zhao (2006), Andersen and Benzoni (2010), Joslin, Priebsch, and Singleton (2014), Filipovic, Larsson, and Trolle (2017), and Joslin (2018), but from a markedly different perspective. Collin-Dufresne and Goldstein (2002) present evidence that interest-rate volatility risks are not spanned by bonds, whereas Andersen and Benzoni (2010) show that quadratic yield variation is not spanned by bond prices or bond yields.

Our theoretical treatment, which incorporates a salient role for market incompleteness and volatility — via the formalism of Tanaka's formula — unmasks the conditions for a negative risk premium on local time (we provide corroborative evidence), a concept decoupled from unspanned stochastic volatility. Our analytical results, with economic inspiration drawn from the concept of local time risk premiums, hold, regardless of the number of state variables. We explore model designs that synthesize negative expected excess returns to holding OTM options, which uncovers and introduces suitable economic restrictions on the unspanned components of the pricing kernel.

2 Theoretical framework

This section achieves three objectives. First, we develop theoretical characterizations, under general diffusion processes, concerning the expected excess return to holding a put or a call option written on the bond futures. Second, we feature the implications of Tanaka's formula that respect the interlinkages between the bond price, futures price, and option prices across strikes and maturities. Third, we derive the pattern of expected excess returns of OTM options, probing the properties of local time under the physical probability measure and under the risk-neutral probability measure.

Our theoretical results, developed using Tanaka's formula, are new, allowing for the pricing kernel to have spanned and unspanned components, and are informative about the empirical consistency of interest-rate models. The theory is developed with an eye toward empirical predictions and tests, using options data on Treasury bond futures. Later, we study parameterized frameworks.

2.1 Dynamics of the pricing kernel and the spot interest-rate

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq \mathfrak{T}}, \mathbb{P})$ be a filtered probability space, with $\mathfrak{T}$ being a fixed, finite time. The filtration $(\mathcal{F}_t)_{0\leq t\leq \mathfrak{T}}$ satisfies the usual conditions. Let $\mathbb{P}$ denote the physical probability measure, and $\mathbb{E}_t^{\mathbb{P}}(\bullet) \equiv \mathbb{E}^{\mathbb{P}}(\bullet|\mathcal{F}_t)$ be the expectation under $\mathbb{P}$, conditional on $\mathcal{F}_t$ (i.e., the information set available at time t). All stochastic processes with a subscript t are $\mathcal{F}_t$-measurable.

We consider a frictionless market in which zero coupon bonds and a bond futures contract trade. Denote the time t price of a zero coupon bond maturing at time T by B_t^T. We do not stipulate that bonds of *all* maturities trade, but we do assume that a bond with maturity T_O trades, where T_O denotes the maturity date of the option, written on the bond futures.

Additionally, we denote the time t bond futures price by $F_t^{T_F}$, where T_F denotes the expiration date of the futures contract. It is understood that

$$t \leq T_O \leq T_F \leq T_B, \tag{1}$$

where T_B denotes the maturity of the zero coupon bond underlying the futures contract.

The spot interest-rate r_t, at time t, is defined as $r_t \equiv \lim_{T \downarrow t} \; -\frac{1}{T-t} \log B_t^T$. Let $\boldsymbol{\omega}_t^{\mathbb{P}}$ and $\mathbf{u}_t^{\mathbb{P}}$ be vectors of independent standard Brownian motions under the probability measure $\mathbb{P}$.

Assume that the stochastic differential equation (SDE) of the pricing kernel M_t, of r_t, of bond prices B_t^T, and of bond futures prices $F_t^{T_F}$, under $\mathbb{P}$, can be depicted as

$$
\begin{aligned}
d \log M_t &= -r_t dt - \frac{1}{2} \boldsymbol{\lambda}[t, \mathbf{X}]' \boldsymbol{\lambda}[t, \mathbf{X}] dt - \frac{1}{2} \boldsymbol{\alpha}[t, \mathbf{X}]' \boldsymbol{\alpha}[t, \mathbf{X}] dt + \boldsymbol{\lambda}[t, \mathbf{X}]' \, d\boldsymbol{\omega}_t^{\mathbb{P}} + \boldsymbol{\alpha}[t, \mathbf{X}]' \, d\mathbf{u}_t^{\mathbb{P}}, && (2) \\
dr_t &= \mu_r[t, \mathbf{X}] dt + \boldsymbol{\sigma}_r[t, \mathbf{X}]' d\boldsymbol{\omega}_t^{\mathbb{P}}, && (3) \\
\frac{dB_t^T}{B_t^T} &= \mu_B[t, T, \mathbf{X}] dt - \boldsymbol{\sigma}_B[t, T, \mathbf{X}]' d\boldsymbol{\omega}_t^{\mathbb{P}}, \quad \text{and} && (4) \\
\frac{dF_t^{T_F}}{F_t^{T_F}} &= \mu_F[t, T_F, \mathbf{X}] dt - \boldsymbol{\sigma}_F[t, T_F, \mathbf{X}]' d\boldsymbol{\omega}_t^{\mathbb{P}}. && (5)
\end{aligned}
$$

The standard Brownian motions $\mathbf{u}_t^{\mathbb{P}}$ are only present in the SDE for M_t and capture risks not spanned by bond or bond futures returns but are, potentially, spanned by options.

The incorporation of unspanned risks, with $\boldsymbol{\alpha}[t, \mathbf{X}]' \, d\mathbf{u}_t^{\mathbb{P}} \neq 0$, is consistent with the effects considered by, among others, Collin-Dufresne and Goldstein (2002), Heidari and Wu (2003), Li and Zhao (2006), Andersen and Benzoni (2010), Joslin, Priebsch, and Singleton (2014), Filipovic, Larsson, and Trolle (2017), and Joslin (2018). Our innovation is to show that the presence of $\boldsymbol{\alpha}[t, \mathbf{X}]' \, d\mathbf{u}_t^{\mathbb{P}}$ is central to resolving the observed pattern of returns to options on bond futures.

In equations (2)–(5), $\mu_r[t, \mathbf{X}]$, $\mu_B[t, T, \mathbf{X}]$, and $\mu_F[t, T_F, \mathbf{X}]$ are drift coefficients, whereas $\boldsymbol{\sigma}_r[t, \mathbf{X}] > \mathbf{0}$ (indicating that each and every element of the vector $\boldsymbol{\sigma}_r[t, \mathbf{X}]$ is strictly positive), $\boldsymbol{\sigma}_B[t, T, \mathbf{X}] > \mathbf{0}$, $\boldsymbol{\sigma}_F[t, T_F, \mathbf{X}] > \mathbf{0}$, $\boldsymbol{\lambda}[t, \mathbf{X}]$, and $\boldsymbol{\alpha}[t, \mathbf{X}]$ are vectors of diffusion coefficients.

The drift and diffusion coefficients may depend upon a vector of variables $\mathbf{X}$ (we drop the t subscript) and are adapted to $\mathcal{F}_t$. Although not done here to maintain sparsity of equation presentation, it is understood that the vector $\mathbf{X}$ can, if desired, be partitioned into two sets of variables: one that is spanned by bonds and bond futures returns and the other that is unspanned.

We assume that $\mu_r[t, \mathbf{X}]$, $\mu_B[t, T, \mathbf{X}]$, $\mu_F[t, T_F, \mathbf{X}]$, $\boldsymbol{\sigma}_r[t, \mathbf{X}]$, $\boldsymbol{\sigma}_B[t, T, \mathbf{X}]$, $\boldsymbol{\sigma}_F[t, T_F, \mathbf{X}]$, $\boldsymbol{\lambda}[t, \mathbf{X}]$, and $\boldsymbol{\alpha}[t, \mathbf{X}]$ are sufficiently differentiable that Ito's lemma can be applied, and they are sufficiently regular so that the SDEs in (2)–(5) have a unique solution.

2.2 Expected return of futures on the bond and volatility of futures return

The absence of arbitrage ensures the existence of a (risk-neutral) equivalent martingale measure $\mathbb{Q}$ (e.g., Harrison and Kreps (1979)). By Girsanov's theorem, under $\mathbb{Q}$, $d\boldsymbol{\omega}_t^{\mathbb{Q}}$ and $d\mathbf{u}_t^{\mathbb{Q}}$, defined by

$$d\boldsymbol{\omega}_t^{\mathbb{Q}} = d\boldsymbol{\omega}_t^{\mathbb{P}} \; - \; \boldsymbol{\lambda}[t, \mathbf{X}]dt \qquad \text{and} \qquad d\mathbf{u}_t^{\mathbb{Q}} = d\mathbf{u}_t^{\mathbb{P}} \; - \; \boldsymbol{\alpha}[t, \mathbf{X}]dt, \tag{6}$$

are standard Brownian increments. The Radon-Nikodym derivative, denoted by $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{t,T}$, is

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{t,T} &= \frac{M_T}{M_t} e^{\int_t^T r_\ell d\ell} \qquad \text{for all } T \geq t, \qquad (7) \\ &= \exp\Big(\int_t^T \{-\frac{1}{2}\boldsymbol{\lambda}[\ell, \mathbf{X}]^{'}\boldsymbol{\lambda}[\ell, \mathbf{X}]d\ell + \boldsymbol{\lambda}[\ell, \mathbf{X}]^{'} d\boldsymbol{\omega}_\ell^{\mathbb{P}} - \frac{1}{2}\boldsymbol{\alpha}[\ell, \mathbf{X}]^{'}\boldsymbol{\alpha}[\ell, \mathbf{X}]d\ell + \boldsymbol{\alpha}[\ell, \mathbf{X}]^{'} d\mathbf{u}_\ell^{\mathbb{P}}\}\Big). \end{aligned}$$

It holds that (i) $\mathbb{E}_t^{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{t,T}) = 1$, and (ii) the state-price density, $d\mathbb{Q}_{t,T}\, e^{-\int_t^T r_\ell d\ell}$, satisfies $d\mathbb{Q}_{t,T}\, e^{-\int_t^T r_\ell d\ell} = \frac{M_T}{M_t}\, d\mathbb{P}_{t,T}$. Assume that $\mathbb{E}_t^{\mathbb{P}}(\{\frac{M_T}{M_t} e^{\int_t^T r_\ell d\ell}\}^2) < \infty$.

The drift and diffusion coefficients of bond price and futures price dynamics are restricted by the requirements that

$$\begin{aligned} B_t^T &= \mathbb{E}_t^{\mathbb{P}}(\frac{M_T}{M_t}) = \mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^T r_\ell d\ell}), \qquad \text{for all } t \leq T, \text{ and} \qquad (8) \\ F_t^{T_F} &= \mathbb{E}_t^{\mathbb{P}}(\frac{M_s}{M_t} e^{\int_t^s r_\ell d\ell} F_s^{T_F}) = \mathbb{E}_t^{\mathbb{Q}}(F_s^{T_F}), \text{ for all } t \text{ and } s \text{ satisfying } t \leq s \leq T_F, \qquad (9) \end{aligned}$$

where $\mathbb{E}_t^{\mathbb{Q}}(\bullet)$ is the expectation under the equivalent martingale measure $\mathbb{Q}$, conditional on $\mathcal{F}_t$.

By Ito's lemma, we have, for all t, T_B, and T_F (e.g., Cox, Ingersoll, and Ross (1981)), that the expected excess returns of bonds and bond futures are

$$\mu_B[t, T_B, \mathbf{X}] - r_t = \boldsymbol{\lambda}[t, \mathbf{X}]^{'}\boldsymbol{\sigma}_B[t, T_B, \mathbf{X}] \qquad \text{and} \qquad \mu_F[t, T_F, \mathbf{X}] = \boldsymbol{\lambda}[t, \mathbf{X}]^{'}\boldsymbol{\sigma}_F[t, T_F, \mathbf{X}]. \tag{10}$$

For our results on the expected return of options on bond futures, we note that the return volatility of bond futures is

$$\boldsymbol{\sigma}_F[t, T_F, \mathbf{X}] \; = \; \boldsymbol{\sigma}_B[t, T_B, \mathbf{X}] \; - \; \boldsymbol{\sigma}_B[t, T_F, \mathbf{X}]. \tag{11}$$

Such a relation follows, by Ito's lemma, since (i) bond futures and forward prices have the same instantaneous volatility (e.g., Cox, Ingersoll, and Ross (1981)), and (ii) forward bond prices have the form $\frac{B_t^{T_B}}{B_t^{T_F}}$ (e.g., Jarrow and Oldfield (1981, equation (13))).

By equations (6) and (7), we have, for all $s \geq t$,

$$\frac{M_t}{M_s} e^{-\int_t^s r_\ell d\ell} = \exp\left(\int_t^s \{-\frac{1}{2}\boldsymbol{\lambda}[\ell, \mathbf{X}]'\boldsymbol{\lambda}[\ell, \mathbf{X}]d\ell - \boldsymbol{\lambda}[\ell, \mathbf{X}]' d\boldsymbol{\omega}_\ell^{\mathbb{Q}} - \frac{1}{2}\boldsymbol{\alpha}[\ell, \mathbf{X}]'\boldsymbol{\alpha}[\ell, \mathbf{X}]d\ell - \boldsymbol{\alpha}[\ell, \mathbf{X}]' d\mathbf{u}_\ell^{\mathbb{Q}}\}\right). \tag{12}$$

For later use, we note that $\mathbb{E}_t^{\mathbb{Q}}(\frac{M_t}{M_s} e^{-\int_t^s r_\ell d\ell}) = 1$.

2.3 Expected excess return of puts and calls on bond futures

We are interested in computing the expected excess return to holding, over the time period t to T_O, an option written on the bond futures price, with moneyness $k \equiv \frac{K}{F_t^{T_F}}$, for strike price K.

Motivated by our empirical investigation, we center our theoretical analysis on out-of-the-money (OTM) options or at-the-money options satisfying $k \leq 1$ for puts and $k \geq 1$ for calls.

Consider the $\mathcal{F}_s$-measurable stochastic process (G_s) defined, for $s \geq t$, by

$$G_s \equiv \frac{F_s^{T_F}}{F_t^{T_F}}, \quad \text{the gross futures return, from } t \text{ to } s, \text{ for } s \text{ satisfying } t \leq s \leq T_F. \tag{13}$$

The notation (G_s) (in parentheses) is meant to emphasize the stochastic *process* for any arbitrary time s. The former stands in contrast to G_s, which instead reflects its *actual value* at time s.

The dynamics of (G_s) under $\mathbb{P}$ are

$$\frac{dG_s}{G_s} = \frac{dF_s^{T_F}}{F_s^{T_F}} = \mu_F[s, T_F, \mathbf{X}]\, ds - \boldsymbol{\sigma}_F[s, T_F, \mathbf{X}]' \underbrace{d\boldsymbol{\omega}_s^{\mathbb{P}}}_{\text{spanned}}, \qquad \text{for all } s \text{ satisfying } t \leq s \leq T_F. \tag{14}$$

Additionally, let the binary variable $\hbar$ be defined by

$$\hbar = -1 \text{ for a put option on bond futures and } \hbar = +1 \text{ for a call option on bond futures.} \tag{15}$$

Denoting $[a]^+ = \max(a, 0)$, the expected return $\mu_t^{[\hbar]}$ of holding the option over t to T_O satisfies

$$1 + \mu_t^{[\hbar]} \equiv \frac{\mathbb{E}_t^{\mathbb{P}}([\hbar(F_{T_O}^{T_F} - K)]^+)}{\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O} r_\ell d\ell}\,[\hbar(F_{T_O}^{T_F} - K)]^+)} = \frac{\mathbb{E}_t^{\mathbb{P}}([\hbar(G_{T_O} - k)]^+)}{\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O} r_\ell d\ell}\,[\hbar(G_{T_O} - k)]^+)}. \tag{16}$$

Tanaka's formula enables analytical tractability in our setting of stochastic interest rates. As shown next, it allows us to decompose the payoff $[\hbar(F_{T_O}^{T_F} - K)]^+$ in an economically appealing and parsimonious fashion. In so doing, we formalize the $\mathbb{P}$ measure expectation of $[\hbar(F_{T_O}^{T_F} - K)]^+$, the $\mathbb{Q}$ measure expectation of $e^{-\int_t^{T_O} r_\ell d\ell}\,[\hbar(F_{T_O}^{T_F} - K)]^+)$, and then consider the expected excess return of options without explicit parameterizations.

2.3.1 Implications of Tanaka's formula for expectations under $\mathbb{P}$ and $\mathbb{Q}$

Tanaka's formula for continuous semimartingales (e.g., Borodin and Salminen (2015)) implies that

$$[\hbar(G_{T_O} - k)]^+ = \underbrace{[\hbar(G_t - k)]^+}_{\text{Intrinsic value } \equiv\, \mathbb{I}_t[\hbar,k]} + \underbrace{\hbar \int_t^{T_O} \mathbb{1}_{\{\hbar G_\ell > \hbar k\}} dG_\ell}_{\text{Gain/loss process}} + \underbrace{\mathbb{L}_t^{T_O}[k; \langle G \rangle]}_{\text{Local time}}. \tag{17}$$

While equation (17) appears abstract, it conveys a financial interpretation (e.g., Carr and Jarrow (1990)). Consider the case of a put option with $\hbar = -1$.

- First, $\mathbb{I}_t[-1, k] \equiv [-(G_t - k)]^+ = [k - G_t]^+$ is the intrinsic value of the option (scaled by $1/F_t^{T_F}$) at inception. This quantity is zero for out-of-the-money and at-the-money options.
- Next, the term $-\int_t^{T_O} \mathbb{1}_{\{G_\ell < k\}} dG_\ell = -\frac{1}{F_t^{T_F}}(\int_t^{T_O} \mathbb{1}_{\{F_\ell^{T_F} < K\}} dF_\ell^{T_F})$ is a stochastic integral that represents the gains/losses to a dynamic trading strategy which takes a short position of magnitude $\frac{1}{F_t^{T_F}}$ at time ℓ, in the futures, if, and only if, $G_\ell < k$ (i.e., $F_\ell^{T_F} < K$).
- Finally, the quantity $\mathbb{L}_t^{T_O}[k; \langle G \rangle]$ is the *local time* of G at k, which, in turn, is related to the quadratic variation $\langle G \rangle_t$ of (G_t) (e.g., Borodin and Salminen (2015, Chapter 2)).

A. Interpretation of local time and its risk premium. The form of the local time $\mathbb{L}_t^{T_O}[k; \langle G \rangle]$ is

$$\mathbb{L}_t^{T_O}[k; \langle G \rangle] = \frac{1}{2} \int_t^{T_O} \delta[G_\ell - k]\, d\langle G \rangle_\ell, \quad \text{where } \delta[\bullet] \text{ denotes the Dirac delta function.} \tag{18}$$

$\mathbb{L}_t^{T_O}[k;\langle G\rangle]$ is a non-negative and non-decreasing (in $(T_O - t)$) stochastic process.

For an Ito process (G_t) of the form $\frac{dG_t}{G_t} = \tilde{\mu}[t, G_t]\, dt + \tilde{\sigma}[t, G_t]\, d\omega_t$, where ω_t denotes standard Brownian motion and $\tilde{\sigma}[t, G_t]$ is the instantaneous volatility of $\frac{dG_t}{G_t}$, its quadratic variation is

$$\langle G\rangle_u \equiv \int_t^u \{\tilde{\sigma}[\ell, G_\ell]\, G_\ell\}^2\, d\ell, \qquad \text{which reflects integrated variance.} \tag{19}$$

One may view local time as a measure of integrated variance computed when the process (G_t) is *precisely equal* to k, which contrasts with measures of integrated variance computed on, for example, the upside or downside, or unconditionally or with the use of integrated *log* variance (i.e., computed using the volatility of the log of an asset price; see Protter (1990)).

Quadratic variation and local time are sample path properties and do not vary with the probability measures $\mathbb{P}$ or $\mathbb{Q}$. However, their expectations may differ under $\mathbb{P}$ and $\mathbb{Q}$.

Suppose the transition probability density functions for G_t at time t to transition to k at time ℓ, under $\mathbb{P}$ and $\mathbb{Q}$, exist and are given by $p^{\mathbb{P}}[t, \ell; k, G_t]$ and $p^{\mathbb{Q}}[t, \ell; k, G_t]$, respectively. Then, the expectation of the local time can be evaluated as (e.g., Borodin and Salminen (2015, page 21))

$$\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) = \frac{1}{2}\int_t^{T_O} \{\tilde{\sigma}[\ell, G_\ell]G_\ell\}^2\Big|_{G_\ell = k} p^{\mathbb{P}}[t, \ell; k, G_t]d\ell = \frac{1}{2}\int_t^{T_O} \{\tilde{\sigma}[\ell, k]\}^2 k^2 p^{\mathbb{P}}[t, \ell; k, G_t]d\ell, \tag{20}$$

$$\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) = \frac{1}{2}\int_t^{T_O} \{\tilde{\sigma}[\ell, G_\ell]G_\ell\}^2\Big|_{G_\ell = k} p^{\mathbb{Q}}[t, \ell; k, G_t]d\ell = \frac{1}{2}\int_t^{T_O} \{\tilde{\sigma}[\ell, k]\}^2 k^2 p^{\mathbb{Q}}[t, \ell; k, G_t]d\ell. \tag{21}$$

The quantity

$$\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) \;-\; \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) \;\; \text{defines the risk premium on local time.} \tag{22}$$

As we show, studying the Tanaka decomposition under the probability measures $\mathbb{P}$ and $\mathbb{Q}$, and the allied expected excess return of the option on the bond futures enables economic insights. Specifically, the attributes of the risk premium on local time can offer cues to differentiating between interest-rate models, heeding to the notion that all interest-rate claims are priced consistently under the same $\mathbb{Q}$. Our theoretical innovation pertains to the non-zero contribution of the unspanned risks $\boldsymbol{\alpha}[t, \mathbf{X}]'\, d\mathbf{u}_t^{\mathbb{P}}$, which implies market incompleteness and may solicit local time risk premiums. These features are our bridge to aligning theory and empirical evidence from interest-rate claims.

B. Expected option payoff under $\mathbb{P}$. Using Tanaka's formula and taking expectations under the $\mathbb{P}$ measure, the numerator of equation (16) becomes

$$\mathbb{E}_t^{\mathbb{P}}([\hbar(G_{T_O}-k)]^+) = \mathbb{I}_t[\hbar,k] + \mathbb{E}_t^{\mathbb{P}}(\hbar\int_t^{T_O}\mathbb{1}_{\{\hbar G_\ell>\hbar k\}}dG_\ell) + \underbrace{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}_{=\mathbb{E}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}}e^{-\int_t^{T_O}r_\ell d\ell}\mathbb{L}_t^{T_O}[k;\langle G\rangle])} \tag{23}$$

where we recall $\mathbb{I}_t[\hbar,k]=[\hbar(G_t-k)]^+$. In addition, we use a change of measure $d\mathbb{Q}_{t,T_O}\,e^{-\int_t^{T_O}r_\ell d\ell}\frac{M_t}{M_{T_O}} = d\mathbb{P}_{t,T_O}$ applied to $\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])$.

C. Expected discounted option payoff under $\mathbb{Q}$. We recognize, using $\mathbb{E}_t^{\mathbb{Q}}(\tilde{a}\,\tilde{b})=\mathbb{E}_t^{\mathbb{Q}}(\tilde{a})\,\mathbb{E}_t^{\mathbb{Q}}(\tilde{b})+\mathrm{cov}_t^{\mathbb{Q}}(\tilde{a},\tilde{b})$, for random variables $\tilde{a}$ and $\tilde{b}$, that

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O}r_\ell d\ell}\,[\hbar(G_{T_O}-k)]^+) &= B_t^{T_O}\,\mathbb{E}_t^{\mathbb{Q}}([\hbar(G_{T_O}-k)]^+) + B_t^{T_O}\,\mathcal{C}_t[\hbar,k], && (24)\\ \text{where}\;\; \mathcal{C}_t[\hbar,k] &\equiv \mathrm{cov}_t^{\mathbb{Q}}(\frac{1}{B_t^{T_O}}e^{-\int_t^{T_O}r_\ell d\ell},\,[\hbar(G_{T_O}-k)]^+). && (25)\end{aligned}$$

Plugging equation (17) into equation (24), the expected discounted option payoff under $\mathbb{Q}$ is

$$\begin{aligned}&\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O}r_\ell d\ell}\,[\hbar(G_{T_O}-k)]^+)\\ &= B_t^{T_O}\left\{\mathbb{I}_t[\hbar,k] + \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left(\hbar\int_t^{T_O}\mathbb{1}_{\{\hbar G_\ell>\hbar k\}}dG_\ell\right)}_{\equiv 0} + \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) + \mathcal{C}_t[\hbar,k]\right\},\end{aligned} \tag{26}$$

since $F_t^{T_F}$ is a martingale under $\mathbb{Q}$, so the stochastic integral in equation (26) is zero.

2.3.2 Putting the various parts together

The theoretical characterizations presented in equations (23) and (26) are not *yet fully* transparent about the expected (gross) return of options $1+\mu_t^{[\hbar]}$ in equation (16). Our goal here is to compare the expected option payoff under $\mathbb{P}$, with the expected discounted option payoff under $\mathbb{Q}$, and then describe the pattern of expected excess return of options in relation to their strikes, under some assumptions about the spanned and unspanned components of the pricing kernel.

The results to follow are theoretically inspired and pertain to whether the local time risk premium is zero or negative, which, in turn, is related to the unspanned components of the pricing kernel (that is to say, the components which capture risks not spanned by bond or bond futures returns). We further explain these ideas in the context of featured parameterized frameworks in Sections 4.1 and 4.2 and also textbook models in Section 4.3, in order to place our results on a firmer economic footing.

Hereinafter, we consider OTM options with intrinsic value $\mathbb{I}_t[\hbar, k] = 0$.

By a consequence of the $\mathbb{P}$-measure (respectively, $\mathbb{Q}$-measure) expectation of Tanaka's formula in equation (23) (respectively, equation (26)), we determine, from equation (16), that

$$\underbrace{1 + \mu_t^{[\hbar]} - \frac{1}{B_t^{T_O}}}_{\text{Expected excess return of OTM options}} = \frac{1}{B_t^{T_O}}\left(\frac{\mathbb{E}_t^{\mathbb{P}}(\hbar \int_t^{T_O} \mathbb{1}_{\{\hbar G_\ell > \hbar k\}} dG_\ell) + \mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle]) + \mathcal{C}_t[\hbar, k]} - 1\right). \quad (27)$$

Our transformations using Tanaka's formula imply that the expected excess return of holding the option depends on (a) the expected value of the dynamic trading strategy in futures, (b) the magnitude of the covariance $\mathcal{C}_t[\hbar, k]$ under $\mathbb{Q}$, and (c) the relative magnitudes of expected local time under $\mathbb{P}$ and under $\mathbb{Q}$. Instrumental to our theoretical treatment is the sign (and size, to mimic empirical quantities) of the local time risk premium.

The following results on expected excess return of OTM options on bond futures are introduced without parameterizing the evolution of the spot interest-rate or the form of $\boldsymbol{\lambda}[t, \mathbf{X}]$ or $\boldsymbol{\alpha}[t, \mathbf{X}]$. In so doing, we analytically highlight the channel of local time risk premiums in a general setting. Introducing this effect, in turn, requires assessing a covariance under the $\mathbb{Q}$ measure:

$$\mathrm{cov}_t^{\mathbb{Q}}\left(\frac{M_t e^{-\int_t^{T_O} r_\ell d\ell}}{M_{T_O}}, \mathbb{L}_t^{T_O}[k; \langle G \rangle]\right) = \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left(\frac{M_t e^{-\int_t^{T_O} r_\ell d\ell}}{M_{T_O}} \mathbb{L}_t^{T_O}[k; \langle G \rangle]\right)}_{= \mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle])} - \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left(\frac{M_t e^{-\int_t^{T_O} r_\ell d\ell}}{M_{T_O}}\right)}_{= 1, \text{ using eq. (12)}} \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle]). \quad (28)$$

Now we show the following:

Result 1 (Expected excess returns of OTM options on bond futures) *Assume that*

$$\mathbb{E}_t^{\mathbb{P}}(\textstyle\int_t^{T_O} \mathbb{1}_{\{G_\ell > k\}} dG_\ell) \text{ is positive and } \mathbb{E}_t^{\mathbb{P}}(-\textstyle\int_t^{T_O} \mathbb{1}_{\{G_\ell < k\}} dG_\ell) \text{ is negative.} \quad (29)$$

(a) ***(Absence of unspanned risks; market is complete)****. If*

$$\boldsymbol{\alpha}[t, \mathbf{X}] = \mathbf{0}, \text{ for all } t, \qquad \text{(no unspanned risks)} \tag{30}$$

then $\operatorname{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}, \mathbb{L}_t^{T_O}[k; \langle G \rangle]) = 0$*, and the local time risk premium is zero. Furthermore, the expected excess return of an OTM put (call) option on bond futures is negative (positive), when it holds that* $\mathcal{C}_t[\hbar, k]$ *is negligible in the sense that* $\frac{\mathcal{C}_t[\hbar,k]}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G \rangle])} \approx 0$.

(b) ***(With spanned and unspanned risks; market is incomplete)****. Suppose, for all* t*,*

$$\boldsymbol{\alpha}[t, \mathbf{X}] \neq \mathbf{0} \quad \text{and} \quad \operatorname{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}, \mathbb{L}_t^{T_O}[k; \langle G \rangle]) \; < \; 0. \tag{31}$$

The local time risk premium is negative. In this case, the expected excess return of OTM puts and calls can both *be negative.*

Proof: See Appendix A. ■

The assumption in equation (29), used in the proof of Result 1, strengthens the idea of a positive bond futures risk premium, namely, $\mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O} dG_\ell) > 0$, to $\mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O} \mathbb{1}_{\{G_\ell > k\}} dG_\ell) > 0$ and $\mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O} \mathbb{1}_{\{G_\ell < k\}} dG_\ell) > 0$. Notably, the term $\hbar \int_t^{T_O} \mathbb{1}_{\{\hbar G_\ell > \hbar k\}} dG_\ell$ relates to risks spanned by bond and bond futures returns. If all risks were to be spanned by bond and bond futures returns; that is, if $\boldsymbol{\alpha}[t, \mathbf{X}] = \mathbf{0}$, then $\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle]) = \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle])$ and the expected excess return to holding an option would, in essence, be determined by the sign of the bond futures risk premium.

We underscore that $\boldsymbol{\alpha}[t, \mathbf{X}] = \mathbf{0}$ is a sufficient condition to prove part (a) of Result 1 but it is not necessary. If $\boldsymbol{\alpha}[t, \mathbf{X}] \neq \mathbf{0}$ (so the market is, in fact, incomplete) but the covariance $\operatorname{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}, \mathbb{L}_t^{T_O}[k; \langle G \rangle])$ equals zero, part (a) of Result 1 would still hold.

The big picture is that if there are no unspanned risks present in the pricing kernel process, then the expected excess return of puts and calls cannot *both* be negative.

Our analysis in part (b) of Result 1 incorporates an explicit role for market incompleteness. In order to align the expected excess return to holding an option on bond futures with empirical realities (reported in Section 3), a necessary condition is that there are unspanned risks (i.e., $\boldsymbol{\alpha}[t, \mathbf{X}] \neq \mathbf{0}$) in the pricing kernel process and a negative local time risk premium.

One key implication of Result 1 is that, in order to generate a non-zero local time risk premium, it is not sufficient to feature time-varying volatility of bond futures returns (as in, for example, the model of Cox, Ingersoll, and Ross (1985)). Crucially, the evolution of the return volatility should incorporate components driven by $d\mathbf{u}_t^{\mathbb{P}}$, and the local time should exhibit a non-zero (to match empirics, specifically, a negative) correlation with the unspanned component of the pricing kernel. Emphasizing our treatment of the problem in Sections 4.1 and 4.2, we consider models with market incompleteness and volatility uncertainty, and then develop, in particular, the analytical restrictions that uncover negative local time risk premiums.

It is provable (but omitted) that the risk premium on local time, depicted as a covariance under the $\mathbb{Q}$ measure in equation (28), is also interpretable as a $\mathbb{P}$ measure covariance. Specifically, $\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) - \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) = -\frac{1}{B_t^{T_O}}\mathrm{cov}_t^{\mathbb{P}}(\frac{M_{T_O}}{M_t}, \mathbb{L}_t^{T_O}[k;\langle G\rangle])$. We focus on the $\mathbb{Q}$ measure covariance for two reasons. First, we are interested in identifying and measuring the local time risk premiums through the empirical counterpart to the expected excess return of an options portfolio (our Result 2). Second, we draw the link to the unspanned components of the pricing kernel.

What about the plausibility of the small quantitative effect of the covariance $\mathcal{C}_t[\hbar, k]$; that is, $\mathcal{C}_t[\hbar, k]$ is negligible. By the Cauchy-Schwarz inequality, $|\mathcal{C}_t[\hbar, k]| \leq \{\mathrm{var}_t^{\mathbb{Q}}(\frac{1}{B_t^{T_O}} e^{-\int_t^{T_O} r_\ell d\ell})\, \mathrm{var}_t^{\mathbb{Q}}((G_{T_O} - k)\mathbb{1}_{\{\hbar G_{T_O} > \hbar k\}})\}^{1/2}$. The rationale for a very small $|\mathcal{C}_t[\hbar, k]|$ is that short-dated bond return volatilities (relevant for one month options) are low. We give further support to the notion that $\mathcal{C}_t[\hbar, k]$ is negligible in Bakshi, Crosby, and Gao (2019, Section I, Table Appendix-I).

The dynamics, in equations (2)–(5), assume that bond prices and bond futures prices have continuous sample paths (although the state variables $\mathbf{X}$ may have jumps), but they are sufficiently versatile so that the initial term-structure of interest-rates can be matched (as also noted in the analyses of Heath, Jarrow, and Morton (1992)). Thus, our dynamics and the expectations under $\mathbb{P}$ of $[\hbar(G_{T_O} - k)]^+$ and under $\mathbb{Q}$ of $e^{-\int_t^{T_O} r_\ell d\ell}\, [\hbar(G_{T_O} - k)]^+$, using Tanaka's formula, encompass many of those considered in the literature. However, supporting our empirical findings requires market incompleteness generating negative local time risk premiums.

Our remaining work involves articulating the information content of options data on Treasury bond futures for models, and the evidence that the average returns of OTM puts and calls on the Treasury bond futures are negative. We derive a theoretical result (Result 2) that links expected straddle returns to the local time risk premium when the strike price equals the current futures price, and study its testable implications. Finally (in Section 4), we explore models in which the interplay between dynamics under $\mathbb{P}$ and those under $\mathbb{Q}$ is such that as to achieve qualitative consistency with the average returns of OTM options on bond futures. In so doing, our focus is on the properties of local time and unspanned components.

3 The empirical puzzle in the Treasury market

The empirical finding highlighted in this section is that

$$\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O} r_\ell d\ell}[K - F_{T_O}^{T_F}]^+) \text{ dominates } \mathbb{E}_t^{\mathbb{P}}([K - F_{T_O}^{T_F}]^+) \text{ for } K < F_t^{T_F}, \quad \text{and} \tag{32}$$

$$\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O} r_\ell d\ell}[F_{T_O}^{T_F} - K]^+) \text{ dominates } \mathbb{E}_t^{\mathbb{P}}([F_{T_O}^{T_F} - K]^+) \text{ for } K > F_t^{T_F}. \tag{33}$$

This outcome is consistent with average returns of options on bond futures being negative for both OTM puts and calls. Moreover, this effect is more pronounced for deeper OTM options. We provide support for these empirical findings using options data on futures of both 10- and 30-year Treasury bonds. Our findings can be considered puzzling because many workhorse models (as explained in Section 4.3) manifest the property that the expected excess return to holding OTM puts and calls on bond futures cannot both be negative.

An economic interpretation of these featured findings is that certain investors, such as those who are leveraged or have floating-rate debt, have an incentive to protect their exposures to increases in interest-rates, which coincide with adverse movements in the value of bond portfolios. On the other hand, certain other investors protect their exposures to declining interest-rates. Thus, investors are averse to both interest-rate increases and interest-rate decreases, resulting in negative average returns of OTM puts and calls. Our results pertaining to expected excess return of OTM options on *bond futures* are new and not manifested in the theoretical work on options on Treasury *bonds*.[1]

[1] The papers that develop bond option formulas include, among others, Cox, Ingersoll, and Ross (1985), Jamshidian (1989), Turnbull and Milne (1991), Chen and Scott (1992), Heath, Jarrow, and Morton (1992), Longstaff and Schwartz (1992), Das and Foresi (1996), Attari (1999), Buhler, Uhrig-Homburg, Walter, and Webber (1999), Chacko and Das (2002), Carr, Gabaix, and Wu (2009), Heidari and Wu (2009), Trolle and Schwartz (2009), and Almeida, Graveline, and Joslin (2011).

A. Data on options on the Treasury bond futures. Our investigation focuses on options on the 10-year and 30-year Treasury bond futures over expiration cycles. These option contracts — which are of American style — are actively traded compared to those on 5- and 2-year Treasury bond futures. We collect the daily data of options from the CME group, which includes the strike price, the settlement price of options, the settlement price of futures, and the remaining maturity. At the beginning of each expiration cycle, we select OTM put and call options, where OTM puts are associated with moneyness $\log \frac{K}{F_t^{T_F}} = \log k < 0$, whereas OTM calls are associated with moneyness $\log \frac{F_t^{T_F}}{K} = -\log k < 0$.[2] Table 1 displays the average dollar open interest and dollar trading volume for OTM options on Treasury bond futures. Our comparison with S&P 500 equity index options indicates that options on Treasury bond futures are liquid.

Next, we keep the shortest maturity options, which usually expire on the last Friday at least two business days from the last business day of the next month. These shortest maturity options have an average maturity of 27.3 days.

In our calculations, we choose one put (call) where $\log \frac{K}{F_t^{T_F}}$ ($\log \frac{F_t^{T_F}}{K}$) is closest to 1%, 3%, and 5% OTM, respectively. To maintain the expiration cycle returns, the option and futures sample starts on 7/22/1991 (respectively, 12/24/1990) for futures on 10-year (30-year) Treasury bond and ends on 12/24/2018, for a total of 330 (337) expiration cycles.

B. Futures risk premium on the 10- and 30-year Treasury bonds is reliably positive. Key to our theoretical characterizations is the notion of a positive risk premium on a long futures position. The excess return of a long position (fully collateralized) in the futures contract is

$$z_{t,T_O}^{\text{futures}} = \frac{F_{T_O}^{T_F}}{F_t^{T_F}} - 1, \quad \text{for futures on the 10-year and 30-year Treasury bonds,} \tag{34}$$

where $F_t^{T_F}$ and $F_{T_O}^{T_F}$ are, respectively, the futures price observed at the start and end of the options expiration cycle.

Table 2 shows that the average excess return corresponding to the 10-year (30-year) futures contract is 3.0% (3.5%) annualized, with a standard deviation of 5.3% (8.7%). To assess statistical significance, we jointly bootstrap (via an i.i.d bootstrap) the returns of 10-year and 30-year futures.

[2]The study of Flesaker (1993) indicates that the impact of the early exercise premium on option prices in the fixed-income market is small, and we are mindful that this may be especially true for OTM options.

With 100,000 bootstrap draws, we obtain 90% confidence intervals, for the average return (in %) of [1.3 4.7] and [0.7 6.1], respectively, which do not intersect zero. Thus, a distinctive attribute of the Treasury market is that the average return of bond futures is positive and statistically significant.

C. Pattern of average returns of puts and calls on bond futures across strikes. We compute the time-series of option (net) returns of fixed moneyness as

$$z^{\text{put}}_{t,T_O,\aleph\%} = \frac{[K - F^{T_F}_{T_O}]^+}{P_t[K]} - 1, \text{ where } K \text{ corresponds to } K = F^{T_F}_t e^{-\aleph\%}, \ \aleph = 1\%, 3\%, \text{ and } 5\%,$$
$$z^{\text{call}}_{t,T_O,\aleph\%} = \frac{[F^{T_F}_{T_O} - K]^+}{C_t[K]} - 1, \text{ where } K \text{ corresponds to } K = F^{T_F}_t e^{+\aleph\%}, \ \aleph = 1\%, 3\%, \text{ and } 5\%,$$

and $P_t[K]$ ($C_t[K]$) is the *settlement* price of a put (call) on the Treasury bond futures with strike K, as reported by the CME. The bid and ask option prices are not reported separately.

Table 3 reports the average returns of put and call options. Focusing on the results from options on the futures on 10-year Treasury bonds, the first observation is that the 5% OTM put returns and the 5% OTM call returns are both highly negative. In particular, the 5% OTM puts generate an average of -93% and the 5% OTM calls an average of -91% (over 27.3 days, not annualized).

The second observation is that the average returns become less negative as the moneyness declines for both puts and calls. For example, the average returns of the 3% OTM puts are -71% and increase (i.e., become less negative) to -41% for the 1% OTM puts. Similarly, the average returns of the 3% OTM calls are -76% and increase to -11% for the 1% OTM calls.

Note further from Table 3 (Panel B) that the pattern of average returns across moneyness is preserved with options on the futures on 30-year Treasury bonds.

Reported also are the 90% bootstrap confidence intervals on the average option returns. Our procedure is to jointly bootstrap all option returns with replacement. The individual average option returns are generally statistically significant and negative, robustly so for puts. This can be gauged by 11 out of 14 bootstrap confidence intervals on average option returns that do not bracket zero.

Finally, options on the 10-year Treasury bond futures deliver more negative average returns than their 30-year counterparts, which can be attributable to the shorter bond duration for the same fixed moneyness. In other words, the 10-year bond futures are written on an asset that is less sensitive to interest-rate fluctuations. Intuitively, the lower volatility of the underlying asset translates into fewer extreme returns on the option, reflecting a $\mathbb{P}$-measure property that a model should seek compatibility with unconditionally.

D. Bootstrap evidence on adjacent strikes. Table 4 tests the average return differences between adjacent strikes for both puts and calls. The hypothesis is that (analogously for calls)

$$\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0, \text{ or, } \overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,3\%} \geq 0, \text{ or, } \overline{z}^{\text{put}}_{t,T_O,3\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0. \tag{35}$$

We conduct a bootstrap (with 100,000 draws) and compute the frequency, for instance, $\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0$, and report it as empirical p-values. Low p-values imply rejection. Table 4 shows that our evidence is consistent with the feature that deeper OTM options exhibit more negative average returns, and this pattern is stronger for options on the 10-year futures. This pattern also holds for options on the futures of the 30-year bond, with the exception that the 3% OTM call is statistically indistinguishable from the 1% OTM call.

Overall, our evidence suggests that the average returns of deeper OTM options on bond futures tend to be more negative. The average returns are also increasing (i.e., less negative) as one moves from strikes deeper OTM to near-money.

E. Tackling the negative risk premium on local time. We develop the following result in the context of our theory with spanned and unspanned components of the pricing kernel (in equation (2)). This result holds without making parametric assumptions about the spot interest-rate process or the form of $\boldsymbol{\lambda}[t,\mathbf{X}]$ or $\boldsymbol{\alpha}[t,\mathbf{X}]$.

Result 2 *The percentage risk premium on local time when $k = \frac{K}{F_t^{T_F}} = 1$ can be inferred from the expected excess return of straddles on bond futures as*

$$\overbrace{\left.\frac{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}\right|_{k=1} - 1}^{\text{Percentage risk premium on local time}} \approx B_t^{T_O} \overbrace{\left(\frac{\mathbb{E}_t^{\mathbb{P}}([F_{T_O}^{T_F} - F_t^{T_F}]^+ + [F_t^{T_F} - F_{T_O}^{T_F}]^+)}{C_t[F_t^{T_F}] + P_t[F_t^{T_F}]} - \frac{1}{B_t^{T_O}}\right)}^{\text{Expected excess return of straddles}}. \tag{36}$$

Proof: Equation (36) follows by summing Tanaka's formula applied to $[G_{T_O} - 1]^+$ and $[1 - G_{T_O}]^+$. See Appendix B. ■

We assess the local time risk premium for $k = 1$ by constructing the returns of long straddles. In implementation, we search for a put and a call with strike $K \approx F_t^{T_F}$. Table 3 shows that average return of at-the-money straddles of -11% (again, *not* annualized) when the underlying is the futures on the 10-year bond and -9% when it is the futures on the 30-year bond. Thus, equation (36) implies that the (percentage) local time risk premium, when $k = 1$, for futures on the 10-year (30-year) bond is -11% (-9%), and is reliably negative. The unambiguously negative local time risk premium imputed from options brings in an additional piece of evidence that reinforces our theoretical angles.

F. Summary and interpretation. Our documented patterns constitute a yardstick for interest-rate models to consider and match, since models should be consistent with both $\mathbb{P}$-measure and $\mathbb{Q}$-measure cumulative distribution functions. One may envision our empirical results as a form of discrepancy in expectations that looks to align the theory with the data. As such, it imposes a set of restrictions on their joint behavior (i.e., the risk premium on local time) instead of enforcing consistency with one probability measure (say, $\mathbb{P}$) without regard for the other (say, $\mathbb{Q}$). Having data on state-contingent contracts from the Treasury markets is crucial to articulating this disparity and is as fundamental to macro-finance models as state-contingent contracts on the equity market index. Our evidence indicates that investors perceive interest-rate movements to both the downside and the upside as unpalatable.

4 What modeling features are consistent with the empirical data?

We now focus on exploring model designs that can be consistent with the data on average returns of options on Treasury bond futures. Encapsulated within these model classes is market incompleteness combined with sources of volatility uncertainty and negative risk premiums on local time.

4.1 An economy in which bonds do not span local time

In what follows, we partition the vector of state variables into two sets: $\mathbf{X}_t \equiv [\mathbf{Z}_t' \ \mathbf{U}_t']'$, where $\mathbf{Z}_t$ (respectively, $\mathbf{U}_t$) are spanned (unspanned) by bond returns.

Let t_0 be some arbitrary initial date. Define the quantity

$$M_t^{\text{unspanned}} \equiv \exp\Big(\int_{t_0}^{t} -\frac{1}{2}\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]'\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]d\ell \;+\; \boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]'\,d\mathbf{u}_\ell^{\mathbb{P}}\Big). \tag{37}$$

Thus, for $T \geq t$, we have $\frac{M_T^{\text{unspanned}}}{M_t^{\text{unspanned}}} = \exp(\int_t^T -\frac{1}{2}\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]'\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]d\ell + \boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]'\,d\mathbf{u}_\ell^{\mathbb{P}})$.

Under our theoretical setup, $M_t^{\text{unspanned}}$ is a martingale (under $\mathbb{P}$) satisfying

$$\mathbb{E}_t^{\mathbb{P}}(M_T^{\text{unspanned}}) \;=\; M_t^{\text{unspanned}}. \tag{38}$$

Next, we construct the pricing kernel M_t as follows:

$$M_t \;=\; \overbrace{e^{-\nu t}\,(\phi+\boldsymbol{\psi}'\mathbf{Z}_t)}^{\text{as in Filipovic, Larsson, and Trolle (2017)}} \times M_t^{\text{unspanned}}, \quad \text{with} \tag{39}$$

$$d\,Z_{i,t} \;=\; (\boldsymbol{\kappa}_Z(\overline{\mathbf{Z}}-\mathbf{Z}_t))_i\;dt \;+\; \sigma_{i,Z}[\mathbf{Z}_t,\mathbf{U}_t]\;\underbrace{d\omega_{i,t}^{\mathbb{P}}}_{\text{spanned}}, \qquad \text{for} \;\; i=1,\ldots,N. \tag{40}$$

The representation of the pricing kernel in equation (39) complements Filipovic, Larsson, and Trolle (2017), who assume $M_t^{\text{unspanned}} \equiv 1$ (i.e., that $\boldsymbol{\alpha}[t,\mathbf{Z},\mathbf{U}] \equiv 0$). There are N independent spanned state variables, denoted by $\mathbf{Z}_t$, which are driven by $\boldsymbol{\omega}_t^{\mathbb{P}}$ (representing spanned risks). Furthermore, $\boldsymbol{\kappa}_Z$ is a N-dimensional square matrix, and $\overline{Z}_i > 0$ and ψ_i, for $i = 1,\ldots,N$, ν, and ϕ are constant scalars, with ψ_i and ϕ chosen to ensure $\phi + \boldsymbol{\psi}'\mathbf{Z}_t > 0$, almost surely, for all t.

Our approach in equation (39) is motivated by the feature that $\boldsymbol{\alpha}[t,\mathbf{Z},\mathbf{U}]'\,d\mathbf{u}_t^{\mathbb{P}} = \mathbf{0}$ is a sufficient condition for a zero local time risk premium (see Result 1). We formalize $M_t^{\text{unspanned}}$ to target consistency with the observed pattern of average returns of options on bond futures.

The implication of equations (39)–(40) is that zero-coupon bond prices are given by

$$B_t^T \;=\; \mathbb{E}_t^{\mathbb{P}}\Big(\frac{M_T}{M_t}\Big) \;=\; \frac{e^{-\nu T}}{e^{-\nu t}}\,\mathbb{E}_t^{\mathbb{P}}\Big(\frac{\phi+\boldsymbol{\psi}'\mathbf{Z}_T}{\phi+\boldsymbol{\psi}'\mathbf{Z}_t}\Big)\times\mathbb{E}_t^{\mathbb{P}}\Big(\frac{M_T^{\text{unspanned}}}{M_t^{\text{unspanned}}}\Big) \quad \text{(by independence and (38))} \tag{41}$$

$$\;=\; \mathfrak{B}(T-t,\mathbf{Z}_t), \quad \text{where} \;\; \mathfrak{B}:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R} \;\; \text{satisfies} \tag{42}$$

$$\mathfrak{B}(T-t,\mathbf{Z}_t) \;\equiv\; e^{-\nu(T-t)}\;\Big(\frac{\phi+\boldsymbol{\psi}'\overline{\mathbf{Z}}+\boldsymbol{\psi}'(e^{-\boldsymbol{\kappa}_Z(T-t)}(\mathbf{Z}_t-\overline{\mathbf{Z}}))}{\phi+\boldsymbol{\psi}'\mathbf{Z}_t}\Big), \tag{43}$$

and depend upon $\mathbf{Z}_t$ but not $\mathbf{U}_t$.

In essence, we preserve the Filipovic, Larsson, and Trolle (2017) specification of the spot interest-rate and of $\boldsymbol{\lambda}[t, \mathbf{Z}, \mathbf{U}]$ but we additionally incorporate unspanned risks (i.e, allow $\boldsymbol{\alpha}[t, \mathbf{Z}, \mathbf{U}]'\, d\mathbf{u}_t^{\mathbb{P}} \neq \mathbf{0}$) in the pricing kernel dynamics. Ito's lemma applied to (39) reveals

$$\frac{dM_t}{M_t} \;=\; -r_t\, dt \;+\; \boldsymbol{\lambda}[t, \mathbf{Z}, \mathbf{U}]'\, d\boldsymbol{\omega}_t^{\mathbb{P}} \;+\; \underbrace{\boldsymbol{\alpha}[t, \mathbf{Z}, \mathbf{U}]'\, d\mathbf{u}_t^{\mathbb{P}},}_{\text{term not in Filipovic, Larsson, and Trolle}} \qquad \text{with} \tag{44}$$

$$r_t \;=\; \nu - \frac{\boldsymbol{\psi}'\boldsymbol{\kappa}_Z(\overline{\mathbf{Z}} - \mathbf{Z}_t)}{\phi + \boldsymbol{\psi}'\mathbf{Z}_t} \quad \text{and} \quad \boldsymbol{\lambda}[t, \mathbf{Z}, \mathbf{U}] \;=\; \frac{\sum_{i=1}^{N} \sigma_{i,Z}[\mathbf{Z}_t, \mathbf{U}_t]\psi_i}{\phi + \boldsymbol{\psi}'\mathbf{Z}_t}. \tag{45}$$

We note that ($\bigtriangledown_{\mathbf{Z}}$ denotes partial derivatives)

$$\frac{dB_t^T}{B_t^T} \;=\; (r_t + \boldsymbol{\lambda}[t, \mathbf{Z}, \mathbf{U}]'\boldsymbol{\sigma}_B[t, T_B, \mathbf{Z}, \mathbf{U}])dt \;-\; \boldsymbol{\sigma}_B[t, T, \mathbf{Z}, \mathbf{U}]'d\boldsymbol{\omega}_t^{\mathbb{P}}, \qquad \text{where} \tag{46}$$

$$\sigma_{B,i}[t, T, \mathbf{Z}, \mathbf{U}] \;=\; -\,\sigma_{i,Z}[\mathbf{Z}, \mathbf{U}]\,\Big(\frac{\bigtriangledown_{\mathbf{Z}}\mathfrak{B}(T-t, \mathbf{Z})}{\mathfrak{B}(T-t, \mathbf{Z})}\Big)_i. \tag{47}$$

We will specify a form of $\sigma_{i,Z}[\mathbf{Z}_t, \mathbf{U}_t]$ which is tractable and desirable from economic standpoints.

Under the constructions in equation (44), with a form of incompleteness due to $\boldsymbol{\alpha}[t, \mathbf{Z}, \mathbf{U}]'\, d\mathbf{u}_t^{\mathbb{P}} \neq \mathbf{0}$, our value-added is to develop the restrictions on $\boldsymbol{\alpha}[t, \mathbf{Z}, \mathbf{U}]$ such that (i) the local time risk premium is negative, and (ii) the expected excess returns of OTM puts and calls on bond futures can both be negative.

Working toward our goals, first, we specify that the M unspanned state variables $U_{i,t}$ follow

$$d\,U_{i,t} = \kappa_i^{\mathbb{P}}(\overline{U}_i - U_{i,t})\,dt + \sigma_{i,U}\,\sqrt{U_{i,t}}\;\; \underbrace{du_{i,t}^{\mathbb{P}}}_{\text{unspanned}}\;, \quad \text{for} \;\; i = 1, \ldots, M, \qquad \text{with} \;\; M \leq N, \tag{48}$$

where $\overline{U}_i > 0$, $\kappa_i^{\mathbb{P}} > 0$, and $\sigma_{i,U} > 0$ are constants.

Completing the picture, second, we consider $\sigma_{i,Z}[\mathbf{Z}_t, \mathbf{U}_t]$, in equation (40), for $i = 1, \ldots, N$, to be of the form

$$\sigma_{i,Z}[\mathbf{Z}_t, \mathbf{U}_t] \;=\; \sqrt{\zeta_i Z_{i,t} + \beta_i U_{i,t}}, \qquad \text{(i.e., shaped by both spanned and unspanned variables)} \tag{49}$$

for constant scalars $\zeta_i \geq 0$ and $\beta_i \geq 0$, with at least one inequality for β_i strict, and $\beta_i = 0$ if $i > M$.

Consider next the futures contract and its dynamics: $\frac{dF_t^{T_F}}{F_t^{T_F}} = \boldsymbol{\lambda}[t, \mathbf{Z}, \mathbf{U}]' \boldsymbol{\sigma}_F[t, T_F, \mathbf{Z}, \mathbf{U}]\, dt - \boldsymbol{\sigma}_F[t, T_F, \mathbf{Z}, \mathbf{U}]' d\boldsymbol{\omega}_t^{\mathbb{P}}$. By Ito's lemma, equation (47), and equation (11), the volatility of futures returns $\boldsymbol{\sigma}_F[t, T_F, \mathbf{Z}, \mathbf{U}]$, needed for calculating $\mathbb{L}_t^{T_O}[k; \langle G \rangle]$, takes the form

$$\sigma_{F,i}[t, T_F, \mathbf{Z}, \mathbf{U}] \;=\; \sigma_{B,i}[t, T_B, \mathbf{Z}, \mathbf{U}] - \sigma_{B,i}[t, T_F, \mathbf{Z}, \mathbf{U}] \;\equiv\; \mho_{i,t} \sqrt{\zeta_i Z_{i,t} + \beta_i U_{i,t}}, \tag{50}$$

$$\text{where} \qquad \mho_{i,t} \;\equiv\; \Big(\frac{\nabla_{\mathbf{Z}} \mathfrak{B}(T_F - t, \mathbf{Z}_t)}{\mathfrak{B}(T_F - t, \mathbf{Z}_t)} - \frac{\nabla_{\mathbf{Z}} \mathfrak{B}(T_B - t, \mathbf{Z}_t)}{\mathfrak{B}(T_B - t, \mathbf{Z}_t)} \Big)_i. \tag{51}$$

Here, $\mho_{i,t}$ depends on T_B, T_F, and $\mathbf{Z}_t$ but we suppress the dependence, for brevity.

As a consequence, the dynamics of (G_s) under $\mathbb{Q}$ are

$$\frac{dG_t}{G_t} \;=\; -\sum_{i=1}^{N} \sigma_{F,i}[t, T_F, \mathbf{Z}, \mathbf{U}]\, d\omega_{i,t}^{\mathbb{Q}} \;=\; -\sum_{i=1}^{N} \sqrt{\zeta_i Z_{i,t} + \beta_i U_{i,t}}\, \mho_{i,t}\, \underbrace{d\omega_{i,t}^{\mathbb{Q}}}_{\text{spanned}}. \tag{52}$$

Pertinent to our theory, and for tractability and sparsity of specification, we set

$$\alpha_i[t, \mathbf{U}] \;=\; b_i \sqrt{U_{i,t}}, \quad \text{for each } i = 1, \ldots, M, \quad \text{where } b_1, \ldots, b_M \text{ are constants.} \tag{53}$$

Then, by Girsanov's theorem, for $i = 1, \ldots, M$, we obtain the $U_{i,t}$ dynamics under $\mathbb{Q}$ as follows

$$d\,U_{i,t} = \kappa_i^{\mathbb{Q}} (\overline{U}_i - U_{i,t})\, dt \;+\; \sigma_{i,U} \sqrt{U_{i,t}}\, du_{i,t}^{\mathbb{Q}}, \qquad \text{where} \quad \kappa_i^{\mathbb{Q}} \equiv \kappa_i^{\mathbb{P}} - b_i\, \sigma_{i,U}. \tag{54}$$

With spanned and unspanned components in the pricing kernel in equation (39), we now proceed to the specification of local time. By equation (18), we determine $\mathbb{L}_t^{T_O}[k; \langle G \rangle] = \frac{1}{2} \int_t^{T_O} \sum_{i=1}^{N} \delta[G_\ell - k]\, (\zeta_i Z_{i,\ell} + \beta_i U_{i,\ell})\, \mho_{i,\ell}^2\, G_\ell^2\, d\ell$.

We are motivated by part (b) of Result 1. For tractability, we consider the sign of the $\mathbb{Q}$ covariance $\operatorname{cov}_t^{\mathbb{Q}}(\log(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}), \mathbb{L}_t^{T_O}[k; \langle G \rangle])$. This is sufficient, by Ito's lemma, as our theory hinges on the *sign* of $\operatorname{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}, \mathbb{L}_t^{T_O}[k; \langle G \rangle])$. Because $\log(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}) = \int_t^{T_O} \{ -\frac{1}{2} \boldsymbol{\lambda}[s, \mathbf{Z}, \mathbf{U}]' \boldsymbol{\lambda}[s, \mathbf{Z}, \mathbf{U}] ds - \boldsymbol{\lambda}[s, \mathbf{Z}, \mathbf{U}]'\, d\boldsymbol{\omega}_s^{\mathbb{Q}} - \frac{1}{2} \boldsymbol{\alpha}[s, \mathbf{U}]' \boldsymbol{\alpha}[s, \mathbf{U}] ds - \boldsymbol{\alpha}[s, \mathbf{U}]'\, d\mathbf{u}_s^{\mathbb{Q}} \}$, we condition on $\mathcal{I}_s$, the sub-filtration of $\mathcal{F}_s$ generated by $\boldsymbol{\omega}_s^{\mathbb{Q}}$ and $\boldsymbol{\lambda}[s, \mathbf{Z}, \mathbf{U}]' \, d\boldsymbol{\omega}_s^{\mathbb{Q}}$ (see the proof of Result 1). Then,

$$
\begin{aligned}
&\operatorname{cov}_t^{\mathbb{Q}}\Big(\int_t^{T_O} -\boldsymbol{\alpha}[s,\mathbf{U}]' \, d\mathbf{u}_s^{\mathbb{Q}},\ \frac{1}{2}\int_t^{T_O} \sum_{i=1}^{N} \delta[G_\ell - k](\zeta_i Z_{i,\ell} + \beta_i U_{i,\ell})\, \mho_{i,\ell}^2\, G_\ell^2\, d\ell \ \ |\, \mathcal{I}_{T_O}\Big) \\
&= \operatorname{cov}_t^{\mathbb{Q}}\Big(\int_t^{T_O} -\boldsymbol{\alpha}[s,\mathbf{U}]' \, d\mathbf{u}_s^{\mathbb{Q}}, \frac{1}{2}\int_t^{T_O} \sum_{i=1}^{M} \int_t^{\ell} \beta_i\, \sigma_{i,U}\, e^{\kappa_i^{\mathbb{Q}}(s-\ell)} \delta[G_\ell - k]\sqrt{U_{i,s}}\, du_{i,s}^{\mathbb{Q}}\, \mho_{i,\ell}^2\, G_\ell^2\, d\ell \ |\, \mathcal{I}_{T_O}\Big) \\
&= \operatorname{cov}_t^{\mathbb{Q}}\Big(\int_t^{T_O} -\boldsymbol{\alpha}[s,\mathbf{U}]' \, d\mathbf{u}_s^{\mathbb{Q}},\ \int_t^{T_O} \sum_{i=1}^{M} \sqrt{U_{i,s}}\, \{ \int_s^{T_O} \frac{\beta_i \sigma_{i,U}}{2} e^{\kappa_i^{\mathbb{Q}}(s-\ell)} \delta[G_\ell - k] \mho_{i,\ell}^2\, G_\ell^2\, d\ell\}\, du_{i,s}^{\mathbb{Q}} \ |\, \mathcal{I}_{T_O}\Big) \\
&= -\mathbb{E}_t^{\mathbb{Q}}\Big(\sum_{i=1}^{M} \int_t^{T_O} b_i \sqrt{U_{i,s}}\ \sqrt{U_{i,s}}\ \{ \int_s^{T_O} \frac{\beta_i \sigma_{i,U}}{2} e^{\kappa_i^{\mathbb{Q}}(s-\ell)} \delta[G_\ell - k]\, \mho_{i,\ell}^2\, G_\ell^2\, d\ell\}\, ds \ |\, \mathcal{I}_{T_O}\Big) \\
&= -\mathbb{E}_t^{\mathbb{Q}}\Big(\sum_{i=1}^{M} \int_t^{T_O} b_i U_{i,s}\ \{ \int_s^{T_O} \frac{\beta_i \sigma_{i,U}}{2} e^{\kappa_i^{\mathbb{Q}}(s-\ell)}\, \delta[G_\ell - k]\, \mho_{i,\ell}^2\, G_\ell^2\, d\ell\ \}\ ds \ |\, \mathcal{I}_{T_O}\Big).
\end{aligned} \tag{55}
$$

We unpack the steps above as follows. In going from the first line to the second, we recognize that the term $\zeta_i Z_{i,\ell}$ is irrelevant to the covariance, and, from the second to the third, we used equation (54) for the evolution of $U_{i,\ell}$ (under $\mathbb{Q}$); that is, for $\ell \geq t$, $U_{i,\ell} = U_{i,t}\, e^{\kappa_i^{\mathbb{Q}}(t-\ell)} + \int_t^{\ell} \overline{U}_i \kappa_i^{\mathbb{Q}}\, e^{\kappa_i^{\mathbb{Q}}(s-\ell)} ds + \int_t^{\ell} \sigma_{i,U}\ e^{\kappa_i^{\mathbb{Q}}(s-\ell)} \sqrt{U_{i,s}}\, du_{i,s}^{\mathbb{Q}}$. Then, we performed a change of the order of integration.

We make three observations. First, considering equation (55), from the perspective of part (b) of Result 1, we note that the integral in curly brackets is positive. Second,

$$
\text{if } b_i \geq 0, \text{and at least one inequality is strict, then } \mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) - \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) < 0. \tag{56}
$$

Hence, (by part (b) of Result 1) negative expected excess returns to both OTM puts and calls on bond futures are possible.

If $\boldsymbol{\alpha}[t, \mathbf{U}]$ were to be zero; that is, if $b_i = 0$, for all $i = 1, \ldots, M$, then the $\mathbb{Q}$-measure covariance in equation (55) is zero. As a consequence, $\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) - \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) = 0$, and there would be zero risk premium on local time, contradicting our empirical evidence (e.g., negative average straddle returns). Thus, our result on local time risk premiums relies on a particular parametrization of $\alpha_i[t, \mathbf{U}] \neq 0$ in equation (53), which implies market incompleteness.

What is the intuition that a certain class of depicted models can characterize the observed patterns to holding OTM options on bond futures? The necessary condition, broadly speaking, for empirical viability, is that the volatility of bond futures returns depends upon state variables whose evolution is impacted by the standard Brownian motions $\mathbf{u}_t^{\mathbb{P}}$. Further, the pricing kernel dynamics embed $\boldsymbol{\alpha}[t, \mathbf{X}]' \, d\mathbf{u}_t^{\mathbb{P}}$, which captures risks not spanned by bond or bond futures returns. Moreover, $-\boldsymbol{\alpha}[t, \mathbf{X}]' \, d\mathbf{u}_t^{\mathbb{Q}}$, and, thus $\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}$, is seen negatively correlated with local time under the $\mathbb{Q}$ measure, a feature that results in negative local time risk premiums. In essence, desirable models are at the intersection of those that adequately fit the cross-section of interest-rate yields and volatilities and those that synthesize negative average returns of options on Treasury bond futures.

4.2 Mechanics of another economy in which local time is not spanned by bonds

Our idea throughout is that the workings of incomplete markets, combined with particularly parameterized volatility uncertainty, can result in empirically plausible economies.

Fundamental to the model of Collin-Dufresne and Goldstein (2002, Proposition 8) are the dynamics of the spot interest-rate r_t, the long-run mean interest-rate θ_t, and stochastic variance, v_t. Their model entertains the possibility of both spanned and unspanned variables in the pricing kernel process, with spanned variables $\mathbf{Z}_t \equiv [r_t \; \theta_t]$ and unspanned variables $\mathbf{U}_t \equiv [v_t]$.

Collin-Dufresne and Goldstein (2002, equations (47), (48), and (49)) propose the following dynamics under $\mathbb{Q}$:

$$\begin{aligned}
\text{Spot interest-rate}: \quad dr_t &= \kappa_r(\theta_t - r_t)\, dt \; + \; \sqrt{\alpha_r + v_t}\, d\omega_{r,t}^{\mathbb{Q}} \; + \; \sigma_{r\theta}\, d\omega_{\theta,t}^{\mathbb{Q}}, && (57)\\
\text{Long run mean}: \quad d\theta_t &= (\gamma_\theta - 2\kappa_r\theta_t + \frac{1}{\kappa_r} v_t)\, dt \; + \; \sigma_\theta\, d\omega_{\theta,t}^{\mathbb{Q}}, \quad \text{and} && (58)\\
\text{Variance}: \quad dv_t &= (\gamma_v - \kappa_v^{\mathbb{Q}} v_t)\, dt \; + \; \sigma_v \sqrt{v_t}\, du_{v,t}^{\mathbb{Q}}, && (59)
\end{aligned}$$

with $\gamma_v > 0$, $\kappa_v^{\mathbb{Q}} > 0$, $\sigma_v > 0$, $\gamma_\theta > 0$, $\kappa_r > 0$, and $\alpha_r \geq 0$. Further, bond prices, B_t^T, are given by (i.e., Collin-Dufresne and Goldstein (2002, equations (55) and (56)))

$$\begin{aligned}
B_t^T &= \exp(\mathfrak{a}[T-t] - \mathfrak{b}[T-t]\, r_t - \mathfrak{c}[T-t]\, \theta_t), \qquad \text{where} && (60)\\
\mathfrak{b}[T-t] &= \frac{1}{\kappa_r}(1 - e^{-\kappa_r(T-t)}), \quad \text{and} \quad \mathfrak{c}[T-t] \; = \; \frac{1}{2\kappa_r}(1 - e^{-\kappa_r(T-t)})^2, && (61)\\
\text{and} \qquad & \mathfrak{a}[T-t] \text{ satisfies an ODE.} && (62)
\end{aligned}$$

Our objectives are two fold: First, to consider the mechanics of local time risk premiums and options on bond futures and disentangle their implications for the pattern of expected excess returns of OTM options on bond futures; second, to articulate restrictions on the unspanned components $\boldsymbol{\alpha}[t,\mathbf{U}]'\, d\mathbf{u}_t^{\mathbb{Q}}$ that ensure empirical consistency with data on interest-rate claims.

Focusing on the constant parameter Λ_{cg}, whose sign we impute for empirical consistency, and in line with Collin-Dufresne and Goldstein (2002, equations (51)–(53)) and our (6), we set

$$du_{v,t}^{\mathbb{Q}} = du_{v,t}^{\mathbb{P}} - \underbrace{\{-\Lambda_{\mathrm{cg}}\sqrt{v_t}\}}_{=\,\boldsymbol{\alpha}[t,\mathbf{U}]}\, dt, \qquad d\omega_{r,t}^{\mathbb{P}} - d\omega_{r,t}^{\mathbb{Q}} = 0, \qquad \text{and} \qquad d\omega_{\theta,t}^{\mathbb{P}} - d\omega_{\theta,t}^{\mathbb{Q}} = 0. \tag{63}$$

Equation (63) reveals the link between the standard Brownian motions under $\mathbb{P}$ and $\mathbb{Q}$. Hence, the Girsanov theorem induced change of measure implies the restrictions that

$$\begin{aligned} \boldsymbol{\lambda}[t,\mathbf{Z},\mathbf{U}] &= \mathbf{0}, \quad \text{(zero risk premium on bonds and bond futures), and} \quad &(64)\\ \boldsymbol{\alpha}[t,\mathbf{U}] &= -\Lambda_{\mathrm{cg}}\sqrt{v_t}. \qquad \text{(mapping to our equation (6))} \quad &(65) \end{aligned}$$

It follows that

$$dv_t = (\gamma_v - \{\kappa_v^{\mathbb{Q}} - \sigma_v\,\Lambda_{\mathrm{cg}}\}v_t)\,dt \;+\; \sigma_v\sqrt{v_t}\;\underbrace{du_{v,t}^{\mathbb{P}}}_{\text{unspanned}}\;, \quad \text{under } \mathbb{P}. \tag{66}$$

As Collin-Dufresne and Goldstein (2002) note, bond prices (in equation (60)) do not depend upon v_t (the unspanned variance state variable).

However, since $d\omega_{r,t}^{\mathbb{Q}} = d\omega_{r,t}^{\mathbb{P}}$, $d\omega_{\theta,t}^{\mathbb{Q}} = d\omega_{\theta,t}^{\mathbb{P}}$, and $\boldsymbol{\lambda}[t,\mathbf{Z},\mathbf{U}] = \mathbf{0}$, it follows that $\mathbb{E}_t^{\mathbb{P}}(F_{T_O}^{T_F}) - \mathbb{E}_t^{\mathbb{Q}}(F_{T_O}^{T_F}) = 0$ and, hence, the bond futures risk premium is zero. Write $d\boldsymbol{\omega}_t^{\mathbb{P}} = (d\omega_{r,t}^{\mathbb{P}}, d\omega_{\theta,t}^{\mathbb{P}})'$ and $d\boldsymbol{\omega}_t^{\mathbb{Q}} = (d\omega_{r,t}^{\mathbb{Q}}, d\omega_{\theta,t}^{\mathbb{Q}})'$.

Building on this setup, we additionally derive the dynamics of (G_s). By Ito's lemma and equations (60) and (11), we obtain the form of the volatility of bond futures returns. Thus,

$$\frac{dG_t}{G_t} = -\boldsymbol{\sigma}_F[t,T_F,\mathbf{Z},\mathbf{U}]'\;\underbrace{d\boldsymbol{\omega}_t^{\mathbb{Q}}}_{\text{spanned}}\;, \text{ where } \boldsymbol{\sigma}_F[t,T_F,\mathbf{Z},\mathbf{U}] = \begin{pmatrix} (\mathfrak{b}[T_B - t] - \mathfrak{b}[T_F - t])\sqrt{\alpha_r + v_t} \\ (\mathfrak{c}[T_B - t] - \mathfrak{c}[T_F - t])\sigma_{r\theta} \end{pmatrix}. \tag{67}$$

The not-yet-answered question is: Under what restrictions can this model produce negative expected excess return of OTM puts and calls on bond futures?

Viewing through the prism of equations (23) and (26), we define

$$\tilde{\sigma}_v[\ell] \;\equiv\; \mathfrak{b}[T_B - \ell] - \mathfrak{b}[T_F - \ell] \qquad \text{and} \qquad \tilde{\sigma}[\ell] \;=\; (\mathfrak{c}[T_B - \ell] - \mathfrak{c}[T_F - \ell])\sigma_{r\theta}. \tag{68}$$

Thus, $\{\alpha_r + v_\ell\}(\tilde{\sigma}_v[\ell])^2 + (\tilde{\sigma}[\ell])^2$ is the instantaneous (log) variance, at time ℓ, of G_ℓ.

The dynamics of variance v_t in equation (59) imply that, for all $\ell \geq t$, the level of v_ℓ satisfies

$$v_\ell = v_t e^{\kappa_v^{\mathbb{Q}}(t-\ell)} \;+\; \int_t^\ell \gamma_v e^{\kappa_v^{\mathbb{Q}}(s-\ell)} ds \;+\; \int_t^\ell \sigma_v e^{\kappa_v^{\mathbb{Q}}(s-\ell)} \sqrt{v_s}\, du_{v,s}^{\mathbb{Q}}, \qquad \text{under } \mathbb{Q}. \tag{69}$$

Using our Result 1, we consider the $\mathbb{Q}$ covariance between $\mathbb{L}_t^{T_O}[k; \langle G \rangle]$ and $\log(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}) = \int_t^{T_O} \{-\frac{1}{2}\boldsymbol{\lambda}[s,\mathbf{Z},\mathbf{U}]'\boldsymbol{\lambda}[s,\mathbf{Z},\mathbf{U}]ds - \boldsymbol{\lambda}[s,\mathbf{Z},\mathbf{U}]' d\boldsymbol{\omega}_s^{\mathbb{Q}} - \frac{1}{2}\boldsymbol{\alpha}[s,\mathbf{U}]'\boldsymbol{\alpha}[s,\mathbf{U}]ds - \boldsymbol{\alpha}[s,\mathbf{U}]'\, d\mathbf{u}_s^{\mathbb{Q}}\}$.

Recognizing that $\boldsymbol{\alpha}[t,\mathbf{U}] = -\Lambda_{\mathrm{cg}}\sqrt{v_t}$, we compute

$$\begin{aligned}
&\mathrm{cov}_t^{\mathbb{Q}}\Big(\int_t^{T_O} -\{-\Lambda_{\mathrm{cg}}\sqrt{v_s}\, du_{v,s}^{\mathbb{Q}}\},\; \frac{1}{2}\int_t^{T_O} \delta[G_\ell - k]\{(\alpha_r + v_\ell)\,(\tilde{\sigma}_v[\ell])^2 + (\tilde{\sigma}[\ell])^2\}G_\ell^2\, d\ell \mid \mathcal{I}_{T_O}\Big) \\
&= \mathrm{cov}_t^{\mathbb{Q}}\Big(\int_t^{T_O} \Lambda_{\mathrm{cg}}\sqrt{v_s}\, du_{v,s}^{\mathbb{Q}}, \frac{1}{2}\int_t^{T_O}\int_t^\ell (\alpha_r + \sigma_v e^{\kappa_v^{\mathbb{Q}}(s-\ell)}\sqrt{v_s} du_{v,s}^{\mathbb{Q}})\,\delta[G_\ell - k](\tilde{\sigma}_v[\ell])^2 G_\ell^2 d\ell \mid \mathcal{I}_{T_O}\Big) \\
&= \mathrm{cov}_t^{\mathbb{Q}}\Big(\int_t^{T_O} \Lambda_{\mathrm{cg}}\sqrt{v_s}\, du_{v,s}^{\mathbb{Q}},\; \int_t^{T_O} \sqrt{v_s}\,\{\int_s^{T_O} \frac{\sigma_v}{2} e^{\kappa_v^{\mathbb{Q}}(s-\ell)}\delta[G_\ell - k](\tilde{\sigma}_v[\ell])^2 G_\ell^2\, d\ell\}\; du_{v,s}^{\mathbb{Q}} \mid \mathcal{I}_{T_O}\Big) \\
&= \mathbb{E}_t^{\mathbb{Q}}\Big(\int_t^{T_O} \Lambda_{\mathrm{cg}}\sqrt{v_s}\,\sqrt{v_s}\;\{\int_s^{T_O} \frac{\sigma_v}{2} e^{\kappa_v^{\mathbb{Q}}(s-\ell)}\delta[G_\ell - k](\tilde{\sigma}_v[\ell])^2 G_\ell^2\, d\ell\}\, ds \mid \mathcal{I}_{T_O}\Big) \\
&= \Lambda_{\mathrm{cg}}\; \underbrace{\mathbb{E}_t^{\mathbb{Q}}\Big(\int_t^{T_O} v_s\;\{\int_s^{T_O} \frac{\sigma_v}{2} e^{\kappa_v^{\mathbb{Q}}(s-\ell)}\delta[G_\ell - k](\tilde{\sigma}_v[\ell])^2 G_\ell^2\, d\ell\}\, ds \mid \mathcal{I}_{T_O}\Big)}_{>0}.
\end{aligned} \tag{70}$$

Our steps emphasize that the term $(\tilde{\sigma}[\ell])^2$ is irrelevant to the covariance, and, we use the evolution of variance v_ℓ (under $\mathbb{Q}$) in equation (69). Finally, we change the order of integration.

Accordingly, the $\mathbb{Q}$ measure covariance in equation (70) is negative, provided

$$\Lambda_{\mathrm{cg}} < 0. \qquad \text{(restriction for negative local time risk premium)} \tag{71}$$

Equivalently, when $\Lambda_{\text{cg}} < 0$, it unravels the theoretical restriction that

$$\boldsymbol{\alpha}[t, \mathbf{U}] \equiv -\underbrace{\Lambda_{\text{cg}}}_{<0} \underbrace{\sqrt{v_t}}_{>0} > 0, \text{ which guides the structure of } \boldsymbol{\alpha}[t, \mathbf{U}]' d\mathbf{u}_t^{\mathbb{P}} \text{ and } \boldsymbol{\alpha}[t, \mathbf{U}]' d\mathbf{u}_t^{\mathbb{Q}}. \tag{72}$$

In line with our empirical evidence, the local time risk premium can be negative, and the expected excess return to OTM puts and calls on bond futures can both be negative, provided $\Lambda_{\text{cg}} < 0$ and $\boldsymbol{\alpha}[t, \mathbf{U}] > \mathbf{0}$. The identification of $\Lambda_{\text{cg}} < 0$ is a central implication of our theory.

4.3 Workhorse models and positioning the relevance of our theoretical results

The following examples are presented to position the relevance of our theoretical and empirical results and to hone economic intuition behind our emphasis on the properties of unspanned components of the pricing kernel.

The proofs of our statements with regard to these extant models are sometimes tedious and require intermediate results; they are detailed in an online note (Bakshi, Crosby, and Gao (2019)).

Case 1 (Extended Vasicek (1977)) *The model assumes, in addition to* $\boldsymbol{\alpha}[t, \mathbf{X}] = \mathbf{0}$*, that*

$$\begin{aligned} dr_t &= (\theta^{\mathbb{P}} - \kappa^{\mathbb{P}} r_t)\, dt + \sigma\, d\omega_t^{\mathbb{P}} \qquad \text{under } \mathbb{P}, & (73)\\ dr_t &= (\theta_t^{\mathbb{Q}} - \kappa^{\mathbb{Q}} r_t)\, dt + \sigma\, d\omega_t^{\mathbb{Q}} \qquad \text{under } \mathbb{Q}, \text{ and} & (74)\\ \lambda[t, r_t] &\equiv \frac{1}{\sigma}(\theta_t^{\mathbb{Q}} - \theta^{\mathbb{P}} - (\kappa^{\mathbb{Q}} - \kappa^{\mathbb{P}})\, r_t), \quad \text{where } \theta_t^{\mathbb{Q}} \text{ is time-varying.} & (75) \end{aligned}$$

In this model, the bond futures price $F_s^{T_F}$*, and also* G_s*, for* $s \geq t$*, are lognormally distributed martingales under* $\mathbb{Q}$*, with*

$$\frac{dF_t^{T_F}}{F_t^{T_F}} = -\sigma_F[t, T_F]\, d\omega_t^{\mathbb{Q}}, \quad \text{where} \quad \sigma_F[t, T_F] \equiv \frac{\sigma}{\kappa^{\mathbb{Q}}}((1 - e^{-\kappa^{\mathbb{Q}}(T_F - t)}) - (1 - e^{-\kappa^{\mathbb{Q}}(T_B - t)})). \tag{76}$$

For this model, the expected excess return of holding OTM puts and OTM calls cannot *both* be negative. Key to deciphering this outcome is that this model embeds deterministic bond futures volatilities, resulting in zero local time risk premiums. Here, one can derive closed-form expressions for $\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle])$ and $\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle])$.

Case 2 (Rare disaster macro-finance (nominal version) model) *Typifying a rare disaster approach is the model of Wachter (2013) with Epstein and Zin (1989) utility, entailing the real pricing kernel, denoted by* M_t^{real}*, which satisfies*

$$\frac{dM_t^{real}}{M_{t^-}^{real}} = -r_t^{real}\,dt - \mathfrak{p}_t\,\mathbb{E}_z(e^{-\gamma z}-1)\,dt - \gamma\,\sigma_c\,d\omega_{c,t}^{\mathbb{P}} + b\,\sigma_{\mathfrak{p}}\sqrt{\mathfrak{p}_t}\,d\omega_{\mathfrak{p},t}^{\mathbb{P}} + (e^{-\gamma z}-1)dN_t, \tag{77}$$

$$r_t^{real} = \beta + \mu - \gamma\sigma_c^2 + \mathfrak{p}_t\,\mathbb{E}_\nu(e^{-\gamma z}\{e^z - 1\}). \tag{78}$$

All sources of uncertainty are uncorrelated, β *is the constant rate of time preference,* γ *is the coefficient of risk aversion, and* $\mathfrak{p}$ *is the probability of a disaster. To determine the nominal term-structure, we assume that consumption prices* $p_{c,t}$*, and its expected growth* g_t*, follow*

$$\frac{dp_{c,t}}{p_{c,t}} = g_t dt + \sigma_{p_c}\sqrt{\alpha_g + \beta_g\,g_t}\,d\omega_{p_c,t}^{\mathbb{P}} \quad \text{and} \quad dg_t = \kappa_g\,(\mu_g - g_t)\,dt + \sqrt{\alpha_g + \beta_g\,g_t}\,d\omega_{g,t}^{\mathbb{P}}, \tag{79}$$

where $\omega_{p_c,t}^{\mathbb{P}}$ *and* $\omega_{g,t}^{\mathbb{P}}$ *are independent standard Brownian motions, and* σ_{p_c}*,* α_g*, and* β_g *are non-negative constants.*

Hence, the nominal *pricing kernel* M_t *satisfies*

$$\frac{dM_t}{M_{t^-}} = -\underbrace{r_t}_{=\,r_t^{real}+g_t}\,dt - \mathfrak{p}_t\,\mathbb{E}_z(e^{-\gamma z}-1)dt - \gamma\sigma_c\,d\omega_{c,t}^{\mathbb{P}} + b\,\sigma_{\mathfrak{p}}\sqrt{\mathfrak{p}_t}\,d\omega_{\mathfrak{p},t}^{\mathbb{P}} - \sigma_{p_c}\sqrt{\alpha_g + \beta_g\,g_t}\,d\omega_{p_c,t}^{\mathbb{P}} + (e^{-\gamma z}-1)dN_t, \tag{80}$$

where b *is a derived constant (Wachter (2013, equation (A9), page 1023)).*

Given that there are jumps only *in the pricing kernel, and, crucially, not in bond prices, our results in Sections 2.3.1–2.3.2 go through. Mirroring our framework,* r_t^{real}*, and, hence,* r_t *does not jump and it is a diffusion. This leads to a two-factor term-structure model, of the type presented in Cox, Ingersoll, and Ross (1985).*

We can show that if $\mathbb{E}_t^{\mathbb{P}}(F_{T_O}^{T_F}) > \mathbb{E}_t^{\mathbb{Q}}(F_{T_O}^{T_F})$*, then* $\mathbb{E}_t^{\mathbb{P}}([k - G_{T_O}]^+)$ *is less than* $\mathbb{E}_t^{\mathbb{Q}}([k - G_{T_O}]^+)$*, and* $\mathbb{E}_t^{\mathbb{P}}([G_{T_O} - k]^+)$ *is greater than* $\mathbb{E}_t^{\mathbb{Q}}([G_{T_O} - k]^+)$*. Thus, in this rare disaster model, the expected excess returns of holding OTM puts and calls on bond futures cannot* both *be negative.*

Case 3 (Quadratic term-structure model (e.g., Leippold and Wu (2002))) *The log bond prices are quadratic functions of mean-reverting Gaussian state variables. For this model, the expected excess return to holding OTM puts and calls on bond futures cannot both be negative.*

Case 4 (Long-run risks (nominal version) model) *We consider a version of the long-run risks model of Bansal and Yaron (2004), as in Zhou and Zhu (2015). The model emphasizes recursive preferences and a long-run risk process, which governs the conditional mean of consumption and dividend growths. We transform this model (using a process for inflation as in equation (79)) to study implications for the nominal term-structure. The empirical disparity that arises is that the expected excess return to holding OTM puts and calls on bond futures cannot both be negative.*

In sum, in the considered cases, models are amiss, with respect to our empirical observation that the average return to holding OTM options on Treasury bond futures are negative.

5 Conclusions

Understanding the intricacies of the term-structure of bonds has been the staple of intellectual activity among finance scholars. It has lead to the development of innovative models of interest-rate sensitive securities, which has also transformed the management of bond portfolios and interest-rate risk by fund managers and corporations.[3]

In this paper, we employ Tanaka's formula for continuous semimartingales, while recognizing that the valuation of all interest-rate claims is linked by the same equivalent martingale measure. Developing the model-free expectation of option-like payoffs using Tanaka's formula, under the physical measure and the equivalent martingale measure, synthesizes the risk premiums on *local*

[3]We refer the reader to Cox, Ingersoll, and Ross (1985), Heath, Jarrow, and Morton (1992), Hull and White (1990), Constantinides (1992), Longstaff and Schwartz (1992), Dai and Singleton (2000), Ahn, Dittmar, and Gallant (2002), Collin-Dufresne and Goldstein (2002), Leippold and Wu (2002), Cochrane and Piazzesi (2005), Feldhütter and Lando (2008), Joslin, Priebsch, and Singleton (2014), Filipovic, Larsson, and Trolle (2017), Cieslak and Povala (2018), Joslin and Konchitchki (2018), Fleckenstein and Longstaff (2018). Cieslak and Povala (2016) decompose conditional volatilities of Treasury yields into components due to short-rate expectations and to term premia. They find that investors pay a large premium for hedging variance risk with derivatives. Joslin (2018) studies conditions under which a class of affine term structure models exhibits unspanned stochastic volatility (USV). He shows that the USV conditions restrict the mean-reversion rates of risk factors and the cross-section of conditional yields in such a way that affine USV models are unlikely to generate the observed cross section of yield volatilities. The study of Cremers, Fleckenstein, and Gandhi (2018) extracts implied volatilities from options on Treasury bond futures and show that it predicts the level and volatility of macroeconomic activities. The mindset of our theoretical and empirical exercises, conducted in the context of unspanned components of the pricing kernel and Treasury bond options, also departs from Breeden and Litzenberger (2014), Wright (2017), and Bretscher, Schmid, and Vedolin (2018).

time. This newly introduced quantity is informative about the expected excess returns to holding options, and our approach allows us to connect the term-structure of interest rates together with the contingent claims on Treasury bond futures. It places the focus on the issue of market incompleteness and the properties of the unspanned components of the pricing kernel (i.e., capturing risks not spanned by bonds or bond futures).

We feature three contributions. First, we construct the returns to holding a long position in options on the futures of the 10- and 30-year Treasury bonds over expiration cycles. Our empirical exercises show that the average returns of out-of-the-money puts and calls on bond futures are both negative, and this effect is more pronounced for deeper out-of-the-money options.

Second, we present results under general diffusion processes for the pricing kernel and interest-rates, that formalize negative expected excess return of options across strikes. Central to achieving theoretical reconciliation is the modeling of market incompleteness and sources of volatility uncertainty, which can plausibly parameterize the negative risk premium on local time.

Third, we consider parameterized frameworks (our Sections 4.1 and 4.2), which are broadly consistent with our empirical findings under suitable restrictions on the unspanned components of the pricing kernel dynamics. To offer a contrast, we also consider a variety of workhorse models and show that the expected excess return to holding a call (respectively, put) on bond futures has the same sign (opposite sign) as the risk premium on bond futures. Thus, the expected excess return to holding puts and the expected excess return to holding calls cannot both be negative (which is inconsistent with our empirical findings and with our featured parameterized frameworks incorporating unspanned risks in the pricing kernel).

Our theoretical results and empirical observations about the plausibility of negative risk premium on local time can pave the path for models that are consistent with the evolution of the yield curve as well as the data on options on Treasury bond futures. The desirability of using Tanaka's formula lies in its ability to link the developed theory to realities of Treasury options data, without imposing any parametric assumptions on the pricing kernel or the factor structure of interest-rates.

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Appendix

Throughout the paper, we employ the following notations.

- T_B is the maturity of the zero coupon Treasury bond that underlies the bond futures contract.
- T_F is the maturity of the futures contract on the zero coupon bond.
- T_O is the maturity of the option on the bond futures (with $T_O \leq T_F \leq T_B$).
- B_t^T denotes the time t price of a zero coupon bond maturing at a given date T.
- $F_t^{T_F}$ is the time t price of the bond futures (maturity date T_F (with $T_F \leq T_B$)).
- $\hbar$ is 1 if the payoff is $(F_{T_O}^{T_F} - K)^+$ (i.e., a call) and -1 if the payoff is $(K - F_{T_O}^{T_F})^+$ (i.e., a put).
- $\mathbb{1}_{\{F_{T_O}^{T_F} > K\}}$ is an indicator function that takes a value of 1 if $F_{T_O}^{T_F} > K$ and zero otherwise.
- $\delta[\bullet]$ is the Dirac delta function.
- $G_s \equiv \frac{F_s^{T_F}}{F_t^{T_F}}$ is the gross return on the bond futures price over the time period t to s.

A Appendix A: Proof of Result 1 on the expected excess return of options on the bond futures

For the proof, we denote by $\mathcal{I}_s$, the sub-filtration of $\mathcal{F}_s$, generated by $\boldsymbol{\omega}_s^{\mathbb{Q}}$ and $\boldsymbol{\lambda}[s, \mathbf{X}]' \, d\boldsymbol{\omega}_s^{\mathbb{Q}}$ in the context of the dynamics in equations (2)–(5). We first establish a result on the covariance $\text{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}, \mathbb{L}_t^{T_O}[k; \langle G \rangle])$ by conditioning on $\mathcal{I}_{T_O}$. By the law of total covariance,

$$
\begin{aligned}
&\text{cov}_t^{\mathbb{Q}}(\overbrace{\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}}^{\text{from equation (12)}}, \mathbb{L}_t^{T_O}[k; \langle G \rangle]) \\
&= \mathbb{E}_t^{\mathbb{Q}}(\text{cov}_t^{\mathbb{Q}}(e^{\int_t^{T_O} \{-\frac{1}{2}\boldsymbol{\lambda}[s,\mathbf{X}]'\boldsymbol{\lambda}[s,X]ds - \boldsymbol{\lambda}[s,\mathbf{X}]' d\boldsymbol{\omega}_s^{\mathbb{Q}} - \frac{1}{2}\boldsymbol{\alpha}[s,\mathbf{X}]'\boldsymbol{\alpha}[s,\mathbf{X}]ds - \boldsymbol{\alpha}[s,\mathbf{X}]' d\mathbf{u}_s^{\mathbb{Q}}\}}, \mathbb{L}_t^{T_O}[k; \langle G \rangle] \big| \mathcal{I}_{T_O})) \\
&\qquad + \text{cov}_t^{\mathbb{Q}}(\underbrace{\mathbb{E}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell} \big| \mathcal{I}_{T_O})}_{= \text{ a constant}}, \ \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle] \mid \mathcal{I}_{T_O})), \qquad \text{(A1)} \\
&= \mathbb{E}_t^{\mathbb{Q}}(e^{\int_t^{T_O} \{-\frac{1}{2}\boldsymbol{\lambda}[s,\mathbf{X}]'\boldsymbol{\lambda}[s,X]ds - \boldsymbol{\lambda}[s,\mathbf{X}]' \, d\boldsymbol{\omega}_s^{\mathbb{Q}}\}} \ \times \\
&\qquad \text{cov}_t^{\mathbb{Q}}(e^{\int_t^{T_O} \{-\frac{1}{2}\boldsymbol{\alpha}[s,\mathbf{X}]'\boldsymbol{\alpha}[s,\mathbf{X}]ds - \boldsymbol{\alpha}[s,\mathbf{X}]' \, d\mathbf{u}_s^{\mathbb{Q}}\}}, \mathbb{L}_t^{T_O}[k; \langle G \rangle] \big| \mathcal{I}_{T_O})), \qquad \text{(A2)}
\end{aligned}
$$

because the covariance of the two expectations vanishes since one term is a constant.

Equation (A2) holds *generally.*

Now set $\boldsymbol{\alpha}[t,\mathbf{X}]=\mathbf{0}$ (i.e., omitting unspanned risks). The covariance term in the final line of equation (A2) becomes zero. This is because with $\boldsymbol{\alpha}[t,\mathbf{X}]=\mathbf{0}$, it holds that $\exp(\int_t^{T_O}\{-\frac{1}{2}\boldsymbol{\alpha}[s,\mathbf{X}]'\boldsymbol{\alpha}[s,\mathbf{X}]ds-\boldsymbol{\alpha}[s,\mathbf{X}]'\,d\mathbf{u}_s^{\mathbb{Q}}\})=1$.

The consequence is that $\mathrm{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}}e^{-\int_t^{T_O}r_\ell d\ell},\mathbb{L}_t^{T_O}[k;\langle G\rangle])=0$. Via equation (28), therefore,

$$\underbrace{\mathbb{E}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}}e^{-\int_t^{T_O}r_\ell d\ell}\mathbb{L}_t^{T_O}[k;\langle G\rangle])}_{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])} \;-\; \underbrace{\mathbb{E}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}}e^{-\int_t^{T_O}r_\ell d\ell})}_{=1}\,\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) \;=\; 0. \tag{A3}$$

Hence, when $\boldsymbol{\alpha}[t,\mathbf{X}]=\mathbf{0}$, we have the result that

$$\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) \;-\; \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle]) \;=\; \underbrace{0.}_{\text{Zero risk premium for local time}} \tag{A4}$$

Return now to the expression for the expected excess return of the option in equation (27). Set $f[\mathcal{C}]\equiv\frac{\mathbb{E}_t^{\mathbb{P}}(\hbar\int_t^{T_O}\mathbb{1}_{\{\hbar G_\ell>\hbar k\}}dG_\ell)+\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])+\mathcal{C}}\;-\;1$. The Taylor series expansion of $f[\mathcal{C}]$ around zero is

$$\begin{aligned} f[\mathcal{C}] \;&=\; \frac{\mathbb{E}_t^{\mathbb{P}}(\hbar\int_t^{T_O}\mathbb{1}_{\{\hbar G_\ell>\hbar k\}}dG_\ell)+\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}\;-\;1 \\ &\quad -\;\frac{\mathbb{E}_t^{\mathbb{P}}(\hbar\int_t^{T_O}\mathbb{1}_{\{\hbar G_\ell>\hbar k\}}dG_\ell)+\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}\frac{\mathcal{C}}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}+O[\mathcal{C}^2]. \end{aligned} \tag{A5}$$

If it were to hold that $\mathcal{C}_t[\hbar,k]$ is negligible; that is, $\frac{\mathcal{C}_t[\hbar,k]}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}\approx 0$, the sign of the expected excess option return is revealed, since $\frac{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}-1=0$ from equation (A4). Specifically,

$$1+\mu_t^{[\hbar]}-\frac{1}{B_t^{T_O}}\approx\frac{1}{B_t^{T_O}}\begin{cases} \overbrace{\frac{\mathbb{E}_t^{\mathbb{P}}(-\int_t^{T_O}\mathbb{1}_{\{G_\ell<k\}}dG_\ell)}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}}^{<\;0\quad\text{by (29)}}+\{\overbrace{\frac{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}-1\}}^{=\;0\text{ when }\boldsymbol{\alpha}[t,\mathbf{X}]=\mathbf{0}} \;\; <0 & \text{for OTM puts,} \\ \frac{\mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O}\mathbb{1}_{\{G_\ell>k\}}dG_\ell)}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}+\{\frac{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k;\langle G\rangle])}-1\} \;\; >0 & \text{for OTM calls,} \end{cases} \tag{A6}$$

by equations (27) and (A5). This concludes our proof of part (a) of Result 1.

With regard to part (b) of Result 1, consider the situation when $\boldsymbol{\alpha}[t, \mathbf{X}] \neq \mathbf{0}$. When $\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle]) < \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k; \langle G \rangle])$, then, in view of equations (27) and (A6), negative expected excess returns to both puts and calls on bond futures are possible. ♣

B Appendix B: Proof of Result 2 (Local time risk premium when $k = \frac{K}{F_t^{T_F}} = 1$ is tied to the expected excess return of straddles on bond futures)

In view of our definitions that, for $s \geq t$, $G_s = \frac{F_s^{T_F}}{F_t^{T_F}}$ and $k = \frac{K}{F_t^{T_F}}$, the at-the-money straddles correspond to $k = 1$.

For brevity of presentation, write $\mathbb{L}_t^{T_O}[k; \langle G \rangle]\big|_{k=1} = \mathbb{L}_t^{T_O}[1; \langle G \rangle]$.

By Tanaka's formula,

$$[G_{T_O} - 1]^+ = [G_t - 1]^+ + \int_t^{T_O} \mathbb{1}_{\{G_\ell > 1\}} dG_\ell + \mathbb{L}_t^{T_O}[1; \langle G \rangle] \qquad \text{and} \tag{B1}$$

$$[1 - G_{T_O}]^+ = [1 - G_t]^+ - \int_t^{T_O} \mathbb{1}_{\{G_\ell < 1\}} dG_\ell + \mathbb{L}_t^{T_O}[1; \langle G \rangle]. \tag{B2}$$

Summing the left-hand sides of equations (B1) and (B2), noting in the present context of at-the-money options that $[G_t - 1]^+ = [1 - G_t]^+ = 0$, and rearranging, we have

$$\begin{aligned} 2\,\mathbb{L}_t^{T_O}[1; \langle G \rangle] &= [G_{T_O} - 1]^+ + [1 - G_{T_O}]^+ - \overbrace{[G_t - 1]^+}^{=\,0} - \overbrace{[1 - G_t]^+}^{=\,0} \\ &\quad - \int_t^{T_O} \mathbb{1}_{\{G_\ell > 1\}} dG_\ell + \int_t^{T_O} \mathbb{1}_{\{G_\ell < 1\}} dG_\ell. \end{aligned} \tag{B3}$$

Therefore,

$$\begin{aligned} \mathbb{L}_t^{T_O}[1; \langle G \rangle] &= \frac{1}{2}[G_{T_O} - 1]^+ + \frac{1}{2}[1 - G_{T_O}]^+ \\ &\quad - \frac{1}{2}\int_t^{T_O} \mathbb{1}_{\{G_\ell > 1\}} dG_\ell + \frac{1}{2}\int_t^{T_O} \mathbb{1}_{\{G_\ell < 1\}} dG_\ell. \end{aligned} \tag{B4}$$

Then, in conjunction with $\mathbb{P}$-measure and $\mathbb{Q}$-measure expectations of $[G_{T_O}-1]^+$ and $[1-G_{T_O}]^+$, and recognizing that $\mathbb{E}_t^{\mathbb{Q}}(\int_t^{T_O} \mathbb{1}_{\{G_\ell>1\}} dG_\ell) = 0$ and $\mathbb{E}_t^{\mathbb{Q}}(\int_t^{T_O} \mathbb{1}_{\{G_\ell<1\}} dG_\ell) = 0$, we obtain

$$
\begin{aligned}
& \mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[1;\langle G\rangle]) \;-\; \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[1;\langle G\rangle]) \\
=\; & \frac{1}{2}\big\{\mathbb{E}_t^{\mathbb{P}}([G_{T_O}-1]^+ \;+\; [1-G_{T_O}]^+) - \mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O} \mathbb{1}_{\{G_\ell>1\}} dG_\ell - \int_t^{T_O} \mathbb{1}_{\{G_\ell<1\}} dG_\ell) \\
& \qquad - \mathbb{E}_t^{\mathbb{Q}}([G_{T_O}-1]^+ + [1-G_{T_O}]^+)\big\}, && \text{(B5)} \\
=\; & B_t^{T_O}\,\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[1;\langle G\rangle])\,\Big(\overbrace{(1+\mu_t^{\text{straddle}}) - \frac{1}{B_t^{T_O}}}^{\text{Expected excess return}} - \frac{\overbrace{\mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O}(\mathbb{1}_{\{G_\ell>1\}} - \mathbb{1}_{\{G_\ell<1\}})dG_\ell)}^{\text{Dynamic futures strategy}}}{B_t^{T_O}\,\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[1;\langle G\rangle])}\Big), && \text{(B6)} \\
& \text{where} \quad (1+\mu_t^{\text{straddle}}) \;\equiv\; \frac{\mathbb{E}_t^{\mathbb{P}}([G_{T_O}-1]^+ + [1-G_{T_O}]^+)}{B_t^{T_O}\,\mathbb{E}_t^{\mathbb{Q}}([G_{T_O}-1]^+ + [1-G_{T_O}]^+)}. && \text{(B7)}
\end{aligned}
$$

If one *were* to assume that $\mathcal{C}_t[\hbar, 1]$ is negligible, then $(1+\mu_t^{\text{straddle}}) - \frac{1}{B_t^{T_O}}$ is the expected excess return to holding an at-the-money straddle, over the time period t to T_O.

The term $\int_t^{T_O}(\mathbb{1}_{\{G_\ell>1\}} - \mathbb{1}_{\{G_\ell<1\}})dG_\ell$ is the gain (loss) from a dynamic trading strategy in the bond futures. However, we are guided by the plausibility that

$$
\mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O} \{\mathbb{1}_{\{G_\ell>1\}} - \mathbb{1}_{\{G_\ell<1\}}\} dG_\ell) \;\approx\; 0, \tag{B8}
$$

because the trading strategy requires, at time ℓ, being long bond futures when $G_\ell > 1$ and short when $G_\ell < 1$. Thus, if the distribution of futures returns is approximately "symmetrical" and the futures risk premium is not too large, there may be a partial offset of the futures risk premium.

When the expression in equation (B8) is zero and $\mathcal{C}_t[\hbar, 1]$ is negligible, equation (B6) becomes

$$
\begin{aligned}
& \frac{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[1;\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[1;\langle G\rangle])}\bigg|_{k=1} \;-\; 1 \\
\approx\; & B_t^{T_O}\,\Big(\overbrace{\frac{\mathbb{E}_t^{\mathbb{P}}([F_{T_O}^{T_F} - F_t^{T_F}]^+ \;+\; [F_t^{T_F} - F_{T_O}^{T_F}]^+)}{\underbrace{\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O} r_\ell d\ell}[F_{T_O}^{T_F} - F_t^{T_F}]^+)}_{C_t[F_t^{T_F}]} \;+\; \underbrace{\mathbb{E}_t^{\mathbb{Q}}(e^{-\int_t^{T_O} r_\ell d\ell}[F_t^{T_F} - F_{T_O}^{T_F}]^+)}_{P_t[F_t^{T_F}]}} - \frac{1}{B_t^{T_O}}}^{\text{Expected excess return of straddles}}\Big). && \text{(B9)}
\end{aligned}
$$

Our conclusion in equation (36) then follows. ♣

Table 1: **Dollar open interest and dollar trading volume for options on Treasury bond futures and the S&P 500 equity index**

Reported is the average dollar open interest (in $ billions) and average dollar trading volume (in $ billions). The dollar open interest (dollar trading volume) is the number of options contracts outstanding (number of options contracts traded) multiplied by the Treasury bond futures price, observed on the last day of the month. We then average the dollar open interest and dollar trading volume for OTM puts and OTM calls over the sample period. NOBS is the number of monthly observations. The data on options on the S&P 500 equity index is obtained from the CBOE.

Options on	Begin date	End date	NOBS	OTM puts		OTM calls	
				Dollar open interest	Dollar trading volume	Dollar open interest	Dollar trading volume
10-year Treasury bond futures	7/31/1991	12/31/2018	330	66.9	8.9	55.9	7.2
30-year Treasury bond futures	12/31/1990	12/31/2018	337	27.6	3.6	27.1	3.4
S&P 500 equity index	1/31/1990	12/31/2018	348	63.5	8.4	53.1	6.9

Table 2: **Properties of the futures risk premiums and bootstrap confidence intervals** Reported are return properties of futures on the 10- and 30-year Treasury bonds. We compute $\frac{F_{T_O}^{T_F}}{F_t^{T_F}} - 1$, which is the excess return corresponding to a fully collateralized long futures position. The return of futures on the 10-year (30-year) Treasury bond are over the sample period of 07/22/1991 to 12/24/2018 (12/24/1990 to 12/24/2018). Shown is the annualized mean, annualized standard deviation, skewness, excess kurtosis, minimum, and maximum. We report the 90% bootstrap confidence intervals based on an i.i.d. bootstrap. The Ljung-Box test, at lags up to 12, rejects autocorrelation in these time-series, hence our focus on the i.i.d. bootstrap. ACF(1) is the first-order autocorrelation, and "Cycles" denotes the number of expiration cycles.

	Futures on Treasury bonds	
Excess return	10-year	30-year
Mean (annualized)	3.0	3.5
Lower Bootstrap CI	1.3	0.7
Upper Bootstrap CI	4.7	6.1
SD (annualized)	5.3	8.7
Minimum	-4.8	-8.3
Maximum	5.0	10.4
ACF(1)	0.05	0.02
Skewness	0.00	-0.02
Kurtosis (excess)	0.41	1.04
Cycles	330	337

Table 3: **Properties of option returns on the futures of the Treasury bond**

This table reports the properties of put and call option returns on the futures of the Treasury bond. We construct the (net) option returns as

$$z^{\text{put}}_{t,T_O,\aleph\%} = \frac{[K - F^{T_F}_{T_O}]^+}{P_t[K]} - 1, \text{ where } K \text{ corresponds to } K = F^{T_F}_t e^{-\aleph\%},\ \aleph = 1\%, 3\%, \text{ and } 5\%,$$

$$z^{\text{call}}_{t,T_O,\aleph\%} = \frac{[F^{T_F}_{T_O} - K]^+}{C_t[K]} - 1, \text{ where } K \text{ corresponds to } K = F^{T_F}_t e^{+\aleph\%},\ \aleph = 1\%, 3\%, \text{ and } 5\%,$$

where $P_t[K]$ ($C_t[K]$) is the *settlement* price of a put (call) on the Treasury bond futures with strike price K, as reported by the CME. The bid and ask option prices are not reported separately. For each moneyness level, we show the average (AVG.), the standard deviation (SD), the skewness, the (excess) kurtosis, and the maximum of option returns over non-overlapping intervals. Interpreting the averages, the reported entry for straddle on futures on a 10-year Treasury bond implies an average return of -11% over the expiration cycles (average of 27.3 days, not annualized). The statistic $\mathbb{1}_{z>0}$ refers to the number of expiration cycles in which the options return is positive. The sample period for the option returns on the futures of the 10-year (30-year) is 07/22/1991 to 12/24/2018 (12/24/1990 to 12/24/2018). The Ljung-Box test, at lags up to 12, rejects autocorrelation in these time-series, hence we focus on the i.i.d. bootstrap to compute the confidence intervals.

	OTM (%)	AVG.	90% Bootstrap CI Lower	90% Bootstrap CI Upper	SD	Skewness	Kurtosis	Max.	$\mathbb{1}_{z>0}$
Panel A: Options on futures of the 10-year Treasury bond (330 observations)									
Puts	5	-93	-99	-81	110	17	300	18	3
Puts	3	-71	-91	-45	260	13	200	41	9
Puts	1	-41	-54	-26	160	4	18	13	54
Straddle		-11	-18	-3	80	1	3	5	122
Calls	1	-11	-27	6	180	2	5	8	79
Calls	3	-76	-88	-63	140	7	54	13	13
Calls	5	-91	-99	-82	90	13	186	13	5
Panel B: Options on futures of the 30-year Treasury bond (337 observations)									
Puts	5	-80	-96	-61	200	12	169	29	6
Puts	3	-58	-73	-40	180	6	35	16	23
Puts	1	-28	-42	-13	160	3	7	8	72
Straddle		-9	-16	-2	80	1	1	3	134
Calls	1	-3	-22	18	220	4	30	20	90
Calls	3	-17	-59	37	540	12	158	81	30
Calls	5	-58	-92	-10	480	15	255	81	7

Table 4: **Pairwise bootstrap p-values for average return differences across moneyness**

Let $\overline{z}^{\text{put}}_{t,T_O,\aleph\%}$ and $\overline{z}^{\text{call}}_{t,T_O,\aleph\%}$ denote the average return of $\aleph\%$ OTM put and call options on Treasury bond futures, respectively. Reported are the p-values for the following hypothesis

$$\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0, \text{ or, } \overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,3\%} \geq 0, \text{ or, } \overline{z}^{\text{put}}_{t,T_O,3\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0$$

and

$$\overline{z}^{\text{call}}_{t,T_O,5\%} - \overline{z}^{\text{call}}_{t,T_O,1\%} \geq 0, \text{ or, } \overline{z}^{\text{call}}_{t,T_O,5\%} - \overline{z}^{\text{call}}_{t,T_O,3\%} \geq 0, \text{ or, } \overline{z}^{\text{call}}_{t,T_O,3\%} - \overline{z}^{\text{call}}_{t,T_O,1\%} \geq 0.$$

We jointly bootstrap the returns of all options with replacement. Reported is the frequency of the observations, for instance, $\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0$. Lower p-values indicate rejection, implying that deeper OTM options have more negative average returns.

	Underlier of option is futures on	
Hypothesis	10-year bond	30-year bond
$\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0$	0.000	0.000
$\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,3\%} \geq 0$	0.000	0.002
$\overline{z}^{\text{put}}_{t,T_O,3\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0$	0.009	0.000
$\overline{z}^{\text{call}}_{t,T_O,3\%} - \overline{z}^{\text{call}}_{t,T_O,1\%} \geq 0$	0.000	0.235
$\overline{z}^{\text{call}}_{t,T_O,5\%} - \overline{z}^{\text{call}}_{t,T_O,3\%} \geq 0$	0.000	0.038
$\overline{z}^{\text{call}}_{t,T_O,5\%} - \overline{z}^{\text{call}}_{t,T_O,1\%} \geq 0$	0.000	0.018