Investor Sentiment, Behavioral Heterogeneity and Stock Market Dynamics

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Abstract

Recent empirical works have confirmed the importance of sentiment in asset pricing. In this paper, we propose that sentiment may not affect everyone in a homogeneous way. We construct a sentiment indicator taking into consideration behavioral heterogeneity of interacting investors. We find that sentiment contributes to several financial anomalies such as fat tails and volatility clustering of returns. More importantly, investor sentiment could also be a significant source of financial market volatility. Our model with sentiment is also able to replicate different types of crises, in which the severity of crisis intensifies with investors' sentiment sensitivity.

Keywords: Investor sentiment, Social interaction, Heterogeneous beliefs, Regime switching, Financial crises

JEL Classification: C61, D84, G12

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1. Introduction

For a long time, Efficient Market Hypothesis (EMH) has been the cornerstone and mainstream belief of modern asset pricing theory. In recent years, however, skepticism against validity of EMH has grown in light of its failure to explain several ubiquitous financial regularities, such as excess volatility and systemic under- or over-valuation of stock prices relative to their intrinsic values. This gives rise to the alternative behavioral finance theory which aims to provide rationale behind the unexplained market anomalies.

Behavioral finance challenges the fundamental assumption of the EMH, in that investors are assumed to be boundedly rational and human psychology plays a crucial role in investment decisions. In fact, even before behavioral paradigm came into the limelight in finance and economics, investor sentiment has been perceived as a common phenomenon by financial analysts and market participants. As Shiller (2003) mentioned, perhaps one of the oldest financial theories expressed long ago in nonacademic papers is the price-to-price feedback theory. The feedback theory suggests that an increase in speculative prices is further propagated into bubble when financial successes of some investors are envied by the others and lead to public enthusiasm towards the speculative asset despite such upward spiral in prices is unsustainable. While conventional wisdom largely supports the idea that investor sentiment may overcome rational thoughts in trading behavior, the sentiment analysis has only started gaining recognition in financial academic research in the past two decades.

One of the pioneering works that formalizes the role of investor sentiment in financial market is the noise trader model proposed by De Long et al. (1990). In their model, uninformed noise traders are susceptible to the influence of sentiment that is in part unpredictable, while rational investors are wary of the noise trader risk and refrain from aggressive arbitrage, thus contributing to prolonged mispricing in financial market. The subsequent related studies conduct more in-depth analyses on specific channels of investor sentiment. Lux (1995, 1998) explicitly model market mood contagion through social interaction among agents to provide a behavioral explanation for bubbles and crashes. Daniel et al. (1998) and Barberis et al. (1998) construct models of investor sentiment based on psychological evidence to reconcile the empirical findings of over-reaction and under-reaction of stock prices to news. In particulars, Daniel el al. (1998) attribute sentiment to overconfidence and self-attribution, whereas Barberis et al. (1998) concentrate on conservatism and representativeness heuristic. More recently, researchers are trying to quantify the effect of sentiment on financial markets by using empirical data. Baker and Wurgler (2006, 2007) develop a “top down” approach to behavioral finance, by first forming a composite sentiment index and then empirically testing the effects of the sentiment index on different types of stocks. They find that low sentiment can predict higher returns for a subset of stocks. By investigating the interactions between daily media context from *Wall Street Journal* column and stock market from 1984-1994, Tetlock (2007) finds that media content is linked to the behavior of individual investors rather than serving as a proxy for new information about fundamental asset values or proxy for market volatility. Furthermore, he finds that high media pessimism predicts downward pressure on market price and increased market volatility. To date, there have been many studies, both theoretical and empirical, that prove evidence of investor sentiment effects in financial market (See for examples, Brown and Cliff, 2004; Da et al, 2014; Lee et al., 2002; Neal and Wheatley, 1998; Stambaugh et al., 2012). In view of that, several experimental studies have brought this into laboratory environment and find that investor’s psychology, specifically over-optimism (Hüsler et al., 2013), friendship network (Makarewics, 2017) and induced positive mood (Lahav and Meer, 2012), can amplify market price oscillations.

While the consensus is that investor sentiment can affect asset prices, the question of its importance on these prices remains. More specifically, can sentiment explain financial crisis which traditional financial models have failed to rationalize? How does sentiment work in the formation of financial crisis? These are particularly pertinent questions to ask given that we are living in an era with more frequent financial crises[1]. Comparing to the large volume of published works on investor sentiment, few studies have directly linked sentiment to market crises. Among

[1] According to Bordo et al. (2001), crisis frequency since 1973 is twice as many compared to that of the Bretton Woods and classical gold standard periods.

these are Siegel (1992) and Baur et al. (1996) that focus on U.S. stock market crash of 1987 and Zouaoui et al. (2011) who use a panel data of international stock markets to find the contribution of investor sentiment in raising the probability of crises within one-year horizon.

Against such backdrop, we aim to investigate the role of investor sentiment on dynamics of asset price as well as market crises within the framework of a heterogeneous agent model (HAM). HAM is a burgeoning framework under behavioral finance which incorporates interacting agents with heterogeneous trading beliefs. Instead of following standard representative agent assumption, HAM assumes the differentiation of traders, especially with respect to their expectations or beliefs on future price. A strand of HAM literature[2] which based on a dichotomy of fundamentalist and chartist beliefs has been proven useful in accommodating market features that seem not easily reconcilable under the traditional financial market paradigm. These features include fat tail, volatility clustering, bubbles and crises (see pioneering works by Beja and Goldman, 1980; Brock and Hommes, 1998; Chiarella and He, 2003; Day and Huang, 1990; He and Westerhoff, 2005; Lux, 1995). While HAM has gained increased popularity in recent years, only a small number of HAM studies have taken into account investor sentiment (Chiarella et al., 2017; Lux, 2012), let alone a rigorous research on the role of sentiment in financial market. To fill this gap in the literature, in this paper, we propose a HAM model with sentiment indicator that captures memory of sentiment, social interaction and sentiment shock. Our idea of social interaction is inspired by the work of Lux (1995, 1998) such that agents are not isolated units. Speculators, who are also commonly known as chartists, will rely on both actual price movements as well as the behavior of their competitors in forming their expectations. As such, we suggest that sentiment may not affect everyone in a homogeneous way. This conjecture is founded on the abundant evidence of behavioral heterogeneity between individual traders and institutional traders, in which the former lacks access to insider information and thus

[2] We follow the model of Brock and Hommes (1998) with both fundamentalists and chartists. There are also other kinds of heterogeneous agent model in economics and finance study, such as Barberis et al. (1998), Hong and Stein (1999), and Scheinkman and Xiong (2003).

is more susceptible to market sentiment (see De Long et al., 1990; Kumar and Lee, 2006). Under our HAM setting, fundamentalists represent rational arbitrageurs that possess information about fundamental asset value, whereas chartists are ill-informed speculators extrapolating on market trend. We thus discriminate between these two groups, in which the chartists' expectations are liable to market sentiment. An endogenous mechanism between sentiment and agent's belief switching is developed with investors switching between fundamentalist and chartist beliefs according to past performance while the sentiment index is contingent on the fraction of adopted beliefs in the market.

Our contributions are mainly threefold. First, to our knowledge, we are the first to model heterogeneous responses to sentiment under a fundamentalist-chartist framework. By analyzing the equilibria of deterministic models, we find the existence of sentiment-related non-fundamental steady states in the systems. Second, through explicitly modelling the sentiment index, we find that investor sentiment is indeed a significant source of financial market volatility. Third, complementing findings of Huang and Zheng (2012)[3], the sentiment channel provides an explanation to the mechanism of regime switching as well as different types of financial crises.

Some highlights of our simulation findings include: (1) sentiment contributes to stylized facts such as fat tails, volatility clustering and long memory dependence of daily returns that are commonly observed in actual stock market; (2) we find that market volatility increases with the presence of sentiment and investor sentiment may help to explain the excess volatility puzzle; and (3) our model with sentiment is able to replicate different types of crises, including sudden crisis, disturbing crisis and smooth crisis as in Huang et al. (2010) and show that sentiment sensitivity of investors is positively correlated with the frequency and the magnitude of crisis.

This paper is organized as follows. Section 2 presents the model. Section 3 explores the dynamic of the deterministic skeleton and the stability of fundamental steady states. Section 4

[3] The authors manage to reproduce sudden, smooth and disturbing crises from the simple market-maker, regime-dependent HAM framework with fundamentalist and chartist beliefs.

analyses and discusses the results of the model simulation, with special focuses on stylized facts, market volatility and crises. Section 5 carries out several robustness checks by assuming different behaviors of investors. Section 6 concludes.

2. The Model

In this section, we set up an asset pricing model with a single risky asset to characterize time series momentum and investor sentiment in the financial market. The modelling approach follows closely the current HAM framework by incorporating bounded rationality, belief heterogeneity and adaptive learning process.

In the HAM literature, the market fractions of different types of traders play a pivotal role in determining the market price behavior. According to Lux (1998), the time-varying market fraction of investors is the source of market mood or market sentiment, which may introduce complicated dynamics in the financial market. Based on both theoretical and empirical evidence, our model extends early models by introducing investor sentiment into the decision making process of agents. In each trading period, population of agents is assumed to be distributed among three groups, each relying upon different behavioral rules. These include fundamentalists who trade according to fundamental analysis as well as momentum traders and contrarian traders who trade differently based on historical price trend. In particular, we postulate that sentiment affects different types of agents in a heterogeneous way. We assume that fundamentalist group is more rational and is immune from market sentiment, while the chartists, both momentum traders and contrarian traders, are susceptible to market sentiment. Moreover, momentum and contrarian traders also react differently with respect to positive and negative sentiment. An endogenous mechanism between sentiment and agents' belief switching is developed, in that investors are allowed to switch their beliefs according to past performance while the sentiment index is contingent on the fraction of adopted beliefs in the market. As in Day and Huang (1990), market price in each period is determined by a market maker who adjusts price as a function of excess demand.

2.1. Fundamentalists. We assume that fundamentalists have more knowledge about the economy and have a notion about fundamental price. Hence**,** fundamentalists make decision based on fundamental price, μ_t. They believe that market price p_t is mean-reverting to the fundamental price and hence will buy (sell) the stock when the current price is below (above) the fundamental price of the stock. They estimate the fundamental price based on various types of fundamental information, such as expected dividends, earnings, price-earnings ratios, economic growth and so forth. In each period, the fundamental price is updated with new information arrival which is accessible to the public. The prior of μ_t is governed by:

$$\mu_t = \mu + e_t \tag{2.1}$$

where μ is the mean of fundamental value and the noise term e_t is independently and normally distributed with mean 0 and standard deviation σ. Instead of deriving the demand functions from expected utility maximization, we adopt simple demand functions for all the three types of agents. As shown in some literature, such as Beja and Goldman (1980), Chiarella et al. (2006), and Day and Huang (1990), these seemingly 'ad hoc' assets demand functions can be reconciled with the underlying expected utility maximization. The excess demand of fundamentalists D_t^f is based on the spread between the latest market price p_t and the fundamental price μ_t, which can be written as:

$$D_t^f = A(x_t)(\mu_t - p_t) \tag{2.2}$$

The reaction function $A(x_t)$ captures the behavior of the fundamentalists when price is around the fundamental price. It is assumed to be a nonlinear smooth function of price deviation from fundamental value. Let $x_t = p_t - \mu_t$ denotes this price deviation, then:

$$A(x_t) = \frac{ax_t^2}{1 + bx_t^4} \tag{2.3}$$

$A(x_t)$ could mimic the change in confidence of fundamentalists. We assume their confidence continuously increases with absolute price deviation $|x_t|$ in a reasonable zone $x_t \in (-Z_t, Z_t)$, and the range is determined by parameter b. Within this reasonable zone, fundamentalists firmly hold the fundamental strategy, and the rise of price misalignment makes them feel more confident that price will revert to fundamental value soon, so they should grasp the opportunity to buy or sell stocks to maximize their gain. As suggested by Day and Huang (1990), such behavior is justified by increasing profit opportunities within a reasonable zone. However, if the misalignment further increases and exceeds the reasonable zone, with high uncertainty in the market, fundamentalists may wrongly predict the trend of asset price and gradually lose confidence. The nonlinear and non-monotonic reaction function is not a crucial assumption for the dynamics generated from the model, but it is more realistic assumption of fundamentalist behavior in financial market. We will further discuss it in section 5.1. The properties of the function A are discussed in Appendix A.

2.2. Chartists. There are two types of chartists in the financial market, namely momentum traders and contrarian traders. Unlike the fundamentalists, both momentum traders and contrarian traders focus only on their short-term estimated market value, v_t, albeit different strategies are adopted by both parties. Similar to fundamentalists, chartists also have time-varying extrapolation rate, but their trading behaviors or confidence levels are sensitive to market sentiment.

2.2.1. Momentum traders. When the current market price is above the short-term value, momentum traders expect future market price to rise and choose to take a long position; conversely, they take a short position. Excess demand of momentum traders is assumed to evolve over time based on the current short-term value by:

$$D_t^{mo} = \beta_1 m_t (p_t - v_t) \tag{2.4}$$

where $\beta_1 m_t$ is the time-varying extrapolation rate of price trend. $\beta_1 > 0$, represents the base extrapolation rate without sentiment effect. m_t is the time-varying sentiment factor, which is updated each period and will affect the trading decision of investors for next period. The sentiment factor is constructed as:

$$m_t = 1 + tanh(\kappa(p_t - v_t)) * h_1 * S_t \tag{2.5}$$

where the sentiment index S_t is derived from social interaction of different types of agents and random sentiment-related information such as news and policies. $S_t \in [-1,1]$ and the construction of sentiment index will be introduced in Section 2.4. $h_1 \in [0,1]$ measures sensitivity of momentum traders to market sentiment. If $h_1 = 0$, it means investor is totally immune to sentiment, and the extrapolation rate is only determined by β_1 as $m_t = 1$. On the contrary, if $h_1 = 1$, this means that traders are very sensitive to sentiment that they perceive in financial market.

Both positive and negative sentiments are expected to stimulate opposite impacts on chartists' long position and short position. More specifically, positive sentiment can enhance the confidence level of momentum traders in long position and raise their cautiousness in short position, while negative sentiment can bolster confidence of momentum traders on short position and weaken it on long position. If the price is above v_t while current market sentiment is positive, momentum traders will feel more confident to follow the price trend and buy in. Hence, their confidence level increases with further price deviation given positive market sentiment. On the other hand, if price trend is upward but market sentiment is negative, momentum traders will still follow the trend but with less confidence. This contradiction of momentum traders' prediction against market sentiment makes them more cautious and reduce their demand for the speculative asset. Besides the market sentiment, the price deviation from the short-term value $p_t - v_t$ also exerts an impact on the investor's sentiment. To standardize price deviation $p_t - v_t$ within the range of $(-1, 1)$, we introduce a $tanh$ function with a scaling factor κ. Thus, the range of m_t is $[0, 2]$.

We now look more closely at the short-term asset value v_t. We assume that all chartists, both momentum traders and contrarian traders, hold on to an identical short-term asset value. As in Huang et al. (2010) and Huang and Zheng (2012), we assume that chartists adopt the adaptive belief mechanism where they update their expectations on short-term asset value according to different price regimes. They believe in support and resistance levels which are derived from common rules of technical analysis[4]. Accordingly, we assume that chartists divide price domain $P = [P_{min}, P_{max}]$ into n regimes such that:

$$\mathbb{P} = \bigcup_{j=1}^{n} \mathbb{P}_j = [\bar{p}_0, \bar{p}_1) \cup [\bar{p}_1, \bar{p}_2) \cup \cdots \cup [\bar{p}_{n-1}, \bar{p}_n] \tag{2.6}$$

where $\bar{p}_j\ for\ j = 1,2,\cdots,n$ represents the different support and resistance levels set by the chartists.

The short-term asset value can be simply extrapolated as the average of the top and the bottom threshold prices:

$$v_t = (\bar{p}_{j-1} + \bar{p}_j)/2 \ \ if\ p_t \in [\bar{p}_{j-1}, \bar{p}_j) \tag{2.7}$$

When price fluctuates within the current regime, there are enough reasons for chartists to believe that the short-term asset value will remain unchanged. However, once the price breaks through either the support or resistance lines, chartists will adjust their expectation on the short-term asset value according to Equation 2.7. This regime dependent phenomenon is commonly found in stock market with chartist's beliefs evolve with regime switching. According to Huang et al. (2010), the short-term asset value for each period is estimated as:

$$v_t = \left(\left\lfloor \frac{p_t}{\lambda} \right\rfloor + \left\lceil \frac{p_t}{\lambda} \right\rceil\right) \cdot \frac{\lambda}{2} \ \ if\ p_t \in \left[\bar{p}_{j-1}, \bar{p}_j\right)\ and\ j = 1,2 \cdots n \tag{2.8}$$

2.2.2. Contrarian traders. Unlike momentum traders, contrarian investors trade stock based on the hypothesis of market overreaction. Specifically, when current price is higher than short-term value v_t, they believe that future market price will drop and therefore take a short position; conversely, they take the long position. We assume contrarian traders use the same method as

[4] Donaldson and Kim (1993) have provided empirical evidence of the existence of support and resistance levels in Dow Jones Industrial Average index.

momentum traders to calculate short-term value v_t, hence the demand function of contrarian traders can be expressed as:

$$D_t^{co} = \beta_2 c_t (p_t - v_t) \tag{2.9}$$

Similarly, $\beta_2 c_t$ is the time-varying extrapolation rate of price trend for contrarian traders. $\beta_2 < 0$, represents the base extrapolation rate without sentiment effect and c_t is the time-varying sentiment factor for contrarian traders, which can be written as:

$$c_t = 1 - \tanh(\kappa(p_t - v_t)) * h_2 * S \tag{2.10}$$

Similar to h_1, h_2 is sensitivity of contrarian traders to market sentiment with a range $[0, 1]$. Although contrarian traders adopt trading strategy opposite to that of momentum traders, they are affected by market sentiment in the same way. When market price is above v_t, contrarian traders expect price to decline. If the market sentiment is negative, contrarian traders will be more confident to take the short position. However, positive market sentiment will decrease the confidence level of contrarian investors. Besides the market sentiment, price deviation is another element that contributes to contrarian traders' sentiment factor c_t.

2.3. Belief switching regime. One of the important features underlying the model is the belief switching regime of agents, which has become widely adopted since it was first proposed by Brock and Hommes (1997). They assume that, at the end of each trading period, agents may switch their belief type or prediction strategy conditional on the past performance of three rules. Specifically, the performance measure depends on realized profitability, which is defined as:

$$\pi_{n,t} = (p_t - p_{t-1}) D_{t-1}^n \tag{2.11}$$

For simplicity, we assume that the gross interest rate between period $t - 1$ and period t is one and there is no cost for fundamentalists to acquire additional information. We further introduce additional memory into the performance measure that can be taken as the geometrically declining weighted average of the realized profits, given by:

$$U_{n,t} = \varphi U_{n,t-1} + \pi_{n,t} \tag{2.12}$$

where $0 \leq \varphi \leq 1$ represents the strength of memory put into the last-period performance, n denotes different types of agents. The memory component could slow down the switching dynamic, as it causes agents to react less quickly to the profitability of a particular strategy.

We use $\omega_{i,t}$ to denote market fraction of three different types of investors. The fractions of three groups vary endogenously over time according to the choice model with multinomial logit probabilities as introduced by Manski and McFadden (1981) as well as Brock and Hommes (1997, 1998):

$$\omega_{n,t} = \frac{\exp(\rho U_{n,t})}{\sum_{h=1}^{3} \exp(\rho U_{n,t})} \tag{2.13}$$

Note that, the new fractions of traders are determined on the basis of the most recent performance measure $U_{n,t}$. The parameter $\rho > 0$ is the intensity of choice measuring the sensitivity of agents with respect to the difference of past performance. The higher is ρ, the quicker agents will respond to difference in performance by switching to the most profitable strategy. For finite ρ, $\omega_{i,t}$ is always positive which implies that not all agents are going for strategy that gives highest profit.

2.4. Sentiment index. As pointed out by Baker and Wurgler (2007), many factors could be used to construct sentiment index of a market, such as investor mood, news from media, mutual fund flows, trading volume, dividend premium and government policies. We construct the sentiment index by focusing on three main sources such as last-period sentiment index (also known as memory of sentiment), investor mood from social interaction, and sentiment shock such as news, polices, firm innovations and so forth. The index function can be written as:

$$S_t = \eta_1 S_{t-1} + \eta_2 SI_t + \eta_3 \epsilon_t \tag{2.14}$$

where η_1, η_2, η_3 are weights assigned to different factors, such that $\eta_1 + \eta_2 + \eta_3 = 1$. ϵ_t is the sentiment shock, which we assume to follow a uniform distribution with range [-1, 1]. SI is social interaction of different types of investors, which influences the current market mood. The idea is inspired by majority opinion formation in Kirman (1993), Lux (1995), and Lux and Marchesi (1999), in which the majority opinion index is computed as the difference between optimistic and pessimistic individuals. Different from Lux and Marchesi (1999), we assume that the opinion index not only contains the opinion of optimistic and pessimistic chartists, but also includes the opinion of fundamentalists. The social interaction should exist among all types of investors, including both fundamentalist and chartist groups. Fundamentalists are optimistic (pessimistic) investor when market price is below (above) their fundamental values. Two different types of chartist may change their opinion under different market states. For instance, when current market price is above the short-term value v_t, momentum traders believe price will continue to go up and thus they belong to the optimistic group. In contrast to momentum traders, contrarian traders believe price trend will reverse and hence they belong to the pessimistic group. However, in the next period, if the market price falls below the short-term value v_{t+1}, momentum traders will become pessimistic while contrarian traders will switch to optimistic opinion, we construct the social interaction index as:

$$SI_t = tanh(\kappa(\mu_t - p_t)) * \omega_t^f + tanh(\kappa(p_t - v_t) * (\omega_t^{mo} - \omega_t^{co})) \qquad (2.15)$$

where $\omega_f, \omega_{mo}, \omega_{co}$ are fractions of fundamentalists, momentum traders and contrarian traders in financial market, respectively. $tanh$ function and κ are used to scale the price deviation, so both the range of SI_t and the range of sentiment index S_t can be constrained to $[-1, 1]$.

2.5. Market clearing mechanism. Instead of using Walrasian auctioneer clearing mechanism, we adopt market maker mechanism, which is akin to the role of specialists in New York Stock Exchange. We assume net zero supply of the risky asset, and market price in each trading period is determined by a market maker who adjusts the price as a function of excess demand. The aggregate market's excess demand is weighted by population fraction. Hence, for a three-agent model (including fundamentalists, momentum traders and contrarian traders), the price p_{t+1} is set by market maker according to the aggregate excess demand, that is:

$$P_{t+1} = P_t + \gamma(\omega_t^f D_t^f + \omega_t^{mo} D_t^{mo} + \omega_t^{co} D_t^{co}) \tag{2.16}$$

where γ represents the speed of price adjustment by the market maker.

3. Analysis of the Model's Deterministic Skeleton

3.1. Market stability analysis. The core of the HAM framework is the existence of heterogeneous agents with different trading strategies. Comparing with standard representative agent models, HAMs are more complicated due to heterogeneous agents' interaction. It would be more interesting to explore how market achieves its steady states and what are the necessary conditions for stable equilibria. By removing the perturbation of noise, the stability analysis could provide an insight into the effect of trading activities as well as interaction among the different types of investors on market stability. Specifically, we will focus on market stability with investor sentiment effect.

In HAM study, both two-agent and three-agent designs are popular. Two-agent setup only considers two types of agents in the model, and it includes cases of fundamentalist versus momentum trader, fundamentalist versus contrarian trader and momentum trader versus contrarian trader. For the three-agent design, all the three types of investors such as fundamentalists, momentum traders and contrarian traders are included into the system. This design has been widely investigated in HAM literature. These include the works of Brock and Hommes (1998), Chiarella and He (2003), and He and Li (2015). In this section, we give an extensive exploration of two-agent and three-agent deterministic models by focusing on the local stability of the fundamental steady states (denoted by $\bar{\mu}$ and $\bar{v}$). To account for the diversity of investors' behaviors in the real financial market, we choose three-agent model as a benchmark for the following numerical analysis. The dynamics of two-agent models will be discussed in Appendix B. We discuss the conditions for the existence of fundamental steady state (FSS) and non-fundamental steady state (NFSS) only for the single regime case[5]. Multiple regime or regime-switching features of the model have been investigated in Huang and Zheng (2012). The system of three-agent market can be modelled to a five-dimensional dynamic map such that:

$$
\begin{aligned}
p_{t+1} &= p_t + \gamma[\omega_t^f A(\mu_t - p_t) + (\beta_1 \omega_t^{mo} m_t + \beta_2 \omega_t^{co} c_t)(p_t - v_t)] \\
u_{t+1}^f &= \varphi u_t^f + (p_{t+1} - p_t) A_t (\mu_t - p_t) \\
u_{t+1}^{mo} &= \varphi u_t^{mo} + (p_{t+1} - p_t) \beta_1 m_t (p_t - v_t) \\
u_{t+1}^{co} &= \varphi u_t^{co} + (p_{t+1} - p_t) \beta_2 c_t (p_t - v_t) \\
S_{t+1} &= \eta_1 S_t + \eta_2 [\tanh(\kappa(\mu_{t+1} - p_{t+1}))\omega_{t+1}^f + \tanh(\kappa(p_{t+1} - v_{t+1}))\,(\omega_{t+1}^{mo} - \omega_{t+1}^{co})]
\end{aligned}
\tag{3.1}
$$

where

$$A = \frac{a(\mu_t - p_t)^2}{1 + b(\mu_t - p_t)^4},$$

$$\omega_{h,t} = \frac{\exp(\rho U_{h,t})}{\sum_{h=1}^{2} \exp(\rho U_{h,t})},$$

$$m_t = 1 + \tanh(\kappa(p_t - v_t)) * h_1 * S_t,$$

$$c_t = 1 - tanh(\kappa(p_t - v_t)) * h_2 * S_t$$

The dynamics in system (3.1) are stochastic and there are two sources of noise. The first source, e_t is the noise term of the fundamental value. The second source, θ_t is the stochastic component of market sentiment derived from random news, innovations and policies. When both noise terms are zero, we are able to study the stability of the system, and we adopt the same way to generate deterministic skeletons for the following scenarios. We investigate the occurrence of the steady state of system (3.1) in one regime case, in which short-term value v_t is assumed as constant $\bar{v}$ over time. The system has multiple steady states with different stability properties.

[5] Regime here refers to the price window set by chartists. One regime case means short-run asset price v_t is constant, and support and resistance levels approach infinity. Multiple-regime case means v_t is updated in each period.

Proposition 1. *The system has*

a) *a unique fundamental steady state with* $\left(p^*, u_f^*, u_{mo}^*, u_{co}^*, S^*\right) = \left(\mu, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ *if* $\beta_1 = -\beta_2$. *The Jacobean matrix of this system has five eigenvalues with* $\lambda_1 = 1$, $\lambda_2 = \varphi$ *. Fundamental steady state is asymptotically stable for* $|\lambda_3|, |\lambda_4|, |\lambda_5| < 1$.

b) *a fundamental steady state with* $\left(p^*, u_f^*, u_{mo}^*, u_{co}^*, S^*\right) = \left(\mu, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ *if* $\beta_1 \neq -\beta_2$ *and* $\mu = \bar{v}$. *FSS is asymptotically stable for* $-6 < \gamma(\beta_1 + \beta_2) < 0$; *Two types of non-fundamental steady states with the form* $\left(p^*, u_f^*, u_{mo}^*, u_{co}^*, S^*\right) = \left(p_1^*, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, S_1^*\right)$, $\left(p^*, u_f^*, u_{mo}^*, u_{co}^*, S^*\right) = \left(p_2^*, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, S_2^*\right)$, *and* $p_1^* < \mu, S_1^* > 0$; $p_2^* > \mu, S_2^* < 0$ *if* $\beta_1 \neq -\beta_2$ *and* $\mu \neq \bar{v}$.

Proof. See Appendix C.

As all the three types of agents can be regarded to have equal forces in financial market, their behavior, especially the extrapolating power of them, may influence the existence and stability of steady state. When $\beta_1 = -\beta_2$, only fundamental steady state exists. In this case, demands of momentum traders and contrarian traders offset each other if we do not consider sentiment effect, and demand of fundamentalists solely determines the market price. With sentiment effect, the price dynamic generated by this model would become richer, but there is a unique steady state with $p = \mu$. To analyze the stability of this system, we use the Jacobean matrix. We find that $\lambda_1 = 1$, $\lambda_2 = \varphi$ and $\lambda_3, \lambda_4, \lambda_5$ are functions of standard parameters such as γ, ρ, φ and sentiment related parameters η_1, η_2, h_1, h_2.Fundamental steady state is asymptotically stable if $|\lambda_3|, |\lambda_4|, |\lambda_5| < 1$, so sentiment is a factor to influence the stability of the steady state. When $\beta_1 \neq -\beta_2$, both fundamental steady state and non-fundamental steady states exist under some specific conditions. The sufficient condition for fundamental steady state is $\mu = \bar{v}$, and this FSS is asymptotically stable for $-6 < \gamma(\beta_1 + \beta_2) < 0$, which means that contrarian traders need to have a stronger extrapolation power to stabilize the market. We have two types of non-fundamental steady state for $\beta_1 \neq -\beta_2$ *and* $\mu \neq \bar{v}$. The high degree of nonlinearity of the equations makes it difficult to explore the stability of the non-fundamental steady states. The features of them are shown in **Lemma 1**.

Lemma 1. *For non-fundamental steady states, the system can achieve both positive and negative sentiment equilibria. If* $\beta_1 > -\beta_2$*, positive (negative) sentiment equilibrium exists at* $p^* < (>)\mu$ *and* $p^* < (>)\bar{v}$*. If* $\beta_1 < -\beta_2$*, positive (negative) sentiment equilibrium exists at* $p^* < (>)\mu$ *and* $p^* > (<)\bar{v}$*.*

Proof. See Appendix C.

For non-fundamental steady states, the position of p^* with respect to μ determines the types of equilibrium. The sentiment generated by two types of chartist will be offset by each other and their effects are cancelled out in the equilibrium, so the sentiment of fundamentalists is the main source of equilibrium sentiment. A positive (negative) sentiment equilibria must be below (above) fundamental price μ. Before the equilibrium, if momentum traders extrapolate stronger , a positive (negative) sentiment steady state would emerge at $p^* < (>)\bar{v}$. If contrarian traders extrapolate stronger, a positive (negative) sentiment steady state would occur at $p^* > (<)\bar{v}$.

4. Numerical Simulation with Stochastic Model

The analysis performed in the previous section confirms that heterogeneity of investor belief and perception of sentiment can drive the market toward regimes characterized by optimistic or pessimistic market sentiment. To study more features in the financial market such as fat tails, negative skewness, and long memory of the distributions of returns, we run simulation based on stochastic models with three-agent in system (3.1). We begin this section by briefly reviewing some of the stylized facts in real financial market. Next, we conduct simulation practice on our stochastic models with and without sentiment separately, thereby enable us to compare the fitness between both models to real financial market in terms of its capability to generate the well-documented stylized facts. More importantly, we are able to explore the role of sentiment effect on these stylized facts. In addition, we also investigate whether sentiment is the source of excess market volatility and the different types of crises, and how sentiment could influence the emergence of crises.

4.1. Stylized facts. We calibrate our sentiment model such that it mimics some well-documented stylized facts of financial markets following the practice of existing literature (See for example, Schmitt and Westerhoff, 2017). As summarized by Westerhoff and Dieci (2006), the five salient characteristics of real-world speculative prices include (1) price distortions in the forms of bubbles and crashes; (2) excess price volatility; (3) leptokurtic distribution of returns (characterized by kurtosis exceeding 3); (4) negligible autocorrelation of daily returns; and (5) strong autocorrelation of absolute daily returns. These patterns can be seen in Fig.4.1, which depicts the dynamics of US stock prices and returns[6] based on daily S&P500 index from Jan 3, 2000 to Feb 12, 2019. As seen from the top panel, the evolution of daily stock prices shows both strong price appreciations and crashes in some periods. The second panel displays daily log returns which show evidence of volatility clustering. Distribution of returns in the third panel indicates the presence of fat tails. The last panel plots the autocorrelation functions (ACF) using both raw returns and absolute returns. We can observe the absence of autocorrelations in raw returns and the slow-dampening autocorrelations in absolute returns, implying long memory of daily returns. We conduct simulation analysis on the basis of the standard parameter setting as shown in Table 4.1

To examine the effect of sentiment on price movements, we compare model with sentiment to model without sentiment by setting the sentiment sensitivity parameter, h, from 1 to 0, while holding all other parameter values constant. Simulation results of 17,000 periods (daily) price trajectories are shown in Fig.4.2. Table 4.2 summarizes the descriptive statistics of both actual and artificial market returns.

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[6] Return at time t as denoted by r_t is computed by $r_t = \log(p_t) - \log(p_{t-1})$.

Table 4.1: Standard Parameter setting

Parameter	Value	Definition
μ	1014	Mean of fundamental prices
σ	1	SD of fundamental prices
β_1	1.75	Momentum extrapolation rate
β_2	-1.25	Contrarian extrapolation rate
ϕ	0.1	Performance memory strength
ρ	0.5	Intensity of choice
γ	0.845	Speed of price adjustment
η_1	0.4	Last-period sentiment weight
η_2	0.5	Social Interaction weight
η_3	0.1	Sentiment shock weight
λ	12	Support and resistance level
a	1.11×10^{-5}	Confident function factor
b	1×10^{-8}	Confident function factor
κ	1000	Scaling factor
$h = h_1 = h_2$	0/1	Without sentiment/with sentiment

Overall, we find the simulated series with investor sentiment exhibiting much more realistic statistical properties as compared to series without investor sentiment. As shown by the top left panel of Fig.4.2, the model with sentiment is capable to produce more volatile prices (indicated by dark line) relative to rather stable fundamental values (indicated by blue line). Unlike the case without sentiment where prices mostly fluctuate closely around the fundamentals, prolonged bubbles and crashes can be generated when investor sentiment is included in the framework. Meanwhile, by allowing for sentiment, the return trajectories show volatility clustering similar to the actual S&P500 returns, while the distribution of returns from the model with sentiment demonstrates leptokurtic behavior given the presence of fat tails. From Table 4.2, the kurtosis and skewness for model with sentiment are 6.68 and -0.099, respectively; whereas model without sentiment have underestimated kurtosis and skewness of 2.14 and -0.017, respectively.

Table 4.2: Summary statistics of returns

Variable	Kurtosis	Skewness	Mean	SD	Min	Max
Actual Market						
S&P500	11.481	-0.215	0.000	0.012	-0.095	0.110
Artificial Market						
With sentiment	6.677	-0.099	0.000	0.010	-0.095	0.064
Without sentiment	2.140	-0.017	0.000	0.005	-0.010	0.010

To check whether there exists long memory dependence in daily returns, we plot ACFs for both raw daily returns (blue line) and absolute daily returns (red line). Without sentiment, the absolute returns have fast-decaying ACF, suggesting there is no long-range dependence for daily returns. However, with sentiment, ACF of absolute returns is strong and persistent even after 50 lags. This finding matches the stylized facts of real stock return, that cross-correlation should be weak for raw returns but strong for absolute returns.

This qualitative finding holds for simulation of deterministic model as well and we can exclude the contribution of stochastic noise to these statistical properties. We also conduct robustness check by evaluating several statistical properties of simulated prices and returns for 1000 simulation runs across a range of sentiment sensitivity parameter values. More details can be found in Table 4.3.

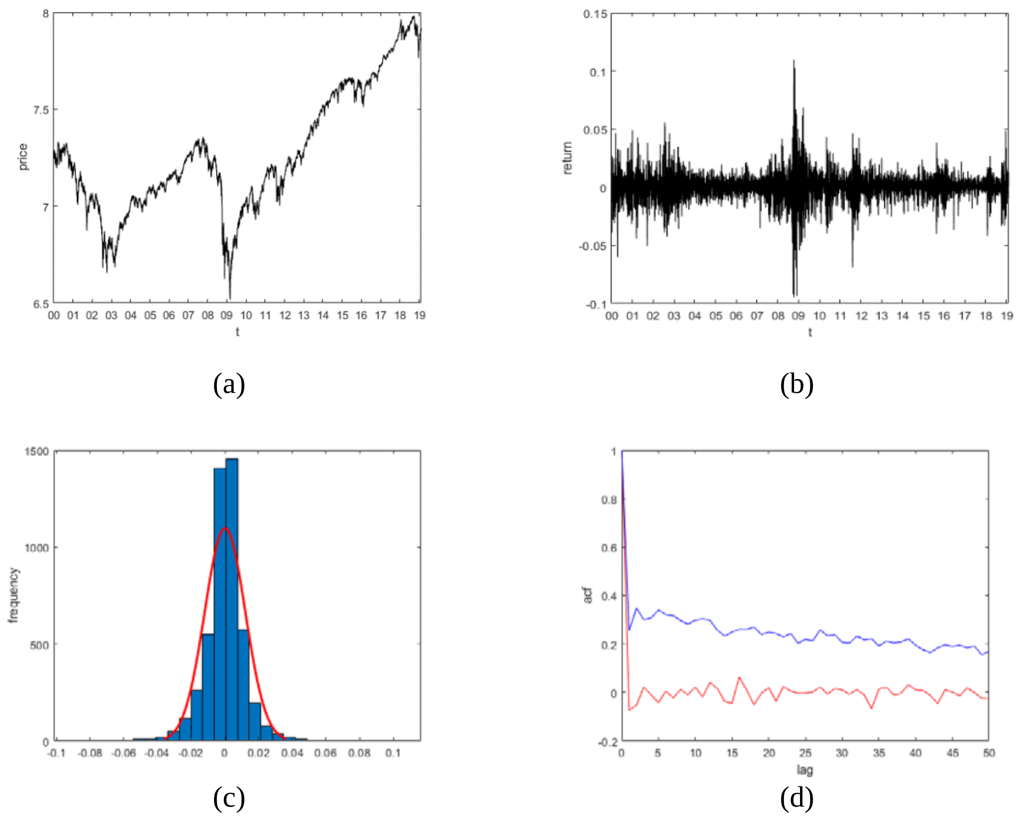


FIGURE 4.1. The dynamics of daily S&P500 index between Jan 3, 2000 and Feb 12, 2019. The panel shows (a) the evolution of the stock price index (b) the returns, (c) the histogram of returns overlaid by normal curve and (d) the autocorrelation function of raw returns (red line) together with the autocorrelation function of absolute returns (blue line).

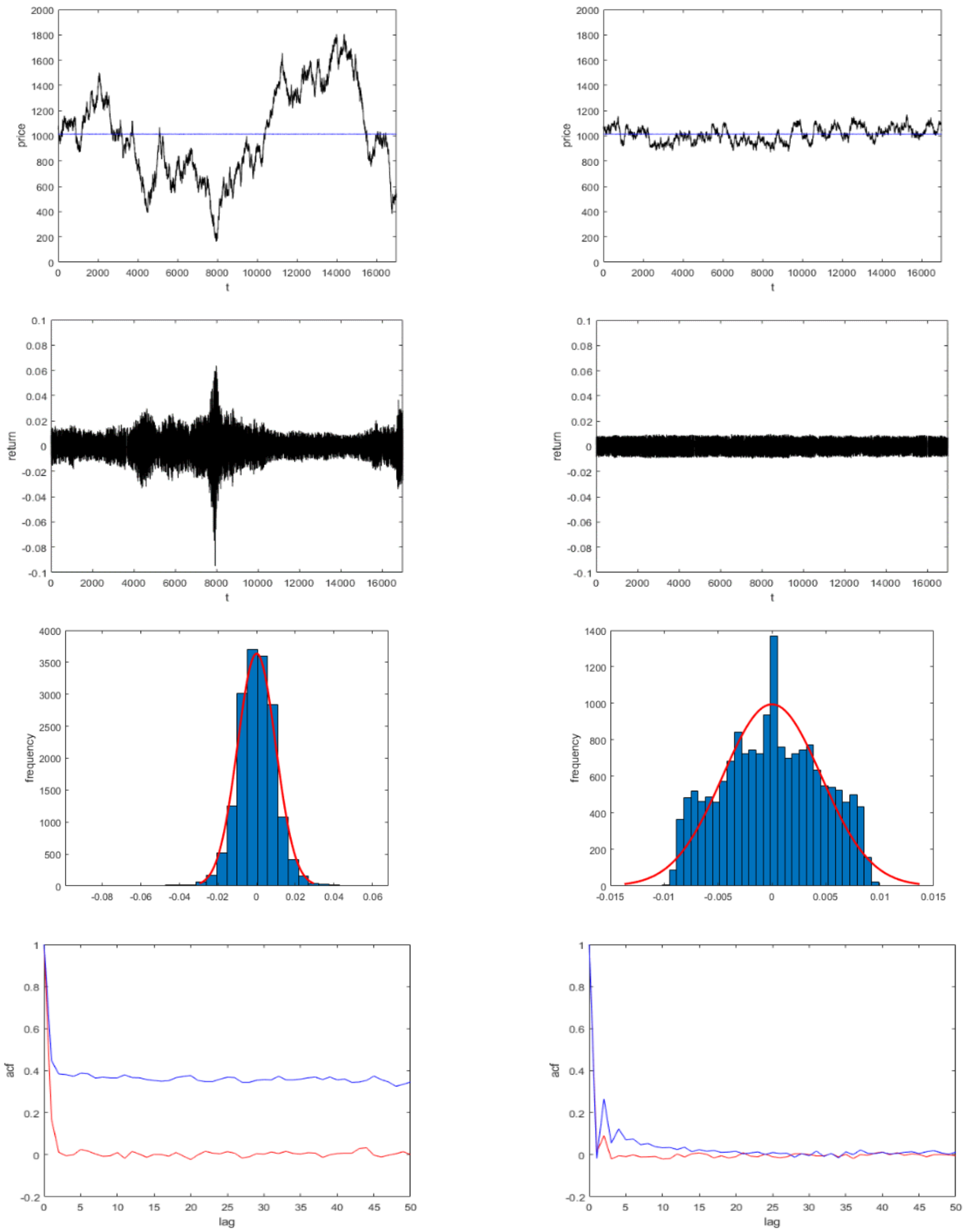


FIGURE 4.2. The dynamics of the models with sentiment (left) and without sentiment (right). The panels show, from top to bottom, the evolution of the stock prices, the returns, the histogram of returns overlaid by normal curve and the ACF of raw returns (red line) together with the ACF of absolute returns (blue line), respectively. The simulation run is based on 17,000 observations.

4.2. Sentiment and excess volatility. Prices of financial assets are typically more volatile than can be justified within standard asset pricing model. This result, known as excess volatility, has been documented in many studies, such as Campbell and Shiller (1987), LeRoy and Porter (1981), Shiller (1981). Recently, some studies attempt to use behavioral approach to tackle excess volatility puzzle. (see for examples, Dumas et al.,2009; Li, 2007; Lof, 2015; Xiong, 2013). In order to study the effect of sentiment on excess volatility, we use the standard deviation of the market prices from the fundamental values $SD_{p-\mu}$ as a quantitative measure of market volatility. Standard deviation $SD_{p-\mu}$ is computed as:

$$SD_{p-\mu} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(p_t - \mu_t)^2}$$

Keeping the same standard parameter set, the deviation of the market prices from the fundamental values $SD_{p-\mu}$ is 358.95 for the model with sentiment and 58.46 for the model without sentiment. This implies that the market is more volatile when sentiment effect is included. In general, we find that $SD_{p-\mu}$ becomes larger as sentiment sensitivity parameter h increases from 0 to 1.

To check the robustness of our result, we run the three-agent model 1,000 times by using Monte Carlo simulation using the standard parameter values with different levels of sentiment sensitivity from 0 to 1 with an interval of 0.2. Fig.4.3 shows the average standard deviations $SD_{p-\mu}$ for 1,000 simulation runs at different h values. As we can see, the average value of $SD_{p-\mu}$ increases with the sentiment sensitivity, suggesting that the volatility of market is positively related to the sensitivity of investors to market sentiment.

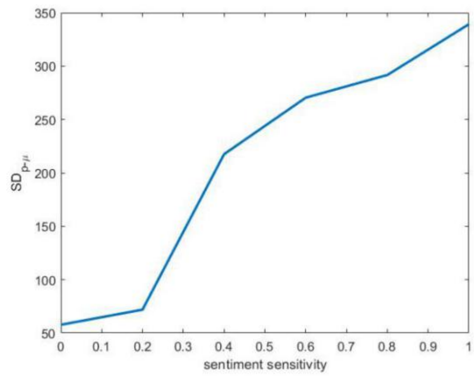


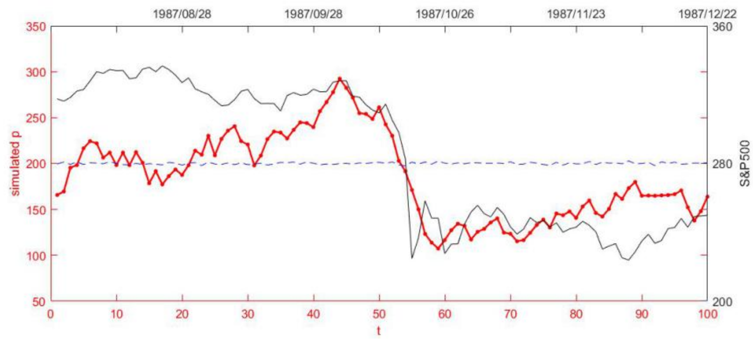
FIGURE 4.3 The effect of sentiment sensitivity on market volatility with 1,000 simulations.

4.3. Sentiment and crises. Financial crisis is another aspect we are interested to investigate under the HAM framework. Many researchers have proven and indicated that standard economic theory failed to envisage the occurrence of financial crises, and heterogeneous beliefs and interaction of heterogeneous agents should be taken into account to understand financial crises. Some literature has documented that models with heterogeneous beliefs have extraordinary power in explaining financial crises, and these studies include but not limit to De Jong et al. (2009), Huang at el. (2010), Lux and Westerhoff (2009) and Xiong (2013). In this section, we investigate different financial crises under the HAM framework as in Huang et al. (2010), but we focus on the role of sentiment on the formation of crises.

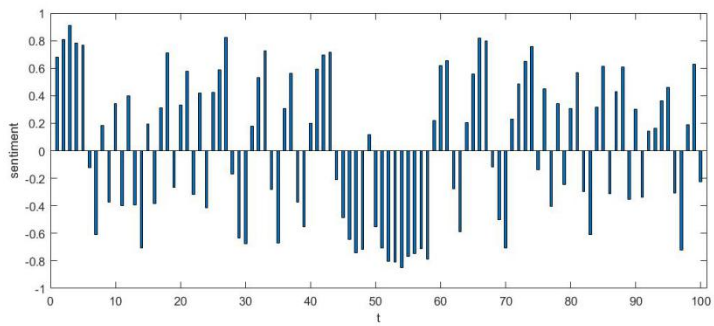
In Huang et al. (2010), they have successfully replicated three typical types of crises, namely sudden crisis, smooth crisis and disturbing crisis, by using HAM. They find that endogenous price dynamic from agents' interaction might be the reason of these crises, and both fundamentalists and chartists could potentially contribute to the financial crises. Following their logic, we also generate the same three typical types of financial crises using our three-agent model with fundamentalists, momentum traders and contrarian traders. For more accurate replication of crisis patterns, we set $\mu = 200, \rho = 0.7, \gamma = 1.8, \lambda = 10, b = \left(\frac{1}{150}\right)^4$ while keeping other parameters in line with the standard values. The difference between our model and theirs is that we have a stochastic model while their model is deterministic.

In a sudden crisis, price abruptly drops from the peak (or near peak) straight down to bottom within a short timeframe. According to Huang et al. (2010), when the price is at the peak, it is highly overvalued. Observing opportunities for profit, investors switch to fundamental strategy excessively and execute great selling forces that cause the steep fall in price. Contrary to their results, we find no contribution of fundamentalists before and during the sudden crisis. As illustrated in Fig.4.4, there is a dramatic price fall between t=50 and t=60, during which, the market is dominated by momentum traders without any switching to fundamental strategy. It is the strong negative sentiment that drives the market crash. Due to either exogenous cause or regime switching, price starts to go down at around t=45, which makes the momentum traders change from buy to sell and create a bearish sentiment. The strong negative sentiment further accelerates selling forces of the momentum traders and subsequently, leads to market panic and causes a sharp decline in price.

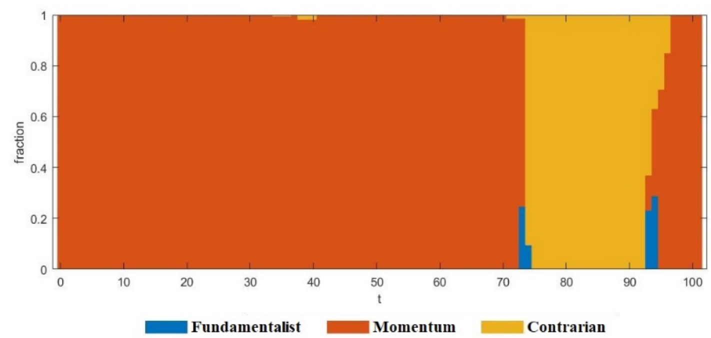
In a smooth crisis, price declines moderately but persistently over a period of time without a visible crash. As shown from Fig.4.5, the downward trend starts somewhere between $t = 30$ and $t = 40$ with an increase in the fraction of fundamentalists. After the first selling from fundamentalists, there are some counter-movements in price caused by contrarian traders. When the downward trend becomes more observable, investors cluster to momentum trading strategy and execute more selling forces to push the price further down. During the decline period, sentiment switches between bullish and bearish with more negative values overall. From $t = 30$ to $t = 70$, the average value of sentiment is -0.3751.



(a)

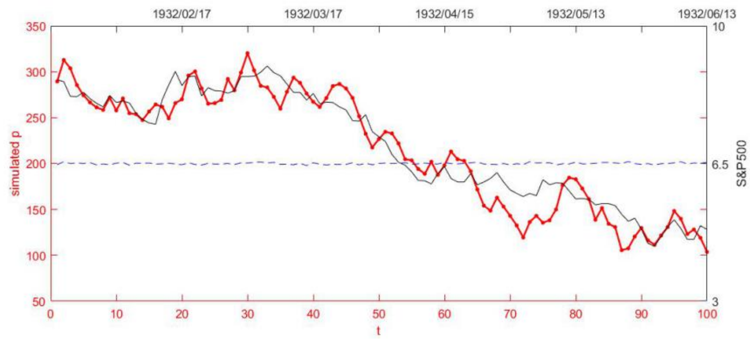


(b)

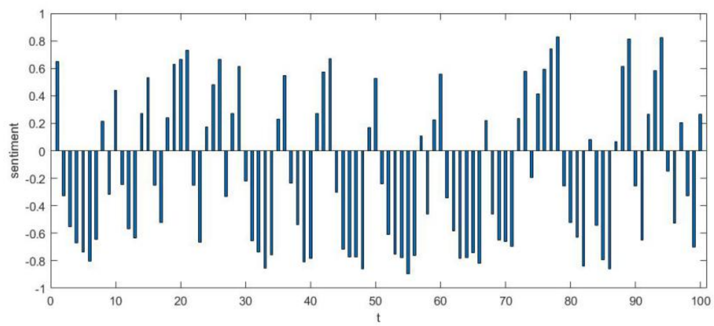


(c)

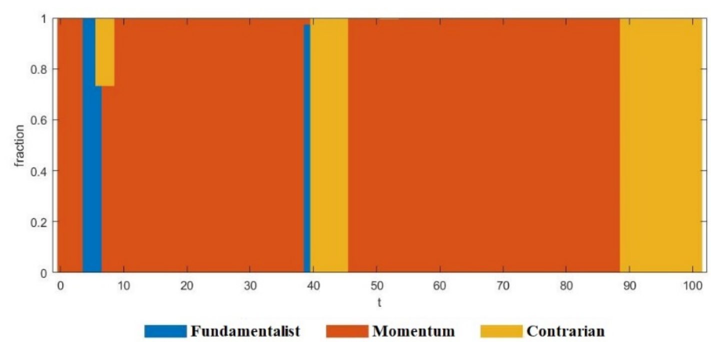
FIGURE 4.4. Sudden crisis modelling. The panels show, from top to bottom (a) comparison between simulated price with S&P 500 from 1987/8/3 to 1987/12/22, (b) simulated sentiment, and (c) fractions of 3 types of investors,



(a)

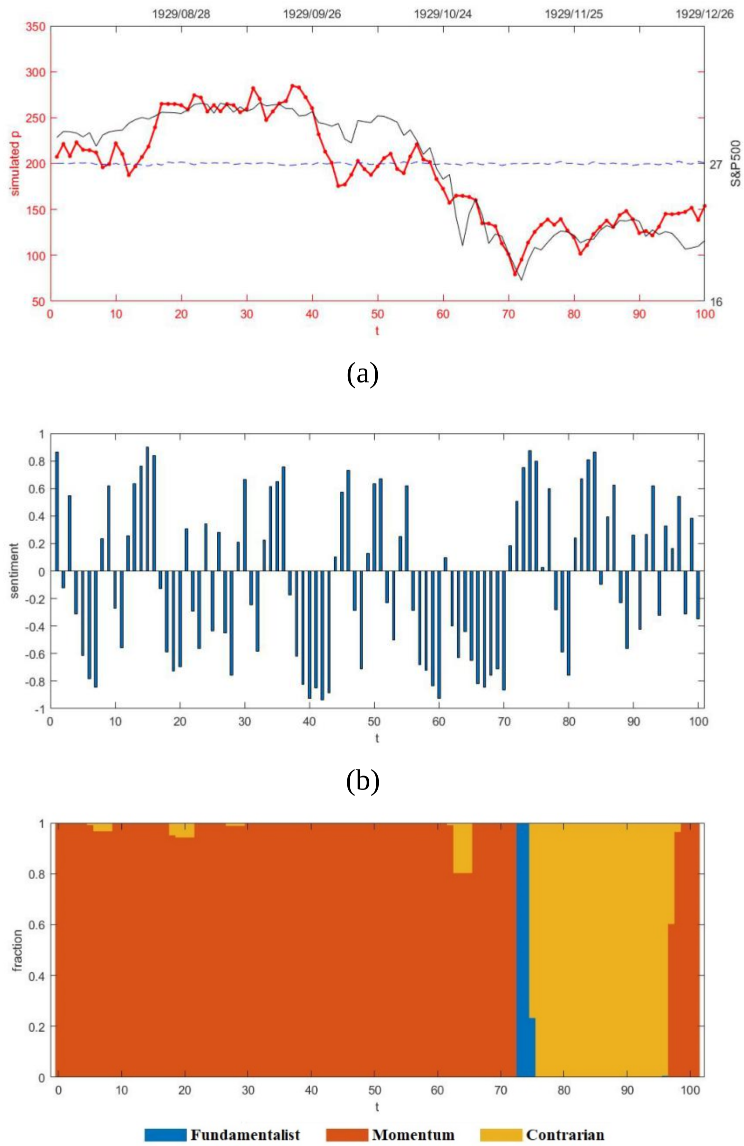


(b)



(c)

FIGURE 4.5. Smooth crisis modelling. The panels show, from top to bottom, (a) comparison between simulated price with S&P 500 from 1932/1/20 to 1932/6/13, (b) simulated sentiment and (c) fractions of 3 types of investors.



(a)

(b)

(c)

FIGURE 4.6. Disturbing crisis modelling. The panels show, from top to bottom, (a) comparison between simulated price with S&P 500 from 1929/8/1 to 1929/12/26 (b) simulated sentiment and (c) fractions of 3 types of investors.

A disturbing crisis is characterized by volatile fluctuations with a downward trend and possible moderate crashes in price. The period of disturbing crisis is somewhere between sudden crisis and smooth crisis. Fig.4.6 shows a simulation of disturbing crisis in our model. Price fluctuates disturbingly before starting to drop sharply at around $t = 37$ due to significant negative sentiment. From $t = 45$ to $t = 55$, price becomes volatile again as the sentiment fluctuates between positive and negative. After that, the downward trend continues with another strong bearish sentiment. Similar to the other types of crisis, momentum traders are responsible for the downward trend. However, during the crisis, some investors shift to contrarian strategy because of high market volatility.

To further investigate the effect of sentiment on crises, we compare the frequency and magnitude[7] of crises in simulated series from model with and without sentiment effect in 1000 simulations. To identify the crisis in financial market, we adopt a crisis indicator called CMAX used in Patel and Sarkar (1998) and Zouaoui (2011) with some adjustments. In this method, CMAX is a ratio calculated by dividing current value by the maximum price over the previous T periods, usually T is one to two years.

$$CMAX_t = \frac{P_t}{\max(P_{t-T} \cdots P_t)}$$

where P_t is the stock market index at time t. CMAX equals one if price rise over the period considered, indicating a bullish market. If price declines over a period, CMAX goes less than one, and crisis is detected each time CMAX drops below a threshold set at the mean of CMAX minus two standard deviations. Both mean and standard deviation are calculated on the whole sample. However, this method may mistakenly identify the bubble correction as a crisis. To fix this problem and make it more suitable for our simulated data, we add one compulsory condition to detect the crisis, which is the current price must be lower than fundamental prices for a certain threshold value. Therefore, the crisis indicator C_t is defined as following:

[7] Following Patel and Sarkar (1998) and Zouaoui (2011), we define the magnitude of a crisis as the percentage drop from the peak to the trough. The date of the peak is the month when price reaches its maximum value over T-period window prior to the crisis identification, and the date of the trough is month when price reaches its minimum during the crisis.

$$C_t = 1 \;\; if\; CMAX_t < \overline{CMAX} - 2\sigma \; and \; P_t < \tau * \mu_t$$

$$C_t = 0, \dots otherwise$$

where σ is the standard deviation of whole sample CMAX, and $0 < \tau < 1$ is the threshold to determine the minimum magnitude to be defined as a crisis. To check the efficiency of this method, we use the S&P 500 monthly data from 1950m1 and 2018m12 and corresponding fundamental value constructed from monthly dividend to detect the occurrence crisis in US stock market[8]. More specifically, we calculate fundamental value using static Gordon growth model with constant discount rate r and growth rate g of the dividend flows D_t, for which

$$P_t^* = \frac{1+g}{r-g} D_t$$

where P_t^* denotes the fundamental price of the stock index. We set $T = 12\; months$ and $\tau = 0.9$.

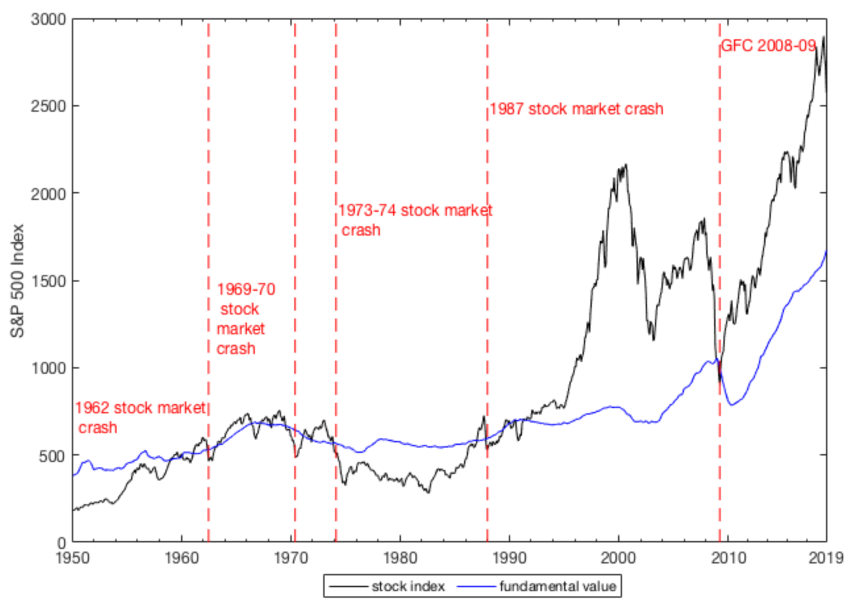
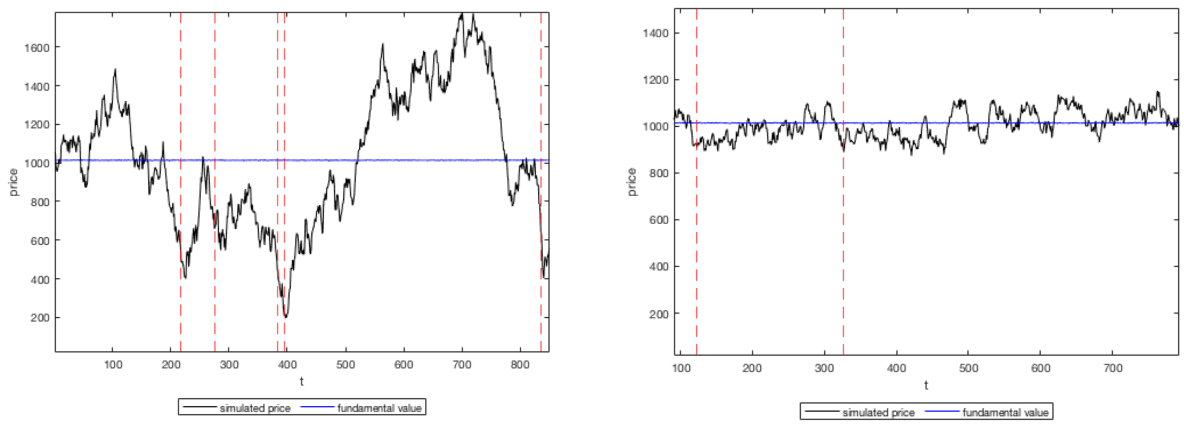


FIGURE 4.7. Crises detected from 1950 using real S&P 500 data.

[8] In accordance to Hommes and Veld (2017), fundamental value is estimated using S&P500 data from 1950 onwards. Both the real S&P500 prices and dividends on monthly frequency are provided by Shiller (2005).

As shown in Fig.4.7, five periods of crises are identified during the period 1950-2018. The first crash occurs in 1962 known as Kennedy Slide, followed by the second tech-stock crash in 1970 and third in 1973-1974 after the end of Bretton Woods monetary system and oil crisis. Most recent two market crashes are detected in 1987 known as Black Monday and during 2008-2009 global financial crisis.



(a) (b)

FIGURE 4.8. Identify crisis in simulated data with sentiment effect $h_1 = h_2 = 1$ (a), without sentiment effect $h_1 = h_2 = 0$ (b)

As CMAX indicator has been proven reliable in identifying crisis in real financial market, we are confident to use it in our simulated data. To be consistent with case of real financial market, we convert our simulated daily data to monthly data and choose $T = 12\ month$ and $\tau = 0.9$ to detect crisis. To obtain a visual impression on effect of sentiment on crisis, we try to detect the crisis in the previous two simulated time series with and without sentiment, respectively. From Fig.4.8, we find more crises in with-sentiment case than without-sentiment case. To check the robustness of our result and investigate the magnitude of crisis in these two cases, we again conduct Monte Carlo simulation with the three-agent model using the standard parameter values across different levels of sentiment sensitivity with interval of 0.2. As shown in Fig.4.9, number of crisis increases with sentiment sensitivity h.[9], which illustrates the significant positive relationship between sentiment sensitivity and frequency of crisis occurrence. We further investigate whether sentiment affects the magnitude of crisis. As we expected, the average magnitude of crisis rises with higher h which corroborates the effect of sentiment on depth of crisis. Hence, we can conclude that sentiment has a significant effect on crisis in terms of frequency and magnitude in our simulations. Our result is consistent with findings in Zouaoui (2011), who find investor sentiment positively influences the probability of the occurrence of stock market crises by using panel data of 16 countries.

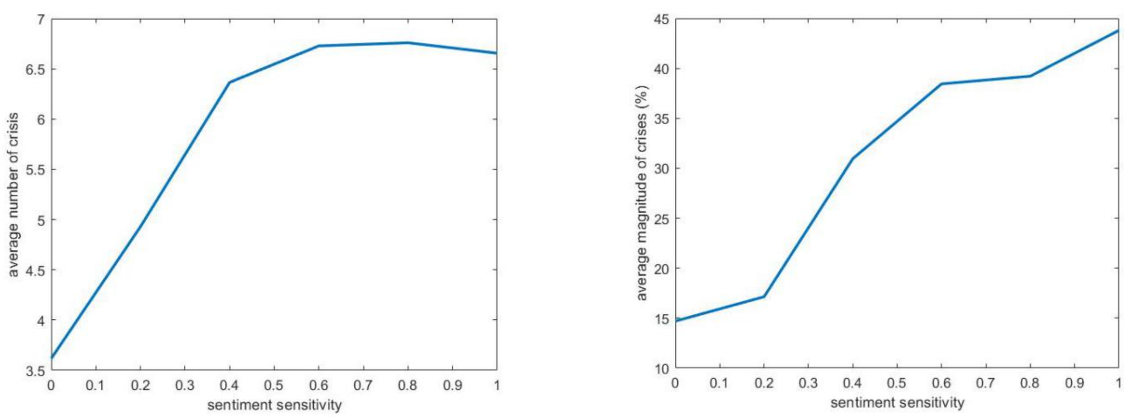


FIGURE 4.9. The effect of sentiment sensitivity on frequency of crisis (left) and average magnitude of crises (right) with 1000 simulations

In summary, we have demonstrated that investor sentiment contributes to more realistic stylized facts and our simulation results also show that the market will be more volatile if investors are very sensitive to market sentiment. More importantly, we successfully replicate different crises, and find that sentiment could be an important source of crisis formation.

[9] There is a slight decrease from $h = 0.8$ to $h = 1$. One explanation is that the extrapolating power of momentum traders becomes stronger because of high sentiment sensitivity, which could draw price to deviate from fundamental value for a long time. It may increase the magnitude and duration of crisis but decrease the frequency of crisis.

Through 1000 Monte Carlo simulations, we find the evidence that sentiment could amplify both the frequency and magnitude of financial crises.

Table 4.3: Simulated market dynamics with different sentiment sensitivities (average for 1000 simulations)

	h=0	h=0.2	h=0.4	h=0.6	h=0.8	h=1
kurtosis	2.155	2.075	3.396	4.615	4.730	5.804
skewness	-0.010	-0.015	-0.026	-0.036	-0.037	-0.040
AC r_1	0.010	0.041	0.166	0.219	0.189	0.166
AC r_5	-0.006	-0.004	0.004	0.001	0.007	0.004
AC r_{10}	-0.003	-0.004	-0.002	-0.003	-0.002	-0.003
AC r_{20}	-0.003	-0.002	-0.002	-0.001	-0.001	-0.001
AC $\|r_1\|$	-0.014	-0.003	0.172	0.344	0.372	0.412
AC $\|r_5\|$	0.070	0.075	0.179	0.252	0.253	0.298
AC $\|r_{10}\|$	0.039	0.046	0.167	0.241	0.243	0.290
AC $\|r_{20}\|$	0.015	0.025	0.158	0.231	0.235	0.282
$SD_{p-\mu}$	57.796	71.944	217.539	270.436	291.671	339.284
# of crisis	3.618	4.929	6.365	6.728	6.759	6.655
magnitude of crisis (%)	14.703	17.141	30.961	38.443	39.213	43.804

5. Robustness analysis

The goal of this section is to perform robustness checks by relaxing several nonlinearity assumptions that we previously imposed on the investor behavior. By excluding possible influences from other factors, such as nonlinear confidence of fundamentalist and regime-switching trading rules of chartist, we further confirm the significance of sentiment in contributing to the formations of stylized facts and crises in our model. In section 5.1, we first consider different setups for fundamentalist's demand function, in the forms of both linear as in Westerhoff and Dieci (2006) and nonlinear as in Day and Huang (1990). In section 5.2, we set different trading rules for chartists to examine our model's robustness to chartist's strategy. The main outcomes of our sentiment model still hold even when we simplify these assumptions albeit the benchmark model is still the best in term of replicating the stylized facts and crises.

5.1. Linear and nonlinear behaviors of fundamentalist. Our model is robust to the alternative assumptions of fundamentalist behavior. Specifically, the main results hold even if we change the confidence function in the benchmark model into two commonly used forms in the extant HAM literature, which are the linear function as in Westerhoff and Dieci (2006) and the nonlinear monotonous function as in Day and Huang (1990). The linear and nonlinear functions are shown in Equations 5.1 and 5.2, respectively.

$$D_t^f = a^f(\mu_t - p_t) \tag{5.1}$$

$$D_t^f = (p_t - a_1\mu_t)^d(a_2\mu_t - p_t)^d(\mu_t - p_t) \tag{5.2}$$

In Equation 5.1, a^f is a positive constant and fundamentalists are assumed to react with the same speed of mean reversion given any price deviation. Equation 5.2 is generalized version of demand function for fundamentalist following Day and Huang (1990), in which $d < 0, 0 \leq a_1 < 1, a_2 > 1$. The function implies that the more price deviates from its fundamental value, the more quickly the price trend is going to reverse within the range $(a_1\mu_t, a_2\mu_t)$,. As can be seen from Table 5.1, after replacing the chance function with linear or nonlinear monotonous function, we can still replicate most of the stylized facts, and the amplification effect of sentiment on volatility and crises remains. It suggests that the exact specification of fundamental demand component is not crucial for the model's ability to explain the stylized facts and crises in the financial markets. However, our nonlinear confidence function of fundamentalist is preferred as it can generate more realistic stylized facts with a wider range of sentiment sensitivity. For instance, the heavy-tail presents starting from $h = 0.4$ under the benchmark model, whereas under the linear chance function, heavy-tail only becomes evident at $h = 1$. In addition, it also helps to explain why the boom and bust cycles in the actual financial markets last longer than expected, since the fundamentalists may gradually lose their confidence when price deviation exceeds a certain threshold. We try to provide some preliminary evidence using google trend to proxy the confidence level of fundamentalists in stock market (see Appendix E). As shown in Fig.A3, google trend index closely follows the price deviation when deviation is below 1100 from 2013 to 2018. Nonetheless, the google trend index experience a big decline and stay at a low level even though the price deviation further increases since 2018. While more rigorous statistical exercises are needed for further conclusion, this hints a possible nonmonotonic correlation between price deviation and fundamentalist's mean-reverting behavior.

5.2. **Different technical rules of chartist.** Finally, we check the robustness of our benchmark model dynamic regarding the technical trading rule of chartists. Instead of deriving the short-run value from supporting and resistance levels, we assume chartists use moving average to calculate their reference price. Following Chiarella et al. (2007) with an exponential moving average, the short-run value can be written as:

$$v_t = kp_t + (1-k)v_{t-1} \tag{5.3}$$

Another common trading strategy is trend extrapolative rule, the chartists base their orders on the most recent price change as in Westerhoff and Dieci (2006). The larger the price trend, the stronger the demand. Take the momentum traders for example, the excess demand can be calculated as:

$$D_t^{mo} = \beta_1 m_t (p_t - p_{t-1}) \tag{5.4}$$

Table 5.2 presents the simulated results for the two alternative trading strategies with a linear demand function of the fundamentalists. It shows that parts of stylized facts can still be replicated. More importantly, the impact of sentiment on excess volatility, leptokurtosis of returns, volatility clustering and magnitude of crisis still persists

Table 5.1. Simulated market dynamics with different sentiment sensitivities using linear and Day and Huang (1990) setups for fundamentalist demand function (average for 1000 simulations)

	Linear demand function						Day and Huang (1990) demand function					
	h=0	h=0.2	h=0.4	h=0.6	h=0.8	h=1	h=0	h=0.2	h=0.4	h=0.6	h=0.8	h=1
kurtosis	2.378	2.312	2.657	2.931	3.053	3.377	2.537	2.551	3.442	4.397	4.807	5.309
skewness	-0.018	-0.020	-0.057	-0.091	-0.088	-0.109	-0.009	-0.012	-0.043	-0.077	-0.086	-0.106
AC r_1	0.013	0.041	0.159	0.218	0.188	0.165	0.022	0.050	0.183	0.225	0.191	0.166
AC r_5	-0.004	0.000	0.006	0.002	0.007	0.004	-0.003	0.000	0.003	-0.002	0.002	-0.002
AC r_{10}	-0.001	0.000	-0.001	-0.001	-0.001	-0.002	0.000	0.000	-0.002	-0.002	-0.002	-0.003
AC r_{20}	0.000	0.000	-0.001	-0.002	-0.001	-0.001	0.000	0.000	-0.001	-0.001	-0.002	-0.001
AC $\|r_1\|$	0.035	0.063	0.172	0.299	0.332	0.352	0.087	0.123	0.248	0.370	0.403	0.418
AC $\|r_5\|$	0.103	0.126	0.183	0.209	0.214	0.238	0.134	0.165	0.231	0.276	0.291	0.315
AC $\|r_{10}\|$	0.074	0.097	0.166	0.195	0.202	0.227	0.086	0.116	0.194	0.250	0.270	0.298
AC $\|r_{20}\|$	0.058	0.084	0.158	0.188	0.196	0.223	0.058	0.088	0.175	0.235	0.259	0.290
$SD_{p-\mu}$	141.601	164.813	239.047	266.797	277.417	293.900	145.606	164.613	234.051	281.749	304.278	336.214
# of crisis	5.560	5.872	6.774	7.376	7.268	7.353	5.452	5.837	6.590	6.760	6.842	6.800
magnitude of crisis (%)	20.166	22.440	30.059	35.723	36.732	38.883	20.927	23.511	33.006	38.715	40.851	42.916

Notes: We set $a^f = 0.01$ for linear demand function; $d = -0.4$, $a_1 = 0$ and $a_2 = 2.4$ for Day and Huang (1990) demand function.

Table 5.2. Simulated market dynamics with different sentiment sensitivities using exponential moving average and extrapolative trend strategy for chartists demand function (average for 1000 simulations)

	Exponential moving average ($k = 0.5$)						Extrapolative trend strategy					
	h=0	h=0.2	h=0.4	h=0.6	h=0.8	h=1	h=0	h=0.2	h=0.4	h=0.6	h=0.8	h=1
kurtosis	3.500	3.549	3.617	3.723	3.913	4.259	3.557	3.593	3.651	3.748	3.939	4.375
skewness	-0.002	-0.003	-0.004	-0.006	-0.010	-0.017	-0.006	-0.007	-0.010	-0.014	-0.020	-0.033
AC r_1	0.113	0.129	0.148	0.170	0.196	0.226	0.095	0.111	0.131	0.155	0.185	0.221
AC r_5	0.021	0.029	0.039	0.051	0.066	0.086	0.002	0.007	0.014	0.022	0.034	0.049
AC r_{10}	0.000	0.001	0.003	0.006	0.010	0.017	-0.002	-0.002	-0.002	-0.002	-0.001	0.001
AC r_{20}	-0.003	-0.003	-0.004	-0.004	-0.005	-0.006	-0.002	-0.003	-0.003	-0.004	-0.005	-0.006
AC $\|r_1\|$	0.070	0.079	0.090	0.105	0.124	0.150	0.106	0.121	0.140	0.164	0.194	0.232
AC $\|r_5\|$	0.055	0.059	0.063	0.070	0.080	0.096	0.059	0.061	0.064	0.069	0.077	0.090
AC $\|r_{10}\|$	0.053	0.056	0.060	0.065	0.072	0.084	0.057	0.059	0.062	0.066	0.072	0.082
AC $\|r_{20}\|$	0.051	0.054	0.058	0.062	0.069	0.079	0.056	0.058	0.060	0.064	0.069	0.078
$SD_{p-\mu}$	169.035	172.429	176.441	181.499	188.128	196.864	176.808	178.924	181.320	184.624	189.657	197.421
# of crisis	9.162	9.286	9.364	9.413	9.494	9.596	9.078	9.203	9.287	9.403	9.487	9.564
magnitude of crisis (%)	39.815	41.130	42.697	44.513	46.699	49.425	40.234	41.410	42.838	44.543	46.671	49.301

Notes: We set $a^f = 0.01$ for linear demand function and change the parameter of chartists as $\beta_1 = 0.35, \beta_2 = -0.25$. To generate stochastic movement in price, we add another random variable in the equation of market maker: $P_{t+1} = P_t + \gamma\left(\omega_t^f D_t^f + \omega_t^{mo} D_t^{mo} + \omega_t^{co} D_t^{co}\right) + \epsilon_M$, where $\epsilon_M \sim N(0, \sigma_M)$. In the simulations, we set $\sigma_M = 10$.

6. Conclusion

In this paper, we set up a sentiment model constituted with memory strength of sentiment, social interaction and sentiment shock components with three different types of agents investing in a financial market. We conjecture that investor sentiment has heterogeneous influence on different types of investors and the sentiment effect is transmittable through interaction among agents. We then carry out an in-depth analysis on the role of sentiment under HAM framework.

The key findings of this paper are in two respects. First, we show that our model could produce both fundamental steady state featured with neutral sentiment corresponding to that of standard paradigm as well as non-fundamental steady states with polarized sentiment (either positive or negative) in two-agent and three-agent models. In three-agent model, sentiment related parameters are able to affect the stability of fundamental steady state when two types of chartists have equal extrapolating power ($\beta_1 = -\beta_2$). Second, reconciling our model with actual US stock market, we discover that by supplementing the standard model with investor sentiment, it dramatically improves the model's ability to replicate stylized facts in financial markets. In particular, numerous key indicators, such as negative skewness, leptokurtosis and long memory of returns, market volatility, as well as number of crisis detected increase with sentiment sensitivity. We also try to deduce theoretical underpinning of different types of crises. The key implication that differentiates our finding from the extant HAM literature is that financial crisis can be triggered even without mean-reverting action of fundamentalist. With presence of investor sentiment, just the momentum traders alone can initiate sudden crisis when there is abrupt downward pressure in market sentiment. In other words, the sentiment channel provides an additional explanation underlying different types of financial crises.

While our sentiment HAM model is capable of replicating and even explaining several important features of financial market, it is not without limitation. One constraint is our model is based on single market framework. Nonetheless, given that international financial markets are becoming more and more integrated nowadays, it is imperative to account for sentiment spillover between countries. Our future research may involve extending the sentiment model to a multi-market setup, such that we are able to draw deeper understanding on investor sentiment as well as financial crisis on an international scale.

Appendix A

Confidence function for fundamentalists

$$A(x_t) = \frac{ax_t^2}{1 + bx_t^4}$$

is a nonlinear smooth and symmetric function of price deviation x_t. The fundamentalists react to the positive and negative trading signals equally and they are less confident when price deviation x_t is extremely large. The shape of $A(x_t)$ function is shown in Fig.A1, and it follows some general properties:

Property 1

$$A(0) = 0 \; and \; A(x_t) > 0 \; \forall \; x_t \neq 0$$

When there are price deviations, the fundamentalists will have positive confidence in their mean-reverting strategy.

Property 2

If $x_t > 0$, $A'(x) > 0 \; for \; x_t < b^{-\frac{1}{4}} \; and \;\; A'(x) < 0 \; for \; x_t > b^{-\frac{1}{4}}$

If $x_t < 0$, $A'(x) < 0 \; for \; x_t > -b^{-\frac{1}{4}} \; and \;\; A'(x) > 0 \; for \; x_t < -b^{-\frac{1}{4}}$

Range of increasing confidence level is [0, A_{max}], $A_{max} = A\left(b^{-\frac{1}{4}}\right) = A\left(-b^{-\frac{1}{4}}\right) = \frac{1}{2}ab^{-\frac{1}{2}}$.

The reasonable zone $(-Z_t, Z_t) = (-b^{-\frac{1}{4}}, b^{-\frac{1}{4}})$.

The confidence level of fundamentalists increases with $|x_t|$ within reasonable zone $(-b^{-\frac{1}{4}}, b^{-\frac{1}{4}})$, and decreases with $|x_t|$ outside this zone. The maximum confidence of fundamentalists is achieved when $x_t = \pm b^{-\frac{1}{4}}$, which is $\frac{1}{2}ab^{-\frac{1}{2}}$.

Property 3

$$\lim_{x_t \to \pm\infty} A(x_t) = 0$$

If the price deviation from fundamental value is too large, the fundamentalists will lose confidence and exit the market.

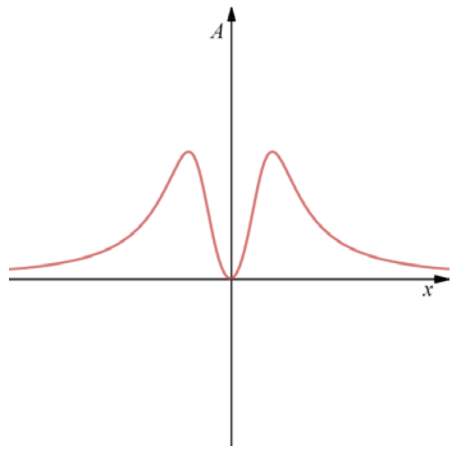


FIGURE A1: Confidence function for fundamentalist

Appendix B

We consider three market structure scenarios, which are fundamentalist versus momentum trader, fundamentalist versus contrarian trader, and momentum trader versus contrarian trader. We focus on the dynamic of these models with sentiment component. We first consider a two-agent model with market populated only by fundamentalists and momentum traders. This scenario has been investigated by many HAM studies within the standard fundamentalist-chartist framework. In this case, the system can be modelled as a three-dimensional nonlinear system such that:

$$\begin{aligned} p_{t+1} &= p_t + \gamma[\omega_t^f A(\mu_t - p_t) + \beta_1 \omega_t^{mo} m_t(p_t - v_t)] \\ U_{t+1} &= \varphi U_t + (p_{t+1} - p_t)[(A_t(\mu_t - p_t) - \beta_1 m_t(p_t - v_t)] \\ S_{t+1} &= \eta_1 S_t + \eta_2\left[\tanh(\kappa(\mu_t - p_{t+1}))\, \omega_{t+1}^f + \tanh(\kappa(p_{t+1} - v_{t+1}))\omega_{t+1}^{mo}\right] \end{aligned} \tag{B.1}$$

where

$$A = \frac{a(\mu_t - p_t)^2}{1 + b(\mu_t - p_t)^4}$$

$\omega_t^f = \frac{\exp(\rho U_t)}{\exp(\rho U_t)+1}$ and $\omega_t^{mo} = \frac{1}{\exp(\rho U_t)+1}$

$$m_t = 1 + \tanh(\kappa(p_t - v_t)) * h_1 * S_t$$

$$U_t = U_t^f - U_t^{mo}$$

The system has multiple steady states with different stability properties.

Proposition 2. *The system has*

a) *an unstable fundamental steady state (FSS) with* $(p^*, U^*, S^*) = (\mu, 0, 0)$ *if* $\mu = \bar{v}$*; two types of non-fundamental steady states (NFSS) with the form* $(p, U, S) = (p_1^*, 0, 0)$, $(p, U, S) = (p_2^*, 0, 0)$ *and* $p_1^* < \mu$*,* $p_2^* > \mu$ *if* $\mu = \bar{v}$*.*

b) *two types of non-fundamental steady states (NFSS) with the form* $(p^*, U^*, S^*) = (p_1^*, 0, S_1^*)$, $(p, U, S) = (p_2^*, 0, S_2^*)$ *and* $p_1^* < \mu, p_2^* > \mu$ *if* $\mu \neq \bar{v}$*.*

In line with Proposition 1, we find the fundamental steady state $p^* = \mu$ with a sufficient condition $\mu = \bar{v}$, and it is proven to be an unstable steady state. With the introduction of sentiment, we also find that when market sentiment is allowed to influence decision-making process of momentum traders, the system can be driven towards several steady states with either smaller (p_1^*) or greater (p_2^*) price than fundamental price. These complex attractors are characterized by persistently polarized levels of positive sentiment, negative sentiment and zero sentiment as we find in benchmark model. **Lemma 2** shows the conditions for these sentiment-persistent steady states.

Lemma 2. *For non-fundamental steady states***,** *the system could achieve positive sentiment equilibria* $(S^* > 0)$ *if* $\mu - \bar{v} > 0$*, negative sentiment equilibria* $(S^* < 0)$ *if* $\mu - \bar{v} < 0$*, zero sentiment equilibria* $(S^* = 0)$ *if* $\mu - \bar{v} = 0$ *and* $\beta_1 \leq \frac{1}{2} a b^{-\frac{1}{2}}$.

Proof of Proposition 2. A steady state (p^*, U^*, S^*) of the deterministic system (B.1) in one regime case must satisfy the following equations:

$A(\mu - p^*) + \beta_1 m^*(p^* - \bar{v}) = 0$

$S^* = \frac{\eta_2}{2(1-\eta_1)}[\tanh(\kappa(\mu - p^*)) + \tanh(\kappa(p^* - \bar{v}))]$

$m^* = 1 + \tanh(\kappa(p^* - \bar{v})) * h_1 * S^*$

If $\mu = \bar{v}$, we have one fundamental steady state $p^* = \mu = \bar{v}$. In this case, $S^* = 0$ and $m^* = 1$. The stability of this fundamental steady state can be inferred from Jacobian matrix J of the system (3.1) as

$$J = \begin{pmatrix} 1 + \frac{1}{2}\gamma\beta_1, & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}$$

The eigenvalues of diagonal matrix lie on the main diagonal of matrix, so

$$\lambda_1 = 1 + \frac{1}{2}\gamma\beta_1,\ \lambda_2 = \varphi,\ \lambda_3 = \eta_1$$

The fundamental steady state is asymptotically stable when all the eigenvalues of Jacobian matrix at the steady state are less than one. Hence, fundamental steady state is unstable due to $\lambda_1 > 0$ (both $\gamma\ and\ \beta_1$ are positive values).

Non-fundamental steady states may also coexist with fundamental steady state when $\mu = \bar{v}$. At the NFSS, $S^* = 0$ and $m^* = 1$, the first equation of system (B.1) can be simplified as $(A - \beta_1)(\mu - p^*) = 0$. In this case, $\mu \neq p^*$ and the solutions are from $A - \beta_1 = 0$. According to shape of A, the possible number of solutions is 0, 2, 4. If β_1 is larger than A_{max} which is $\frac{1}{2}ab^{-\frac{1}{2}}$, there is no solution; If $\beta_1 = A_{max}$, there are two symmetric solutions around fundamental value

μ (positive x_t and negative x_t). If $\beta_1 < A_{max}$, four steady states occur with two above μ and the other two below μ.

Proof of Lemma 2. If $\mu \neq \bar{v}$, non-fundamental steady states could also emerge with persistent positive or negative sentiment. p^* must be lower or higher than both μ and $\bar{v}$ to satisfy the first equation of system (3.1). According to the positions of μ and $\bar{v}$, we can also derive the market sentiment equilibria well as the relation between the confidence level of fundamentalists and momentum traders' extrapolation rate at steady state. When $\mu > \bar{v}, S^*$ must be positive due to $\tanh(\kappa(\mu - p^*)) + \tanh(\kappa(p^* - \bar{v})) > 0$ for both lower and higher p^*, so we have a positive sentiment equilibrium. At the same time, $(p_1^*) < \beta_1 m^*$ $(A(p_2^*) > \beta_1 m^*)$ must be satisfied to get lower (higher) equilibrium p_1^* (p_2^*) . Conversely, when $\mu > \bar{v}$, $S^*$is always negative and we have negative sentiment equilibria. Lower (higher) equilibrium p_1^* (p_2^*) is associated with $A(p_1^*) > \beta_1 m^*$ $(A(p_2^*) < \beta_1 m^*)$.

Next, we consider the scenario with only fundamentalist and contrarian trader. This design has been explored in Chiarella and He (2003), He and Li (2015), Agliari et al. (2018) and they confirm the stabilizing role of contrarian traders. Unlike these studies, we incorporate investor sentiment into the system. We assume that only contrarian traders are sensitive to market sentiment, and the three-dimensional nonlinear system can be written as:

$$\begin{aligned}
&p_{t+1} = p_t + \gamma[\omega_t^f A(\mu_t - p_t) + \beta_2 \omega_t^{co} c_t(p_t - v_t)] \\
&U_{t+1} = \varphi U_t + (p_{t+1} - p_t)[(A_t(\mu_t - p_t) - \beta_2 c_t(p_t - v_t)] \\
&S_{t+1} = \eta_1 S_t + \eta_2\left[\tanh(\kappa(\mu_t - p_{t+1}))\omega_{t+1}^f - \tanh(k(p_{t+1} - v_{t+1}))\omega_{t+1}^{co}\right]
\end{aligned} \tag{B.2}$$

where

$$A = \frac{a(\mu_t - p_t)^2}{1 + b(\mu_t - p_t)^4}$$

$$\omega_t^f = \frac{\exp(\rho U_t)}{\exp(\rho U_t)+1} \text{ and } \omega_t^{co} = \frac{1}{\exp(\rho U_t)+1}$$

$$c_t = 1 - tanh(p_t - v_t) * h_2 * S_t$$

$$U_t = U_t^f - U_t^{co}$$

We also start by investigating the stability of the system.

Proposition 3. *The system has*

a) *a unique fundamental steady state with* $(p^*, U^*, S^*) = (\mu, 0, 0)$ *if* $\mu = \bar{v}$. *Fundamental steady state is asymptotically stable for* $-4 < \gamma\beta_2 < 0$ *, and it undergoes a flip bifurcation for* $\gamma\beta_2 = -4$.

b) *three types of non-fundamental steady states with the form* $(p^*, U^*, S^*) = (p_1^*, 0, S_1^*)$ *,* $(p^*, U^*, S^*) = (p_2^*, 0, S_2^*), (p^*, U^*, S^*) = (p_3^*, 0, S_3^*)$ *and* $p_1^* < \frac{1}{2}(\mu + \bar{v}), p_2^* > \frac{1}{2}(\mu + \bar{v}), p_1^* = \frac{1}{2}(\mu + \bar{v})$ *if* $\mu \neq \bar{v}$.

We find a unique fundamental steady state if $\mu = \bar{v}$, and it is asymptotically stable for $-4 < \gamma\beta_2 < 0$. For $\mu \neq \bar{v}$, we find three different types of NFSS, and these complex attractors are characterized by persistently polarized levels of positive, negative and neutral sentiment. In line with benchmark model, both FSS and sentiment-orientated NFSS are found in this two-agent model. **Lemma 3** shows these sentiment-persistent steady states in detail.

Lemma 3 *For non-fundamental steady states, the system achieves positive sentiment equilibria* $(S^* > 0)$ *if* $p^* < \frac{1}{2}(\mu + \bar{v})$*, negative sentiment equilibria* $(S^* < 0)$ *if* $p^* > \frac{1}{2}(\mu + \bar{v})$ *neutral sentiment equilibria* $(S^* = 0)$ *if* $p^* = \frac{1}{2}(\mu + \bar{v})$ *and* $\beta_1 \leq \frac{1}{2}ab^{-\frac{1}{2}}$.

Proof of Proposition 3. Besides the heterogeneous extrapolation rates for momentum traders and contrarian traders, they also react differently to market sentiment. The steady state (p^*, U^*, S^*) of the deterministic system (B.2) in one regime case must satisfy the following equations:

$$A(\mu - p^*) + \beta_2 c^*(p^* - \bar{v}) = 0$$

$$S^* = \frac{\eta_2}{2(1 - \eta_1)}[\tanh(\kappa(\mu - p^*)) - \tanh(\kappa(p^* - \bar{v}))]$$

$$c^* = 1 - \tanh(\kappa(p^* - \bar{v})) * h_1 * S^*$$

If $\mu = \bar{v}$, we have unique fundamental steady state $p^* = \mu = \bar{v}$. In this case, $S^* = 0$ and $c^* = 1$. The stability of this fundamental steady state can be inferred from Jacobian matrix J of the system (B.2) as

$$J = \begin{pmatrix} 1 + \frac{1}{2}\gamma\beta_2, & 0 & 0 \\ 0 & \varphi & 0 \\ -\eta_2(1 + \frac{1}{2}\gamma\beta_2) & 0 & \eta_1 \end{pmatrix}$$

The eigenvalues of Jacobian matrix are

$$\lambda_1 = 1 + \frac{1}{2}\gamma\beta_2,\ \lambda_2 = \varphi,\ \lambda_3 = \eta_1$$

As $0 < \varphi < 1$ and $0 < \eta_1 < 1$, the stability of the system is determined by λ_1. It is possible to have λ_1 inside the unit circle due to $\beta_2 < 0$. When $|\lambda_1| < 1$ or $-4 < \gamma\beta_2 < 0$, the fundamental steady state is locally stable. Note that setting $\gamma\beta_2 = -4$ gives us the flip bifurcation as $\lambda_1 = -1$.

Proof of Lemma 3. If $\mu \neq \bar{v}$, three types of non-fundamental steady states may emerge with persistent positive, negative sentiment or zero sentiment. The equilibrium p^* must between the fundamental value μ and short-term value $\bar{v}$. If price is lower or higher than both μ and $\bar{v}$, it will be attracted to the zone constructed by $\min(\mu, \bar{v})\ and\ \max(\mu, \bar{v})$. According to the position of p^*, we can derive three different market steady states with positive, negative or neutral sentiment. As the equilibrium price is within $(\min(\mu, \bar{v}), \max(\mu, \bar{v}))$, the distances of p^* to μ and $\bar{v}$ decide the sign of equilibrium sentiment S^*. If $p^* < \frac{1}{2}(\mu + \bar{v})$, $S^* > 0$ and a positive sentiment equilibrium is achieved. To know the source of this positive market sentiment, we need to analyze the relative position of μ and $\bar{v}$. If $\mu > \bar{v}$ and $p_1^* < \frac{1}{2}(\mu + \bar{v})$, we need to have $A(p_1^*) < -\beta_2 c^*$ to satisfy the first equation of system (B.2), and market sentiment is mainly from fundamentalists' optimistic beliefs. Otherwise, if $\mu < \bar{v}$ and $p_1^* < \frac{1}{2}(\mu + \bar{v})$, $A(p_1^*) > -\beta_2 c^*$ must be satisfied to get the positive sentiment equilibrium, and optimism is derived from relatively aggressive contrarian traders. Another negative sentiment steady state is featured with

$p_2^* > \frac{1}{2}(\mu + \bar{v})$. If $\mu > \bar{v}$, equilibrium could be achieved with $A(p_2^*) > -\beta_2 c^*$. If $\mu > \bar{v}$, equilibrium could emerge with $A(p_2^*) < -\beta_2 c^*$. A special case is when $p_3^* = \frac{1}{2}(\mu + \bar{v})$, then $S^* = 0$, and equilibrium is achievable if $A(p_3^*) = -\beta_2$. It means we must have $\beta_2 = -\frac{4a(\mu+\bar{v})^2}{16+b(\mu+\bar{v})^4}$ to get this steady state.

The last two-agent model has been rarely explored within HAM framework. In this case, market only consists of two types of chartist, momentum traders and contrarian traders. One possible situation for this setup is that the market is highly speculative or it is difficult to access the fundamental value of the asset (such as Bitcoin market), then most of the investors would choose chartist strategies. By only including momentum traders and contrarian traders, we can model the system as:

$$\begin{aligned}
p_{t+1} &= p_t + \gamma(\beta_1 \omega_t^{mo} m_t + \beta_2 \omega_t^{co} c_t)(p_t - v_t) \\
U_{t+1} &= \varphi U_t + (\beta_1 m_t - \beta_2 c_t)(p_{t+1} - p_t)(p_t - v_t) \\
S_{t+1} &= \eta_1 S_t + \eta_2 \tanh(\kappa(p_{t+1} - v_{t+1}))\, G_{t+1}
\end{aligned} \tag{B.3}$$

where

$$\omega_t^{mo} = \frac{\exp(\rho U_t)}{\exp(\rho U_t)+1} \text{ and } \omega_t^{co} = \frac{1}{\exp(\rho U_t)+1}$$

$$m_t = 1 + tanh(\kappa(p_t - v_t)) * h_1 * S_t$$

$$c_t = 1 - tanh(\kappa(p_t - v_t)) * h_2 * S_t,$$

$$U_t = U_t^{mo} - U_t^{co},$$

$$G_t = \omega_t^{mo} - \omega_t^{co} = tanh\left(\frac{\rho\kappa}{2}\left(\varphi U_t + (\beta_1 m_t - \beta_2 c_t)(p_{t+1} - p_t)(p_t - v_t)\right)\right)$$

As momentum traders have belief opposite to that of contrarian traders, the heterogeneous extrapolation rate of these two chartist groups may affect the price dynamic and stability of steady state.

Proposition 4. *The system (B.3) has a unique steady state* $(p^*, U^*, S^*) = (\bar{v}, 0, 0)$. *The steady state is locally and asymptotically stable provided that* $-4 < \gamma(\beta_1 + \beta_2) < 0$, *and it undergoes a flip bifurcation for* $\gamma(\beta_1 + \beta_2) = -4$.

The unique steady state in this case is very similar to FSS in the bench market model, but we fail to find the sentiment-orientated NFSS. In the steady state, the positive and negative sentiment from two types of chartists offset each other due to the identical fraction of two groups. We further investigate the ability of these two-agent models to explain the stylized facts in financial market, and we find they can duplicate most of these facts in market. Through the robustness check, we confirm our model are robust in terms of producing market dynamics with FSS and sentiment-orientated NFSS and stylized facts in markets.

Proof of Proposition 4. The steady state (p^*, U^*, S^*) of the deterministic system (B.3) in one regime case must satisfy the following equations:

$$(\beta_1 m^* + \beta_2 c^*)(p^* - \bar{v}) = 0$$

$$S^* = 0$$

$$m^* = c^* = 1$$

If $\beta_1 \neq -\beta_2$, we have unique steady state $p^* = \bar{v}$. The corresponding Jacobian matrix can be derived as

$$J = \begin{pmatrix} 1 + \frac{1}{2}\gamma(\beta_1 + \beta_2), & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}$$

The eigenvalues of Jacobian matrix are

$$\lambda_1 = 1 + \frac{1}{2}\gamma(\beta_1 + \beta_2),\ \lambda_2 = \varphi,\ \lambda_3 = \eta_1$$

As $0 < \varphi < 1$ and $0 < \eta_1 < 1$, the stability of the system is determined by λ_1. For absolute value of λ_1 to be less than 1, we must have $-4 < \gamma(\beta_1 + \beta_2) < 0$ to make the system locally stable. The setting $\gamma(\beta_1 + \beta_2) = -4$ gives us the flip bifurcation as $\lambda_1 = -1$.

Appendix C

Proof of Proposition 1. The relationship between β_1 and β_2 could determine the number and type of steady states. If $\beta_1 = -\beta_2 = \beta$, only fundamental steady state is possible with the following equations:

$$\omega_f^* A^*(\mu - p^*) + \beta(\omega_{mo}^* m^* - \omega_{co}^* c^*)(p^* - \bar{v}) = 0$$

$$\omega_f^* = \omega_{mo}^* = \omega_{co}^* = \frac{1}{3}$$

$$u_f^* = u_{mo}^* = u_{co}^* = 0$$

$$S^* = 0$$

$$m^* = c^* = 1$$

The corresponding Jacobian matrix with $\beta_1 = -\beta_2 = \beta$ can be derived as

$$J = \begin{pmatrix} 1 & 0 & B & -B & C \\ 0 & \varphi & 0 & 0 & 0 \\ 0 & 0 & \varphi + \beta\delta B & -\beta\delta B & \beta\delta C \\ 0 & 0 & -\beta\delta B & \varphi + \beta\delta B & \beta\delta C \\ 0 & 0 & D & -D & E \end{pmatrix}$$

where

$$\delta = \mu - \bar{v}, B = \frac{1}{3}\gamma\rho\beta(\mu - \bar{v}), C = \frac{1}{3}\gamma\beta(\mu - \bar{v})(h1 + h2)\tanh(\mu - \bar{v})$$

$$D = -\frac{1}{9}\rho\eta_2[\gamma\beta(\mu - \bar{v}) - 3\varphi\tanh(\kappa(\mu - \bar{v}))],$$

$E = \eta_1 - \frac{1}{9}\,\eta_2\gamma\beta(\mu - \bar{v}\,)(h1 + h2)\tanh(\kappa(\mu - \bar{v}))$

There are five eigenvalues for this Jacobian matrix, which are

$$\lambda_1 = 1,\ \lambda_2 = \varphi,\ \lambda_3 = \varphi + \beta\delta B,\ \lambda_4 = \frac{\varphi^2 + 2\varphi\beta\delta B}{\varphi + \beta\delta B},$$

$$\lambda_5 = E - \frac{2\beta\delta CD}{\varphi} + \frac{4\beta\delta CD(\varphi + \beta\delta B)}{\varphi^2 + 2\varphi\beta\delta B} - \frac{2\varphi\beta\delta CD(\varphi + \beta\delta B)}{(\varphi^2 + 2\varphi\beta\delta B)(\varphi + \beta\delta B)}$$

To make the system locally stable, we need to have $|\lambda_3|, |\lambda_4|, |\lambda_5| < 1$. For a special case $\mu = \bar{v}$, the Jacobian matrix could be simplified to

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varphi & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 & 0 \\ 0 & 0 & 0 & \varphi & 0 \\ 0 & 0 & D & 0 & \eta_1 \end{pmatrix}$$

The corresponding five eigenvalues for this diagonal matrix are

$$\lambda_1 = 1,\ \lambda_2 = \varphi,\ \lambda_3 = \varphi,\ \lambda_4 = \varphi,\ \lambda_5 = \eta_1$$

As $0 < \varphi < 1$ and $0 < \eta_1 < 1$, the system is always stable for this special case.

If $\beta_1 \neq -\beta_2$, both FSS and NFSS arepossible, and steady state $\left(p^*, u_f^*, u_{mo}^*, u_{co}^*, S^*\right)$ of the deterministic system (3.1) in one regime case must satisfy the following equations:

$\omega_f^* A^*(\mu - p^*) + (\omega_{mo}^*\beta_1 m^* + \omega_{co}^*\beta_2 c^*)(p^* - \bar{v}) = 0$

$\omega_f^* = \omega_{mo}^* = \omega_{co}^* = \frac{1}{3}$

$u_f^* = u_{mo}^* = u_{co}^* = 0$

$S^* = \frac{\eta_2}{3(1 - \eta_1)}\tanh(\kappa(\mu - \ p^*))$

$m^* = 1 + \tanh(\mathrm{k}(p^* - \bar{v})) * h_1 * S^*$

$c^* = 1 - \tanh(\mathrm{k}(p^* - \bar{v})) * h_1 * S^*$

For fundamental steady state, we have $p^* = \mu = \bar{v}$, and Jacobian matrix can be derived as

$$J = \begin{pmatrix} 1 + \frac{1}{3}\gamma(\beta_1 + \beta_2), & 0 & 0 & 0 & 0 \\ 0 & \varphi & 0 & 0 & 0 \\ 0 & 0 & \varphi & 0 & 0 \\ 0 & 0 & 0 & \varphi & 0 \\ -\frac{1}{3}\eta_2 & 0 & 0 & 0 & \eta_1 \end{pmatrix}$$

Five eigenvalues for this Jacobian matrix will be

$$\lambda_1 = 1 + \frac{1}{3}\gamma(\beta_1 + \beta_2),\ \lambda_2 = \varphi,\ \lambda_3 = \varphi,\ \lambda_4 = \varphi,\ \lambda_5 = \eta_1$$

We can have a locally stable system provided that $-6 < \gamma(\beta_1 + \beta_2) < 0$.

For NFSS, we focus on the features of the equilibria. There are two types of steady states, positive and negative sentiment steady states. From the sentiment function, it is clear that when $p^* < \mu$, $S^* > 0$ and vice versa. But p^* must satisfy the first equation of system (3.1).

$A^*(\mu - p^*) + (\beta_1 + \beta_2)(p^* - \bar{v}) + (\beta_1 h_1 - \beta_2 h_2)[S^* \tanh(\kappa(p^* - \bar{v}))](p^* - \bar{v}) = 0$

If $p^* < \mu$, we have S^* and the first and third term of this equation are positive, so the second term must be negative. When $\beta_1 > -\beta_2$ ($\beta_1 < -\beta_2$), we have $p^* < \bar{v}$ ($p^* > \bar{v}$). For negative sentiment non-fundamental steady state $S^* < 0$, we have $p^* > \mu$ and $p^* > \bar{v}$ ($p^* < \bar{v}$) when $\beta_1 > -\beta_2$ ($\beta_1 < -\beta_2$).

Appendix D

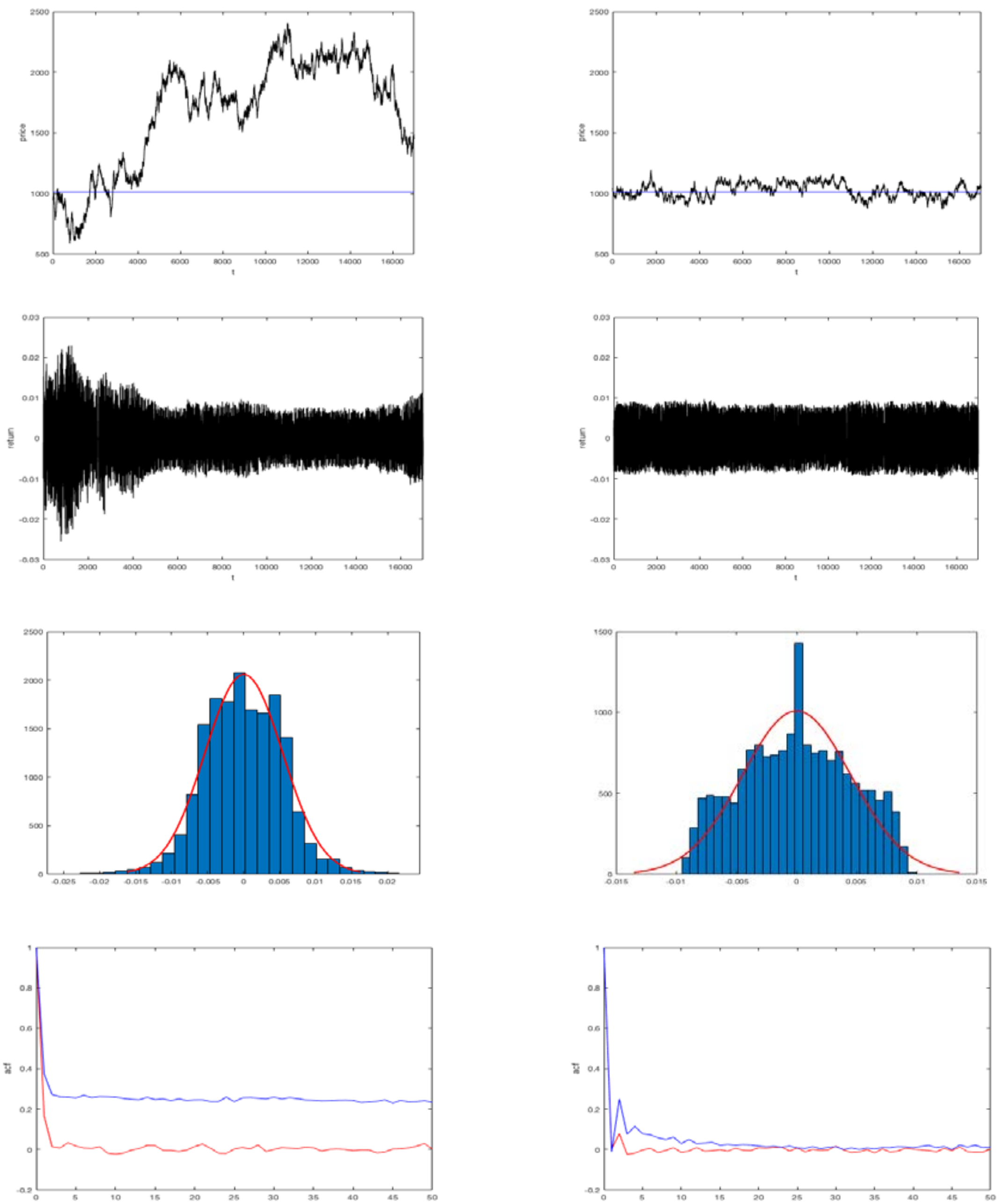


FIGURE A2. The dynamics of the deterministic models with sentiment (left) and without sentiment (right). The panels show, from top to bottom, the evolution of the stock prices, the returns, the histogram of returns overlaid by normal curve and the ACF of raw returns (red line) together with the ACF of absolute returns (blue line), respectively. The simulation run is based on 17000 observations.

Appendix E

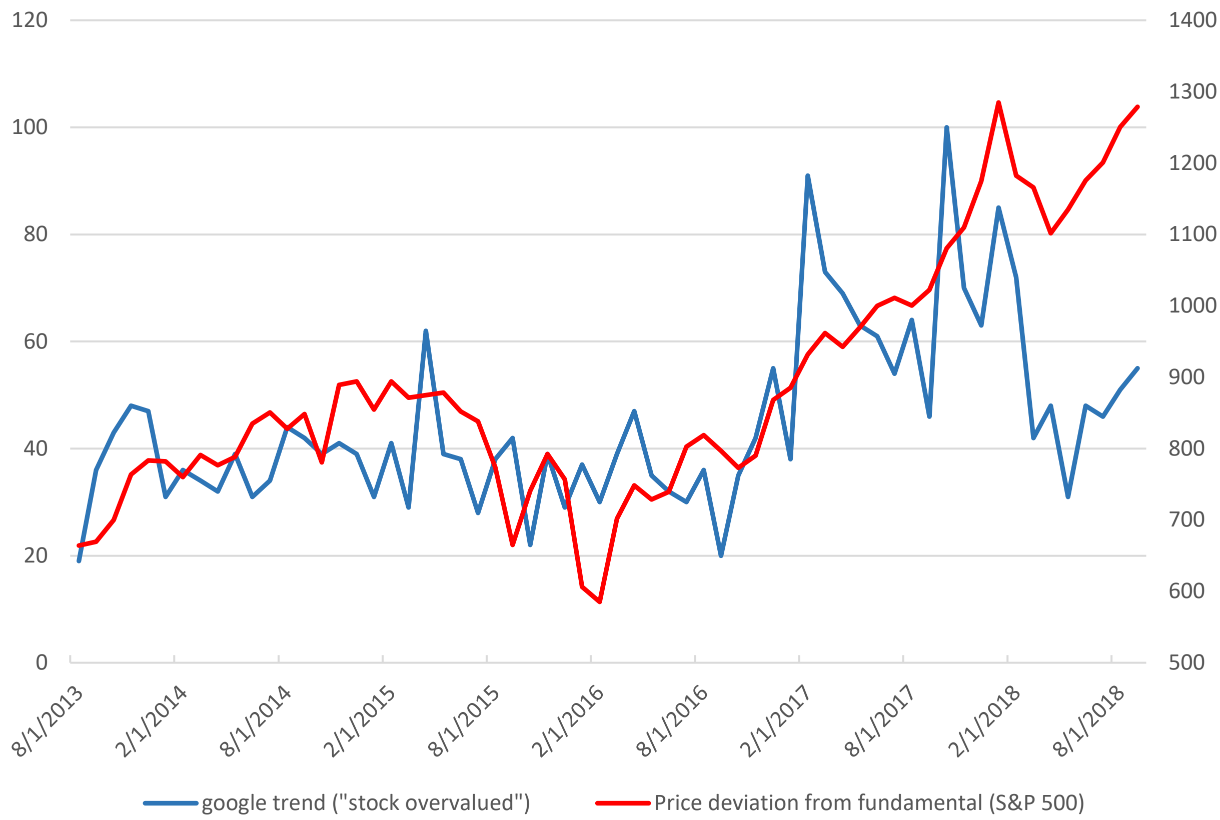


FIGURE A3. Price deviation from fundamental value and google trend index on "stock overvalued" from 2013 to 2018. Google trend index is used as a proxy for fundamentalist confidence, the more people search for "stock overvalued", the higher the google trend index, the more confidence of fundamentalists.

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