Directed Search with Phantom Vacancies

James Albrecht

Georgetown University and IZA

Bruno Decreuse

Aix-Marseille Univ. (Aix-Marseille School of Economics), EHESS, CNRS and Centrale Marseille

Susan Vroman

Georgetown University and IZA

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Abstract

When vacancies are filled, the ads that were posted are often not withdrawn, creating "phantom" vacancies. The existence of phantoms implies that older job listings are less likely to represent true vacancies than are younger ones. We assume that job seekers direct their search based on the listing age and so equalize the expected benefit of a job application across listing age. Since wages do not depend on listing age, this is equivalent to equalizing the job finding rate across listing ages. Forming a match with a vacancy of age a creates a phantom of age a with probability β and this leads to a negative informational externality that affects all vacancies of age a or older. Thus, the magnitude of this externality decreases with a. Relative to the constrained efficient search behavior, the directed search of job seekers leads them to over-apply to younger listings. We illustrate the model using US labor market data. The contribution of phantoms to overall frictions is large, but, conditional on the existence of phantoms, the social planner cannot improve much on the directed search allocation.

1 Introduction

This paper is based on two premises. First, many listings for job openings that are advertised on job boards or newspapers or are heard about from friends and acquaintances are out of date. We use the concept of *phantom vacancies* to model this out-of-date information, where by a phantom vacancy we mean a job listing that continues to be advertised even though the vacancy has already been filled. We present evidence below that this type of stale information exists on job boards such as Craigslist.

Second, job seekers are aware of this stale information and adjust their search behavior accordingly. On job boards, searchers can observe the posting date for job listings. They understand that older listings are more likely to be phantoms. They also understand that other searchers also understand this so there is likely to be more competition at younger listings. Workers take these countervailing forces into account when directing their search based on listing age. In the directed search equilibrium that we analyze, workers follow a mixed strategy with respect to listing age that trades off the probability the listing is a phantom against the extent of competition for the position.

We argue that phantom vacancies are an important source of labor market frictions and hence unemployment. Why is there unemployment? A job seeker may fail to find an advertised position that matches his or her skills. Alternatively, appropriate positions may be advertised, but the job seeker's application may be met with the response, "Sorry, but the job has already been filled." Job search theory has ignored the latter friction, i.e., phantom vacancies, which we emphasize here. And, as online job search becomes more common, we argue that the frictions caused by phantoms may become relatively more important as a source of unemployment. From the individual job seeker's perspective, with online search, it should be easier to identify appropriate advertised positions, but it may become more difficult to be sure that they haven't already been filled.

The concept of phantom vacancies was introduced in Chéron and Decreuse (2017), and we use the matching function developed in that paper. Chéron and Decreuse (2017) is a model of *random search* in the sense that a job seeker is just as likely to apply to one listing as to any other; i.e., the unemployed are assumed to be unable to adapt to the existence of phantoms. In contrast, ours is a model of *directed search* – job seekers can direct their search based on listing age and can thus take the existence of phantoms into account.

We therefore have two objectives in our paper. The first is to characterize the directed search equilibrium. In equilibrium, searchers allocate themselves across "submarkets" that are defined by listing age. Worker directed search satisfies a no-arbitrage condition, namely, that the expected payoffs associated with searching in the various submarkets must be equalized. On the other side of the market, firms decide how many new vacancies to list. We also allow firms to "renew" their listings – for a fee, a firm can relist its vacancy, thereby effectively resetting its age to zero. The wage is determined by Nash bargaining in the decentralized equilibrium.

Our second objective is to characterize the constrained efficient allocation and to understand how and why it differs from the equilibrium allocation. The nature of the constrained efficient allocation depends on the tools we allow the social planner to employ. We first suppose that the social planner can choose firms' vacancy posting and listing renewal behavior. Second, we suppose that the social planner can also choose the allocation of searchers across submarkets; i.e., we allow the social planner to direct job seekers' search.

Before we address these questions, we offer some evidence for our two premises. Are phantoms important in real-world labor markets? To the extent that workers apply for jobs that are already filled,[1] and casual empiricism suggests this is often the case, phantoms matter. More formally, Chéron and Decreuse (2017) presents evidence of phantoms from an online job site. Using Craigslist data, they show that the distribution of job listings by age over one month (the time at which Craigslist destroys ads) is uniform by week. This implies that ads are not withdrawn as soon as the corresponding jobs are filled; instead, job listings persist for some time as phantoms. We are not arguing that all job boards fail to remove every obsolete listing immediately, but the Craigslist evidence clearly suggests the existence of stale information in the labor market. Similarly, Davis and Samaniego de la Parra (2018) argue that "...'stale postings' are a major concern about measuring posting durations in empirical studies that rely on data from some prominent online platforms for posting job vacancies." (p.8) See also footnote 10 (p.10) in their paper, which discusses the existence of phantoms on CareerBuilder.com and Monster.com.[2]

There is also evidence that job seekers direct their search towards recently posted job listings. First, as the following query to AskaManager.com suggests, this is what some job seekers say they do:

> I am currently on the job hunt and I had a question about applying to jobs online. You know how most websites will tell you the job has been posted 1 day ago, 28 days ago, etc. For some reason, I have concluded that I need to apply to a job the first week they post the position to have the best chances of being hired. Although I heard that it can take up to a month for the company to hire anyone for the position, I feel that applying to a job that was posted 3 weeks ago isn't that promising. What is your take on this situation?

Second, data that link applications to vacancy postings indicate that job applicants direct their search towards younger listings. The DHI Vacancy and Flow Applications Database, which

[1] In Acharya and Wee (2018), firms that have filled their vacancies continue to advertise those positions in an effort to find more productive "replacement" workers. The longer a job has been advertised, the less likely it is that an application for that position will be successful. In this sense, these listings are similar to phantoms. The important difference between our model and that of Acharya and Wee (2018) is that we allow job seekers to respond to the existence of phantoms by directing their search.

[2] There is also evidence of phantoms in other markets with search frictions. Fradkin (2017) documents that on Airbnb, about 15% of first attempts to make a booking by prospective renters in his sample fail due to "stale vacancies," i.e., because the hosts failed to block out specific dates on their calendars promptly even though those dates were in fact not available.

links 77 million applications to nearly 7 million vacancy postings, dating from January 2012, is an example of such a linked dataset. Using data from Dice.com, the primary source for the DHI database, Davis and Samaniego (2018) state (p.13) that "job seekers exhibit a striking propensity to target new and recently posted vacancies: 39 percent of applications flow to vacancies posted in the last 48 hours, and 59 percent go to those posted in the last 96 hours. Older postings attract relatively few applications." The results of Belot, Kircher and Muller (2018, pp.16-17) show the same pattern. In an audit study in which paired vacancies differed only in the posted wage and the listing date, they find that job seekers are substantially more likely to save the younger listing, even if the difference in the listing date was only one or two days.

Another piece of evidence that workers react to listing age is that firms choose to repost their vacancies even though there is a cost to doing so. Listing renewal appears to be quite common. Employers relist their vacancies in order to let workers know that the jobs are still unfilled. In Appendix A1, we show that listing renewal is quite common on Craigslist.

There are other explanations for why job seekers target younger listings. In stock-flow models such as Coles and Smith (1998), new job seekers flowing into the market search through the entire stock of listed vacancies.[3] A job seeker who fails to find a match in this first search step, having already examined the extant stock, is then limited to searching through the inflow of new listings. As with phantoms, stock-flow implies that most applications go to younger listings. Two pieces of evidence suggest that stock-flow matching may not be the sole explanation for the way that job seekers direct their search. First, as we noted above, listing renewal appears to be quite common. In a stock-flow model, employers have no incentive to relist. Second, the stock-flow model can be tested by looking at how worker application behavior changes with elapsed duration of unemployment. All else equal, recently unemployed workers should be equally likely to apply to a young listing as to an old one, while workers who have been unemployed for a longer time should only apply to young listings. Data on how worker application behavior varies over an unemployment spell are scarce, but there is one study that addresses this question. Using data from SnagAJob, Faberman and Kudlyak (2014) find (p.4) that "The fraction of applicants to a newly-posted vacancy rises with duration, consistent with a stock-flow model, but it does so only slightly ..."

Another possible explanation for why worker search is concentrated on younger listings is job heterogeneity. Some jobs are "lemons" – a worker contacting such a job is unlikely to find it acceptable or they may find the employer is overly picky – while other jobs are "plums." Lemons would then be over-represented in the stock of old listings, and workers would respond

[3] This assumption seems most applicable when an intermediary is present to help job seekers identify all appropriate job openings in the stock of vacancies. In Coles and Smith (1998), that intermediary role is played by the UK Jobs Centre.

by directing their applications towards young listings.[4] However, it is difficult to imagine that the existence of lemons explains why almost 40% of the applications observed in the Dice.com data go to listings that are at most 48 hours old. While stock-flow matching and lemons may be part of the reason that young listings receive more applications than old listings do, the evidence suggests that there is room for other explanations, e.g., phantom vacancies.

In the model we present below, we focus on phantoms and abstract from considerations of stock-flow matching and lemons. We use a model of sequential search in which unemployed workers apply for one job at a time. An alternative would be to allow for multiple applications, but as both Albrecht, Gautier, and Vroman (2006) and Galenianos and Kircher (2008) show, this would introduce other inefficiencies, and in this paper, we are interested in concentrating on the externality caused by phantom vacancies. Another alternative would be to assume nonsequential search in which firms actively recruit and screen workers,[5] e.g., Wolthoff (2018), but, in order to focus clearly on the role that phantoms play in labor market equilibrium, we have chosen to follow the main thread of the literature and to assume that search is sequential.[6]

Our basic results are as follows. In the decentralized equilibrium allocation, workers follow a mixed strategy with respect to the choice of submarket. Specifically, workers allocate their applications across submarkets so that the job-finding rate is constant with respect to listing age. Younger listings, i.e., new listings and recently renewed listings, receive a high weight and a disproportionate number of applications while older listings receive relatively few applications. If the social planner is limited to choosing how many new vacancies are posted and the age at which job listings are renewed, we show there is a modified Hosios condition that implements the constrained efficient allocation. In this case, if firms can post and commit to a wage rather than engage in Nash bargaining, a competitive search equilibrium decentralizes the social planner allocation.[7] However, when the social planner can also choose the allocation of job seekers across submarkets, then the equilibrium allocation is generically inefficient. In equilibrium, job

[4] The idea that time on the market may be used as a signal of quality when search frictions are present has been explored in several papers, e.g., Kim (2017).

[5] Based on their analysis of the Dice.com data, Davis and Samaniego (2018) suggest that models of nonsequential search are the better tool for analyzing labor market outcomes. However, their data pertain to high-skill workers seeking technical jobs where screening is likely to be particularly important.

[6] We argue that phantoms are likely to be important even if firms search nonsequentially. With nonsequential search, firms advertise a position and then collect a list of applicants. At some point – when the applicant list is long enough – the firm makes an offer. The longer a position has been advertised, the more likely it is that an offer has been made and accepted, i.e., the more likely it is that the listed vacancy is a phantom.

[7] Note that ours is a model of directed search rather than competitive search. In our baseline model, workers direct their search based on listing age. Listing age is a variable that firms can't control (except by renewal), i.e., they cannot compete by choosing listing age. In competitive search equilibrium, however, firms can compete for worker applications through their posted wages.

seekers direct their search more toward younger listings than the social planner would have them do. The equilibrium allocation of worker applications across listing ages generates a dynamic congestion externality. When matches are formed, phantoms congest the market, and a phantom that is created when a younger listing leads to a match is more costly than one that is created when an older listing does so. That is, the magnitude of the externality decreases with the listing age of the vacancy, and workers have no incentive to incorporate this dynamic effect into their decision calculus.[8]

We supplement our theoretical results with numerical illustrations (WORK IN PROGRESS). We do this in three steps. First, we calibrate our model to US labor market outcomes over the period 2000-2008. In doing this, we make a baseline parameter assumption, namely, that the fraction of ads that are not removed when the corresponding vacancy is filled equals 1/2. We then examine the sensitivity of our result to changes in this key parameter. Second, we numerically solve for the constrained efficient allocation when the social planner can choose the level of new vacancy creation and the listing renewal age but is unable to direct worker search. Finally, we numerically solve for the full constrained efficient allocation; i.e., we allow the social planner to allocate job seekers across listing ages in addition to setting vacancy creation and listing age.

Discussion of numerical results.

In the next section, we lay out our model of directed search with phantom vacancies. We then discuss the social planner's problem in Section 3. Section 4 contains our numerical simulations and conclusions are given in Section 5.

2 Decentralized Equilibrium

In this section we develop an equilibrium search and matching model of unemployment with directed search by listing age, phantom vacancies, endogenous job creation and endogenous listing renewal. We set up the model and analyze the decentralized allocation.

[8]A related effect is present in the frictionless dynamic matching model of Board, Meyer-ter-Vehn and Sadzik (2017). In their model, each firm hires a worker from a pool of applicants. Firms with more talented incumbent workers are better at recruiting (distinguishing good from bad hires) and offer higher wages in order recruit before other firms. This leads to dynamic adverse selection – the further down a firm is in the recruiting queue, the lower is the average quality of the applicant pool it faces. The analogy with our result is that the adverse selection that a firm creates through its hiring decision is greater the higher the firm's position in the recruiting queue. The externality created thus has a cascading quality as does our phantom externality.

2.1 The model setup

We focus on the stationary state of a continuous time model. The unit of time is a month. There is a continuum of workers of unit mass, and each worker can be either employed or unemployed. The endogenous mass of unemployed is u. There is also a continuum of vacancies of endogenous mass v. There is a listing for each vacancy, and these differ by age, $a \geq 0$.

Creating a new vacancy comes at a one-time cost c. Once the vacancy is created, the firm has a listing that gradually ages with calendar time. The listing can be renewed at any time at cost k, $0 < k < c$. In exchange for the renewal cost, the firm has a new listing of age 0. We will endogenize the age, A, at which a firm that has failed to fill its vacancy renews its listing.

Filled jobs produce y. All jobs, filled or vacant, are destroyed at Poisson rate λ, and newly separated workers join the pool of unemployed.

The labor market is segmented by listing age. In submarket a, $u(a)$ unemployed try to match with $v(a)$ vacancies. Each time there is a match, with probability β, the corresponding ad is not withdrawn and a phantom vacancy is created. Phantoms, once created, persist in the market.

The matching process is frictional. In addition to the usual search frictions, information persistence about vacancies that have already been filled but are still advertised creates an added friction. The flow of new matches in submarket a is

$$M(a) = \pi(a)m(u(a), v(a) + p(a)),$$

where $p(a)$ is the phantom flow, $\pi(a) = \frac{v(a)}{v(a)+p(a)}$ is the nonphantom proportion. The contact function, $m(\cdot)$, is strictly increasing in both arguments, strictly concave and has constant returns to scale.[9]

Workers cannot distinguish between phantoms and nonphantoms. Therefore the number of contacts $m(\cdot)$ depends on the number of listings $v(a) + p(a)$. As no one can match with a phantom, $m(\cdot)$ is multiplied by the fraction of contacts that are with unfilled vacancies.

We denote market tightness for listings of age a by $\theta(a) = (v(a)+p(a))/u(a)$. The job-finding rate for submarket a is $\mu(a) = m(1, \theta(a))\pi(a)$ and the job-filling rate is $\eta(a) = m(1, \theta(a))/\theta(a)$. We write $m(\theta)$ for $m(1, \theta)$ below when there is no risk of confusion.

Each time a vacancy is filled or destroyed, a phantom is created with probability $\beta \in [0, 1]$. For all $a \in [0, A]$, phantoms and vacancies evolve as follows:

$$\begin{aligned}
\dot{v}(a) &= -(\eta(a) + \lambda)v(a), \\
\dot{p}(a) &= \beta(\eta(a) + \lambda)v(a),
\end{aligned}$$

[9] The form of the matching function is taken from Chéron and Decreuse (2017). They derive their matching function from first principles in discrete time and then extend it to continuous time. As noted in the introduction, their analysis is based on random search, whereas ours is a model of directed search based on listing age.

where a dot over a variable denotes its derivative with respect to the listing age. These laws of motion imply $p(a) = \beta(v(0) - v(a))$.

The resulting nonphantom proportion has the following law of motion:

$$\dot{\pi}(a) = -(m(\theta(a))/\theta(a) + \lambda)\pi(a)(1 - (1 - \beta)\pi(a)).$$

The nonphantom proportion decreases with age. Phantoms accumulate as employers gradually fill their jobs and as vacancies are destroyed at exogenous rate λ.

2.2 Equilibrium

Workers direct their search by listing age. The values of being unemployed, U, and employed, W, are defined as follows:

$$rU = \max_{a \in [0,A]} \{b + \mu(a)[W - U]\}, \tag{1}$$

$$rW = w + \lambda[U - W]. \tag{2}$$

In equilibrium, job seekers allocate themselves across the different listing ages so that the job-finding rate $\mu(a) = \pi(a)m(\theta(a))$ stays constant; i.e., since $\pi(0) = 1$, $\pi(a)m(\theta(a)) = m(\theta(0))$ for all a. Since the nonphantom proportion decreases with listing age, this condition implies that market tightness increases with listing age.

Let $V(a)$ be the value of a vacancy of age a and let J be the value of a filled job. We have

$$rV(a) = \eta(a)[J - V(a)] - \lambda V(a) + V'(a), \tag{3}$$

$$rJ = y - w - \lambda J. \tag{4}$$

The value of a vacancy changes with the listing age, reflecting the rate of applications by vacancy age and the length of time until renewal. Immediately after renewal, the listing age is reset to 0. By continuity of the value function, we have $V(A) = V(0) - k$. Let $\overline{F}(a) = \exp(-\int_0^a (\eta(s) + \lambda)ds)$ be the survival probability for a vacancy, i.e., the probability that the vacancy is neither filled nor destroyed by age a. Integrating equation (3) forward with this boundary condition gives

$$V(0) = \frac{J \int_0^A \eta(s)e^{-rs}\overline{F}(s)ds - ke^{-rA}\overline{F}(A)}{1 - e^{-rA}\overline{F}(A)}. \tag{5}$$

To derive the optimal renewal age, we need to discuss the firms' coordination problem. Consider a firm with the belief that all other firms set renewal age $\tilde{A} > 0$ and that rational workers also hold this belief. In this case, workers would not consider job listings older than $\tilde{A}$. They would suppose that these must be phantoms. It follows that the firm must either set a renewal age lower than $\tilde{A}$ or equal to it. Keeping the listing after $\tilde{A}$ would be pointless because the job-filling rate is 0 after $\tilde{A}$.

Suppose first that $\tilde{A}$ is very large. The optimal renewal age $\hat{A}$, the firm's best-response to $\tilde{A}$, results from the condition $V'(\hat{A}) = 0$. Using equation (3), we obtain

$$V(\hat{A}) = V(0) - k = \frac{\eta(\hat{A})}{r + \lambda + \eta(\hat{A})} J. \tag{6}$$

Firms renew their listings when the rate of applications to their vacancies becomes sufficiently small. The threshold rate, $\eta(\hat{A})$, increases as the value of a filled job falls. Therefore parameters that decrease the value of a filled job such as λ raise the threshold rate $\eta(\hat{A})$.

Now suppose that $\tilde{A}$ is small, i.e., smaller than the $\widehat{A}$ given by equation (6). Then the firm sets $A = \tilde{A}$. It follows that any $\tilde{A}$ belonging to $[0, \hat{A}]$ can be an equilibrium of the renewal game. Hereafter, we only focus on equilibria such that $\tilde{A} = \hat{A}$.[10] As we show later, an equilibrium of this type produces the choice of renewal age, A, that the social planner would choose when the social planner can select the renewal age and the number of vacancies to create but cannot direct the job seekers' choice of submarket.

The wage is determined by Nash bargaining over the match surplus. We assume this wage can be renegotiated at any time, which explains why it is not conditional on the listing age. We also assume that the job is destroyed if the firm and the worker do not reach an agreement. This means that the firm's outside option is 0. If $\gamma \in (0, 1)$ denotes workers' bargaining power, then the wage that maximizes the Nash product solves $(1 - \gamma)(W - U) = \gamma J$. Using equations (1) to (4) and solving for the wage gives

$$\begin{aligned} w &= \gamma y \frac{r + \lambda + m(\theta(0))}{r + \lambda + \gamma m(\theta(0))} + (1 - \gamma) b \frac{r + \lambda}{r + \lambda + \gamma m(\theta(0))}, \\ J &= \frac{(1 - \gamma)(y - b)}{r + \lambda + \gamma m(\theta(0))}. \end{aligned}$$

Finally, free entry implies that firms create vacancies until $V(0) = c$.

[10] Such equilibria are symmetric. There cannot be asymmetric equilibria. In an asymmetric equilibrium, there would be at least two distinct $\hat{A}$ solving the first-order condition (6), say A_1 and A_2, with $A_1 < A_2$ without loss of generality. This would imply that $\theta(A_1) = \theta(A_2)$, with $\theta(a) > 0$ for $a \in [A_1, A_2]$. This is impossible because the no-arbitrage condition states $\pi(a)m(\theta(a)) = m(\theta(0))$, whereas $\pi(A_1) > \pi(A_2)$.

The equilibrium allocation is characterized by the following system of equations:

$$\begin{aligned}
m(\theta(0)) &= \pi(a)m(\theta(a)), & (7)\\
\pi'(a) &= -\left[\frac{m(\theta(a))}{\theta(a)}+\lambda\right]\pi(a)[1-(1-\beta)\pi(a)], & (8)\\
c(1-e^{-rA}\overline{F}(A))+ke^{-rA}\overline{F}(A) &= \frac{(1-\gamma)(y-b)}{r+\lambda+\gamma m(\theta(0))}\int_0^A \eta(a)e^{-ra}\overline{F}(a)da, & (9)\\
c-k &= \frac{\eta(A)}{r+\lambda+\eta(A)}\frac{(1-\gamma)(y-b)}{r+\lambda+\gamma m(\theta(0))}, & (10)\\
\mathsf{u} &= \lambda/(\lambda+m(\theta(0))), & (11)
\end{aligned}$$

with $\pi(0)=1$ and $\eta(a)=m(\theta(a))/\theta(a)$.

In the free-entry condition (9), the left-hand side is the mean stock cost of a new listing. The cost of renewal is weighted by the discounted probability $e^{-rA}\overline{F}(A)$ that the job is still available at the renewal age. The cost of a new vacancy is weighted by the complementary term, $1-e^{-rA}\overline{F}(A)$. The right-hand side is the value of a filled job, $\frac{(1-\gamma)(y-b)}{r+\lambda+\gamma m(\theta(0))}$, multiplied by the discounted probability that the job is filled before the renewal age is reached.

From the no-arbitrage condition (7), we can express tightness in submarket a as a function of initial tightness and the nonphantom proportion, i.e., $\theta(a)=m^{-1}(m(\theta(0))/\pi(a))$. We then insert this into the law of motion of the nonphantom proportion (8) to obtain the following Cauchy problem:

$$\dot{\pi}(a)=-\left[\frac{m(\theta(0))/\pi(a)}{m^{-1}(m(\theta(0))/\pi(a))}+\lambda\right]\pi(a)[1-(1-\beta)\pi(a)],\pi(0)=1.$$

We write the solution to this problem as $\pi(a,\theta(0))$ to highlight the dependence on $\theta(0)$. Solving for equilibrium reduces to finding the initial tightness $\theta(0)$ and the optimal renewal age such that the free-entry condition (9) and the optimal renewal condition (10) hold. Lastly, unemployment is determined by the Beveridge curve (11).

There exists an equilibrium provided $(1-\gamma)(y-b)/(r+\lambda)>c$. The left-hand side is the maximum value of a filled job. This maximum value must exceed the cost of job creation. Initial tightness qualitatively responds to changes in the economic environment in a standard way. It decreases with workers' bargaining power, γ, unemployment income, b, the discount rate, r, the job destruction rate, λ, and the vacancy creation cost, c. It increases with output per worker, y. Unemployment varies accordingly.

The density function of listings by listing age is

$$\phi_{v+p}(a)=(v(a)+p(a))/\int_0^A (v(s)+p(s))ds$$

and the associated cumulative distribution function is

$$\Phi_{v+p}(a)=\int_0^a (v(s)+p(s))ds/\int_0^A (v(s)+p(s))ds.$$

The growth rate of the density is $\dot{\phi}_{v+p}/\phi_{v+p} = -(1-\beta)(\eta+\lambda)\pi \leq 0$. The growth rate is generally declining, reflecting the listing stock depletion as workers gradually find jobs. However, phantom formation reduces the rate of depletion. The listing density does not vary with age when $\beta = 1$ because the job-filling rate is exactly offset by the rate at which phantoms are created.

The density function of applications by listing age is $\phi_u(a) = u(a)/\mathsf{u} = (v(a)+p(a))/(\theta(a)\mathsf{u})$ and the associated cumulative distribution function is $\Phi_u(a) = \mathsf{u}^{-1}\int_0^a (v(s)+p(s))\theta(s)^{-1}ds$. The growth rate of the density is $\dot{\phi}_u/\phi_u = \dot{\phi}_{v+p}/\phi_{v+p} - \dot{\theta}/\theta < \dot{\phi}_{v+p}/\phi_{v+p}$ when $\beta > 0$, i.e., when there are phantoms in the market. Since the growth rate of the density of applications with age is less than the corresponding growth rate of listings, the model predicts a concentration of job seekers' efforts at younger listing ages than would occur in the absence of phantoms.

To conclude this section, we briefly discuss the case without phantoms. This corresponds to $\beta = 0$. The nonphantom proportion is one at all ages and the no-arbitrage condition becomes $m(\theta(a)) = m(\theta(0))$. Tightness does not change across submarkets. As the job-filling rate does not decrease with age, firms have no incentive to renew their listings. Formally, the value $V(A) = V(0)$ for all possible renewal ages so it is not worth paying the cost k, as small as it may be, to get a new listing. Thus, $A \to \infty$ and the free-entry condition becomes

$$c = \frac{\eta(\theta(0))}{r+\eta(\theta(0))+\lambda}\frac{(1-\gamma)(y-b)}{r+\lambda+\gamma m(\theta(0))}. \tag{12}$$

This equation is analogous to the standard free-entry condition with a flow cost of posting jobs. The left-hand side is a stock cost and therefore the right-hand side is a fraction of the value of a filled job. The economic mechanism behind entry, though, is fundamentally the same as in the standard model. An increase in tightness has two effects that reduce the value of a vacancy. First, it increases the bargained wage, thereby lowering the value of a filled job. Second it decreases the job-filling rate.

3 Constrained efficient allocation

To focus on steady states, we study the constrained efficient allocation in the case in which the discount rate, r, tends to 0. We first consider the case in which the planner is unable to direct worker search so that the job-finding rate does not vary across submarkets. We show that the decentralized allocation can generate the constrained efficient allocation provided a modified Hosios condition holds. We then turn to the case in which the planner allocates job seekers across the different submarkets. In this case, we highlight a novel externality: in the decentralized equilibrium, due to phantoms, job seekers over-apply to young listings relative to the social planner optimum.

In both cases, the planner maximizes net flow output less vacancy creation costs and listing renewal costs, i.e.,

$$\Omega = b + (1 - \mathbf{u})(y - b) - c\underbrace{[v(0) - v(A)]}_{\text{new vacancies}} - k \underbrace{v(A)}_{\text{renewals}} . \tag{13}$$

3.1 Social Planner Problem 1

In this subsection, we assume that the social planner can choose the flow of new vacancies, $v(0) - v(A)$, and the renewal age, A, but is unable to allocate job seekers across listing ages. This means that the job-finding rate must be the same in all submarkets, i.e., $\pi(a)m(\theta(a)) = m(\theta(0))$ for all $a \in [0, A]$.

It is useful to express $v(0)$ and $v(A)$ as functions of θ_0 and A, where θ_0 is the planner's choice of initial market tightness. This allows us to write the planner's objective in a more convenient form. To do this, we use the steady-state condition that the rate at which vacancies are filled equals the rate at which workers find jobs, i.e.,

$$\begin{aligned} v(0) \int_0^A \eta(a)\overline{F}(a)da &= m(\theta_0)\mathbf{u} \text{ or} \\ v(0) &= m(\theta_0)\mathbf{u} / \int_0^A \eta(a)\overline{F}(a)da. \end{aligned}$$

Then, using $v(A) = v(0)\overline{F}(A)$, we rewrite the social planner problem as

$$\max_{\theta_0, A} \left\{ \frac{m(\theta_0)}{\lambda + m(\theta_0)}(y - b) - \frac{\lambda}{\lambda + m(\theta_0)} m(\theta_0) Z(\theta_0, A) \right\},$$

where $Z(\theta_0, A) = [c(1 - \overline{F}(\theta_0, A)) + k\overline{F}(\theta_0, A)] / \int_0^A \eta(\theta_0, a)\overline{F}(\theta_0, a)da$. This notation highlights the dependence of the survivor function and the job-filling rate across submarkets on θ_0.

This social planner objective differs from the standard one in three ways. First, absent phantoms, in the standard model, there is never an incentive to renew a vacancy, i.e., $A \to \infty$. Second, since $A \to \infty$ and $\theta(a)$ is a constant in the standard model, the denominator of $Z(\theta_0, A)$, i.e., $\int_0^A \eta(\theta_0, a)\overline{F}(\theta_0, a)da$, is simply $m(\theta_0)/\theta_0$, so $v(0) = \theta_0\mathbf{u}$. In our problem, the social planner needs to take into account the fact that the choice of θ_0 determines the profile of the job-filling rate $\eta(\theta_0, a)$ across submarkets. Finally, in the usual formulation, the cost associated with posting and maintaining vacancies is a constant independent of market tightness. Here, in contrast, the vacancy cost depends directly on θ_0 since an increase in initial market tightness makes it more likely that the vacancy renewal cost will be incurred.

To characterize the social planner optimum, let $\alpha_0 = \theta_0 m'(\theta_0)/m(\theta_0)$ be the elasticity of the contact function with respect to initial market tightness, let $\sigma_0 = \theta_0 Z_{\theta_0}(\theta_0, A)/Z(\theta_0, A)$ be the elasticity of $Z(\theta_0, A)$ with respect to θ_0, and define $\varepsilon_0 = \alpha_0/(\alpha_0 + \sigma_0)$. The first-order condition of the social planner problem with respect to θ_0 gives

$$\begin{aligned} Z(\theta_0, A) &= \varepsilon_0 \frac{y-b}{\lambda + (1-\varepsilon_0)m(\theta_0)}; \text{ i.e.,} \\ c(1-\overline{F}(\theta_0, A)) + k\overline{F}(\theta_0, A) &= \varepsilon_0 \frac{y-b}{\lambda + (1-\varepsilon_0)m(\theta_0)} \int_0^A \eta(\theta_0, a)\overline{F}(\theta_0, a)da. \end{aligned} \tag{14}$$

The left-hand side is the average cost of a new listing, which is a weighted average of the cost of a new vacancy, c, and the cost of renewal, k. The weights correspond to the respective shares of new vacancies and renewals in each new cohort of listings. The right-hand side is the social value of a filled job, $\varepsilon_0 \frac{y-b}{\lambda + (1-\varepsilon_0)m(\theta_0)}$, multiplied by the probability that the listing is filled before A is reached, $\int_0^A \eta(\theta_0, a)\overline{F}(\theta_0, a)da$.

The first-order condition with respect to A gives

$$(c-k)[\eta(\theta_0, A) + \lambda] \int_0^A \eta(\theta_0, a)\overline{F}(\theta_0, a)da = \eta(\theta_0, A)[c(1-\overline{F}(\theta_0, A)) + k\overline{F}(\theta_0, A)]. \tag{15}$$

The left-hand side is the increase in average listing cost induced by a higher A. The right-hand side is the marginal welfare gain due to the higher probability of vacancy filling. In computing this condition, we use the fact that $f(\theta_0, a) = (\eta(\theta_0, a) + \lambda)\overline{F}(\theta_0, a)$.

Combining equations (14) and (15) gives

$$c - k = \frac{\eta(\theta_0, A)}{\eta(\theta_0, A) + \lambda} \varepsilon_0 \frac{y-b}{\lambda + (1-\varepsilon_0)m(\theta_0)}. \tag{16}$$

A listing is renewed when the capital gain induced by renewal, $c-k$, is equal to the opportunity cost of keeping the listing alive, i.e., a term measuring the probability of finding a worker multiplied by the social value of a filled job.

There is a modified Hosios condition under which the decentralized equilibrium gives the constrained efficient allocation. Comparing equations (14) and (15) to equations (9) and (10) when $r \to 0$ implies that the two allocations coincide, i.e., the initial labor market tightness and the renewal date in the decentralized equililbrium equal those chosen by the social planner when the bargaining power $\gamma = 1 - \varepsilon_0$. That is, the elasticity of the matching function in the standard Hosios condition must be replaced by $\varepsilon_0 = \alpha_0/(\alpha_0 + \sigma_0)$.

This condition differs from the standard Hosios condition for several reasons. First, as noted above, the cost per vacancy varies with θ_0. This generates a modified Hosios condition for a reason similar to the one discussed in Julien and Mangin (2019).[11] Second, also as discussed

[11] Julien and Mangin (2019) derive a modified Hosios condition that achieves the constrained efficient outcome. Their modified Hosios condition accounts for the possibility that net flow output per filled vacancy may vary with market tightness. The similarity here is that the cost per vacancy varies with (initial) market tightness.

above, θ_0 determines the shape of market tightness across submarkets, and hence the job-filling rate, across submarkets, a feature that is absent in the model without phantoms. Finally, and less essentially, our model differs from the standard one insofar as (i) there is a fixed cost of vacancy creation rather than a flow cost and (ii) vacancies are destroyed at a constant rate. Neither (i) nor (ii) are due to the presence of phantoms. In Appendix A, we show that when firms can post and commit to wages rather than engaging in Nash bargaining, the corresponding competitive search equilibrium decentralizes the social planner solution in this case.

If the phantom birth rate is set to zero, firms do not renew their listings and $A \to \infty$. In this case, the job-filling rate, η, is a constant over listing age and depends only on initial labor market tightness, θ_0, i.e., the job-filling rate is $\eta(\theta_0)$ and $Z(\theta_0) = c(\eta(\theta_0) + \lambda)/\eta(\theta_0)$. This in turn implies that the elasticity of $Z(\theta_0)$ with respect to θ_0 is $\sigma_0 = (1 - \alpha_0)\lambda/(\eta(\theta_0) + \lambda)$, and so

$$\varepsilon_0 = \frac{\alpha_0}{\alpha_0 + (1 - \alpha_0)\lambda/(\eta(\theta_0) + \lambda)}. \tag{17}$$

Again, the decentralized equilibrium allocation is constrained efficient provided that $\gamma = 1 - \varepsilon_0$.

Obsolete information and listing renewal only affect the modified Hosios condition through the elasticity σ_0. Vacancy creation and listing renewal do not in themselves generate new externalities.

3.2 Social Planner Problem 2

We now turn to the case in which the planner can allocate job seekers across submarkets. We fix the replacement age, A, and the flow of new vacancies, $n_0 = v(0) - v(A)$, to their values in the decentralized equilibrium. This allows us to focus on the externality associated with phantoms. We then suppose that the planner chooses tightness by listing age to maximize the social criterion, Ω.

We first rewrite Ω in a more convenient form. Noting that

$$\lambda(1 - \mathbf{u}) = \int_0^A \mu(a)u(a)da$$

and

$$\int_0^A \mu(a)u(a)da = \int_0^A \eta(a)v(a)da = \int_0^A (\eta(a) + \lambda)v(a)da - \lambda \int_0^A v(a)da = n_0 - \lambda \int_0^A v(a)da,$$

the planner's objective can be rewritten as

$$\Omega = b + n_0 \left(\frac{y - b}{\lambda} - c \right) - (y - b) \int_0^A v(a)da - kv(A). \tag{18}$$

The constrained efficient allocation is then the solution to the following optimal control problem:

$$\max_{\theta(\cdot)} \left\{ -(y - b) \int_0^A v(a)da - kv(A) \right\} \tag{$*$}$$

subject to

$$
\begin{aligned}
\dot{v} &= -(\eta(\theta)+\lambda)v && (c1)\\
v(0) &= n_0 + v(A) && (c2)\\
\dot{p} &= \beta(\eta(\theta)+\lambda)v && (c3)\\
p(0) &= 0, && (c4)\\
\int_0^A \left[(v(a)+p(a))/\theta(a) - v(a)\right] da &= 1 - n_0/\lambda. && (c5)
\end{aligned}
$$

The planner is constrained by the evolution of vacancies by age, $(c1)$ - $(c2)$, the evolution of phantoms by age, $(c3)-$ $(c4)$, and the equality between the flows into and out of unemployment $(c5)$. Equation $(c5)$ can be rewritten as

$$n_0 = \lambda(1-u) + \lambda \int_0^A v(a)da$$

and can be interpreted as a resource constraint in the sense that new vacancies just equal destroyed jobs plus destroyed vacancies.

Let $\sigma_1(a)$ and $\sigma_2(a)$ be the costates associated with the state variables $v(a)$ and $p(a)$, respectively. In the discussion that follows, we use normalized versions of these costates, $s_1(a)$ and $s_2(a)$. The normalizing constant is such that the value of applying to a new listing equals one. Also, let $\theta^{\text{eff}} : [0, A] \to \mathbb{R}_+$, $\pi^{\text{eff}} : [0, A] \to [0, 1]$, and $u^{\text{eff}} : [0, A] \to \mathbb{R}_+$ denote the social planner's solution.[12]

In Appendix B, we prove that the solution to the social planner problem has the following properties for all $a \in [0, A]$:

(i) $(\beta s_2(a) - s_1(a))(1-\alpha(\theta^{\text{eff}}))\pi^{\text{eff}} m(1, \theta^{\text{eff}}) = 1$ where $s_1(a) < \beta s_2(a) \le \beta s_2(A) = 0$.

The functions $s_1 : [0, A] \to \mathbb{R}$ and $s_2 : [0, A] \to \mathbb{R}$ satisfy

$$
\begin{aligned}
\dot{s}_1 &= B + \frac{1}{\theta^{\text{eff}}} - \frac{1}{(1-\alpha(\theta^{\text{eff}}))\pi^{\text{eff}}\theta^{\text{eff}}} - \frac{\lambda}{(1-\alpha(\theta^{\text{eff}}))\pi^{\text{eff}} m(\theta^{\text{eff}})}, && (19)\\
\dot{s}_2 &= \frac{1}{\theta^{\text{eff}}}, && (20)
\end{aligned}
$$

with

$$s_1(A) = s_1(0) - (B+1)k/(y-b) = -[(1-\alpha(\theta^{\text{eff}}(A)))\pi^{\text{eff}}(A) m(1, \theta^{\text{eff}}(A))]^{-1}$$

$$\beta s_2(0) = [(1-\alpha(\theta^{\text{eff}}(0))) m(1, \theta^{\text{eff}}(0))]^{-1} - [(1-\alpha(\theta^{\text{eff}}(A)))\pi^{\text{eff}}(A) m(1, \theta^{\text{eff}}(A))]^{-1} + (B+1)k/(y-b)$$

[12] Note that in an effort to keep the equations manageable, we suppress the dependence of θ^{eff}, π^{eff}, and u^{eff} on a.

and

$$B = \frac{\alpha(\theta^{\text{eff}}(0)) + \lambda\theta^{\text{eff}}(0)/m(1,\theta^{\text{eff}}(0))}{1-\alpha(\theta^{\text{eff}}(0))}\frac{u^{\text{eff}}(0)}{n_0} - \frac{\alpha(\theta^{\text{eff}}(A)) + \lambda\theta^{\text{eff}}(A)/m(1,\theta^{\text{eff}}(A))}{1-\alpha(\theta^{\text{eff}}(A))}\frac{u^{\text{eff}}(A)}{n_0} > -1.$$

(ii) Market tightness and the nonphantom proportion evolve according to

$$-\frac{\theta^{\text{eff}}\alpha'(\theta^{\text{eff}})}{1-\alpha(\theta^{\text{eff}})}\frac{\dot{\theta}^{\text{eff}}}{\theta^{\text{eff}}} + \alpha(\theta^{\text{eff}})\frac{\dot{\theta}^{\text{eff}}}{\theta^{\text{eff}}} + \frac{\dot{\pi}^{\text{eff}}}{\pi^{\text{eff}}} = \left(B + \frac{1-\beta}{\theta^{\text{eff}}}\right)(1-\alpha(\theta^{\text{eff}}))\pi^{\text{eff}}m(1,\theta^{\text{eff}}) - \left(\frac{m(1,\theta^{\text{eff}})}{\theta^{\text{eff}}} + \lambda\right) \quad (21)$$

$$\frac{\dot{\pi}^{\text{eff}}}{\pi^{\text{eff}}} = -\left(\frac{m(1,\theta^{\text{eff}})}{\theta^{\text{eff}}} + \lambda\right)\left(1-(1-\beta)\pi^{\text{eff}}\right). \quad (22)$$

The initial conditions are $\pi^{\text{eff}}(0) = 1$ and $\theta^{\text{eff}}(0) = \theta_0$, where θ_0 is defined implicitly by the resource constraint ($c5$).

(iii) The nonphantom proportion $\pi^{\text{eff}}(a)$ is everywhere decreasing in a, and, provided $\alpha'(\theta) \leq 0$, the job-finding rate is higher when searching listings of age 0 than when searching listings of age A, i.e., $\mu(0) > \mu(A)$.

Property (i) describes the efficient allocation of job seekers across listing ages. Allocating an additional job seeker to a listing of age a increases expected employment by $(1-\alpha(\theta))\pi m(1,\theta)$, but this increase in expected employment entails an opportunity cost, namely, the creation (with probability β) of a phantom with shadow value $s_2(a)$ and the elimination of a vacancy with shadow value $s_1(a)$.

Both shadow values are negative and the phantom value is larger (less negative) than the vacancy value. The shadow value of a vacancy is negative because of the resource constraint. Adding an additional age a vacancy means one fewer filled job and hence less output. The shadow value of a phantom is also negative since phantoms hinder the matching process. The shadow value $s_2(a)$ quantifies the externality associated with phantoms. Integrating equation (20) forward gives $s_2(a) = -\int_a^A \theta(b)^{-1}db$. At given tightness $\theta = (v+p)/u$, having an additional phantom increases the number of job seekers by $1/\theta$, a social loss because these job seekers could be allocated to different submarkets. As the phantom ages, this impact persists, though its magnitude varies with tightness. The shadow value of a phantom is the cumulative impact from its current age to A.

The shadow value $s_1(a)$ accounts for the social costs and benefits of a vacancy. Integrating equation (19) forward gives

$$s_1(a) = s_1(A) - \int_a^A \{B + \theta(b)^{-1}\}db + \int_a^A \{[(1-\alpha(\theta(b)))\pi(b)\theta(b)]^{-1} + \lambda[(1-\alpha(\theta(b)))\pi(b)m(1,\theta(b))]^{-1}\}db$$

That is, the shadow value of a vacancy of age a equals the shadow value it will have if it remains unfilled at age A minus any costs or benefits associated with the vacancy in the interim. There is a fixed opportunity cost B at each instant in time, and given the tightness θ, the vacancy attracts $1/\theta$ job seekers, just as phantoms do. The benefits are represented by the final integral. Recalling that $(\beta s_2(a) - s_1(a))(1-\alpha(\theta))\pi m(1,\theta) = 1$ and that the vacancy is converted into a job at rate $\eta(\theta(a)) = m(1,\theta(a))/\theta(a)$ and gets destroyed at rate λ, the last term can be rewritten as

$$\int_a^A \{[\eta(\theta(b)) + \lambda][\beta s_2(b) - s_1(b)]\}db.$$

That is, it is the rate at which the vacancy is filled or destroyed times the associated change in value.

Property (ii) shows the laws of motion for tightness and the nonphantom proportion in the social planner solution. By differentiating the allocation rule in (i) and using (19) and (20), we can eliminate the two costate variables. Equation (21) shows the law of motion for $(1-\alpha(\theta(a)))\pi(a)m(\theta(a))$ the marginal productivity of an age a vacancy, and equation (22) shows the law of motion for the nonphantom proportion. Combining the two equations, we obtain the following differential equation characterizing the change in tightness:

$$[-\frac{\theta^{\text{eff}}\alpha'(\theta^{\text{eff}})}{1-\alpha(\theta^{\text{eff}})} + \alpha(\theta^{\text{eff}})]\frac{\dot{\theta}^{\text{eff}}}{\theta^{\text{eff}}} = B(1-\alpha(\theta^{\text{eff}}))\pi^{\text{eff}}m(1,\theta^{\text{eff}}) - (1-\beta)\pi^{\text{eff}}(\alpha(\theta^{\text{eff}})\eta(\theta^{\text{eff}}) + \lambda).$$

Property (iii) describes some of the characteristics of the efficient allocation. The nonphantom proportion falls as listings age and the marginal productivity of job seekers is larger at age A than at age 0. When the elasticity α is well-behaved, i.e., when $\alpha'(\theta^{\text{eff}}) \leq 0$, this also implies that the job-finding rate tends to decrease with listing age.

The directed search allocation is not constrained efficient. The rules allocating job seekers across listing ages in the directed search (decentralized) and the constrained efficient allocations are, respectively

$$\pi^{\text{ds}}(a)m(1,\theta^{\text{ds}}(a)) = m(1,\theta^{\text{ds}}(0)) \tag{23}$$

$$(\beta s_2(a) - s_1(a))(1-\alpha(\theta^{\text{eff}}(a)))\pi^{\text{eff}}(a)m(1,\theta^{\text{eff}}(a)) = 1. \tag{24}$$

Comparing equations (23) and (24) reveals three differences.

The first difference is due to the presence of an intertemporal externality that is internalized in the social planner allocation. The social planner accounts for phantom birth and phantom persistence. Allocating a job seeker to listings of a given age translates into more matches at that age, which fuels the phantom stock. The magnitude of this informational externality decreases with age. This occurs for two reasons. First, a young phantom persists for a relatively long time and thus has the potential to affect many job seekers during its lifetime. Second, given the concentration of job seekers at young listing ages, the phantom has the potential to affect a large number of job seekers. Symmetrically, an old phantom has a short life and only impacts the distribution of job seekers where the density is low. Formally, the shadow value of a phantom is negative and strictly increases with listing age. It is equal to 0 when A is reached. After this age, all vacancies have been relisted and phantoms are irrelevant, which explains why phantoms have a smaller effect as A gets closer.

The second difference is unrelated to phantoms. In the directed search allocation, what matters is the job-finding rate $\pi m(1,\theta)$, which must be constant over listing age. This implies that its growth rate, $\dot{\pi}/\pi+\alpha\dot{\theta}/\theta$, equals 0. In the efficient allocation, what matters is the marginal productivity of job seekers $(1-\alpha)\pi m(1,\theta)$, i.e., the job-finding rate times the elasticity $1-\alpha$. This elasticity may vary with age, which is reflected in the term $-[\theta\alpha'/(1-\alpha)]\dot{\theta}/\theta$ on the left-hand side of equation (21). Of course, in the special case of a Cobb-Douglas contact function $\alpha'=0$.

The third difference is also not related to phantoms. Instead, it is due to the relisting cost that must be paid once the listing reaches A. Suppose $\beta=0$ so that there are no phantoms and, for simplicity, suppose also that α does not change with θ. Evaluating equation (24) at $a=A$ and $a=0$ gives

$$-s_1(A)(1-\alpha)m(1,\theta^{\text{eff}}(A)) \;=\; 1, \tag{25}$$

$$-s_1(0)(1-\alpha)m(1,\theta^{\text{eff}}(0)) \;=\; 1. \tag{26}$$

As $s_1(A)=s_1(0)-(B+1)k/(y-b)<s_1(0)$, we have $\mu(A)<\mu(0)$. The relisting cost implies that the shadow value of a vacancy is discontinuous in A. It is then efficient to allocate more job seekers per listing at age A than at age 0. Intuitively, the best way to minimize the duration of a vacancy is to allocate evenly the job seekers across job listings. However, the planner also wants to minimize vacancy renewal spending. The planner is then willing to increase the average duration of a vacancy in exchange for a reduction in the probability of reaching age A.

4 Numerical illustrations

We solve the model when the contact function is Cobb-Douglas, i.e., $m(\theta)=\zeta\theta^{\alpha}$, $0<\alpha<1$. We choose parameters so that the decentralized allocation reproduces various US labor market outcomes. We then compute the constrained efficient allocation, first when the job-finding rate is constrained to be constant across listing ages and second when it can vary optimally with a. Lastly, we discuss the role played by the phantom birth probability.

4.1 Baseline parametrization

The key to simulating the decentralized allocation is that knowledge of θ_0 and A makes it possible to find the functions $\pi^{\mathrm{ds}}(a)$ and $\theta^{\mathrm{ds}}(a)$. (Here the superscript '*ds*' denotes the decentralized, i.e., directed search, allocation.) For our baseline parameterization, we set $A = 1$, corresponding to the maximum age of an ad on Craigslist. We now explain how we set θ_0.

We begin by defining the auxilliary function $\tau(a) = v(a)/v(0)$. Since (i) $\dot{\tau} = -(\eta(\theta) + \lambda)\tau$, (ii) $\theta = \theta_0 \pi^{-1/\alpha}$ (using the assumption that the contact function is Cobb-Douglas), and (iii) $\pi = v/(v+p) = \dfrac{\tau}{\beta + (1-\beta)\tau}$, we have

$$\dot{\tau}^{\mathrm{ds}} = -\left[\zeta \theta_0^{\alpha-1} \left(\frac{\tau^{\mathrm{ds}}}{\beta + (1-\beta)\tau^{\mathrm{ds}}}\right)^{\frac{1-\alpha}{\alpha}} + \lambda\right] \tau^{\mathrm{ds}}, \tag{27}$$

$$\tau^{\mathrm{ds}}(0) = 1. \tag{28}$$

Given a trial value for θ_0, once we set values for α, β, λ and ζ, we can solve this differential equation numerically. We then choose θ_0 to match the average duration of a vacancy, which can be written as

$$d = \int_0^A \tau^{\mathrm{ds}}(a, \theta_0)(1 - \tau^{\mathrm{ds}}(A, \theta_0))^{-1} da.$$

Davis et al (2013) estimate that the mean vacancy duration is between 14 and 25 days. However they also report that the work only starts a couple of weeks later. Thus we set d to one month. This implies an updated value for θ_0, and we iterate to a solution.

To set the parameters α, β, λ and ζ, we proceed as follows. First, for our baseline calibration, we set $\beta = 0.5$, i.e., filled vacancies become phantoms with probability 1/2. We then choose the search parameters α, λ and ζ. Using Shimer's (2005) methodology, we compute the mean job-finding rate and the mean job separation rate for the period 2000-2008. This gives $\mu = 0.5$ and $\lambda = 0.03$. The corresponding unemployment rate is $\mathsf{u} = \lambda/(\lambda + \mu) = 0.0563$.

We set α and ζ to match an aggregate elasticity of hires to the aggregate ratio v/u of about .3. This is the typical estimate when regressing the log job-finding probability on the log vacancy-to-unemployed ratio. Since $\zeta\theta_0^{\alpha} = \mu$, fixing α also sets $\zeta = \mu\theta_0^{-\alpha}$. This elasticity is

$$\frac{d \ln \mu}{d \ln \mathsf{v}/\mathsf{u}} = \alpha \frac{d \ln \theta_0}{d \ln \mathsf{v}/\mathsf{u}}. \tag{29}$$

Phantoms may impact the number of reported vacancies so that econometricians use $(\mathsf{v}+\mathsf{p})/\mathsf{u}$ in lieu of v/u. Therefore we numerically compute the two elasticities.

Once we have solved for θ_0, we have our numerical solution for the differential equation that determines $\tau^{\mathrm{ds}}(a, \theta_0)$, we can solve for π^{ds} and θ^{ds}. We know that $\pi = \tau/(\beta + (1-\beta)\tau)$ and that the equilibrium condition $\pi m(\theta) = m(\theta_0)$ implies $\theta = \theta_0 \pi^{-1/\alpha}$. This gives π^{ds} and θ^{ds} for all $a \in [0, A]$, namely,

$$\pi^{\text{ds}}(a, \theta_0) = \frac{\tau^{\text{ds}}(a, \theta_0)}{\beta + (1-\beta)\tau^{\text{ds}}(a, \theta_0)}, \tag{30}$$

$$\theta^{\text{ds}}(a, \theta_0) = \theta_0 \pi^{\text{ds}}(a, \theta_0)^{-1/\alpha}. \tag{31}$$

We now set the remaining model parameters: r, y, b, γ, c and k. We normalize $y = 1$ and set $r = 0$ to be able to compute the constrained efficient allocation. We set $b = 0.7$, a standard value in the literature.

Last, we set the parameters γ, c and k so that the free-entry condition holds, $A = 1$ is optimally chosen and the bargaining power decentralizes the constrained efficient allocation. That is γ, c and k are chosen to solve

$$\begin{aligned}
\gamma &= 1 - \varepsilon_0, \\
c &= J(\theta_0) \left\{ \int_0^1 \eta(a) e^{-ra} \tau^{\text{ds}}(a, \theta_0) da + \frac{\eta(1)}{r + \lambda + \eta(1)} e^{-rA} \tau^{\text{ds}}(1, \theta_0) \right\}, \\
k &= J(\theta_0) \left\{ \int_0^1 \eta(a) e^{-ra} \tau^{\text{ds}}(a, \theta_0) da - \frac{\eta(1)}{r + \lambda + \eta(1)} \left(1 - e^{-rA} \tau^{\text{ds}}(1, \theta_0)\right) \right\}.
\end{aligned}$$

Table 1 gives the baseline parameters.

λ	ζ	α	y	b	γ	r	c	k	β	θ_0	A
0.03	0.70	0.15	1	0.7	0.15	0.00	2.29	0.11	0.5	0.12	1.0

Table 1: Calibrated parameters

The elasticity of the matching function with respect to the total number of advertised jobs $v + p$ is $\alpha = 0.15$. The corresponding elasticity of the job-finding rate with respect to the vacancy-to-unemployed ratio is $d \ln \mu / d \ln(\mathsf{v}/\mathsf{u}) = 0.29$. When v/u is replaced by $(\mathsf{v} + \mathsf{p})/\mathsf{u}$, this elasticity is 0.37. Workers' bargaining power is $\gamma = 0.15$. It is low because the elasticity $\sigma_0 = 0.026$ is also small below α. This can be expected in the case without phantoms where $\sigma_0 = (1 - \alpha)\lambda/(\eta(\theta_0) + \lambda)$. The job-filling rate $\eta(\theta_0) >> \lambda$, so that σ_0 is low. This implies that the optimal $\gamma = \sigma_0/(\alpha + \sigma_0)$ is also low. The cost of creating a new job is about twice one-month value-added. The cost of renewing ads is 11% of one-month value added. This is larger than the direct cost of ads, but seems in the correct order of magnitude once added wage and utility costs of editing and managing these ads.

We now show some features of the decentralized allocation. Figure 1 compares the distributions of listings, vacancies and job-seekers by listing age. It highlights the bias of the job-seekers

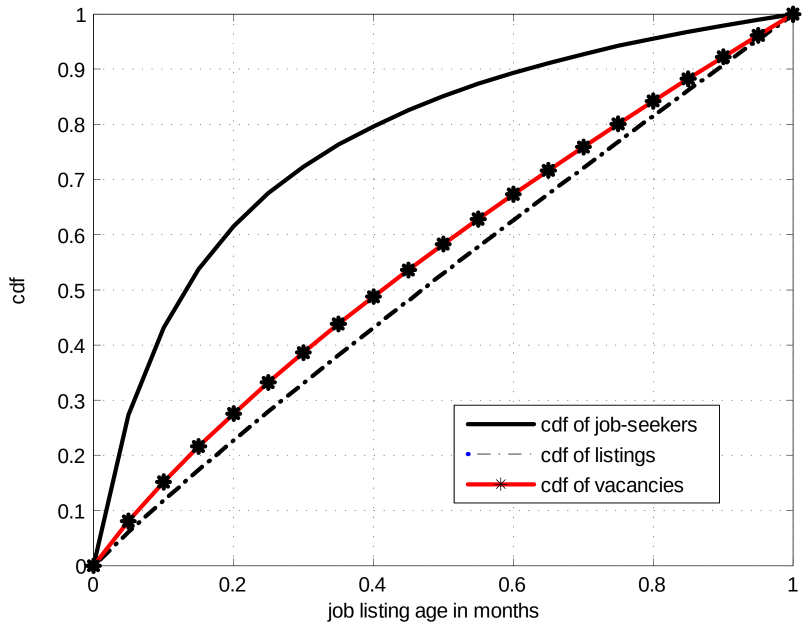


Figure 1: Distributions of listings and job-seekers by listing age

towards young listings. The distribution of vacancies (red crosses) reflects the pattern of hirings, which progressively deplete the stock of vacancies. The distribution of listings (discontinuous blue line) is closer to uniform because half the listings survive job filling and become phantoms. The distribution of job-seekers (black line) is heavily distorted towards young listings because workers fear the phantoms contaminating older listings. In the absence of phantoms, the three distributions would coincide.

A way to quantify the bias towards young listings is to compute job queues at various listing ages. The mean job queue between two dates, say a_1 and a_2, is $(a_2 - a_1)^{-1} \int_{a_1}^{a_2} \theta(a)^{-1} da$. In the directed search allocation, Table 2 shows that the mean job queue is 6.58 job seekers per listing for listings aged less than 24 hours. It falls to 4.51 for listings aged between 24 and 48 hours. The first week concentrates 68% of the job-seekers' efforts. These figures are broadly in line with evidence reported in Davis and Samaniego (2018).

	Density of job seekers	Mean job queue
Day 1	0.21	6.58
Day 2	0.14	4.51
Day 3	0.10	3.35
Day 4	0.07	2.62
Week 1	0.68	0.96
Week 2	0.18	0.52
Week 3	0.09	0.57

Table 2: Mass of job seekers and job queues at various intervals of listing age

4.2 Phantom birth probability

We now examine some of the effects of changes in the phantom birth probability, β. We consider three scenarios: the standard one where $\beta = 0.5$, the pessimistic one where $\beta = 0.75$ and the optimistic one where $\beta = 0.25$. Otherwise, we use the parameter values set in our baseline calibration with two important exceptions. First, changing β deeply modifies the competitive search allocation. This translates into a different workers' bargaining power γ. Formally the elasticity σ_0 is impacted by the new pattern of phantom birth and this also affects the optimal $\gamma = \sigma_0/(\alpha + \sigma_0)$. Varying β while holding A fixed would, of course, imply a different value for θ_0, but $A = 1$ would no longer be the optimal relisting age.

To take this into account, we adapt the solution method used for the baseline calibration. For given θ_0, we numerically find the functions π^{ds} and θ^{ds} and compute the associated optimal renewal age A using equation (10). Then we update θ_0 to ensure that the free-entry condition (9). Lastly we update γ to find the competitive search allocation, i.e., $\gamma = 1 - \varepsilon_0$.

	$\beta = 0.25$	$\beta = 0.50$	$\beta = 0.75$
γ	0.08	0.15	0.26
θ_0	2.77	0.12	0.01
A	1.49	1.000	2.35
u	0.036	0.056	0.112

Table 3: The three scenarios. The model is calibrated with $\beta = 0.5$. Then β is decreased to 0.25 or increased to 0.75, all other parameters being the same but the bargaining power, which is set to its optimal value $\gamma = 1 - \varepsilon_0$.

Table 3 shows the differences between the three scenarios in terms of workers' bargaining power, initial tightness, renewal age and unemployment rate. The picture is dominated by

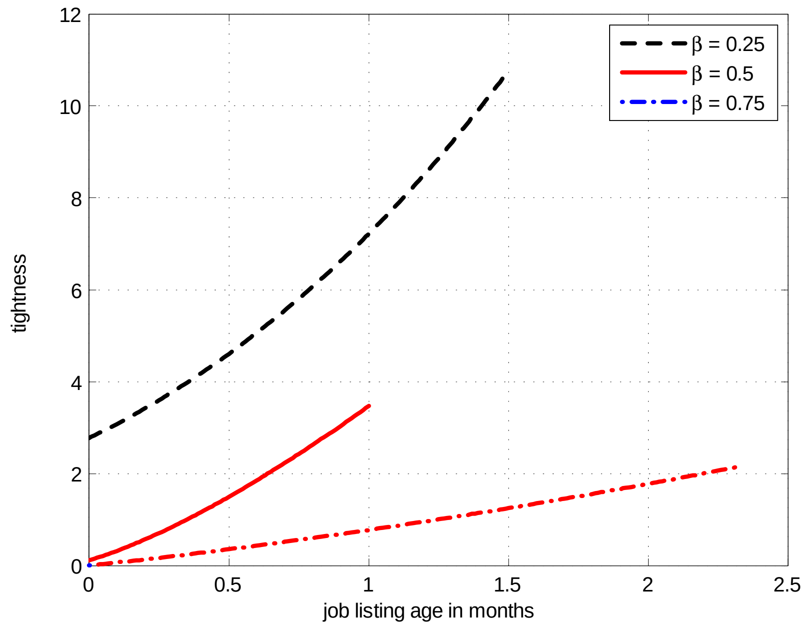


Figure 2: Tightness as a function of listing age and workers' bargaining power

changes in workers' bargaining power, which is halved in the optimistic scenario and doubled in the pessimistic one. The reason is the elasticity σ_0 increases with the magnitude of obsolete information. These differences translate into similar differences in unemployment rate, which more than triple between the pessimistic and the optimistic scenarios.

Initial tightness strongly decreases with β, reflecting the increase in workers' bargaining power and the prevalence of phantoms at higher listing ages. The renewal age is ambiguously impacted. On the one hand, job-seekers prefer younger vacancies when β is large and the rate of filling jobs declines more rapidly with a. On the other hand, workers' bargaining power is higher and this tends to lengthen the renewal age. This reasoning explains why the renewal age is longer both in the case where $\beta = 0.25$ and $\beta = 0.75$.

Figure 2 shows tightness as a function of listing age in the three scenarios. Initial tightness decreases with β. Workers concentrate their search on young listings when β increases.Figure 3 shows the vacancy proportion as a function of listing age in the three scenarios. As expected, this proportion decreases with β. The decline with age is very strong when $\beta = 0.75$. This is due to the combination of having more phantoms associated to match formation and a spectacular decrease in initial tightness due to the much higher workers' bargaining power. The latter effect implies that the job-filling rate is impressively large in the beginning of the listing existence. In

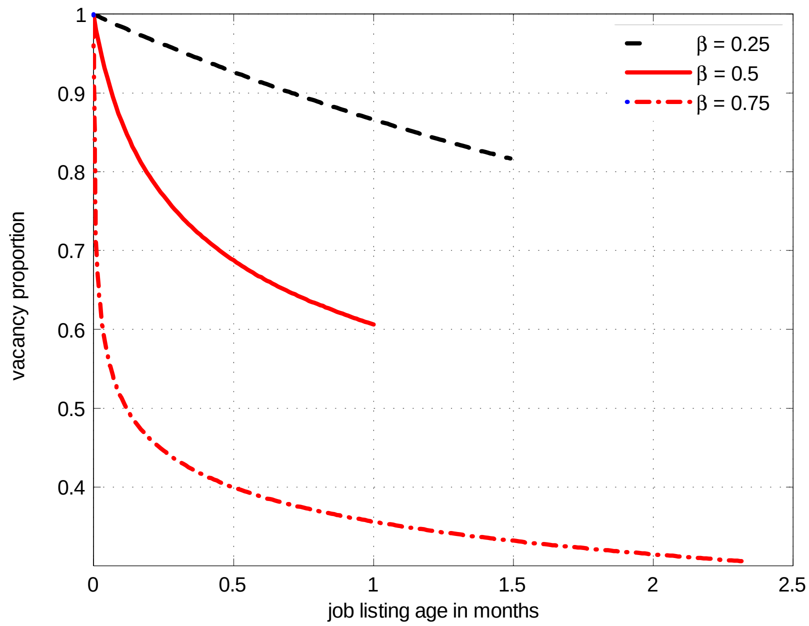


Figure 3: Vacancy proportion as a function of listing age and workers' bargaining power

tun, many phantoms are created.Figure 4 shows the density of listings by age decreases with β. This naturally implies that job-seekers are more interested in recent listings in this case.Finally, Figure 5 shows the ratio of the pdf of job-seekers to the pdf of listings by listing age. Figure 5 shows that job seekers very much focus on young listings when $\beta = 0.5$ and $\beta = 1$. As β decreases, the distribution gets closer to the distribution of listings by age. Under random search the two distrbution would coincide and the ratio would always be equal to one.

4.3 Constrained efficient allocation

We turn to the case where the planner can optimally allocate the job-seekers to the different cohorts of listings.

We use again the auxiliary variable τ. We start with a guess on θ_0 and then numerically integrate the following system of differential equations:

$$\dot{\tau} = -(B\theta^{\alpha-1} + \lambda)\tau, \tag{32}$$

$$\tau(0) = 1, \tag{33}$$

$$\alpha\frac{\dot{\theta}}{\theta} = \frac{\theta\pi(B\theta^{\alpha-1} + \lambda)(1 - (1 - \beta)\pi)}{\theta\pi + (1 - \alpha)B\theta_0^\alpha/(\alpha B\theta_0^\alpha + \lambda)} \tag{34}$$

$$\theta(0) = \theta_0. \tag{35}$$

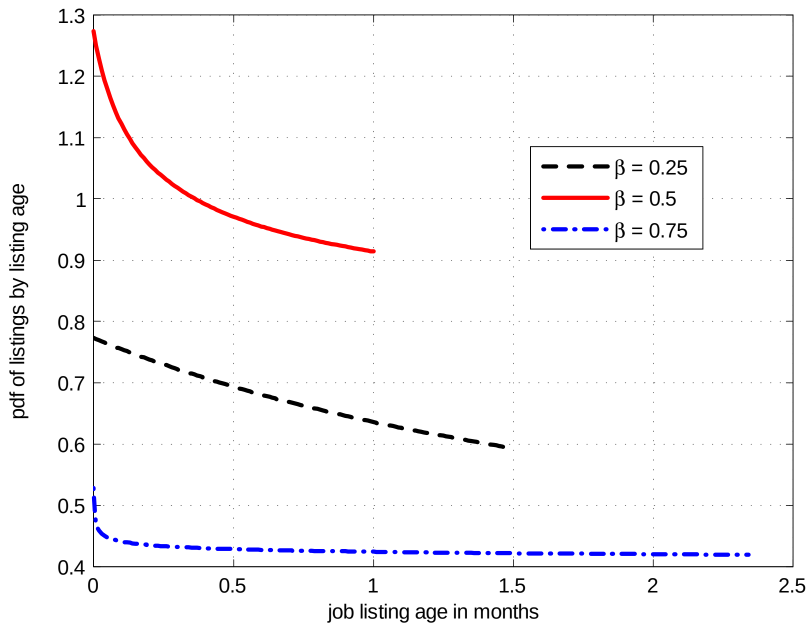


Figure 4: Density of listings by listing age

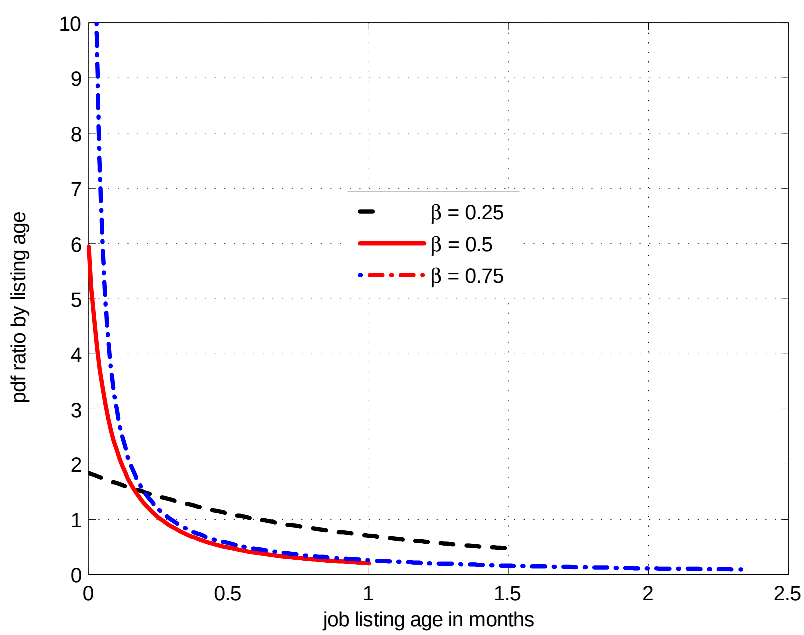


Figure 5: Job-seeker to listing pdf ratio by listing age

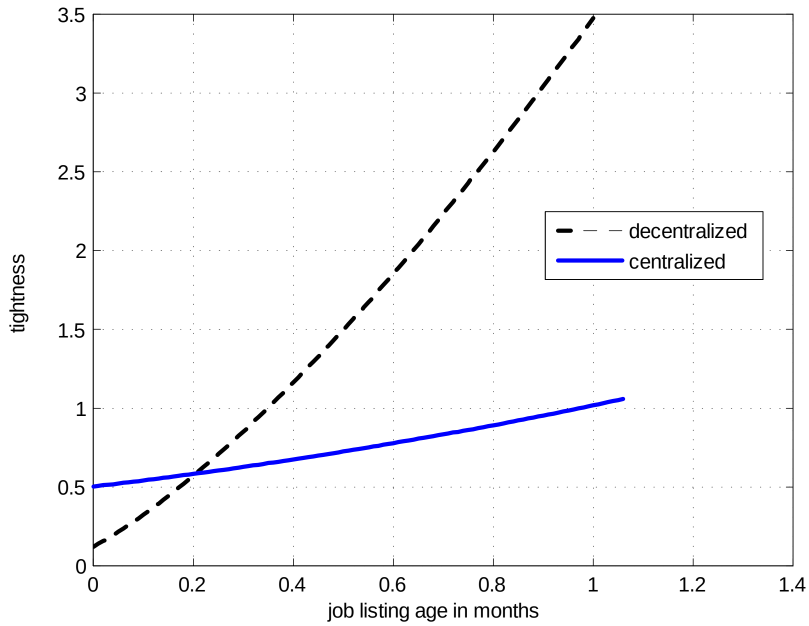


Figure 6: Tightness in the centralized vs decentralized allocations

We then find the associated renewal age using equation (equation defining the efficient age) and update θ_0 so that equation (equation replacing the free-entry condition) holds. In practice we update θ_0 so as to maximize stationary consumption.

	Ω	A	θ_0	u
decentralized	0.905	1.0	0.12	0.056
centralized	0.913	1.06	0.50	0.056

Table 4: Centralized vs decentralized allocations

Table 4 displays the key differences between both allocations. Stationary aggregate consumption, Ω, is increased by 0.8%. This gain comes from lower costs of vacancy management. The unemployment rate is the same in both allocations (slightly lower in the centralized one). The table features two additional findings: the renewal age is larger by 6% in the centralized case and initial tightness is four times larger.

Figure 6 confirms that the planner has a lower preference for young listings than the job-seekers have in the decentralized allocation. Tightness is initially larger in the centralized case

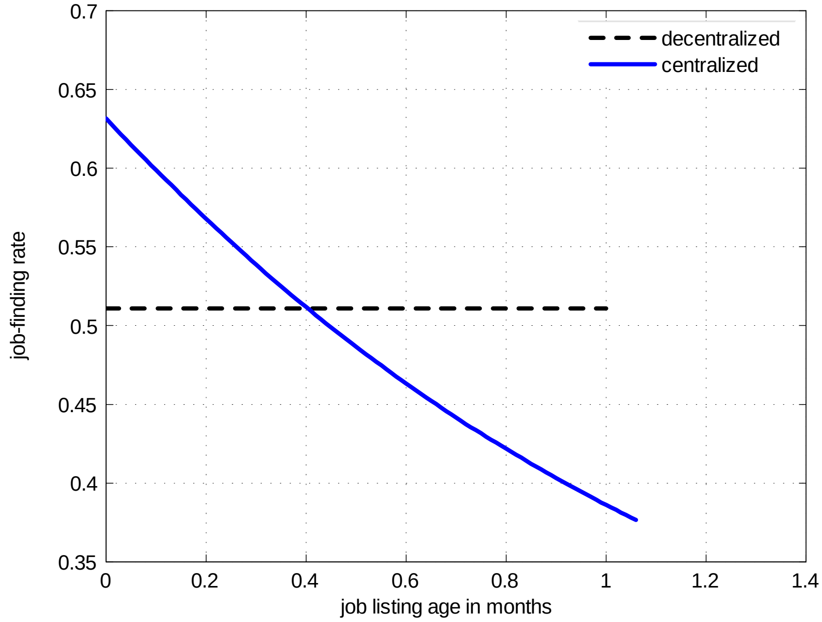


Figure 7: Job-finding rate by listing age in the centralized vs decentralized allocations

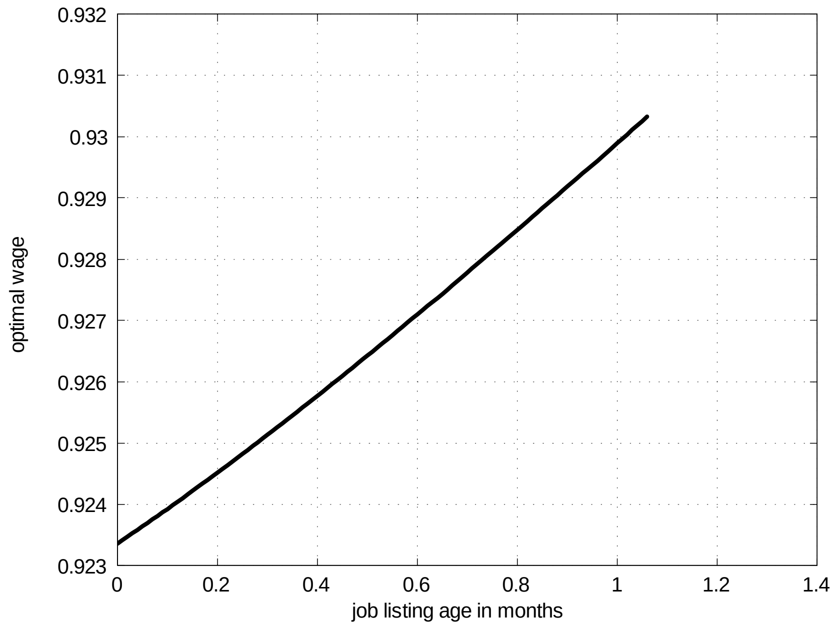
and becomes lower after a week. This pattern implies that workers search for jobs at older listings. In turn this justifies that the planner sets a longer renewal age.

Figure 7 shows the job-finding rate by listing age. The equilibrum condition $\mu(a) = \mu(0)$ for all $a \leq A$ is not satisfied in the centralized allocation. Instead, workers who search for jobs at low listing age have a higher chance of finding one.

Figure shows the pattern of optimal wage by listing age. Here again, we warn the reader that this pattern is not seen as realistic. Instead this is a theoretical exercise showing competitive wage function decentralizing the efficient allocation. The wage increases by 0.7% in one month. This is needed to make the job-seekers accept lower job-finding rates when they search for older listings.

5 Appendix A

In this appendix, we consider the decentralization of the social planner solution when the planner can choose θ_0 and A but is unable to direct worker search. In this case, we show that the social planner allocation can be decentralized in competitive search equilibrium. Suppose firms post and commit to wages rather than engaging in Nash bargaining. Let $U(w, \theta_0)$ be the value of



unemployment for a worker who sends his or her application to a firm posting a wage of w with initial labor market tightness θ_0. Let $V(0; w, \theta_0, A)$ be the value of a new listing with wage w, tightness θ_0, and listing renewal age A. The competitive search equilibrium can be described as the solution to

$$\max_{w,\theta_0} U(w, \theta_0) \text{ subject to } V(0; w, \theta_0, A) = c,$$

where A is given by equation (6).

The unemployment value is

$$U(w, \theta_0) = \frac{1}{r} \left(\frac{(r + \lambda)b + m(\theta_0)w}{r + \lambda + m(\theta_0)} \right),$$

and, from equation (5), the value of a new listing is

$$V(0; w, \theta_0, A) = \left(\frac{y - w}{r + \lambda} \right) \frac{\int_0^A \eta(a)e^{-ra}\overline{F}(a)da - ke^{-rA}\overline{F}(A)}{1 - e^{-rA}\overline{F}(A)}.$$

Using the constraint gives

$$\left(\frac{y - w}{r + \lambda} \right) \int_0^A \eta(a)e^{-ra}\overline{F}(a)da = c(1 - e^{-rA}\overline{F}(A)) + ke^{-rA}\overline{F}(A).$$

Solving for w gives

$$w = y - (r + \lambda)\frac{c(1 - e^{-rA}\overline{F}(A)) + ke^{-rA}\overline{F}(A)}{\int_0^A \eta(a)e^{-ra}\overline{F}(a)da} \equiv y - (r + \lambda)\Gamma(\theta_0, A)$$

Substituting this into $U(w, \theta_0)$ gives

$$U(\theta_0) = \frac{1}{r}\left(\frac{(r+\lambda)b + m(\theta_0)[y - (r+\lambda)\Gamma(\theta_0, A)]}{r + \lambda + m(\theta_0)}\right).$$

We can now maximize $U(\theta_0)$ to find the competitive search equilibrium. The necessary condition for this problem can be written as

$$m'(\theta_0)[y - b - (r+\lambda)\Gamma(\theta_0, A)] - m(\theta_0)\Gamma_{\theta_0}(\theta_0, A)(r + \lambda + m(\theta_0)) = 0 \tag{36}$$

To show that this is equivalent to the necessary condition in the social planner problem 1, we let r go to 0. Now we have

$$\Gamma(\theta_0, A) = \frac{c(1 - e^{-rA}\overline{F}(A)) + ke^{-rA}\overline{F}(A)}{\int_0^A \eta(a)e^{-ra}\overline{F}(a)da} \rightarrow \frac{c(1 - \overline{F}(A)) + k\overline{F}(A)}{\int_0^A \eta(a)\overline{F}(a)da} = Z(\theta_0, A),$$

and equation (36) becomes

$$m'(\theta_0)[y - b - \lambda Z(\theta_0, A)] - m(\theta_0)Z_{\theta_0}(\theta_0, A)(\lambda + m(\theta_0)) = 0.$$

Multiplying through by $\dfrac{\theta_0}{m(\theta_0)}$ gives

$$\begin{aligned} \alpha_0[y - b - \lambda Z(\theta_0, A)] - \sigma_0 Z(\theta_0, A)(\lambda + m(\theta_0)) &= 0. \\ \alpha_0(y - b) - (\alpha_0 + \sigma_0)\lambda Z(\theta_0, A)] - \sigma_0 Z(\theta_0, A)m(\theta_0) &= 0. \end{aligned}$$

Finally, dividing by $\alpha_0 + \sigma_0$ (and recalling that $\varepsilon_0 = \alpha_0/(\alpha_0 + \sigma_0)$) gives

$$\varepsilon_0(y - b) - \lambda Z(\theta_0, A) + (1 - \varepsilon_0)Z(\theta_0, A)m(\theta_0) = 0.$$

This is equivalent to equation (14), i.e.,

$$Z(\theta_0, A) = \frac{\varepsilon_0(y - b)}{\lambda + (1 - \varepsilon_0)m(\theta_0)}.$$

6 Appendix B

Proof of Proposition 3:

Planners' problem in raw form.—The planner's problem is the following

$$\max_{\theta(\cdot)} -(y-b)\int_0^A v(a)da - kv(A) \tag{$*$}$$

subject to

$$\begin{aligned}
\dot{v} &= -(\eta(\theta)+\lambda)v, \\
\dot{p} &= \beta(\eta(\theta)+\lambda)v, \\
v(0)-v(A) &= n_0, \\
p(0) &= 0, \\
1-\int_0^A [(v(a)+p(a))/\theta(a)]da + \int_0^A v(a)da &= n_0/\lambda.
\end{aligned}$$

There are two state variables v and p, and one control variable θ.

We now prove the following result.

Theorem 1 In the efficient allocation, the following properties hold:

$$(\beta s_2 - s_1)(1-\alpha(\theta))\pi m(1,\theta) = 1, \tag{37}$$

where the functions $s_1 : [0,A] \to \mathbb{R}$ and $s_2 : [0,A] \to \mathbb{R}$ are such that

$$\dot{s}_1 = Z - \frac{1-(1-\alpha)\pi}{(1-\alpha)\pi\theta} - \frac{\lambda}{(1-\alpha(\theta))\pi m(1,\theta)}, \tag{38}$$

$$\dot{s}_2 = 1/\theta, \tag{39}$$

where $Z \in \mathbb{R}$ and $s_1(a) < \beta s_2(a) \le \beta s_2(A) = 0$ for all $a \in [0,A]$.

Planner's problem in standard form.—To solve ($*$), we transform the integral constraint into a differential equation with associated boundary conditions. Let $x(a) = \int_0^a \{[v(s)+p(s)]/\theta(s) - v(s)\}ds$ for all $a \in [0,A]$. The planner's problem is now:

$$\max_{\theta(\cdot)} -\int_0^A v(a)da(y-b) - kv(A) \tag{$*$}$$

subject to

$$\begin{aligned}
\dot{v} &= -(\eta(\theta)+\lambda)v, \\
\dot{p} &= \beta(\eta(\theta)+\lambda)v, \\
\dot{x} &= (v+p)/\theta - v, \\
v(0)-v(A) &= n_0, \\
p(0) &= 0 \\
x(0) &= 0, \\
x(A) &= 1-n_0/\lambda.
\end{aligned}$$

Solving.—Let $H : \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian such that:

$$H(y, \sigma, \theta) = -v(y - b) + (\beta\sigma_2 - \sigma_1)(\eta(\theta) + \lambda)v + \sigma_3[(v + p)/\theta - v],$$

where σ_1 and σ_2 are the costates associated with the state variables v and p, and σ_3 is the costate associated with the auxiliary variable x, $z = (v, p, x)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. We define $\phi_1(z(0), z(A)) = v(0) - v(A) - n_0$, $\phi_2(z(0), z(A)) = p(0)$, $\phi_3(z(0), z(A)) = x(0)$, $\phi_4(z(0), z(A)) = x(A) - 1 + n_0/\lambda$.

We now introduce the main result that we need, a standard theorem in optimal control problems, which accounts for equality constraints on the state variables at the beginning and at the end of possible ages. A version can be found in Theorem 11.1 in Hestenes (1966).

Theorem 2 (*Maximum principle*) Suppose $\theta^*(\cdot)$ is optimal for the optimization problem $(*)$ and let z^* be the corresponding trajectory of the state variables. Then there exists $\sigma^* : [0, A] \to \mathbb{R}^3$ and $\rho^* \in \mathbb{R}^4$ such that for all $a \in [0, A]$:

A. Maximization principle

$$H(z^*(a), \sigma^*(a), \theta^*(a)) = \max_{\theta \geq 0} H(z^*(a), \sigma^*(a), \theta) \tag{40}$$

B. Adjoint equations

$$\dot{\sigma}_i^* = -\frac{\partial H(z^*(a), \sigma^*(a), \theta^*(a))}{\partial z_i(a)}, i = 1, 2, 3 \tag{41}$$

C. Transversality conditions

$$\sigma_i^*(0) = \sum_{j=1}^{4} \rho_j^* \frac{\partial \phi_j}{\partial z_i(0)}, i = 1, 2, 3, \tag{42}$$

$$\sigma_1^*(A) = -\sum_{j=1}^{4} \rho_j^* \frac{\partial \phi_j}{\partial v(A)} - k, \tag{43}$$

$$\sigma_2^*(A)p(A) = 0, \tag{44}$$

$$\sigma_3^*(A) = -\sum_{j=1}^{4} \rho_j^* \frac{\partial \phi_j}{\partial x(A)}. \tag{45}$$

The transversality conditions feature the Lagrange multiplier ρ^*, which is associated with the constraints $\phi_i(\cdot) = 0$, $i = 1, ..., 4$. The number of vacancies at termination age 0 directly enters the planner's objective, which explains the additive term in the transversality condition associated to $\sigma_1^*(A)$. There are no constraints involving the initial or final number of phantoms. The associated transversality condition is $\sigma_2(A)p(A) = 0$.

Applying Theorem 2.—Hereafter we neglect the star * notation. We first focus on the adjoint equations and transversality conditions. This gives

$$
\begin{aligned}
\frac{\partial H}{\partial v} &= -(y-b) + (\beta\sigma_2 - \sigma_1)(\eta(\theta) + \lambda) + \sigma_3(\theta^{-1} - 1) = -\dot{\sigma}_1, && (46)\\
\frac{\partial H}{\partial p} &= \sigma_3/\theta = -\dot{\sigma}_2, && (47)\\
\frac{\partial H}{\partial x} &= 0 = -\dot{\sigma}_3, && (48)\\
\sigma_1(0) &= \rho_1, && (49)\\
\sigma_1(A) &= \rho_1 - k, && (50)\\
\sigma_2(A)p(A) &= 0, && (51)\\
\sigma_3(0) &= \rho_3, && (52)\\
\sigma_3(A) &= -\rho_4. && (53)
\end{aligned}
$$

It follows that $\sigma_3(a) = -\rho_4$ for all $a \in [0, A]$.

We now turn to the maximization principle given in A above. Suppose that the first-order condition is necessary. Then we have

$$\frac{\partial H}{\partial \theta} = (\beta\sigma_2 - \sigma_1)\eta'(\theta)v - \sigma_3 \frac{v+p}{\theta^2} = 0. \tag{54}$$

Re-arranging terms, we obtain

$$-(\beta\sigma_2 - \sigma_1)(1 - \alpha(\theta))\pi m(1, \theta) = \sigma_3 \tag{55}$$

Taking the second derivative of the Hamiltonian gives

$$
\begin{aligned}
\frac{\partial^2 H}{\partial \theta^2} &= (\beta\sigma_2 - \sigma_1)\eta''(\theta)v + 2\sigma_3 \frac{v+p}{\theta^3}\\
&= \frac{v+p}{\theta^3}\left[(\beta\sigma_2 - \sigma_1)\frac{\theta\eta''(\theta)}{\eta'(\theta)}\frac{\theta\eta'(\theta)}{\eta(\theta)}\theta\eta(\theta)\pi + 2\sigma_3\right]\\
&= \frac{v+p}{\theta^3}\left[-(\beta\sigma_2 - \sigma_1)\frac{\theta\eta''(\theta)}{\eta'(\theta)}(1 - \alpha(\theta))\pi m(1, \theta) + 2\sigma_3\right].
\end{aligned}
$$

Once evaluated at the proposed maximum, we have

$$\frac{\partial^2 H}{\partial \theta^2} = \sigma_3 \frac{v+p}{\theta^3}\left[\frac{\theta\eta''(\theta)}{\eta'(\theta)} + 2\right]. \tag{56}$$

But $\theta\eta''(\theta)/\eta'(\theta) = m_{\theta\theta}(1, \theta)/\eta'(\theta) - 2 > -2$. Thus the candidate solution satisfies the second-order condition provided that $\sigma_3 = -\rho_4 < 0$, which implies $\rho_4 > 0$. By virtue of (55), this also implies $\beta\sigma_2 - \sigma_1 > 0$ for all $a \in [0, A]$.

Putting things together.—From (46) and (47), we obtain

$$\dot{s}_1 = Z - \frac{1}{(1-\alpha)\pi\theta} + \frac{1}{\theta} - \frac{\lambda}{(1-\alpha(\theta))\pi m(1,\theta)}, \tag{57}$$

$$\dot{s}_2 = 1/\theta, \tag{58}$$

where $s_1 \equiv \sigma_1/\rho_4$ and $s_2 \equiv \sigma_2/\rho_4$ are the normalized costates and $Z = (y-b)/\rho_4 - 1 > -1$.

Equation (58) combined with the boundary condition $s_2(A) = 0$ implies that $s_2(a) < 0$ for all $a < A$. In turn, equation (55) implies that $s_1(a) < \beta s_2(a) \leq 0$ for all $a \in [0, A]$.

Proof of Proposition 3. (i) is an immediate implication of Theorem 1. We have $s_1(0) = -\rho_1/\rho_4$ and $s_1(A) = -\rho_1/\rho_4 - k/\rho_4$. Using the different boundary constraints together with equation (55), we obtain $s_1(A) = -[(1-\alpha(\theta(A)))\pi(A)m(1,\theta(A))]^{-1}$, $s_1(0) = -[(1-\alpha(\theta(A)))\pi(A)m(1,\theta(A))]^{-1}$ $+(Z+1)k/(y-b)$ and $\beta s_2(0) = [(1-\alpha(\theta(0)))m(1,\theta(0))]^{-1}$ $-[(1-\alpha(\theta(A)))\pi(A)m(1,\theta(A))]^{-1}$ $+(Z+1)k/(y-b)$.

To find $Z = B$, we use the fact that the Hamiltonian stays constant over age. Then we derive B from the equality $H(z(0), \sigma(0), \theta(0)) = H(z(A), \sigma(A), \theta(A))$.

(ii) Differentiating equation (55) with respect to age, it comes

$$(\beta\dot{s}_2 - \dot{s}_1)(1-\alpha)\pi m(1,\theta) + \left[\frac{-\dot{\alpha}}{1-\alpha} + \frac{\dot{\pi}}{\pi} + \alpha\frac{\dot{\theta}}{\theta}\right] = 0. \tag{59}$$

Using (57) and (58), we have $\beta\dot{s}_2 - \dot{s}_1 = (\beta-1)/\theta - B$ $+[(1-\alpha)\pi\theta]^{-1}$ $+\lambda[(1-\alpha)\pi m(1,\theta)]^{-1}$. Therefore

$$\frac{-\dot{\alpha}}{1-\alpha} + \frac{\dot{\pi}}{\pi} + \alpha\frac{\dot{\theta}}{\theta} = B(1-\alpha)\pi m(1,\theta) - \frac{m(1,\theta)}{\theta} - \lambda + (1-\beta)(1-\alpha)\pi\frac{m(1,\theta)}{\theta}. \tag{60}$$

Noting that $\dot{\alpha} = \alpha'(\theta)\dot{\theta}$ and $\dot{\pi}/\pi = -(m(1,\theta)/\theta + \lambda)(1-(1-\beta)\pi)$, we finally have

$$\left[\alpha - \frac{-\theta\alpha'}{1-\alpha}\right]\frac{\dot{\theta}}{\theta} = B(1-\alpha)\pi m(1,\theta) - (1-\beta)\pi(\lambda+\alpha)m(1,\theta)/\theta). \tag{61}$$

(iii) We know $\dot{\pi} < 0$. We have $(1-\alpha(\theta(0)))m(1,\theta(0)) = 1/(\beta s_2(0) - s_1(0)) > (1-\alpha(\theta(A)))\pi(A)m(1,\theta(A)) = -1/s_1(A)$. When $\alpha'(\cdot)$ is sufficiently small, this implies that $\mu(0) = m(1,\theta(0)) > \mu(A) = \pi(A)m(1,\theta(A))$.

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