Optimal Currency Exposure Under Risk and Ambiguity Aversion

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Abstract

This paper addresses the choice of optimal currency exposure for a risk and ambiguity averse international investor. Robust mean-variance preferences, explicitly capturing an investor's dislike for model uncertainty, are used in order to derive the model-free optimal currency exposure in the presence of both risk and ambiguity aversion. We show that the sample efficient currency demand can be expressed as a vector of generalized ridge regression coefficients of fully hedged portfolio returns on the excess currency returns. Moreover, the underlying model uncertainty corresponds to the penalty term in the regression. The empirical analysis of the derived currency overlay strategy employs the foreign exchange, stock, and bond returns over the period from 1999 to 2018. We find that our proposed hedging strategy leads to significant improvements of the portfolio performance and examine the effect of model uncertainty on optimal currency allocations.

Keywords: Currency exposure, ambiguity aversion, model uncertainty, regularization, ridge regression, hedging strategy, currency overlay, international asset allocation.

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1 Introduction

A natural approach to improve portfolio performance is to invest internationally. Besides potential diversification benefits, international asset allocation also poses challenges to investors. One of the main issues is foreign currency exposure: investors have to decide on the amount of currency exposure to hedge. A number of competing approaches are available in the existing literature. However, to the best of our knowledge, no work presented in the area of currency hedging deals with model uncertainty of the probability laws governing the stochastic processes of asset and currency returns. In this paper, we present a novel approach for determining the optimal currency exposure in an international portfolio under uncertainty. The model takes into account investor's aversion to risk as well as her aversion to ambiguity, and jointly incorporates them directly into the currency exposure decision. Furthermore, it sheds new light on optimal currency allocation under a framework of model uncertainty.

Adding the ambiguity component into the decision of optimal currency exposure is valuable for at least two reasons. First, there is a general dissatisfaction with the empirical and predictive performance of the expected utility/rational expectations paradigm. The ambiguity literature is currently perceived as one of the promising solutions. We aim to incorporate the uncertainty associated with predictive models, as we explicitly account for model uncertainty in forecasts. Second, several papers have analyzed key events from the financial crisis of 2008–2009. The general conclusion is that such events are connected to ambiguity in the form of poorly understood information and are related to investors' aversion to difficult-to-quantify uncertainty, as opposed to risk. Events such as Swiss franc unpeg, Brexit, and other incidents of political uncertainty and trade tensions, fuel the foreign exchange market with a high level of ambiguity. However, existing currency hedging models do not capture agents' aversion to such uncertainty. In this work, we explicitly account for investor's ambiguity aversion and tractably capture uncertainty directly in the optimal currency allocation decision.

The paper starts with introducing a general framework for international asset allocation in a model-free setting, without any assumptions on the dynamics of returns. Accounting for model uncertainty directly in the agent's currency allocation decision, we then employ robust mean-variance preferences which explicitly capture the agent's risk and ambiguity aversion. Building on that, we derive closed form solutions characterizing the optimal currency exposure for a risk and ambiguity averse investor.

We show that the optimal in-sample currency exposure for a risk and ambiguity averse agent can be found by a generalized ridge regression. The demeaned hedged portfolio returns are regressed on the demeaned currency excess returns and shrunk towards the infinitely ambiguity averse optimal exposure. The shrinkage is distorted by the level and direction of model uncertainty. Such artificial regression recovers the sample efficient currency exposures and enables a geometric interpretation of obtained results. In the presence of ambiguity, the generalized ridge penalty term corresponds to the utility loss arising from model uncertainty.

In the empirical part of the paper we first investigate whether our theoretical model confirms the findings in the existing currency hedging literature. Additionally, we employ recent market data ranging from 1999 to 2018. In line with previous findings we conclude that the US dollar, the Japanese yen, the Swiss franc and the euro tend to appreciate when international stock markets fall. Hence, these currencies are attractive stores of value for international equity investors. Conversely, we find that the Australian dollar, the Canadian dollar, and the British pound exhibit positive correlation with world stock markets. Moreover, our results suggest a growing acceptance of the Japanese yen as a reserve currency after the global financial crisis and the opposite for the Swiss franc. We also confirm that bond investors' risk management demands for currencies are small or zero, regardless of the investors' home country.

Next, we investigate the empirical effect of ambiguity aversion on the estimator of optimal currency exposure. We show that ambiguity induces an increase in bias and a simultaneous shrinkage of confidence intervals of the proposed estimator. This result gives rise to a possible bias-variance trade-off. Thus, acknowledging uncertainty can lead to an improved estimator of optimal currency exposure, measured in terms of mean squared error. This improvement arises even though ambiguity enters an investor's robust utility function as a strictly negative value.

In order to investigate whether volatility reductions come at the cost of lower expected returns per unit of portfolio risk we compute realized Sharpe ratios. In the minimum variance case, an investor enters into a positive exposure to reserve currencies. These currencies exhibit lower average returns and their underlying positive exposure decreases realized Sharpe ratios. The volatility and Sharpe ratio for the ambiguity adjusted currency overlay strategy lie between the minimum variance and mean-variance cases.

Our work gives rise to numerous possible extensions in various theoretical and empirical directions. In addition, the research topic is practically relevant and widely discussed in the financial services industry, especially in the areas of strategic asset allocation and wealth management.

The paper is organized as follows. Section 2 reviews the existing literature and Section 3 introduces the theoretical model. Section 4 addresses the empirical performance of the proposed theoretical model and analyzes the empirical effect of ambiguity aversion on optimal currency exposures. Section 5 concludes. Additional proofs and figures are presented in the Appendix.

2 Literature Review

A number of currency hedging strategies have been presented in the literature. Starting already with the seminal papers that aimed to address this research question, opinions among different authors were divided. Perold and Schulman (1988) proposed 100% hedging as the optimal strategy. They argued that since a currency trade is a zero-sum game, the hedging reduces volatility without loss of expected return over the long run. Froot (1993), on the other hand, concluded that full hedging can reduce risk over short horizons, however, over the long term it may actually increase risk without an adequate return compensation. More specifically, in the long run, hedged returns are dominated by surprises in inflation and real interest rates. Therefore, hedging currency exposure does not provide a protection against risk factors affecting long-term exchange rates, and the hedging ratio should be zero. Under certain (strong) assumptions, Black (1989) showed that all investors should apply a universal hedging policy, irrespective of the portfolio composition and the reference currency. Glen and Jorion (1993) showed that international diversification decreases portfolio risk whether or not the assets are hedged. Although these papers fundamentally disagree about the optimal hedging policy, they do share one common trait—their results represent extreme outcomes and can be seen as corner solutions of the optimal currency exposure problem.

Other researchers obtained rather mixed results. Solnik (1993) found that, in the short term, the optimal currency hedging is specific for each investor. The proposed hedging policy is a function of the portfolio structure and the percentage of foreign assets. More recent literature shares a broader consensus that currency hedging tends to lower the portfolio volatility. Additionally, conditional hedging outperforms strategies that employ fixed hedge ratios. Haefliger, Waelchli and Wydler (2002) proposed full hedging for fixed-income portfolios, while equity portfolios can only be partially hedged (or even unhedged), depending on the correlations between equity and currency returns. Jorion (1994) considered a global mean-variance optimization, where positions in assets and currencies can be determined simultaneously or separately. Either way, the optimal currency exposures depend on the portfolio reference currency. As shown by Eun and Resnick (1988), a low accuracy of estimated input parameters—in particular the mean returns—is the main driver of poor ex-ante performance of the joint optimization of asset allocations and currency hedge ratios.

On the other hand, hedging for the purpose of risk minimization mitigates estimation risk, as the covariance structure is found to be estimated with higher precision. Schmittman (2010) analyzed constant hedging strategies in comparison to the static variance minimizing hedging ratios calculated with ordinary least squares, while De Roon et al. (2011) included currency positions as a further asset class and pointed out that risk hedging and speculative benefits are two motivations for internationally diversified portfolios. Overall, the studies provide supporting evidence showing that currency hedging reduces risk in multi-currency portfolios.

The seminal work in this area is arguably the paper *Global Currency Hedging* of Campbell, Serfaty-de Medeiros and Viceira (2010). The authors proved that it is possible to find optimal hedging ratios minimizing volatility for general portfolios. Such hedging decision should be made considering the correlations between currencies and equities. The authors use data over the period of 1975 to 2005 and empirically show that the US dollar (particularly in relation to the Canadian dollar), the euro, and Swiss franc (particularly in the second half of the analyzed period) have moved against world equity markets. Thus, these currencies should be attractive to risk-minimizing global equity investors despite their low average returns. A long position in the US-Canadian exchange rate is a particularly effective hedge against equity risk. These results hold for both short and long-term investment horizons. On the other hand, the authors show that most currency returns are almost uncorrelated with bond returns. Thus, risk-minimizing bond investors should avoid holding currencies; that is, they should fully currency-hedge their international bond positions. This is consistent with common practice of institutional investors.

It is important to notice that all of the above noted works in the currency hedging literature assume that investors know perfectly the true probability law governing the stochastic processes of asset returns. However, in many situations, agents are uncertain about the validity of the model, and hence any particular probability law used to describe return processes is subject to potential parameter estimation errors and model misspecification. This drawback can be addressed with the incorporation of different methods that account for decision making under ambiguity in the study of portfolio analysis. Let us review a few key definitions relating to ambiguity, its difference from risk, and how aversion to ambiguity might be measured. The two most prominent approaches of how aversion to uncertainty might be measured and modelled are the use of Bayesian portfolio analysis and the use of ambiguity averse preferences. Bayesian portfolio analysis employs and facilitates the use of fast, intuitive, and easily implementable numerical algorithms which are able to simulate otherwise complex economic quantities. On the other hand, ambiguity averse preferences take a leap from the traditional use of rational expectations and maximization of (subjective) expected utility. Uncertainty about the predictions is derived and captured directly in the utility function and in this way decision making is performed under ambiguity about the financial market outcomes, such as portfolio choices and equilibrium asset prices.

That a distinction might be drawn between standard expected utility and more general decision models has been known since Knight (1921). According to Knight, there are two kinds of uncertainty: the first, called risk, corresponds to situations in which all relevant events are associated with a (objectively or subjectively) uniquely determined probability assignment; the second, called (Knightian) uncertainty, corresponds to situations in which some events do not have an obvious probability assignment. The experimental relevance of the distinction between risk and uncertainty has been formally discussed by Ellsberg (1961), whose findings have shown that agents are not always able to derive a unique probability distribution over the reference state space. After Ellsberg's seminal paper, uncertain environments have become better known as ambiguous and the general dislike for them as ambiguity aversion.

Let us start with the review of the literature of Bayesian portfolio studies. Although recognized by Markowitz (1952), the problem of estimation errors did not receive serious attention until the 1970s, when the first applications in finance are entirely based on uninformative or data-based priors. Later, Jorion (1986) introduced the hyperparameter prior approach in the spirit of the Bayes-Stein shrinkage prior, whereas Black and Litterman (1992) advocated an informal Bayesian analysis with economic views and equilibrium relations. The study by Pastor (2000) centers prior beliefs around values implied by asset pricing theories, and Tu and Zhou (2010) argue that the investment objective provides a useful prior for portfolio selection. All of these studies assume that asset returns are identically and independently distributed through time, whereas Kandel and Stamabugh (1996), Barberis (2000), and Avramov (2002) account for the possibility that returns are predictable by macro variables such as the aggregate dividend yield, the default spread, and the term spread. Moreover, Uppal and Wang (2003) note that the evidence from experimental economics and psychology suggest that, in some situations, agents' uncertainty cannot be expressed using a single probability distribution; that is, ambiguity is present. Uppal and Wang develop a model of intertemporal portfolio choice in which investors account explicitly for this ambiguity. Building on this, Garlappi, Uppal and Wang (2007) present a model for an investor with multiple priors and aversion to ambiguity. Multiple priors are characterized by a confidence interval around the estimated parameters and the ambiguity aversion is modelled via a minimization over the priors.

The other strand of literature captures the ambiguity aversion directly in the decision maker's preferences and consequently also in the utility function itself. When a decision maker has too little information to form a single prior, she may plausibly consider a set of probability distributions and not a unique prior. Schmeidler (1989) formalized this intuition starting from the observation that the probability attached to an uncertain event may not reflect the heuristic amount of information that has led to that particular probability assignment. Motivated by this consideration, Schmeidler suggested to assign non-additive probabilities, or capacities, to allow for the encoding of information that additive probabilities cannot represent. Gilboa and Schmeidler (1989) extended this work by presenting multiple prior preferences. After the seminal paper by Gilboa and Schmeidler had gained popularity, Anderson et al. (1998, 2003) and Hansen and Sargent (2001) noted that multiple-prior criteria also appear in the robust control theory used in engineering. Robust control theory specifies the set of probabilities by taking a single "approximating model" and statistically perturbing it. This reflects a situation wherein agents have a specific model of reference and, acknowledging the possibility of errors, seek robustness against misspecifications.

Klibanoff, Marinacci and Mukerji (2005) proposed that the ambiguity of a risky event can be characterized by a set of subjectively plausible cumulative probability distributions for this event. The decision maker subjectively weights this distributions and resulting (KMM) preference relation describes the investor's attitude towards ambiguity. Building on this, Maccheroni, Marinacci and Ruffino (2013) derive the analogue of the classic Arrow-Pratt approximation of the certainty equivalent (given the underlying KMM preferences) under model uncertainty as described by the smooth model of decision making under ambiguity. They study its scope by deriving a tractable mean-variance model adjusted for ambiguity and solving the corresponding portfolio allocation problem. We use a similar approach in this work and connect this ambiguity aversion adjusted mean-variance preferences to the optimal currency exposure framework. The analytical tractability of the enhanced Arrow-Pratt approximation renders this model especially well suited for calibration exercises aimed at exploring the consequences of model uncertainty on the optimal currency allocations.

Let us have a final note of caution on terminology. In the literature, ambiguity and uncertainty are not always clearly distinguished, nor are they clearly defined. In this paper, we use both terms equivalently. Uncertainty or ambiguity is meant to represent "non-probabilized" uncertainty (situations in which the decision maker is not given probabilistic information about the external events that affect the outcome of a decision) as opposed to risk, which is "probabilized" uncertainty.

3 Model

3.1 Portfolio Return including Currency Forward Contracts

In this section, we present a general framework capturing hedged portfolio returns, where hedging is performed with currency forward contracts. The presented expressions will be used in the subsequent derivation of optimal currency exposures. As we are working with portfolio returns in a cross-section of underlying assets, we throughout the work use simple, and not logarithmic, returns. Allow us to introduce some notation.

Let $P_{i,t}$ denote the asset value expressed in local currency at time t, and $\tilde{R}_{i,t+1}$ the corresponding simple return of asset i over the time period between t to $t+1$. Equivalently, let $S_{c_i,t}$ denote the time t spot foreign exchange rate in domestic currency per foreign currency and $e_{c_i,t+1}$ the return of the exchange rate from t to $t+1$. We index the currency c corresponding to asset i as c_i. Denote by $\lambda_{i,t}$ the number of shares of stock i (or the equivalent for bonds, commodities or any other asset class) held at time t. An asset return expressed in domestic currency is measured as

$$\bar{R}_{i,t+1} = \frac{\lambda_{i,t+1} P_{i,t+1} S_{c_i,t+1}}{\lambda_{i,t} P_{i,t} S_{c_i,t}} - 1 = \tilde{R}_{i,t+1} + e_{c_i,t+1} + \tilde{R}_{i,t+1} e_{c_i,t+1}, \tag{1}$$

where we impose $\lambda_{i,t+1} = \lambda_{i,t}$ in order the second equality holds. In such way, the returns are influenced only by the market fluctuations and not by rebalancing. Equation (1) shows that an asset return in domestic currency can be viewed as a sum of an asset return in local currency, an exchange rate return and a second-order cross product between the two.

We now move from a single asset case to the return of a portfolio. Consider an investor with an arbitrary domestic currency and a portfolio consisting of N assets. These assets can be denominated in domestic or foreign currencies. Assume that a given portfolio admits exposure to k foreign currencies. We index the domestic currency by $c = 1$ and the foreign currencies with implicit exposure in the portfolio with $c = 2, \ldots, k+1$.

Our intent is to group assets by their currency exposure. Denote $x_{i,t}$ as the fraction of wealth invested in asset i, and $\mathcal{A}_{c,t}$ as the set of all assets denominated in currency c held in a portfolio at time t. Then $w_{c,t} := \sum_{j \in \mathcal{A}_{c,t}} x_{j,t}$ establishes a fraction of wealth invested in assets denominated in currency c. Using $\hat{R}_{c,t+1} = \sum_{j \in \mathcal{A}_{c,t}} \frac{x_{j,t}}{w_{c,t}} \tilde{R}_{j,t+1}$, the return of a group of assets denominated in currency c is obtained by

$$R_{c,t+1} = \hat{R}_{c,t+1} + e_{c,t+1} + \hat{R}_{c,t+1} e_{c,t+1}, \tag{2}$$

where we used the fact that $e_{c_i,t+1} = e_{c,t+1}$ for all i in $\mathcal{A}_{c,t}$. For domestic currency, $S_{1,t} = 1$ and $e_{1,t+1} = 0$ trivially hold for all t. The unhedged portfolio return is given by

$$R_{t+1} = \sum_{i=1}^{N} x_{i,t} \bar{R}_{i,t+1} = \sum_{c=1}^{k+1} w_{c,t} R_{c,t+1}, \tag{3}$$

with $\sum_{i=1}^{N} x_{i,t} = \sum_{c=1}^{k+1} w_{c,t} = 1$ for every t. The sum in the second equality, instead of over all individual assets, ranges over all currencies implicitly held in a portfolio. This will be beneficial later since we are interested in obtaining optimal currency exposures given the predetermined instrument weights $x_{i,t}$. Note that this setup also allows for short selling as the only condition on the instrument weights is that they (their absolute values) sum up to 1.

Next, we study a hedged portfolio. Let $F_{c,t}$ denote the time $t+1$ prevailing forward exchange rate in domestic currency per foreign currency c, available on the market at time t. We assume that for any t, there exists in the market such quoted object $F_{c,t}$, with the delivery at $t+1$. The forward rate $F_{c,t}$ is such that the price of the corresponding forward contract equals zero for every c at time t when the contract is obtained. Define $f_{c,t} := (F_{c,t} - S_{c,t})/S_{c,t}$ as forward premium, which can be understood as an equivalent to a rate of return applicable to forward contracts.[1] Again, $F_{1,t} = 1$ and $f_{1,t} = 0$ trivially hold for all t.

Let $\phi_{c,t}$ denote an amount expressed as a fraction of total portfolio value (in domestic currency), which an agent invests to a forward exchange contract for currency c at time t. Therefore, $\phi_{c,t}/S_{c,t}$ represents the value of a notional of a forward contract in local currency c at time t. This value is expressed in relative terms of the total portfolio value. Since an investor wishes to hedge the foreign currency exposure, she sells the foreign currency forward. This corresponds to setting $\phi_{c,t} > 0$ and is by no arbitrage principle analogous to a strategy of shorting foreign bonds and holding domestic bonds.[2] Such short position of a forward contract on currency c yields a pay-off of $F_{c,t} - S_{c,t+1}$ at time $t + 1$.

Remember that the number of foreign currency exposures in a given portfolio equals to k. Suppose that there is a universe of n possible foreign currencies which can be traded on the market, where $n \geq k$. The additional $n - k$ currencies are the ones which could be traded, whereas an investor did not decide to hold an implicit exposure to.[3] We index them by $c = k+2, \dots, n+1$. The return of a portfolio with forwards is equal to

$$R^h_{t+1} = R_{t+1} + \sum_{c=2}^{n+1} \phi_{c,t}(f_{c,t} - e_{c,t+1}), \tag{4}$$

where $f_{c,t} - e_{c,t+1} = (F_{c,t} - S_{c,t+1})/S_{c,t}$ represents the time $t + 1$ normalized pay-off of a short forward contract on currency c. Since $S_{1,t} = F_{1,t} = 1$ for all t, the choice of $\phi_{1,t}$ is completely arbitrary, however, we would like to keep the interpretation of $\phi_{c,t}$ as a fraction of portfolio value corresponding to the notional of a forward contract on currency c. Therefore, we set it so that

$$\phi_{1,t} = 1 - \sum_{c=2}^{n+1} \phi_{c,t}. \tag{5}$$

We will later see that this choice implies that all (net) currency exposures sum up to 0 and hence makes the currency portfolio a zero investment portfolio.

If an investor already holds an implicit exposure to some currency c, meaning $w_{c,t} \neq 0$, then we define a hedge ratio through $h_{c,t} := \phi_{c,t}/w_{c,t}$. This is a case for currencies $c = 2, \dots, k+1$ and does, by construction, not apply to the domestic currency $c = 1$. For example, when $\phi_{c,t} = h_{c,t} = 0$, assets denominated in currency c are left unhedged. Conversely, $\phi_{c,t} = w_{c,t}$ or $h_{c,t} = 1$, implies a unitary hedge ratio, which we throughout this work call a full hedge. One can speak about currency hedging only in the case a portfolio bears an implicit exposure to a certain currency c, for which $w_{c,t} \neq 0$.

Consider a return on a fully hedged portfolio. Using equation (4), this corresponds to an investor who chooses $\phi_{c,t} = w_{c,t}$ for $c = 1, \dots, k + 1$, and $\phi_{c,t} = 0$ for $c = k + 2, \dots, n + 1$. Such investor fully hedges her current implicit currency exposure and does not take on any novel currency exposures, in addition to the ones already present in the portfolio at time t. Using the general expression from (4), the above defined particular choice of $\phi_{c,t}$ corresponding to a full hedge, as well as equations (2) and (3), the return of a fully hedged portfolio is equal to

$$R^{fh}_{t+1} = \sum_{c=1}^{k+1} w_{c,t}\hat{R}_{c,t+1} + \sum_{c=2}^{k+1} w_{c,t} f_{c,t} + \sum_{c=2}^{k+1} w_{c,t}\hat{R}_{c,t+1} e_{c,t+1}. \tag{6}$$

[1] This quantity is deterministic at time t and represents the cost of carry for entering a currency forward contract.

[2] This is also equivalent to borrowing in foreign currency and lending in domestic currency.

[3] Note that $n \leq N$ does not hold in general. The number of assets in the portfolio can be lower than the size of the universe of considered foreign currencies, which corresponds to $n > N$.

Given a foreign currency c, its return $e_{c,t+1}$ is in a fully hedged portfolio replaced by the forward premium $f_{c,t}$, which is a quantity prevailing at time $t+1$, but already known at time t. This is how currency hedging eliminates randomness coming from the exchange rate fluctuations. However, there are still terms $\hat{R}_{c,t+1}e_{c,t+1}$ left, even in a fully hedged portfolio. These terms remained since the hedge happens at time t, at which point the exact exposure at time $t+1$ cannot be known. In order to acquire a better intuition, reflect on a single asset portfolio consisting only of a foreign asset i. An investor, at time t, hedges its full foreign currency exposure of $\lambda_{i,t}P_{i,t}$. At time $t+1$, she gets to exchange this $\lambda_{i,t}P_{i,t}$ back to the domestic currency at an exchange rate $F_{c_i,t}$. She then exchanges the rest, which amounts to $\lambda_{i,t}(P_{i,t+1} - P_{i,t})$, at the spot exchange rate $S_{c_i,t+1}$. This happened since an investor was not able to know her future exposure, which would be, in a case of a perfect hedge, exchanged at a non-random forward rate $F_{c_i,t}$. This is a reason perfect hedging is impossible and the explanation for where the second-order cross product terms in equation (6) come from. Note that this terms are, being products of two returns, generally of small size, but can, for example in cases of market crashes or returns being measured over longer periods, also become relevant.[4]

Reflect on the general expression for a portfolio return with forwards given in equation (4). The presented setup allows for entering an arbitrary currency exposure which is not necessarily present in the unhedged portfolio return R_{t+1}. To capture this fact, we throughout this paper work with currency exposures (which can be altered by taking positions in forward contracts, as described above) and not only with currency hedges. We define an exposure to currency c as $\psi_{c,t} \coloneqq w_{c,t} - \phi_{c,t}$, where $w_{c,t}$ accounts for the implicit (or existing) portfolio currency exposure and $\phi_{c,t}$ corresponds to the change in this exposure coming from the position in a currency forward contract. Therefore, a currency exposure $\psi_{c,t}$ should be understood as an implicit (existing) currency exposure net of the hedging proportion. A fully hedged portfolio then corresponds to $\psi_{c,t} = 0$ for all c, where an investor holds no exposure to any currency c. A positive value of $\psi_{c,t}$ corresponds to a partially hedged portfolio, where an investor holds some exposure to currency c. A completely unhedged portfolio, by the same reasoning, corresponds to $\psi_{c,t} = w_{c,t}$. In the cases of $\psi_{c,t} < 0$ and $\psi_{c,t} > w_{c,t}$ we speak about over- and under-hedging. Notice that given a portfolio of k foreign currency exposures, the remaining $n-k$ currencies available on the market can be accessed by entering a corresponding forward contract. Then it holds that $\psi_{c,t} = -\phi_{c,t}$ for $c = k+2, \dots, n+1$ as $w_{c,t} = 0$ for all such c. Using this notation, equation (4) can be rewritten in terms of currency exposures as

$$\begin{aligned} R^h_{t+1} &= R^{fh}_{t+1} + \sum_{c=2}^{k+1} \psi_{c,t}(e_{c,t+1} - f_{c,t}) - \sum_{c=k+2}^{n+1} \phi_{c,t}(e_{c,t+1} - f_{c,t}) = \\ &= R^{fh}_{t+1} + \sum_{c=2}^{n+1} \psi_{c,t}(e_{c,t+1} - f_{c,t}). \end{aligned} \tag{7}$$

Equation (5) then implies

$$\psi_{1,t} = -\sum_{c=2}^{n+1} \psi_{c,t},$$

so that $\psi_{1,t}$ represents the domestic currency exposure. Since an investor can change her implicit exposure to a particular currency c, she can achieve this only by entering a particular currency forward contract. This is by no arbitrage equivalent to borrowing (or shorting bonds) in the domestic currency and simultaneously investing in bonds denominated in currency c. Hence, the currency portfolio is a zero investment portfolio and the currency exposures add to zero.

[4] Even though one does not have a possibility to directly affect this second-order terms, investors can, for example, perform a re-hedge in the case market conditions between t and $t+1$ change substantially.

Equations (4) and (7) both represent the same quantity. These two representations show that a portfolio return including currency forward contracts can be decomposed into two parts, either an unhedged portfolio return and a return coming from currency hedging, as in equation (4), or a fully hedged portfolio return and a return arising from a currency exposure, as in equation (7). This framework of international asset allocation is completely model-free, we only used a definition of a simple return and a pay-off of a short position in a currency forward contract (no dynamics are assumed for asset or currency returns). One can, however, also suppose that the covered interest rate parity holds. Denoting with $r_{c,t}$ the country c nominal short-term riskless interest rate at time t and using the covered interest parity, we can write $F_{c,t}/S_{c,t} = (1 + r_{1,t})/(1 + r_{c,t})$, where $r_{1,t}$ denotes the domestic nominal short-term riskless interest rate. Hence, the forward premium can be written as $f_{c,t} = (r_{1,t} - r_{c,t})/(1 + r_{c,t}) \approx r_{1,t} - r_{c,t}$. Plugging this approximation[5] for $f_{c,t}$ into equation (7) we obtain

$$R^h_{t+1} - r_{1,t} \approx \sum_{c=1}^{k+1} w_{c,t}(\hat{R}_{c,t+1} - r_{c,t}) + \sum_{c=2}^{k+1} w_{c,t}\hat{R}_{c,t+1}e_{c,t+1} + \sum_{c=2}^{n+1} \psi_{c,t}(e_{c,t+1} - r_{1,t} + r_{c,t}).$$

This expression of hedged portfolio return is based on the domestic and foreign excess returns, the second-order cross products expressing residual hedging risk arising from imperfect hedging, and the exposure returns stemming from the positions in currency forward contracts. Note that expressing returns in their excess over a risk free rate is very common in the asset pricing literature. Our framework nests this representation through the variable $f_{c,t}$. It assumes the existence and availability of $f_{c,t}$, but does not require the existence of a risk free rate, and avoids difficulties around its estimation. Since $f_{c,t}$ is available on the market, our approach avoids any estimation problems around it, and is, allowing for arbitrary domestic currency, number of assets, underlying currencies and a universe of possible currency exposures, entirely general.

3.2 Robust Mean-Variance Preferences

The aim of this section is to introduce a robust mean-variance setting of decision making which accounts for model uncertainty, the situation in which an investor is uncertain about the true probabilistic model governing the occurrence of different states. As always, a decision problem is structured on a state space Ω, which represents all possible realizations of uncertainty. Sets of these states of nature are called events and the outcome space is a σ-algebra $\mathcal{F}$ which contains the random outcomes of agent's decisions. A preference relation is defined over the mapping from Ω to $\mathcal{F}$. In a risk only setting, all agents agree on the prevailing probability measure $\mathbb{P}$, whereas in the setting with ambiguity, there is a set of models $\mathcal{Q}$ that captures the presence of model uncertainty. We impose as little structure as possible on the admissible class of such models and denote with $\mathbb{Q}$ a particular model (probability measure) from the set $\mathcal{Q}$.

We employ a decision utility function that is used to describe robust mean-variance decision preferences and is derived in Maccheroni, Marinacci and Ruffino (2013). The authors formulate a natural and parsimonious extension of the classical mean-variance expected utility model that is able to deal with ambiguity and is given by

$$U(l) = \mathrm{E}_{\bar{\mathbb{Q}}}[\,l\,] - \frac{\lambda}{2}\mathrm{Var}_{\bar{\mathbb{Q}}}(l) - \frac{\theta}{2}\mathrm{Var}_{\mu}(\mathrm{E}_{\mathbb{Q}}[\,l\,]), \tag{8}$$

where l is an uncertain prospect, λ and θ are positive coefficients representing risk and ambiguity

[5] Note that, under the assumption of covered interest rate parity, the expression $(r_{1,t} - r_{c,t})/(1 + r_{c,t})$ represents the sensitivity of forward points $F_{c,t} - S_{c,t}$ with respect to the spot exchange rate $S_{c,t}$ for currency c at time t. Moreover, using logarithmic returns $f_{c,t} = \log \frac{F_{c,t}}{S_{c,t}} = r_{1,t} - r_{c,t}$ holds with equality.

aversion, μ is the agent's prior probability on the space Δ of possible models $\mathcal{Q}$, and $\bar{\mathbb{Q}}$ is the reduced[6] probability $\int_\Delta \mathbb{Q}\, d\mu(\mathbb{Q})$ induced by the prior μ.

Terms $\mathrm{E}_{\bar{\mathbb{Q}}}[\,l\,]$ and $\mathrm{Var}_{\bar{\mathbb{Q}}}(l)$ correspond to the estimates of mean and variance obtained by combining predictions given by different probabilistic models $\mathbb{Q}$ the agent uses to describe the stochastic nature of the problem. The underlying weighting scheme is captured by the information parameter μ. In the same way that $\mathrm{Var}_{\bar{\mathbb{Q}}}(l)$ measures risk, term $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\,l\,])$ measures the model uncertainty that an agent perceives in the evaluation of prospect l. Note that $\mathrm{E}_{\mathbb{Q}}[\,l\,] : \Delta \mapsto \mathbb{R}$ is a random variable $\mathbb{Q} \mapsto \mathrm{E}_{\mathbb{Q}}[\,l\,]$ that associates the expected value $\mathrm{E}_{\mathbb{Q}}[\,l\,]$ to each possible model $\mathbb{Q}$. Its variance $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\,l\,])$ reflects the uncertainty on the expectation $\mathrm{E}_{\mathbb{Q}}[\,l\,]$ with respect to μ, which equals to

$$\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\,l\,]) = \int_\Delta \left(\int_\Omega l(w)\, d\mathbb{Q}(w) \right)^2 d\mu(\mathbb{Q}) - \left(\int_\Delta \left(\int_\Omega l(w)\, d\mathbb{Q}(w) \right) d\mu(\mathbb{Q}) \right)^2 .$$

Thus, higher values of $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\,l\,])$ correspond to investor's poorer information on prospect's outcomes as measured by her underlying models, which in turn corresponds to higher model uncertainly and is explicitly captured in the robust mean-variance utility function. The case in which the support of the prior μ is a singleton $\mathbb{P}$ or when the information condition $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\,l\,]) = 0$, corresponds to the absence of model uncertainty, where the prospect is regarded as purely risky and (8) represents the classic mean-variance utility. The Bayesian approach that is traditionally used to deal with estimation errors assumes that investors are neutral to ambiguity, which is fully captured already in $\bar{\mathbb{Q}}$. Maccheroni, Marinacci and Ruffino (2013) extend this approach to aversion to ambiguity about the estimated expected returns. The resulting separation of taste parameters λ and θ, and uncertainty measures $\mathrm{Var}_{\bar{\mathbb{Q}}}(l)$ and $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\,l\,])$, gives the robust mean-variance preferences an unsurpassed tractability which we utilize in our setting. This is illustrated next, when the optimal currency exposure under risk and ambiguity aversion are derived.

3.3 Optimal Currency Exposure under Model Uncertainty

Consider a risk and ambiguity averse investor who maximizes her utility

$$\max_{\mathbf{\Psi}_t} U(R^h_{t+1}) = \max_{\mathbf{\Psi}_t} \left\{ \mathrm{E}_{\bar{\mathbb{Q}}}[R^h_{t+1}] - \frac{\lambda}{2} \mathrm{Var}_{\bar{\mathbb{Q}}}(R^h_{t+1}) - \frac{\theta}{2} \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[R^h_{t+1}]) \right\}, \tag{9}$$

where the hedged portfolio return R^h_{t+1} from (7) represents a risky and ambiguous prospect for the robust mean-variance utility function given in (8), and $\mathbf{\Psi}_t = (\psi_{2,t}, \ldots, \psi_{n+1,t})'$ denotes the $(n \times 1)$ vector of foreign currency exposures. The maximization is conducted only with respect to currency exposures $\mathbf{\Psi}_t$, after the portfolio weights have already been determined. One can, therefore, interpret this optimization problem as a currency overlay strategy. Overlay strategies are constructed for the management of the existing currency risk in a portfolio where the determination of currency exposures is treated as a separate decision from the overall asset allocation. Since the objective of the above maximization is to optimally manage the risk-ambiguity-return properties which arise exclusively from the contribution of currencies, this can be viewed as a single-period (ambiguity adjusted) currency overlay strategy.

Denoting with $\mathbf{e}_{t+1} = (e_{2,t+1}, \ldots, e_{n+1,t+1})'$ the $(n \times 1)$ vector of foreign currency returns, with $\mathbf{f}_t = (f_{2,t}, \ldots, f_{n+1,t})'$ the $(n \times 1)$ vector of foreign currency premiums, and using the linearity of

[6] The probability measure $\bar{\mathbb{Q}}$ is called a reduction of μ on Ω since it can be interpreted in terms of reduction of compound lotteries. For example, if supp $\mu = \{\mathbb{Q}_1, \ldots, \mathbb{Q}_n\}$ is finite and $\mu(\mathbb{Q}_i) = \mu_i$ for $i = 1, \ldots, n$, then $\bar{\mathbb{Q}}(A) = \mu_1 \mathbb{Q}_1(A) + \cdots + \mu_n \mathbb{Q}_n(A)$, for any event (set of outcomes) $A \in \mathcal{F}$.

expectation and the variance sum law, we express

$$\begin{aligned}
\mathrm{E}_{\bar{\mathbb{Q}}}[R^h_{t+1}] &= \mathrm{E}_{\bar{\mathbb{Q}}}[R^{fh}_{t+1}] + \boldsymbol{\Psi}_t' \,\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] \\
\mathrm{Var}_{\bar{\mathbb{Q}}}(R^h_{t+1}) &= \mathrm{Var}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}) + \boldsymbol{\Psi}_t' \,\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t)\boldsymbol{\Psi}_t + 2\boldsymbol{\Psi}_t' \,\mathrm{Cov}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t) \\
\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[R^h_{t+1}]) &= \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}]) + \boldsymbol{\Psi}_t' \,\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\boldsymbol{\Psi}_t + \\
&\quad + 2\boldsymbol{\Psi}_t' \,\mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]),
\end{aligned}$$

where $\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t)$ and $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ denote the $(n \times n)$ variance-covariance matrices of the $(n \times 1)$ random vectors $(\mathbf{e}_{t+1} - \mathbf{f}_t)$ and $\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]$; and $\mathrm{Cov}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t)$ and $\mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ denote the $(n \times 1)$ vectors of covariances between R^{fh}_{t+1} and $e_{c,t+1} - f_{c,t})$, and between $\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}]$ and $\mathrm{E}_{\mathbb{Q}}[e_{c,t+1} - f_{c,t})]$, for $c = 2, \ldots, n+1$, respectively.

Note that one could also write $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] = \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}] - \mathbf{f}_t$, since $\mathbf{f}_t$ is deterministic at time t. We keep the notation from above and always regard $e_{c,t+1} - f_{c,t} = (S_{c,t+1} - F_{c,t})/S_{c,t}$ as our random variable of interest. This, for each c, highlights that we are concerned in the difference between $S_{c,t+1}$ and $F_{c,t}$, which embodies investor's choice between entering a forward contract (and attaining $F_{c,t}$) or leaving the implicit exposure unhedged (and realizing $S_{c,t+1}$). Hence, $\psi_{c,t}$ represents the relative amount of the currency position between the two. Since forward premium $f_{c,t}$ can be interpreted as a riskless return obtained by entering a forward contract on currency c, we refer to $e_{c,t+1} - f_{c,t}$ as the currency excess return. Taking a vector derivative of (9) with respect to $\boldsymbol{\Psi}_t$ yields the first-order optimality condition

$$\begin{aligned}
&\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] - \lambda\big[\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t)\boldsymbol{\Psi}_t + \mathrm{Cov}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t)\big] - \\
&-\theta\big[\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\boldsymbol{\Psi}_t + \mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\big] = 0,
\end{aligned} \tag{10}$$

where 0 is a $(n \times 1)$ vector of zeros. The Hessian matrix of second-order vector derivatives equals to

$$\mathrm{Hess}(\boldsymbol{\Psi}_t) = -\big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\big]. \tag{11}$$

Since we work with a non-degenerate random vector $(\mathbf{e}_{t+1} - \mathbf{f}_t)$ whose components are linearly independent, its corresponding variance-covariance matrix $\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t)$ is positive definite. As λ and θ are positive scalars, and $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ is a positive semi-definite matrix, the Hessian matrix given in (11) is negative definite. Therefore, the first order condition (10) is a necessary and sufficient condition which characterizes the maximum of the optimization problem (9) with respect to the currency exposure weights $\boldsymbol{\Psi}_t$. Rearranging the terms in (10) establishes the optimal currency exposure for a risk and ambiguity averse international investor

$$\begin{aligned}
\boldsymbol{\Psi}^*_t = &-\Big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\Big]^{-1} \cdot \\
&\cdot \Big[\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t) + \theta \mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]) - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\Big],
\end{aligned} \tag{12}$$

where the matrix in the first bracket is positive definite and its inverse exists. This equation represents the general formula for a future looking (out-of-sample) optimal currency exposure of a risk and ambiguity averse agent, given the estimates for the variance-covariance structure and expected returns in the presence of model uncertainty. Because of a linear relationship between portfolio return and currency exposure (obtained by the construction of hedging with forward contracts), the maximization objective function amounts of a sum of linear and quadratic forms in $\boldsymbol{\Psi}_t$ and the optimal solution is obtained in closed form. Let us analyze limiting cases of equation (12) further.

In the case of a decision maker who minimizes the variance (volatility) of returns, or is, in other words, infinitely risk averse with $\lambda \to \infty$, the optimal exposure becomes

$$\boldsymbol{\Psi}^*_{t,risk} = -\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t)^{-1} \mathrm{Cov}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t). \tag{13}$$

In the case where $\bar{\mathbb{Q}} = \mathbb{P}$, which in our setting corresponds to no model uncertainty with μ being a Dirac measure, equation (13) accords with the existing literature. A negative correlation between hedged portfolio returns and exchange rates implies that the foreign currency tends to appreciate when the hedged portfolio loses value. Therefore, an investor wishes to hold such currencies that are negatively correlated with the hedged portfolio returns and the resulting optimal risk minimizing currency exposure in (13) is positive. In the case of sufficiently large negative correlation, an investor can reduce portfolio risk by under-hedging, that is, by holding foreign currency in excess to the current exposure of $w_{c,t}$. Conversely, if the hedged portfolio returns and exchange rates are positively correlated, the foreign currency tends to depreciate when the value of the hedged portfolio falls. In such case, the optimal risk minimizing currency exposure in (13) becomes negative. The investor can reduce the risk of the portfolio by over-hedging, that is, by shorting foreign currency in excess of what would be required to fully hedge the implicit currency exposure. By the same reasoning, zero correlation between hedged portfolio returns and excess currency returns implies holding no currency exposure at all. In this case currency exposure only adds risk to the investor's portfolio and (since in the limit $\lambda \to \infty$ this risk is not compensated) the investor is better off fully hedging her portfolio.

On the other hand, the optimal exposure of an infinitely ambiguity averse agent $\theta \to \infty$ is given as

$$\mathbf{\Psi}^*_{t,amb} = -\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t])^{-1}\mathrm{Cov}_\mu(\mathrm{E}_\mathbb{Q}[R^{fh}_{t+1}], \mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]), \tag{14}$$

where matrix $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ has to be positive definite in order its inverse exists. Similarly as how risk is captured in the reduced probability $\bar{\mathbb{Q}}$ for $\lambda \to \infty$ in equation (13), here we obtain an analogous result for $\theta \to \infty$, where the uncertainty is captured in μ, the agent's prior probability on the space of possible models. Since $\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]$ is a random variable associated to each model $\mathbb{Q}$, this unpredictability is the only uncertainty that matters in the limiting case of $\theta \to \infty$. If the underlying models exhibit a negative correlation between the predictions of hedged portfolio returns and exchange rates, this implies a positive optimal ambiguity minimizing currency exposure given in (14), and the converse for the positive correlation. The intuition is the same as for the risk only minimizing case from above. The only difference is that, for $\theta \to \infty$, the ambiguity is completely captured in μ, compared to the risk, for $\lambda \to \infty$, being entirely captured in $\bar{\mathbb{Q}}$.

Mark that the general solution given in (12) allows for an arbitrary combination of purely risky and ambiguous[7] assets with the variance-covariance matrix $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ being positive semi-definite. However, in a limiting case (14), where $\theta \to \infty$, this only makes conceptual sense when all assets are ambiguous and their corresponding variance-covariance matrix is positive definite.[8]

Take an ambiguity neutral agent with $\theta \to 0$. This is equivalent to the mean-variance setting where all uncertainty is captured in $\bar{\mathbb{Q}}$ and the optimal currency exposure is obtained as

$$\begin{aligned}\mathbf{\Psi}^*_{t,mv} &= -\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t)^{-1}\Big[\mathrm{Cov}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t) - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\Big] = \\ &= \mathbf{\Psi}^*_{t,risk} + \frac{1}{\lambda}\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t)^{-1}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t].\end{aligned} \tag{15}$$

Notice that the optimal mean-variance currency exposure can be expressed as a sum of the optimal minimum variance exposure from (13) and the market price of currency risk adjusted for risk aversion λ. The market price of currency risk represents the trade-off between the expected excess return on currencies and the variance of these returns. Its effect to the optimal mean-variance currency exposure vanishes as $\lambda \to \infty$. Under the risk-neutral measure, a forward

7 Throughout the work we use the term ambiguous for assets who are risky and ambiguous simultaneously and purely risky for assets which are risky but not ambiguous.

8 This is conceptually similar to the non-existence of a variance matrix of an (excess) return vector with a riskless asset as at least one of the vector components.

rate is an unbiased predictor of the future spot rate. However, in the real world measure, this does not hold in general and investor's predictive models aim to capture that. In the case of $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] > 0$, the optimal mean-variance currency exposure increases for the risk aversion adjusted market price of currency risk. An agent increases her foreign currency exposure since she expects to get a higher risk adjusted return from the unhedged position compared to return from the hedged position. Since the additional risk is compensated, she becomes, in the mean-variance sense, better off. The converse is true for $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] < 0$. In the case where $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] = 0$, hedging does, in expectation, not affect the currency returns and the optimal mean-variance exposure equals the optimal exposure (13) affected by risk only. We can argue that starting with the infinitely risk averse agent from (13), equation (15) presents a generalization to the case where risk adjusted deviations from the uncovered interest parity are explicitly captured in the optimal currency exposure.

We can look at the optimal mean-variance currency exposure as an unambiguous currency overlay strategy. The focus of the minimum variance currency exposure is strictly to eliminate risk, no part of $\mathbf{\Psi}^*_{t,risk}$ seeks to add an additional return to the portfolio. The optimal mean-variance currency exposure then introduces an additional component seeking to add a source of excess return (commonly referred to as "alpha"). This is also a characteristic of a currency overlay strategy. An initial currency exposure $\mathbf{\Psi}^*_{t,risk}$ is determined and the portion of exposure on top of that is then seeking to generate alpha by exploiting (predicting) currency movements, which then leads to $\mathbf{\Psi}^*_{t,mv}$. Importantly, the decision of how to optimize the currency risk-return spectrum is independent of the decision of how to allocate a portion of the portfolio to different foreign denominated assets. It is also essential to understand that a currency overlay (optimal currency exposure) is not a direct investment. It is a risk management strategy, in our case implemented with forward contracts, that does not require any investment or additional asset allocation.

Rewriting the general version of optimal currency exposure given in (12), one can show that

$$\begin{aligned}\mathbf{\Psi}^*_t &= \mathbf{\Psi}^*_{t,mv} + \\ &+ \left[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\right]^{-1} \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\,(\mathbf{\Psi}^*_{t,amb} - \mathbf{\Psi}^*_{t,mv}),\end{aligned} \tag{16}$$

where the expression from (15) is used and the positive definiteness of $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ is assumed in order to use (14). The detailed proof of (16) can be found in the Appendix. We can use similar reasoning as above and argue that (16) presents an explicit correction of the optimal currency exposure arising from a generalization of an unambiguous (purely risky) mean-variance agent to the decision maker with aversion to ambiguity. The term $(\mathbf{\Psi}^*_{t,amb} - \mathbf{\Psi}^*_{t,mv})$ quantifies the deviation of optimal currency exposures for an infinitely ambiguity averse agent and an unambiguous agent. In case of the two being equal, there is no exposure correction compared to the unambiguous mean-variance agent, and the correction grows larger in the difference of the two. The term in front of $(\mathbf{\Psi}^*_{t,amb} - \mathbf{\Psi}^*_{t,mv})$ accounts for the size and direction of the model uncertainty correction and vanishes for $\theta \to 0$. Hence, it can be understood as the ambiguity correction equivalent to the market price of currency risk from above.

Example. *Here, we present a setup in which we differentiate between a purely risky and an ambiguous asset. This problem is in the ambiguity aversion literature known as a natural extension of the standard (riskless and risky asset) portfolio problem to the setting of model uncertainty. International portfolio allocation problems provide a natural application of this setting, with domestic government bonds viewed as risk-free, other domestic assets viewed as purely risky, and foreign assets and exchange rates viewed as ambiguous.*

Take an investor who is fully invested in a portfolio of domestic assets and is considering whether exposure to other currencies would help improving her portfolio risk-ambiguity-return spectrum. We treat the domestic portfolio position as purely risky and the foreign currency positions as ambiguous. This corresponds to $\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}]$ *being a constant for all models* $\mathbb{Q}$ *and implies*

$\mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R^{fh}_{t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]) = 0$. *Using equation* (12), *the optimal currency exposure is obtained as*

$$\mathbf{\Psi}^*_{t,expl} = -\Big[\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1} - \mathbf{f}_t) + \frac{\theta}{\lambda}\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\Big]^{-1} \cdot \cdot\Big[\mathrm{Cov}_{\bar{\mathbb{Q}}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t) - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\Big]. \tag{17}$$

Observe that in the limit when $\lambda \to \infty$, *the optimal currency exposure converges to the minimum variance case (infinitely risk averse agent) given in* (13). *Moreover, in the case when an agent is becoming infinitely ambiguity averse with* $\theta \to \infty$, *the optimal currency exposure converges to zero for all foreign currencies* $c = 2, \ldots, n+1$ *and the entire currency exposure is kept solely in the domestic currency. Maggiori, Neiman and Schreger (2018) demonstrate that investors exhibit home-currency bias in that they disproportionately hold securities denominated in their domestic currency. In the example above, where we treated the domestic portfolio position as purely risky, we showed that large ambiguity towards the uncertain exchange rates can explain why investor holdings are biased toward their own currencies. This shows that the puzzle of insufficient currency diversification can also be driven by investor's ambiguity aversion.*

A In-sample Analysis

The inputs to any kind of portfolio optimization framework are usually estimated with large errors. We presented a setting which accounts for such model uncertainty in the expectation of future returns. This approach can be very useful for asset managers and other practitioners which deal with determining the future looking optimal currency exposure of their portfolio. On the other hand, researchers are also interested in the historical optimality and the role of sampling error in the construction of the ex-post (in-sample) efficient portfolio weights, or in our case currency exposures. For example, Britten-Jones (1999) showed that the ordinary least squares (OLS) regression of a constant onto a set of asset's excess returns, without an intercept term, results in an estimated coefficient vector that is a set of risky-asset-only portfolio weights for a sample efficient mean-variance portfolio. Similar idea applied to the optimal currency exposures is presented in Campbell et al. (2010). The authors derive an approximation to the ex-post minimum variance optimal currency exposures by regressing portfolio logarithmic excess returns on a constant and a vector of currency excess returns, and switching the sign of the slope. Let us analyze how can our setting, with the addition of model uncertainty, be compared to the approaches presented in the existing literature.

In what follows we work with the demeaned historical returns, which come from the sample measure $\mathbb{H}$. Define a loss function $\mathcal{L}(R^h_{t+1}) \coloneqq -U(R^h_{t+1})$ as a negative of robust mean-variance utility function. This choice implies

$$\min_{\mathbf{\Psi}_t} \mathcal{L}(R^h_{t+1}) = \max_{\mathbf{\Psi}_t} U(R^h_{t+1}),$$

which shows the equivalence between minimizing the chosen loss function and maximizing utility. The argument which minimizes the loss function is the optimal in-sample currency exposure. It can be found as

$$\arg\min_{\mathbf{\Psi}_t} \mathcal{L}(R^h_{t+1}) = \arg\min_{\mathbf{\Psi}_t} \Big\{\lambda \mathrm{Var}_{\mathbb{H}}(R^h_{t+1}) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[R^h_{t+1}])\Big\}, \tag{18}$$

where the expectation term from the utility function vanishes because of demeaning, $\mathrm{Var}_{\mathbb{H}}(R^h_{t+1})$ is the unbiased sample variance of hedged portfolio returns, and measures $\mathbb{Q}$ represent the model uncertainty perceived by an agent in the sample period. Note that the usual approach would be to minimize the risk function which is defined as the expectation of the loss function, however, we work with historical (and not future) returns and there is no need to take such expectation.

Let $\mathbf{X}$ denote the $(T \times n)$ matrix of demeaned historical currency excess returns $\mathbf{e}_{t+1} - \mathbf{f}_t$, and let $\mathbf{y}$ denote the $(T \times 1)$ vector of demeaned historical fully hedged portfolio return R_t^{fh}, where T is the number of observations in the sample. Then, we have

$$\mathrm{Var}_{\mathbb{H}}(\mathbf{e}_{t+1} - \mathbf{f}_t) = \frac{1}{T-1}\mathbf{X}'\mathbf{X}$$
$$\mathrm{Cov}_{\mathbb{H}}(R_t^{fh}, \mathbf{e}_{t+1} - \mathbf{f}_t) = \frac{1}{T-1}\mathbf{X}'\mathbf{y}.$$

Let $\mathbf{W} = \frac{\lambda}{T-1}\mathbf{I}$, where $\mathbf{I}$ is a $(T \times T)$ identity matrix, $\mathbf{Z} = \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[R_{t+1}^h])$ and $\mathbf{z_0} = -\mathbf{\Psi}_{t,amb}^*$, where we assumed that the inverse of $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ exists. We can express the loss function as

$$\mathcal{L}(R_{t+1}^h) = \mathbf{\Psi}_t{}'\mathbf{X}'\mathbf{W}\mathbf{X}\mathbf{\Psi}_t + 2\mathbf{\Psi}_t{}'\mathbf{X}'\mathbf{W}\mathbf{y} + \mathbf{\Psi}_t{}'\mathbf{Z}\mathbf{\Psi}_t + 2\mathbf{\Psi}_t{}'\mathbf{Z}\mathbf{z_0} + \mathrm{rest}, \tag{19}$$

where we explicitly write the terms which depend on $\mathbf{\Psi}_t$ and with rest denote other terms which do not affect the optimization. Using equation (19), the minimization can be rewritten as

$$\begin{aligned}\arg\min_{\mathbf{\Psi}_t} \mathcal{L}(R_{t+1}^h) &= \arg\min_{\mathbf{\Psi}_t} \left\{ \Big(\mathbf{y} + \mathbf{X}\mathbf{\Psi}_t\Big)' \mathbf{W} \Big(\mathbf{y} + \mathbf{X}\mathbf{\Psi}_t\Big) + (\mathbf{\Psi}_t + \mathbf{z_0})'\mathbf{Z}(\mathbf{\Psi}_t + \mathbf{z_0}) + \mathrm{rest} \right\} = \\ &= \arg\min_{\mathbf{\Psi}_t} \big\|\mathbf{y} - \mathbf{X}(-\mathbf{\Psi}_t)\big\|_{\mathbf{W}}^2 + \big\|(-\mathbf{\Psi}_t) - (-\mathbf{\Psi}_{t,amb}^*)\big\|_{\mathbf{Z}}^2,\end{aligned} \tag{20}$$

where $\big\|\mathbf{\Psi}_t\big\|_{\mathbf{D}}^2 = \mathbf{\Psi}_t'\mathbf{D}\mathbf{\Psi}_t$ stands for the weighted L^2-norm squared given a positive definite matrix $\mathbf{D}$.

Corollary. *Equation* (20) *shows that the optimal ex-post currency exposure for a risk and ambiguity averse agent can be found as a generalized ridge regression of the demeaned historical hedged portfolio returns on the demeaned currency excess returns and shrunk towards the infinitely ambiguity averse optimal exposure distorted by the level of uncertainty. A generalized ridge regression estimator minimizes a weighted least squares augmented with a generalized ridge penalty. The weighting matrix is given by* $\mathbf{W} = \frac{\lambda}{T-1}\mathbf{I}$ *and accounts for agent's risk aversion* λ*, the generalized penalty is given by* $\mathbf{Z} = \theta\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[R_{t+1}^h])$ *and accounts for model uncertainty and ambiguity aversion* θ*, and the shrinkage target is given by* $-\mathbf{\Psi}_{t,amb}^*$*, which is a negative of optimal currency exposure for an infinitely ambiguity averse agent* (14)*. The optimal ex-post currency exposure minimizing the loss function given in* (18) *is then obtained as a negative of the described generalized ridge regression estimator. We just showed that above stated in-sample loss function minimization problem can be solved using an artificial generalized ridge regression which recovers the sample efficient currency exposures for a risk and ambiguity averse investor.*

In the limiting case of no model uncertainty with $\theta \to 0$, the penalty term vanishes. Optimal risk reductive ex-post currency exposure can then be estimated via an OLS regression $\arg\min_{\mathbf{\Psi}_t} \big\|\mathbf{y} - \mathbf{X}(-\mathbf{\Psi}_t)\big\|_{\mathbf{I}}^2$ of demeaned hedged portfolio returns on the demeaned currency excess returns (in the usual L^2-norm) and obtained by changing the sign of the estimator. This is in line with the existing literature (Campbell et al. (2010), Schmittmann (2010) etc.) where the vector of optimal currency exposures is obtained as the negative of the slopes (without an intercept) of a multiple regression of the excess portfolio return on a constant and the vector of currency excess returns. Notice that this is equivalent to demeaning the returns and regressing without a constant term. A constant would in our approach represent an unambiguous prospect and the corresponding generalized penalty matrix $\mathbf{Z}$ would not be positive definite. It would not generate a proper norm, as well as the shrinkage target would not exist. Therefore, we have to use the demeaned returns and regress without the constant term. This lack of an intercept implies that residuals need not sum to zero. All in all, our approach with the demeaned returns in the case of no model uncertainty exactly corresponds to the OLS based approaches presented in the existing literature.

Note that the reason for the in-sample interpretation of optimal currency exposure as a regression coefficient arises from the linear relationship between the portfolio return and currency exposure (zero value currency portfolio). This linearity is obtained since forward contract is an instrument with a linear pay-off. Moreover, ambiguity is measured in a convex fashion, which enables that the solution is obtained by minimizing the weighted L^2-norm (generalized ridge regression). Hence, the differentiability and smoothness are preserved and the explicit solution exists.

B Geometric Interpretation

The regression in (20) is slightly unusual, there is no intercept and the residual vector can be correlated with the regressors. It can be regarded as an artificial regression which recovers the sample efficient currency exposures and at the same time provides a useful geometric interpretation.

Let us start with a geometric interpretation of the risk only version which corresponds to the ordinary least squares linear regression. The dependent variable $\mathbf{y}$ represents the return on a fully hedged portfolio that has no exposure to currency risk, the independent variables $\mathbf{X}$ involve only the excess returns on currencies, coefficients $\mathbf{\Psi}_t$ represent the currency exposures (weights of a zero value currency portfolio), and the regression residuals show the deviation of the pure currency portfolio $\mathbf{X\Psi}_t$ from the fully hedged portfolio returns $\mathbf{y}$. The estimated optimal in-sample currency weights produce a pure currency exposure which is closest in terms of least squares distance to the fully hedged portfolio returns. The closer the two, the larger amount of risk can be reduced via the currency position, where the optimal risk reductive currency exposure corresponds to the negative of the estimated regression coefficient $-\mathbf{\Psi}_t$. In such way a sample efficient risk reduction is obtained.

The situation gets slightly more complicated in the presence of model uncertainty. The generalized penalty term corresponds to the utility loss arising from model uncertainty. The ordinary least squares regression presents a geometric interpretation as an orthogonal projection of hedged portfolio returns onto the space spanned by the currency excess returns. In the case of generalized ridge regression, the resulting coefficients (optimal exposures) still lie in the span of predictors (currencies), whereas the projection is not orthogonal anymore. The magnitude of penalizing factor θ and the structure of model uncertainty $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[R^h_{t+1}])$ control the distance and direction of shrinkage towards the target $-\mathbf{\Psi}^*_{t,amb}$. The generalized penalty is a quadratic form. It geometrically implies a non-zero centered, ellipsoid parameter constraint. This can be seen in figure 1, where the red point corresponds to the set of currency exposures obtained by the weighted least squares regression, without regularization. The blue point represents the shrinkage target and the dashed ellipsoid contains all sets of currency exposures that are attainable. For larger penalty term θ this set becomes smaller since the tighter constraint binds, and the converse for the smaller penalty. The shape of the ellipsoid is determined by the model uncertainty matrix $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[R^h_{t+1}])$. The optimal solution is given as the tangent of the weighted least squares contour sets and the set of attainable solutions. Note that in the classical ridge regression the penalty matrix is an identity multiplied with the scalar penalty and the target is set at the origin. This implies a sphere at the origin representing the attainable set of points. In the generalized ridge regression this sphere becomes an ellipsoid centered at the particular shrinkage target. The weighted least squares regression sum of squares are convex in $\mathbf{\Psi}_t$, as well as the ridge penalty is convex in $\mathbf{\Psi}_t$. This implies a unique minimizer of the penalized sum of squares. The explicit solution of equation

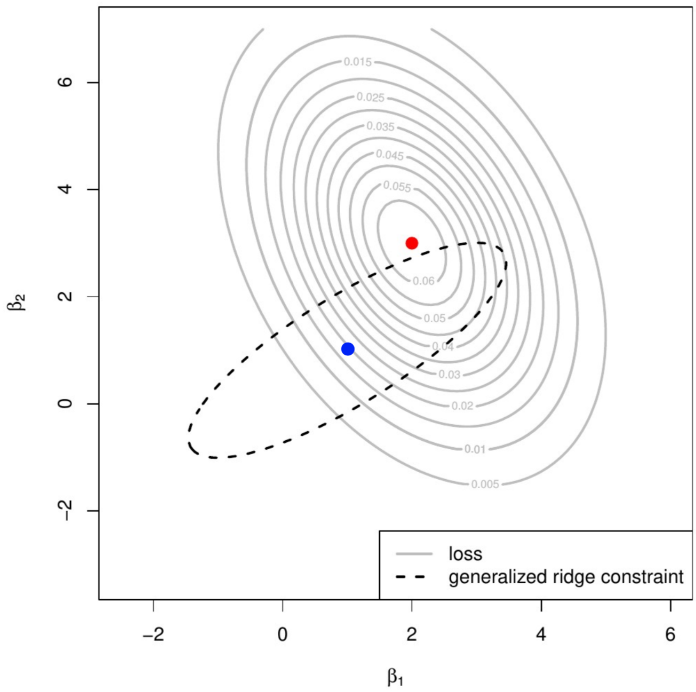

Figure 1 – This figure presents a geometric interpretation of a generalized ridge regression. The red point corresponds to the solution obtained by weighted least squares, around the point the contours of the squared residual sums cost function are presented. The blue point is the shrinkage target and the ellipsoid around that point represents the set of attainable points (given a particular penalty parameter). The set of attainable points shrinks towards the target for a larger penalization parameter. The shape of the ellipsoid is determined by the positive definite matrix in the weighted L^2-norm. The optimal solution of this optimization problem is given as the tangent of the weighted least squares contour sets and the set of attainable points. This plot has been taken from the lecture notes of Wessel N. van Wieringen, VU University Amsterdam, avalilable at: `https://arxiv.org/abs/1509.09169v3`.

(20) yields a sample efficient optimal risk and ambiguity currency exposure and is given by

$$\begin{aligned}\mathbf{\Psi}^*_{t,\mathbb{H}} &= -\big(\mathbf{X}'\,\mathbf{W}\mathbf{X}+\mathbf{Z}\big)^{-1}\big(\mathbf{X}'\,\mathbf{W}\mathbf{y}+\mathbf{Z}\mathbf{z_0}\big) = \\ &= \mathbf{\Psi}^*_{t,\mathbb{H},risk} + \big(\mathbf{X}'\,\mathbf{W}\mathbf{X}+\mathbf{Z}\big)^{-1}\,\mathbf{Z}\,\big(\mathbf{\Psi}^*_{t,amb} - \mathbf{\Psi}^*_{t,\mathbb{H},risk}\big),\end{aligned} \tag{21}$$

where $\mathbf{\Psi}^*_{t,\mathbb{H},risk} = -(\mathbf{X}'\,\mathbf{X})^{-1}\mathbf{X}'\,\mathbf{y}$ is the sample efficient risk only minimizing currency exposure obtained by ordinary least squares. The proof of (21) is given in the Appendix. Observe that this equation represents the in-sample equivalent of the general future looking solution given in (12) and (16). It also provides an intuition for how model uncertainty affects optimal currency exposure. The correction term arising from model uncertainty is affected by the difference between $\mathbf{\Psi}^*_{t,amb}$ and $\mathbf{\Psi}^*_{t,\mathbb{H},risk}$ transformed with $\big(\mathbf{X}'\,\mathbf{W}\mathbf{X}+\mathbf{Z}\big)^{-1}\,\mathbf{Z}$ which vanishes for $\theta \to 0$. In the particular case of $\mathbf{\Psi}^*_{t,amb} = \mathbf{\Psi}^*_{t,\mathbb{H},risk}$, the shrinkage target matches the ordinary least squares solution (this geometrically corresponds to the red and blue point matching). In this case, the risk only sample efficient solution is attainable for any θ and the model uncertainty does not affect the solution obtained by ordinary least squares.

Observe that the penalty term from equation (20) represents the utility loss from model uncertainty in the given sample period. One could think of risk minimizing ex-post efficient currency exposure $\mathbf{\Psi}^*_{t,\mathbb{H},risk}$ as the first-best in the case of no model uncertainty. Then, in the presence of model uncertainty, $\mathbf{\Psi}^*_{t,\mathbb{H}}$ provides the solution to the second-best, taking into account the structure of agent's uncertainty and ambiguity aversion θ. This correction of optimal currency exposure from the first-best to the second-best is geometrically represented in figure 1. Note that the optimal currency exposures (weights of a zero investment currency portfolio) are still acquired in the space spanned by currency excess returns and the deviation from an orthogonal projection is determined by the ridge regularization penalty term.

Example. *Allow us to revisit the example of an agent fully invested in a portfolio of domestic assets who considers other currencies in order to improve her risk-ambiguity-return spectrum. The future looking (out-of-sample) solution is given in* (17). *Treating the domestic position as purely risky and foreign currency positions as ambiguous implies* $\mathrm{Cov}_\mu(\mathrm{E}_\mathbb{Q}[R^{fh}_{t+1}], \mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]) = 0$ *and hence* $\mathbf{\Psi}^*_{t,amb} = 0$. *This geometrically corresponds to the shrinkage target based in the origin. Taking* $\theta \to \infty$ *shrinks the set of attainable currency exposures to a single point in the origin. Since the structure of model uncertainty did not change, this is in line with the result obtained for the future looking case. Such infinite penalization represents a large under-fitting (in a regularization framework) and corresponds to the largest deviation from the first-best to the second-best in the sense of utility loss and optimal exposure weights, given a particular shrinkage target.*

The optimal in-sample exposures can also be found by using the sample estimators of the variance-covariance structure under $\mathbb{H}$ and utilizing equation (12). However, the regression approach provides a simple tool for the empirical analysis which allows for inference procedures for hypotheses about the efficient currency allocation. The use of these inference procedures shows the importance and magnitude of sampling error in estimates of the efficient currency exposure of an international portfolio. Moreover, the regression framework highlights the stochastic nature of the variables estimated from sample data. The regularization part (model uncertainty) induces a bias-variance trade-off of the estimated sample efficient currency allocation. The estimates become biased, whereas obtained confidence intervals are shrunk, given the structure of model uncertainty which controls the distance and direction of shrinkage towards the infinitely ambiguity averse target. A large aversion to ambiguity could then also be interpreted as an in-sample under-fitting, which makes the distribution of optimal currency exposures too regular and thus largely biased.

We showed that there is a direct link, an equivalence, between a robust mean-variance utility representation and the solution obtained with a generalized ridge regression. An interesting question arises, namely, if there exists a class of utility functions which would yield an equivalent representation via a lasso regression (penalization using L^1-norm). Ridge regression corresponds to smooth shrinkage where differentiability is preserved and the solution is obtained in a closed form. On the other hand, lasso regression corresponds to an extreme shrinkage (sparsity), penalization is performed in a non-differentiable L^1-norm and the solution has to be obtained numerically.[9] In the work of Brodie et al. (2009), the authors show that adding a lasso penalty to the optimization problem corresponds to accounting for transaction costs. This gives rise to a suggestion that an elastic net regularization, a regression method that linearly combines the L^1 and L^2 penalties of the lasso and ridge methods, could be a framework that captures both transaction costs and model uncertainty.

4 Empirical Analysis

The goal of this section is to empirically test the derived theoretical model, extract information about the optimal currency exposures for risk and ambiguity averse investor using market data spanning from 1999 to 2018, and compare the results to other approaches in the existing literature.

[9] One can make similar arguments also for the cases where penalization is performed via L^0- or L^∞-norms.

Summary Statistics

	Australia	Canada	Switzerland	Eurozone	UK	Japan	USA
Log interest rates							
Average	4.33	2.38	0.76	1.85	2.85	0.22	2.20
Standard Deviation	1.45	1.54	1.22	1.69	2.22	0.26	1.98
Stock returns							
Average	7.71	7.43	4.31	4.79	5.38	4.00	5.88
Standard Deviation	15.06	16.26	16.60	21.03	17.64	20.77	18.86
Bond returns							
Average	4.88	3.75	2.04	3.52	4.01	1.07	3.73
Standard Deviation	3.26	2.57	1.73	2.44	2.55	1.30	3.47
Currency returns							
Average	0.92	0.77	1.61	−0.03	−1.15	0.09	·
Standard Deviation	12.56	8.99	10.75	9.70	9.27	10.22	·

Table 1 – Data presented in this table extends from 1999:1 to 2018:6. Unless otherwise specified, all following tables use data from the full period. Interest rates are log 3-month government bill rates and the stock (bond) returns are the foreign stocks (bonds) returns of a fully hedged investor. The currency return (in this table) is the log return of a US investor borrowing in dollars and holding foreign currency. Observe the average annualized mean and standard deviation (expressed in percentage points) of short-term nominal interest rates, stock, bond and currency returns (with respect to the US dollar).

We consider seven major developed-market currencies, the US dollar, the euro, Japanese yen, Swiss franc, British pound, Canadian dollar and Australian dollar, over the period 1999 to 2018. Using recent market data enables us to detect possible shifts of optimal currency exposures which happened, for example, after the global financial crisis of 2008. Such shifts have not yet been studied in the existing literature. Moreover, the addition of the ambiguity aversion allows us to examine the effect of model uncertainty on the optimal currency exposure also from the empirical perspective.

4.1 Data and Summary Statistics

Our empirical analysis uses the data of exchange rates, short-term interest rates, equity broad market indices, and fixed income total return indices (for various maturities) obtained from Thomson Reuters Datastream and Bloomberg. The data series are available at a daily frequency, even though we analyze different maturities in order to check the effect of the corresponding hedge maturity on the portfolio risk, return and ambiguity characteristics. We report results for seven developed economies: Australia, Canada, Switzerland, Eurozone, United Kingdom, Japan and United States. Our sample period starts in January 1999, when the euro was introduced to the world financial markets, and ends in June 2018. Table 1 reports the full-sample annualized mean and standard deviation of short-term nominal interest rates, log stock and bond returns, and log currency returns with respect to the US dollar. Annualized average nominal short-term interest rates differ across countries. They are the lowest for Switzerland and Japan, and the highest for Australia, Canada, and the UK. Short-term rates exhibit very low annualized volatility, under 2% for all countries but UK.

Average annualized returns of currency exchange rates with respect to the US dollar over this period are negative for the British pound, close to zero for the euro and Japanese yen, and positive

Cross-country Return Correlations

	Australia	Canada	Switzerland	Eurozone	UK	Japan	USA
			Panel A: Currencies				
Australia	1.00						
Canada	0.53	1.00					
Switzerland	0.24	0.30	1.00				
Eurozone	0.43	0.44	0.68	1.00			
UK	0.41	0.47	0.41	0.57	1.00		
Japan	0.09	0.24	0.41	0.35	0.28	1.00	
USA	0.29	0.54	0.37	0.44	0.50	0.53	1.00
			Panel B: Stocks				
Australia	1.00						
Canada	0.22	1.00					
Switzerland	0.33	0.47	1.00				
Eurozone	0.33	0.53	0.73	1.00			
UK	0.35	0.54	0.81	0.81	1.00		
Japan	0.55	0.20	0.31	0.27	0.30	1.00	
USA	0.14	0.73	0.50	0.54	0.53	0.12	1.00
			Panel C: Bonds				
Australia	1.00						
Canada	0.15	1.00					
Switzerland	0.27	0.40	1.00				
Eurozone	0.12	0.40	0.49	1.00			
UK	0.25	0.49	0.53	0.55	1.00		
Japan	0.22	0.09	0.15	0.10	0.13	1.00	
USA	0.08	0.77	0.34	0.38	0.45	0.08	1.00

Table 2 – This table presents cross-country correlations of currency, stock, and bond returns. Each cell of Panel A reports the correlation of currency returns between the row and column currencies. The correlation reported is an average across all possible base countries. Panels B and C report correlations of fully hedged stock and bond market returns between the row and column countries.

for the Australian dollar, Canadian dollar and the Swiss franc, reflecting an appreciation of these currencies with respect to the US dollar over the analyzed period. Exchange rate volatility relative to the dollar is in the range $9.5\% - 12.5\%$ for all currencies except for the Canadian dollar, which moves closer with the US dollar yielding a slightly lower bilateral volatility of 9%.

Table 2 reports the full-sample correlations of currency returns[10] (Panel A), fully hedged returns of stocks (Panel B), and fully hedged returns on bonds (Panel C). All three panels show that currency, stock and bond market returns are positively cross-correlated.

Some correlations stand out as unusually large. The Canadian dollar exhibits a high correlation with the US dollar, as well as Canadian stock and bond markets are highly correlated with the US stock and bond market. This high correlations reflect the dual role of the Canadian economy as a resource-dependent economy that is simultaneously highly integrated with the United States. Similarly, the correlation between the euro, Swiss franc and British pound, as well as the Swiss,

[10] We report in Panel A the average correlation of each currency pair across all possible base currencies in our data set.

Eurozone and UK stock and bond markets are very high. These correlations demonstrate the integration of the Swiss and UK economies with the European economy.

Generally, the correlations in table 2 are still small enough to suggest the presence of substantial benefits of international diversification in this sample period.

4.2 Optimal Risk Minimizing Currency Exposures for Equity and Bond Investors

This section presents the empirical analysis based on the estimation of optimal in-sample risk only minimizing currency exposures (we add the ambiguity component in the next section) for a set of stock or bond portfolios. This allows us to use equation (20) in the particular case of no model uncertainty. Optimal risk minimizing currency exposure is then found by the ordinary least squares regression of demeaned fully hedged portfolio returns on the demeaned currency excess returns, and switching the sign of the slopes. This approach enables us to easily provide the corresponding test statistics on the estimated optimal currency demands. In order to allow for a simple comparison of the results with the most of the existing literature, we set the domestic country to be the United States, and hence refer to the domestic investor as a US investor, and to the domestic currency as the US dollar.

A Single Country Stock Portfolios

We start our empirical analysis by examining the case of an investor who is fully invested in a single-country equity portfolio and is evaluating whether exposure to other currencies would help reduce the volatility of her quarterly portfolio return. Table 3 reports optimal currency exposures for the case in which the investor is considering one currency at a time (Panel A), and for the case in which she is considering multiple currencies simultaneously (Panel B). In both panels, the reference stock market is reported at the left of each row, whereas the currency under consideration is reported at the top of each column. In all tables we report Newey-West heteroskedasticity and autocorrelation consistent standard errors in parentheses below each optimal currency exposure. Following the standard convention, we mark with one, two, or three stars coefficients for which we reject the null hypothesis of zero at a 10%, 5%, and 1% significance level, respectively.

Panel A of Table 3 studies an investor who is fully invested in a domestic stock portfolio and decides how much exposure to a single currency c to hold in order to minimize the portfolio return volatility. To facilitate interpretation, it is useful to discuss an example in detail. The first non-empty cell in the first row of the table, which corresponds to the Australian stock market and the Canadian dollar, has a value of 0.49. This means that a risk-minimizing Canadian investor should over-hedge the Australian dollar exposure implicit in her Australian stock market investment, and hold a net long 49% exposure to the Canadian dollar.

Panel A of Table 3 shows that optimal demands for foreign currency are large, positive, and statistically significant for two stock markets (rows of the table), those of Australia and Canada. Investors in the Australian and Canadian stock markets are keen to hold foreign currency, because the Australian and Canadian dollars tend to depreciate against all currencies when their stock markets fall. Thus, any foreign currency serves as a hedge against fluctuations in these stock markets.

At the opposite extreme, it is optimal for investors in the Japanese and Swiss stock markets to hold economically and statistically large short positions in all currencies, implying that the Japanese yen and the Swiss franc tend to appreciate against all currencies when the Japanese and Swiss stock markets fall. Observe also that the US stock market generates large negative demand for the Canadian dollar, reflecting the fact that the Canadian dollar tends to depreciate when the US stock market falls and the opposite.

Optimal Currency Exposure for Single-Country Portfolios

	Currency						
Market	Australia	Canada	Switzerland	Eurozone	UK	Japan	USA
Panel A: Single Currency							
Australia		0.49**	0.61***	0.65***	0.39***	0.49***	0.51***
		(0.21)	(0.13)	(0.19)	(0.13)	(0.05)	(0.10)
Canada	−0.70**		0.60***	0.52***	0.23	0.61***	0.94***
	(0.28)		(0.16)	(0.15)	(0.18)	(0.14)	(0.23)
Switzerland	−0.76***	−0.64***		−0.81***	−0.69***	0.25	−0.20
	(0.11)	(0.14)		(0.30)	(0.16)	(0.19)	(0.19)
Eurozone	−0.76**	0.08	1.11**		−0.02	0.87***	1.10***
	(0.31)	(0.21)	(0.57)		(0.32)	(0.17)	(0.26)
UK	−0.70***	−0.49***	0.33*	0.14		0.34***	0.34
	(0.13)	(0.14)	(0.18)	(0.16)		(0.12)	(0.26)
Japan	−0.70***	−0.92***	−0.50	−0.52*	−0.74***		−0.90***
	(0.13)	(0.16)	(0.35)	(0.31)	(0.17)		(0.25)
USA	−0.67***	−0.90***	−0.02	−0.23	−0.58*	0.41**	
	(0.12)	(0.22)	(0.20)	(0.25)	(0.30)	(0.19)	
Panel B: Multiple Currencies							
Australia	−0.40***	−0.29**	0.04	0.34	−0.31**	0.36***	0.25*
	(0.10)	(0.13)	(0.23)	(0.27)	(0.15)	(0.09)	(0.14)
Canada	−0.77***	−0.28	0.14	0.59**	−0.34**	0.22**	0.45***
	(0.14)	(0.18)	(0.20)	(0.23)	(0.13)	(0.10)	(0.13)
Switzerland	−0.58***	−0.17	0.61***	0.10	−0.26*	0.31***	−0.02
	(0.14)	(0.18)	(0.20)	(0.21)	(0.15)	(0.12)	(0.17)
Eurozone	−0.68***	−0.24	0.08	−0.18	−0.52***	0.30**	1.25***
	(0.15)	(0.19)	(0.36)	(0.40)	(0.19)	(0.14)	(0.16)
UK	−0.76***	−0.27**	0.12	0.31	0.15	0.30***	0.15
	(0.12)	(0.13)	(0.21)	(0.24)	(0.12)	(0.09)	(0.14)
Japan	−0.40**	−0.61**	0.15	0.73*	−0.55**	0.81***	−0.14
	(0.19)	(0.28)	(0.34)	(0.39)	(0.26)	(0.31)	(0.31)
USA	−0.72***	−0.33**	0.30*	0.42**	−0.35**	0.32***	0.36**
	(0.15)	(0.13)	(0.17)	(0.21)	(0.14)	(0.12)	(0.16)

Table 3 – This table studies an investor holding a portfolio composed of equity from her own country, who chooses a foreign currency position in order to minimize the variance of her portfolio. Panel A allows the investor to use only one foreign currency. Panel B allows her to choose a vector of positions in all available foreign currencies. Rows indicate the equity being held (as well as the base country), columns the currencies used to manage risk. Cells of Panel A are obtained by regressing the demeaned hedged return of the row country stock market onto the demeaned return of the column country currency. Rows of Panel B (excluding diagonal terms) are obtained by regressing the demeaned return of the row country stock market onto the vector of all demeaned foreign currency returns. The optimal exposure is then obtained by switching the sign of the obtained regression coefficients. Diagonal terms in Panel B are obtained by computing the opposite of the sum of other terms in the same row and the corresponding standard deviation is calculated. Reported currency positions are the amount of dollars invested in foreign currency per dollar in the portfolio. We run monthly regressions on overlapping quarterly returns. Standard errors are corrected for autocorrelation due to overlapping intervals using the Newey-West procedure. We mark with one, two, or three asterisks coefficients for which we reject the null of zero at a 10%, 5%, and 1% significance level, respectively.

Panel B of Table 3 reports optimal currency demands for an investor who is fully invested in a domestic stock portfolio, but who uses the whole range of available currencies to minimize the portfolio return volatility. Rows indicate the equity being held (as well as the base country), columns the currencies used to manage risk.[11]

When single-country stock market investors consider investing in all currencies simultaneously, they always choose positive exposures to the Japanese yen, almost always positive exposures to the US dollar, the euro, and the Swiss franc, always negative exposures to the Australian dollar and Canadian dollar, and almost always negative exposure to the British pound. Relative to Panel A, the optimal currency demands are mostly economically larger for the US dollar, and less statistically significant for the euro and the Swiss franc. This reflects two features of the multiple-currency analysis. First, a position that is long the US dollar and short the Canadian dollar is a highly effective hedge against stock market declines. Thus, allowing investors to use both North American currencies increases the risk management demand for the US dollar. Second, the euro and Swiss franc are both good hedges but they are highly correlated. Thus, the demand for each currency is less precisely estimated when investors are allowed to take positions in both currencies. In this sense the euro and the Swiss franc are substitutes for one another.

These empirical results are in line with the ones presented in Campbell et al. (2010), even though the authors used a data set including the period 1975 up to 2005 and we are using data from 1999 up to 2018. The most notable difference is that in our analysis, optimal exposures for the Japanese yen are positive and highly economically and statistically significant. This shows the growing acceptance for the Japanese yen as a reserve currency in the recent years. The other difference is that the optimal exposures for the Australian dollar are very large in magnitude and highly statistically significant. The negative positions became economically larger and highly statistically significant using more recent data set. This shows that the Australian dollar started to move more consistently in the same direction as the resource dependent Australian economy, making the position in Australian dollars more risky.

B Global Equity and Bond Portfolios

Thus far, we have discussed only investors who are fully invested in a single country stock market, and use currencies to hedge the risk of that stock market. Now, we look at the case where the investor holds an equally weighted global equity portfolio (using all seven stock markets included in our analysis), and who considers the whole vector of available currencies to minimize the portfolio return volatility.

Panel A of Table 4 studies the case in which investors have access to all seven currencies from the countries included in the equally weighted stock portfolio. Panel B considers a case in which investors do not have close currency substitutes available for investment. Specifically, Panel B excludes Canada and Eurozone from the analysis because the Canadian stock market is highly correlated with the US stock market, and the Canadian dollar is also highly correlated with the US dollar; similarly, there is a very high positive correlation between the Swiss stock market and the Eurozone market, and between the Swiss franc and the euro. Comparison of the results in Panel A and Panel B clarifies the roles of the Canadian dollar and the US dollar, and the euro and the Swiss franc, in investors' portfolios. Both panels report estimates of optimal currency demands based first on the full sample, and then on two sub periods, pre-crisis period 1999 to 2008 (Subperiod I) and the post-crisis period 2009 to 2018 (Subperiod II).

The optimal currency portfolio in Panel A has a positive, statistically significant exposure to the Japanese yen and the US dollar, and a large negative exposure to the Australian dollar, the

[11] Notice that the numbers in each row must add up to zero, because the domestic currency exposure must offset the vector of foreign currency demands.

Optimal Currency Exposure for an Equally Weighted Global Equity Portfolio: Multiple Currency Case

	Currency						
Time Period	Australia	Canada	Switzerland	Eurozone	UK	Japan	USA
Panel A: Seven-Country Optimization							
Full Period	−0.64***	−0.31**	0.23	0.31	−0.33**	0.39***	0.35**
	(0.11)	(0.14)	(0.22)	(0.26)	(0.14)	(0.12)	(0.13)
Subperiod I	−0.67***	−0.22	1.58***	−0.65	−0.22	−0.05	0.24
	(0.18)	(0.21)	(0.43)	(0.49)	(0.27)	(0.11)	(0.20)
Subperiod II	−0.51***	−0.25	−0.19	0.10	−0.19	0.65***	0.40**
	(0.12)	(0.17)	(0.22)	(0.23)	(0.13)	(0.09)	(0.17)
Panel B: Five-Country Optimization							
Full Period	−0.75***		0.45***		−0.28**	0.39***	0.19
	(0.10)		(0.17)		(0.13)	(0.13)	(0.12)
Subperiod I	−0.89***		1.08***		−0.37	−0.01	0.20
	(0.15)		(0.15)		(0.24)	(0.10)	(0.16)
Subperiod II	−0.62***		−0.13		−0.20*	0.65***	0.31**
	(0.09)		(0.15)		(0.11)	(0.09)	(0.14)

Table 4 – This table studies an investor holding a portfolio composed of stocks from all countries, with equal weights, who chooses a vector of positions in all available foreign currencies to minimize variance of her portfolio. Panel A considers a case where all seven currencies are available, whereas Panel B excludes the Canadian dollar and the euro. Within each panel, rows indicate the time period over which the optimization is computed, and columns the currencies used to manage risk. The full period runs from 1999 to 2018, the first subperiod covers the years 1999 through 2008, the second subperiod covers the rest of the sample. Reported currency positions are the amounts of dollars invested in foreign currency per dollar in the portfolio. We run monthly regressions of overlapping 3-month returns. Standard errors are corrected for autocorrelation due to overlap intervals using the Newey-West procedure. We mark with one, two, or three asterisks coefficients for which we reject the null of zero at a 10%, 5%, and 1% significance level, respectively.

Canadian dollar and the British pound. The long position in the US dollar and short position in Canadian dollar are not independent of each other: Panel B shows that, once we exclude the Canadian dollar from the list of currencies available to the investor, the optimal exposure to the US dollar becomes small and statistically insignificant. A position that is long the US dollar and short the Canadian dollar helps investors hedge against global stock market movements: the US dollar tends to appreciate relative to the Canadian dollar in periods of stock market weakness.

The optimal currency portfolio in Panel A also has positive but statistically insignificant exposures to the euro and the Swiss franc. This lack of individual economic and statistical significance arises from the euro and the Swiss franc being close substitutes. Panel B shows that when the euro is excluded from the list of currencies, the demand for the Swiss franc increases dramatically and is statistically significant at the 1% level.

Once again, it is useful to review the exact meaning of the numbers we report. The numbers shown in Table 4 are optimal (net) currency exposures. If it is optimal for all investors to fully hedge the currency exposure implicit in their stock portfolios or, equivalently, to hold no currency exposure, the optimal currency demands shown in Table 4 should be equal to zero everywhere. To obtain optimal currency hedging demands from optimal currency exposures, we need only compute the difference between portfolio weights, which in this case is 14.3% for each country stock market (the portfolio is equally weighted among seven different countries), and the optimal currency exposure corresponding to that country.

Remark. *In summary, the risk-minimizing strategy for a global equity investor involves long exposure to the US dollar, Japanese yen and the Swiss franc (or a combination of the euro and the Swiss franc), a short position in the Canadian dollar, Australian dollar and British pound. That is, investors in global equities want to under-hedge their exposure to the dollar, the yen, the euro, and the Swiss franc, and over-hedge their exposure to the other currencies. This strategy minimizes the volatility of overall portfolio returns because the euro, Swiss franc, Japanese yen, and US dollar tend to appreciate when international stock markets decline.*

In comparison to the results reported in the existing literature, most notably the work of Campbell et al. (2010), the positions presented here are very similar. The only notable difference is again the highly statistically significant positive exposure to Japanese yen, in comparison to the slightly negative optimal exposure in the compared work (which used an older data set). The same as in the previous chapter, the negative exposure to the Australian dollar increases in magnitude and becomes highly significant in our case.

The sample period for which we have estimated optimal currency exposures includes an early period before the global financial crisis with high interest rates and generally good performance of the stock markets, followed by the post-crisis period of low interest rates and extremely poor performance of the stock markets. The first subperiod starts with the creation of the euro as a common European currency. It is reasonable to examine whether the results we have shown for the full sample hold across these two markedly different subperiods, so we divide our sample period into the periods 1999 to 2008 and 2009 to 2018.

The bottom two rows in each panel of Table 4 report subsample results for an investor holding an equally weighted global stock portfolio, and using the vector of available currencies to manage risk. It is striking, however, that Swiss franc positions tend to fall rapidly between the first and the second subperiod, whereas the position in the Japanese yen increases substantially. Also the positions in the US dollars and the euro increase from the first to the second subperiod, however, not statistically significantly for the euro. This shows that the yen began to move more consistently against world stock markets in the second subsample and the opposite for the Swiss franc.

Remark. *An important change occurred between the periods 1999 to 2008 and 2009 to 2018: the Japanese yen (and the euro to some extent) became much more competitive with the US dollar as a desirable currency for risk-minimizing global equity investors. This reflects the fact that the yen (and euro) found growing acceptance as a reserve currency for international investors. It is striking that, on the other hand, Swiss franc, after the global financial crisis, massively lost its position as the most desirable reserve currency.*

We conduct the same analysis for global bond portfolios and find that the risk-minimizing currency demands for internationally diversified bond market investors are generally very small and not statistically significant. The full analysis, which is presented in the Appendix, shows that international bond investors should fully hedge the currency exposure implicit in their bond portfolios, with possibly a small short bias towards the Japanese yen.

The results for the global bond investors are consistent with the existing literature which proposes full currency hedging for the international bond investors. Interestingly, full currency hedging is more common among international bond mutual funds than among international equity funds, and is also frequently practiced by institutions investing in international bonds.

4.3 Effect of Ambiguity Aversion

In this section, an empirical analysis of the effect of ambiguity aversion on the optimal currency exposure is performed. Our aim is to statistically investigate how model uncertainty affects the optimal currency exposure in comparison to the case of risk only (certainty), which can be perceived as the current state of the literature.

In the theoretical part of our work we studied an example where an agent solves a currency allocation problem with the robust mean-variance preferences and differentiates between purely risky and ambiguous assets (currencies). Moreover, in the previous section we conducted an empirical analysis examining the case of an investor with en equally weighted global equity portfolio who is determining whether exposure to other currencies would help reduce the volatility (risk) of her quarterly portfolio return. We adopt these two frameworks here by looking at the fully hedged (and hence expressed in domestic terms) portfolio position as purely risky and the foreign currency positions as ambiguous.

As we work with historical (in-sample) returns we throughout this section use the notation of historical measure $\mathbb{H}$. Moreover, we assume that the uncovered interest rate parity holds and hence do not include the expected returns in the calculations. Note that the estimation of sample expected returns is extremely noisy and therefore does not provide any meaningful information. Our setting corresponds to the one in (17) by

$$\mathbf{\Psi}^*_{t,expl} = -\Big[\mathrm{Var}_{\mathbb{H}}(\mathbf{e}_{t+1} - \mathbf{f}_t) + \frac{\theta}{\lambda}\mathrm{Var}_{\mu}(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])\Big]^{-1} \cdot \Big[\mathrm{Cov}_{\mathbb{H}}(R^{fh}_{t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t)\Big].$$

Observe that using an economic argument of fully hedged portfolio return perceived as purely risky, we centered the ridge target at the origin. In order to stay in a completely theoretical framework and avoid specifying particular models for the prediction of currency excess returns, we throughout this section assume that different models are independent with $\mathrm{Var}_{\mu}(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]) = \frac{1}{T-1}\mathbf{I}$, where T is the number of data points in the sample and $\mathbf{I}$ is the identity matrix, scaled in order to be of an appropriate order. In such way we can perform an in-sample empirical analysis without exactly specifying various prediction models. Observe that the underlying assumptions transform our setting to the classical (non-generalized) ridge regression.

Let us start with an analysis of optimal currency exposure in dependence of risk and ambiguity aversion parameters λ and θ. Notice that in order to gain the intuition about the effects of risk and ambiguity aversion parameters in a 3-dimensional plot, we have to work with the univariate case of optimal currency exposure (an exposure to a single foreign currency). Similar reasoning then extends to the general multivariate case of optimal currency exposure, whereas the corresponding plots cannot be presented.

In figure 2 observe the optimal Swiss franc exposure for a euro based investor, calculated under the assumption of no model uncertainty $\mathrm{Var}_{\mu}(\mathrm{E}_{\mathbb{Q}}[e_{t+1} - f_t]) = 0$, which corresponds to the current state of the literature. The optimal currency exposure is found via the ordinary least squares regression (minimum variance case) and is a constant function which does not depend on the choices of λ and θ. The optimal currency exposure of the risk and ambiguity averse agent is plotted in figure 3. Observe the dependence of optimal Swiss franc exposure to for a euro based investor, on the risk and ambiguity aversion parameters λ and θ. When $\lambda \to \infty$, the optimal currency exposure converges to the minimum variance case from figure 2. However, when $\theta \to \infty$, the optimal currency exposure converges to 0 (full hedging), where the convergence for the lower values of λ is faster compared to the convergence for higher values of λ. As an agent is becoming more uncertain, for example, she does not trust her predictive models, her optimal choice is to hold less exposure to such ambiguous currencies.

One has to note that the optimal (in-sample) currency exposure is estimated from a finite sample. Being a function of the data, the optimal exposure estimator is itself a random vari-

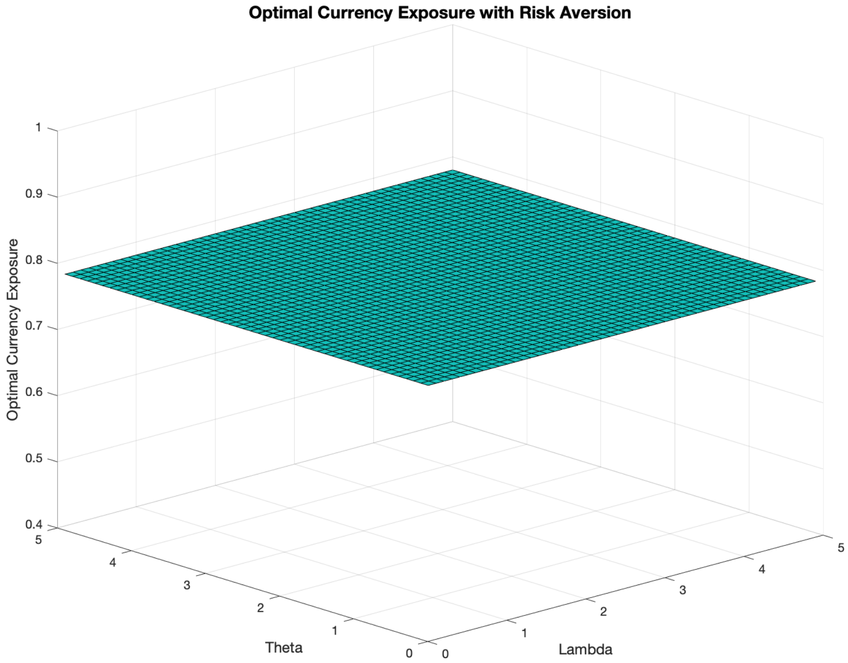

Figure 2 – Optimal currency exposure in CHF (for a EUR based investor) in dependence of risk and ambiguity aversion parameters λ and θ is plotted here. We assume no ambiguity $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[e_{t+1}-f_t]) = 0$ and uncovered interest rate parity to hold. The optimal currency exposure hence corresponds to the minimum variance case and does not depend on λ or θ.

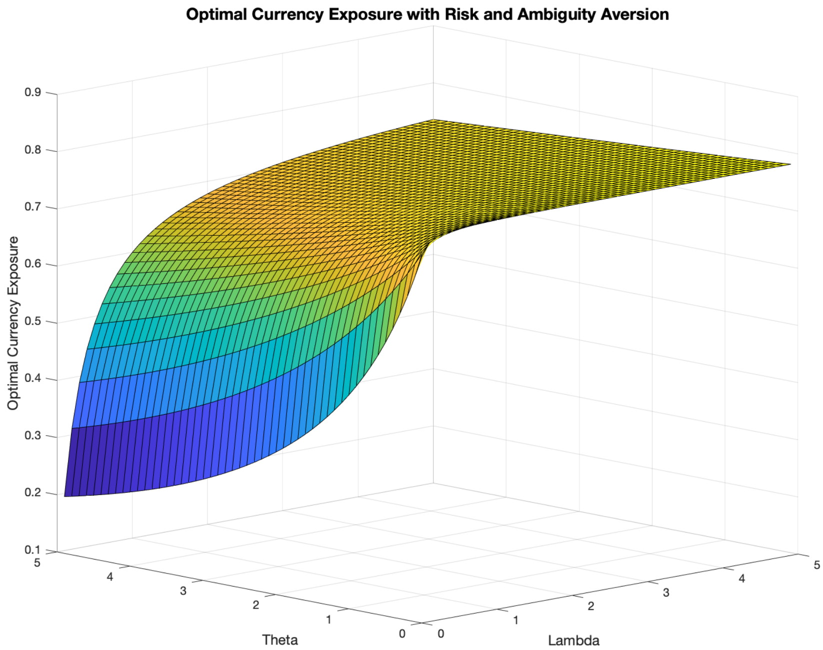

Figure 3 – Optimal currency exposure in CHF (for a EUR based investor) in dependence of risk and ambiguity aversion parameters λ and θ is plotted here. We assume independent prediction models $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]) = \frac{1}{T-1}\mathbf{I}$ and the uncovered interest rate parity to hold. When $\lambda \to \infty$, the optimal currency exposure converges to the minimum variance case and when $\theta \to \infty$, the optimal currency exposure converges to 0 (full hedging).

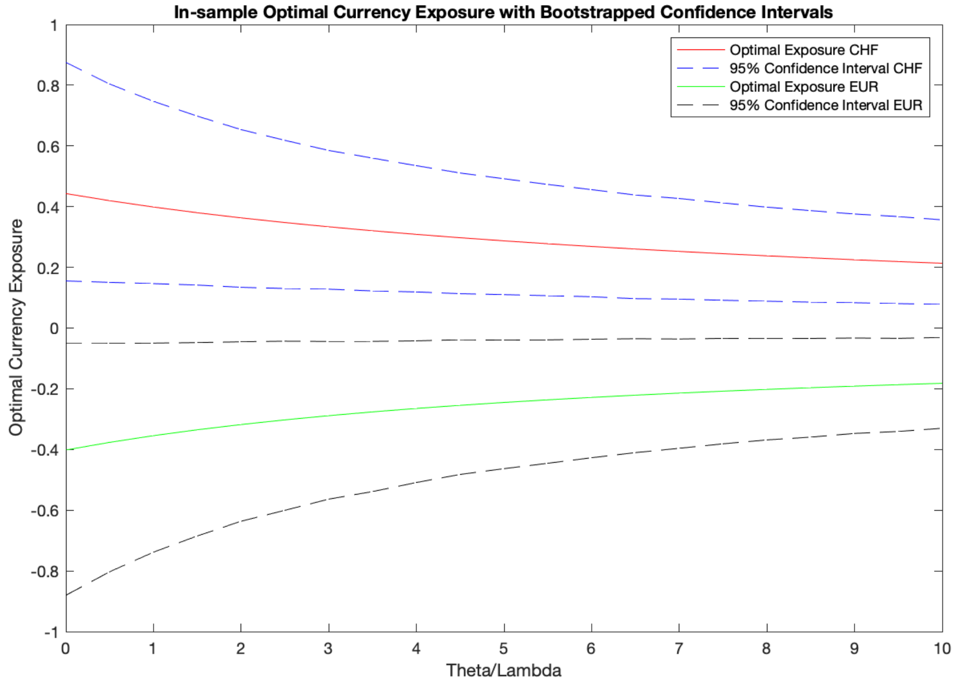

Figure 4 – Optimal currency exposure and the corresponding 95% confidence intervals for CHF and EUR (for a USD based investor) in dependence of risk and ambiguity aversion parameters λ and θ are plotted here. We assume independent prediction models $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]) = \frac{1}{T-1}\mathbf{I}$ and the uncovered interest rate parity to hold. Observe that as the bias increases for larger values of θ, simultaneously, the confidence intervals shrink (variance of the estimator decreases).

able, whose properties can be studied further. Hence, we analyze how different values of λ and θ affect the confidence intervals of historical optimal currency exposure. We form a monthly non-overlapping returns data set and assume these observations arise from an independent and identically distributed population. This allows us to use bootstrapping (random sampling with replacement) in order to infer the confidence intervals of the sample estimate of optimal currency exposure. We take a portfolio based in US dollars with equally weighted exposures to Canadian, Swiss, Eurozone and USA equity and bond markets. Figure 4 depicts the estimated optimal exposure in Swiss francs and euros, together with the bootstrapped 95% confidence intervals for different choices of λ and θ. As the ambiguity aversion parameter corresponds to the penalization parameter of ridge regression, we observe an increase in bias for larger values of θ. Simultaneously, the confidence intervals shrink, representing the decrease in the variance of the optimal exposure estimator. In the limiting case of $\theta \to \infty$, the estimator converges to 0 (given our underlying economic assumptions), which leads to the largest achievable in-sample bias with zero variance. The same phenomenon can be observed in figure 5, where we plot the bootstrapped distribution of optimal Swiss franc exposure. The distribution for $\theta = 0$, which represents the current approaches in the literature (certainty), is extremely wide, showing that the estimator of optimal currency exposure exhibits large parameter uncertainty. For larger values of θ its distribution shifts towards the ridge target (which is equal to zero in our example), and narrows extensively. In the limit when $\theta \to \infty$, the distribution converges to the Dirac measure at the point 0. In figure 6, observe the mean squared error of the estimated optimal Canadian dollar, Swiss franc, euro and US dollar exposures. In order to produce this plot, we assume that in the case of certainty (obtained at $\theta = 0$), the optimal exposure, which is then found via an ordinary least squares regression, is unbiased. Observe that, for large values of ambiguity aversion θ, the mean squared

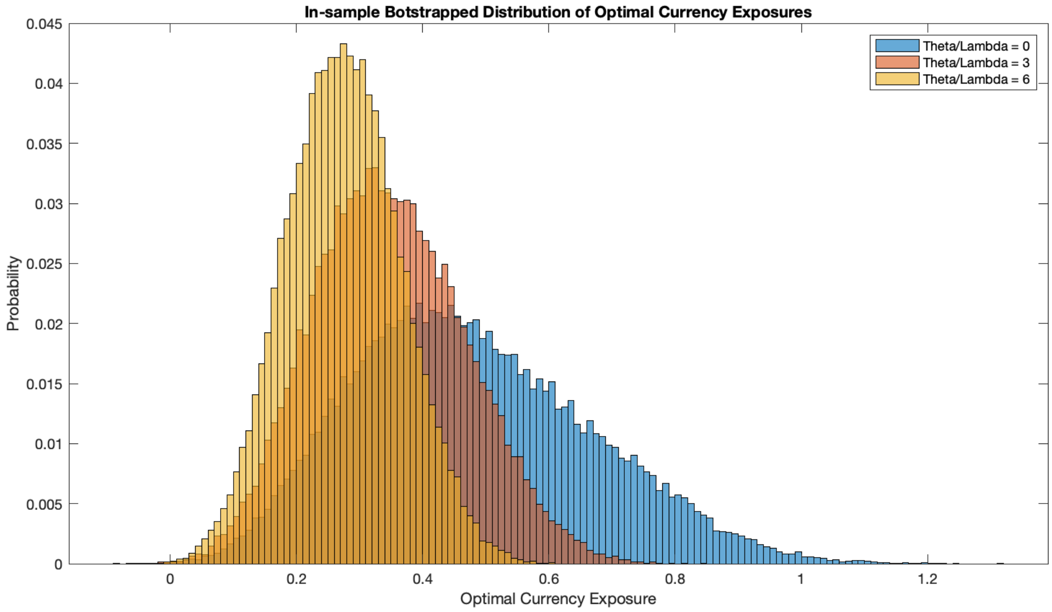

Figure 5 – Bootstrapped distribution of optimal currency exposure in CHF (for a USD based investor) for different values of risk and ambiguity aversion parameters λ and θ is plotted here. We assume independent prediction models $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]) = \frac{1}{T-1}\mathbf{I}$ and the uncovered interest rate parity to hold. Observe the shift in bootstrapped distribution for larger values of θ.

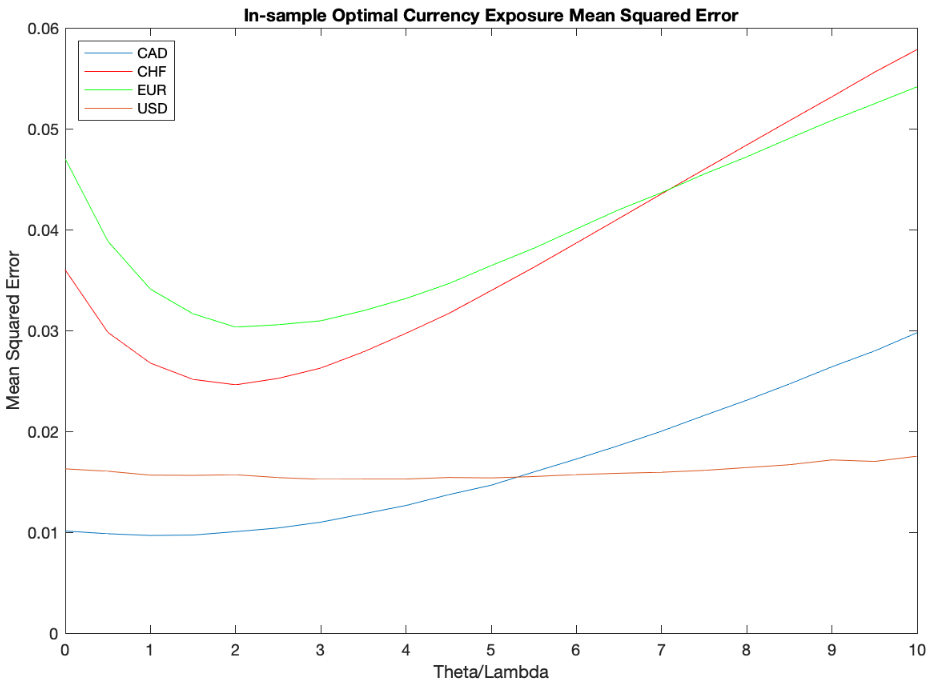

Figure 6 – Bootstrapped mean squared error of optimal currency exposure in CAD, CHF, EUR, and USD (for a USD based investor) for different values of risk and ambiguity aversion parameters λ and θ is plotted here. We assume independent prediction models $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]) = \frac{1}{T-1}\mathbf{I}$ and the uncovered interest rate parity to hold. Observe that accounting for ambiguity can lead to improved mean squared error of the optimal exposure estimates in-sample.

error deteriorates compared to the case of certainty. However, in the instance where θ/λ takes values around 2, the ambiguity adjusted estimator of the in-sample optimal currency exposure for Swiss franc and euro exhibits an improvement in terms of mean squared error compared to the case without model uncertainty. This result can be seen as an example of a bias-variance trade-off. It implies that acknowledging uncertainty can, for particular values of λ and θ, lead to an improved in-sample estimator of optimal currency exposure, measured in the sense of mean squared error. This arises even though ambiguity enters investor's robust utility function as a strictly negative value. Note that the regularization methods for portfolio selection are usually employed in order to obtain improved out-of-sample estimators, whereas we showed that an improvement can be obtained even in-sample.

Above, we gained an intuition of the effect of model uncertainty on the estimation of optimal currency exposure. The question that arises now is how large are the corresponding risk and ambiguity reductions. In order to answer this question, we consider a global equally weighted stock portfolio and perform an in-sample back test analysis for constant hedging, as well as for three different currency overlay strategies: first, the minimum variance agent, second, the mean-variance agent, and third, the robust (ambiguity adjusted) mean-variance agent. In table 5, we report, for each base currency, the annualized standard deviations and Sharpe ratios of in-sample daily returns given these alternative currency overlay strategies, where trading (hedging) is performed at quarterly time intervals. The first three columns report volatilities and Sharpe ratios for unhedged, half hedged, and fully hedged portfolios, and the rest of the columns present these quantities for the three investors described above.

Full-sample results, in Table 5, show that the benefit of full currency hedging depends sensitively on an investor's base currency. It is particularly large for Japanese and Swiss investors because these investors have a risk-reducing base currency, so they gain by hedging back to that currency and out of foreign currencies. The volatility reduction from full currency hedging is particularly small for Australian and Canadian investors because the home currency for these investors is risky in the sense that it is positively correlated with their equity positions. In fact, full currency hedging actually increases risk for the Australian and the Canadian investor. Optimal hedging, in-sample, reduces risk for all investors. An important point is that the benefit of optimal hedging is substantial, both economically and statistically. The gains from currency hedging are also substantial for global bond investors, but in this case almost all the gains can be achieved by full currency hedging. We report these results in the Appendix of the paper. The presented optimal mean-variance currency exposure is calculated assuming $\lambda = 7$ and the optimal ambiguity adjusted currency exposure is calculated using $\lambda = 5$ and $\theta = 5$. Note that the minimum variance agents all agree on the optimal currency exposure (it is not dependent on the risk or ambiguity aversion parameters), whereas the classical mean-variance and robust mean-variance agents may differ according to the subjective risk and ambiguity aversion (parameters λ and θ).

A question of practical importance is whether the volatility reductions come at the cost of lower expected return per unit of portfolio risk. To examine this question, we compute the realized Sharpe ratios of global equity portfolios under the minimum variance currency hedging policy, together with the optimal mean-variance and optimal ambiguity adjusted mean-variance currency overlay strategies. Note that the realized volatilities are independent of the base currency for the cases with full or different kinds of optimal hedging (overlay), but not for the cases with no hedging and half hedging. On the other hand, the Sharpe ratios depend sensitively on the realized average (excess) returns and hence differ across base currencies.

The Sharpe ratios for minimum variance case drop substantially compared to the cases of no, half or full hedging. In order to reduce the variance an investor takes a positive exposure to the, so called, reserve currencies, which exhibit lower expected returns and affect the realized Sharpe ratios. However, this result is not in line with the results obtained by using an older

Global Equity Portfolios with Risk and Ambiguity Aversion

Base Country	No Hedge	Half Hedge	Full Hedge	Opt Min Var Hedge	Opt Mean Var Hedge	Opt Robust Ambiguity Hedge
Volatility						
Australia	11.88%	11.67%	13.12%	11.52%	12.35%	11.65%
Canada	12.65%	12.43%	13.12%	11.52%	12.35%	11.65%
Switzerland	16.31%	14.39%	13.12%	11.52%	12.35%	11.65%
Eurozone	13.97%	13.28%	13.12%	11.52%	12.35%	11.65%
UK	13.86%	13.13%	13.12%	11.52%	12.35%	11.65%
Japan	19.27%	15.80%	13.12%	11.52%	12.35%	11.65%
USA	15.79%	14.17%	13.12%	11.52%	12.35%	11.65%
Sharpe Ratio						
Australia	0.26	0.38	0.43	0.26	0.48	0.41
Canada	0.40	0.43	0.42	0.21	0.48	0.39
Switzerland	0.37	0.40	0.41	0.14	0.41	0.32
Eurozone	0.46	0.45	0.42	0.16	0.45	0.35
UK	0.48	0.46	0.42	0.19	0.48	0.39
Japan	0.44	0.44	0.41	0.10	0.40	0.31
USA	0.40	0.42	0.42	0.16	0.44	0.36

Table 5 – This table reports the standard deviation and Sharpe ratio of portfolios featuring different uses of currencies for risk management. We present results for equally weighted global equity portfolios. Within each panel, rows represent base countries and columns represent the currency overlay strategy. "No Hedge" refers to the simple equity portfolio. "Half Hedge" refers to a portfolio in which half of the implicit currency risk is neutralized. "Full Hedge" refers to a portfolio in which all of the implicit currency risk is neutralized. "Optimal Minimum Variance Hedge" refers to a portfolio in which the currency position is chosen optimally to minimize variance. "Optimal Mean-Variance Hedge" refers to a portfolio of the agent with the classical mean-variance preferences and the risk aversion parameter of $\lambda = 7$. "Optimal Robust Ambiguity Hedge" refers to a portfolio of the agent with the robust (ambiguity adjusted) mean-variance preferences, the risk aversion parameter of $\lambda = 5$ and the ambiguity aversion parameter of $\theta = 5$. All presented results are computed with daily returns and hedging at a quarterly horizon.

data set from Campbell et al. (2010), where, for some base currencies, Sharpe ratios increased following the minimum variance hedging strategy. On the other hand, the Sharpe ratios are (in-sample) much higher for the mean-variance investor, as she is, in her utility function, evaluating the expectation of each currency return in comparison to the corresponding forward premium (obtained from entering the forward contract). The Sharpe ratios presented for the optimal mean-variance currency overlay strategy are, however, not always higher in comparison with the no, half or full hedge, since we are using a fairly large value of the risk aversion parameter $\lambda = 7$. It is also important to note that these results should be interpreted with caution because they are calculated using sample average currency returns, which are noisy estimates of the true mean currency returns.

It is interesting to notice that the volatility and the Sharpe ratios of the optimal ambiguity adjusted currency overlay strategy are lying between the minimum variance and mean-variance cases. This result is intuitive, an ambiguous agent, in comparison to the variance minimizing agent, takes into account also the expected return structure and hence increases the Sharpe ratio and the volatility of returns. Whereas on the other hand, in comparison to the classical mean-variance investor, she is ambiguous and hence closer to full hedging (as discussed before), from where a slight lowering of the Sharpe ratio and the volatility come from. Again, these results are obtained in-sample and calculated using average currency returns, which are noisy estimates of the true mean returns. Take this table only as an intuitive representation of different currency overlay strategies and their effect on the portfolio returns. Moreover, the variance-covariance structure is known to fluctuate over time, therefore, its prediction power becomes vital for the investors trying to manage the risk of their currency exposure. This demonstrates the main reason why an out-of-sample back test analysis is necessary and will be presented in the updated version of this working paper. Nevertheless, high prediction power of future returns (and its corresponding model uncertainty) has already been shown to be of vital importance.

5 Conclusion

The goal of this paper is to study the choice of optimal foreign currency exposure for an investor who is risk and ambiguity averse. We start with presenting a completely general framework for hedged portfolio returns in a model-free setting, meaning that there are no assumptions on the dynamics of asset or currency returns. In order to account for model uncertainty directly in the agent's currency allocation decision, we employ robust mean-variance preferences which explicitly capture the agent's risk and ambiguity aversion. We combine the expressions for hedged portfolio returns (which separate the returns arising from assets in their local currencies and the pure currency exposure) with the robust mean-variance utility representation and derive closed form expressions that characterize the optimal currency exposure for a risk and ambiguity averse investor.

We examine the case of an investor whose portfolio consists of domestic assets—which she regards as purely risky—and is considering whether exposure to other currencies—which she regards as ambiguous—would help improve her risk-ambiguity-return spectrum. We show that in the limit, when the risk aversion parameter $\lambda \to \infty$, the optimal currency exposure converges to the minimum variance case (infinitely risk averse agent). On the other hand, when the ambiguity aversion parameter $\theta \to \infty$, the optimal foreign currency exposure converges to 0 (full hedging), and the investor holds an exposure only to the domestic (purely risky) position. This result shows that the puzzle of insufficient currency diversification (home-currency bias) can be driven by investor's ambiguity aversion.

We show that the optimal in-sample currency exposure for a risk and ambiguity averse agent can be found by a generalized ridge regression of the demeaned hedged portfolio returns on the demeaned currency excess returns and shrunk towards the infinitely ambiguity averse optimal exposure distorted by the level of model uncertainty. Such artificial regression recovers the sample efficient currency exposures, highlights the stochastic nature of optimal exposures calculated from a finite sample, empowers geometric interpretation of obtained results, and enables inference procedures for hypotheses about the efficient currency exposures. The reason for a geometric interpretation of the optimal currency exposure as a solution to the described regression problem is the linear relationship between the currency exposure and the portfolio return (derived in a model-free setting).

The generalized penalty term of the ridge regression corresponds to the utility loss arising from the model uncertainty. This result emphasizes the importance of predictive modelling and risk management for international investors. Moreover, in the first best, the case without model uncertainty, the projection of hedged portfolio returns onto the space spanned by the currency excess returns is orthogonal. With the penalization, which arises from model uncertainty, this projection is not orthogonal anymore. The magnitude of ambiguity aversion parameter (penalizing factor) and the structure of model uncertainty control the distance and direction of shrinkage and in such way characterize the solution of the second best (in the presence of model uncertainty).

This regularization, biasing of the estimate which stabilizes the inference, is here not assumed a priori. It originates as a solution to the robust (ambiguity aversion consistent) mean-variance maximization problem.

In the empirical part of the work we fit the derived theoretical model to historical market data ranging from 1999 to 2018. In comparison to the existing literature, the main empirical results are preserved. We confirm that the US dollar, Japanese yen, Swiss franc and the euro tend to appreciate when international stock markets fall. This negative correlation generates demand for these currencies (attracts flows of capital at times when bad news about the world economy arrive) as a way to reduce the volatility of international stock portfolios. On the other hand, the Australian dollar, Canadian dollar, and British pound, are positively correlated with world stock markets. These patterns imply that international equity investors can minimize their risk by taking short positions in the Australian and Canadian dollars, and British pound, and long positions in the US dollar, Japanese yen, euro, and Swiss franc. Moreover, our results suggest a growing acceptance of the Japanese yen as a reserve currency after the global financial crisis and the opposite for the Swiss franc. This suggests that the yen has partially displaced the franc as a reserve currency.

Using bootstrapping in order to perform statistical inference, we show that in the case of certainty, $\theta = 0$, the distribution of the optimal currency exposure estimator is extremely wide (exhibits large parameter uncertainty). For larger values of θ its distribution shifts towards the ridge target (which is equal to zero in our example), and narrows extensively. This corresponds to an increase in bias and a simultaneous shrinkage of confidence intervals. Building upon the intuition of this bias-variance trade-off we show that acknowledging uncertainty can, for particular values of λ and θ, lead to an improved in-sample estimator of optimal currency exposure, measured in the sense of mean squared error. This arises even though ambiguity enters investor's robust utility function as a strictly negative value.

We computed realized Sharpe ratios in order to check if volatility reductions come at the cost of lower expected returns per unit of portfolio risk. Sharpe ratios for minimal variance cases drop substantially compared to the simple constant hedging strategy. In the minimum variance case, an investor enters into a positive exposure to reserve currencies, which exhibit lower returns and affect the realized Sharpe ratios. The volatility and Sharpe ratios for the ambiguity adjusted currency overlay strategy lie between the minimum variance and mean-variance cases. In our example, ambiguity corresponded to the shrinkage towards zero foreign currency exposure (full hedging), in this way stabilized optimal estimates and provided the optimal in-sample solution for the ambiguity averse agent. This preliminary empirical results suggest an improvement of Sharpe ratios and more robust optimal exposure estimates also for the out-of-sample back test results which will follow in the newer version of this working paper.

This paper is still in the working phase and will be extended in several directions. Most importantly, the currently executed empirical analysis is preliminary. More concise effect of ambiguity on optimal currency exposures will be investigated from an empirical perspective and an out-of-sample back test will be performed in the succeeding version of the paper. From the theoretical perspective, we will express the constrained optimal currency exposure in a form of a (linearly constrained) quadratic program, which alleviates potential non-realistic currency demands.

This working paper allows for possible generalizations in several research directions. Let us start with the theoretical considerations. Variance is a natural benchmark, which we extended by accounting for model uncertainty. Further implications of risk and ambiguity in higher moments can be pursued. Such higher moments can then be accounted for by different risk measures which can also capture the downside risk which is, for example, often present in emerging market currencies. This would require a derivation of robust preferences capturing also the co-skewness and co-kurtosis of asset and currency returns, from where one could study their effect on the optimal currency exposures. Moreover, as forward contracts (used for hedging in the current setup) are linear instruments, one can as well study hedging with options, especially to investigate the effect of mitigating the currency downside risk. The setup would in this case become non-linear, which requires more complicated optimization framework and its solutions could potentially not be presented in closed form. It is also possible to integrate the speculative and risk management components of currency demand by solving the optimal portfolio choice problem for an investor choosing currency positions jointly with stock and bond positions (not treating portfolio weights as given such as in this work). This would lead to a portfolio choice which is efficient, in comparison to determining the asset weights separately from underlying currency exposure in a two-step optimization procedure. One could also include the effect of hedging (trading) transaction costs directly to the optimized utility function. This would correspond to an investor seeking to improve the risk-ambiguity-return characteristics in a cost-efficient manner. The transaction costs would in this case correspond to the L^1-normed penalty (Lasso regression), such as in Brodie et al. (2009). The generalized ridge regression would in such case be extended to the generalized elastic net regularization.

From the empirical point of view, it would be interesting to look at emerging market currencies and indices jointly with the already investigated developed markets. It is also possible to extend an empirical analysis using different asset classes, such as corporate bonds, and investigate a possible difference in optimal currency exposures of the portfolio consisting of investment grade vs high yield bonds of developed and emerging markets. A potential prospect is also an inclusion of commodities (for example broad commodity indices) or studying specific sectors, such as energy, agriculture, precious metals, and observing if different sectors in various market environments covary with currencies in different ways. This relates to the growing academic interest in sector rotation investment strategies, where we can examine how currencies relate to the corresponding macro variables used for predictions and sector rotation decisions. One can as well look at different investment styles, such as value, size, momentum, minimum variance etc., and investigate the relation between currency exposures and different types of factors (factor models). Such analysis can be performed out-of-sample with different sampling frequencies and maturities of hedges in order to investigate the robustness of the proposed currency overlay strategies. The effects of specific events, such as Swiss franc unpeg and Brexit, can be investigated as well.

To sum up, we study the choice of optimal currency exposure for an international investor who is risk and ambiguity averse. The robust mean-variance preferences are employed in order to determine the optimal ambiguity adjusted currency exposure. We connect model uncertainty to the penalized regression and in such way formally connect the area of financial economics (asset allocation) with statistical learning (regularization). The derived theoretical model is tested with data ranging from 1999 to 2018 and the empirical effect of model uncertainty is examined and compared to the existing currency hedging approaches. The next version of this working paper will be extended by an out-of-sample empirical analysis and some minor novel theoretical considerations.

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Appendix

Derivation of equation (16)

The main idea of the following derivation is to start with equation (12) and use matrix algebra in order to rewrite the expression into (16).
The optimal exposure for a risk and ambiguity averse international investor is given in (12) as

$$\begin{aligned}\mathbf{\Psi}_t^* = &-\Big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\Big]^{-1} \cdot \\ &\cdot \Big[\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) + \theta \mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R_{t+1}^{fh}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\Big],\end{aligned}$$

Assuming positive definiteness of $\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])$, one can rearrange the terms as

$$\begin{aligned}\mathbf{\Psi}_t^* = \quad & -\Big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\Big]^{-1} \cdot \\ & \cdot \Big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t)(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t))^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) + \\ & + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t))^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) - \\ & - \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t))^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) + \\ & + \theta \mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R_{t+1}^{fh}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\Big] = \\ = \quad & -\Big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\Big]^{-1} \cdot \\ & \cdot \Big[(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])) \cdot \\ & \quad \cdot (\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t))^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) - \\ & \quad - \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t))^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) + \\ & \quad + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])^{-1} \mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R_{t+1}^{fh}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\Big] = \\ = \quad & - \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t)^{-1}(\mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) + \\ & + \big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\big]^{-1} \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) \cdot \\ & \quad \cdot \big[\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t)^{-1}(\mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) - \\ & \quad - \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])^{-1} \mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R_{t+1}^{fh}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\big] = \\ = \quad & \mathbf{\Psi}_{t,mv}^* + \big[\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t) + \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\big]^{-1} \theta \mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])\,(\mathbf{\Psi}_{t,amb}^* - \mathbf{\Psi}_{t,mv}^*),\end{aligned}$$

where we used

$$\begin{aligned}\mathbf{\Psi}_{t,amb}^* &= -\mathrm{Var}_\mu(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])^{-1} \mathrm{Cov}_\mu(\mathrm{E}_{\mathbb{Q}}[R_{t+1}^{fh}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]), \\ \mathbf{\Psi}_{t,mv}^* &= -\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t)^{-1}\Big[\mathrm{Cov}_{\bar{\mathbb{Q}}}(R_{t+1}^{fh}, \mathbf{e}_{t+1}-\mathbf{f}_t) - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\Big] = \\ &= \mathbf{\Psi}_{t,risk}^* + \frac{1}{\lambda}\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{e}_{t+1}-\mathbf{f}_t)^{-1}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t].\end{aligned}$$

This concludes the derivation of equation (16).

Derivation of equation (21)

The main idea of the derivation is to show that the solution to the generalized ridge regression problem is of the form of first equality in (21), and then to use matrix algebra and rewrite the expression into the form of second equality of (21).

The aim of this derivation is to obtain a closed form solution to the in-sample risk and ambiguity adjusted currency exposure optimization problem. In order to achieve this, we have to solve the generalized ridge regression given in equation (20). We use the same notation as in the main part of the paper. Hence, let $\mathbf{X}$ denote the $(T \times n)$ matrix of demeaned historical currency excess returns $\mathbf{e}_{t+1} - \mathbf{f}_t$, and let $\mathbf{y}$ denote the $(T \times 1)$ vector of demeaned historical fully hedged portfolio return R_t^{fh}, where T is the number of observations in the sample. Moreover, let $\mathbf{W} = \frac{\lambda}{T-1}\mathbf{I}$, where $\mathbf{I}$ is a $(T \times T)$ identity matrix, $\mathbf{Z} = \theta \mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[R_{t+1}^h])$ and $\mathbf{z_0} = -\mathbf{\Psi}_{t,amb}^*$, where we assumed that the inverse of $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ exists. By equation (20) we deal with an optimization problem of the form

$$\begin{aligned}\arg\min_{\mathbf{\Psi}_t} \mathcal{L}(R_{t+1}^h) &= \arg\min_{\mathbf{\Psi}_t} \left\{ \Big(\mathbf{y} + \mathbf{X}\mathbf{\Psi}_t\Big)' \mathbf{W} \Big(\mathbf{y} + \mathbf{X}\mathbf{\Psi}_t\Big) + (\mathbf{\Psi}_t + \mathbf{z_0})' \mathbf{Z} (\mathbf{\Psi}_t + \mathbf{z_0}) + \text{rest} \right\} = \\ &= \arg\min_{\mathbf{\Psi}_t} \left\| \mathbf{y} - \mathbf{X}(-\mathbf{\Psi}_t) \right\|_{\mathbf{W}}^2 + \left\| (-\mathbf{\Psi}_t) - (-\mathbf{\Psi}_{t,amb}^*) \right\|_{\mathbf{Z}}^2,\end{aligned}$$

where we explicitly write the terms which depend on $\mathbf{\Psi}_t$ and with rest denote other terms which do not affect the optimization. Using equation (19) this can be written as

$$\arg\min_{\mathbf{\Psi}_t} \mathcal{L}(R_{t+1}^h) = \arg\min_{\mathbf{\Psi}_t} \left\{ \mathbf{\Psi}_t{}' \mathbf{X}' \mathbf{W} \mathbf{X} \mathbf{\Psi}_t + 2\mathbf{\Psi}_t{}' \mathbf{X}' \mathbf{W} \mathbf{y} + \mathbf{\Psi}_t{}' \mathbf{Z} \mathbf{\Psi}_t + 2\mathbf{\Psi}_t{}' \mathbf{Z} \mathbf{z_0} + \text{rest} \right\}.$$

A vector derivative with respect to $\mathbf{\Psi}_t{}'$ then yields the first order optimality condition given by

$$2\mathbf{X}' \mathbf{W} \mathbf{X} \mathbf{\Psi}_t + 2\mathbf{X}' \mathbf{W} \mathbf{y} + 2\mathbf{Z}\mathbf{\Psi}_t + 2\mathbf{Z}\mathbf{z_0} \overset{!}{=} 0.$$

Rearranging the terms shows the first equality in equation (21), given by

$$\mathbf{\Psi}_{t,\mathbb{H}}^* = -\big(\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z}\big)^{-1} \big(\mathbf{X}' \mathbf{W} \mathbf{y} + \mathbf{Z}\mathbf{z_0}\big).$$

Note that the second order vector derivative is positive definite, which ensures that the unique global minimum of the loss function is attained. Now we can rewrite the obtained equation as

$$\begin{aligned}\mathbf{\Psi}_{t,\mathbb{H}}^* &= -\big(\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z}\big)^{-1} \big(\mathbf{X}' \mathbf{W} \mathbf{X} (\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} + \\ &\quad + \mathbf{Z}(\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} - \mathbf{Z}(\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} + \mathbf{Z}\mathbf{z_0}\big) = \\ &= -(\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} + (\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z})^{-1} \mathbf{Z}((\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} - \mathbf{z_0}) = \\ &= \mathbf{\Psi}_{t,\mathbb{H},risk}^* + \big(\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z}\big)^{-1} \mathbf{Z} \left(\mathbf{\Psi}_{t,amb}^* - \mathbf{\Psi}_{t,\mathbb{H},risk}^*\right),\end{aligned}$$

where we used $\mathbf{z_0} = -\mathbf{\Psi}_{t,amb}^*$ and the definition of $\mathbf{W}$ in order to express

$$\mathbf{\Psi}_{t,\mathbb{H},risk}^* = -(\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} = -(\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{y}.$$

This concludes the derivation of equation (21).

Optimal Currency Exposure for Global Bond Investors

Here we examine the risk-minimizing currency exposures implied by an equally weighted global bond portfolio. Table 6, whose structure is identical to Table 4, reports optimal currency exposures at a one-quarter horizon in the multiple currency case for our full sample period and for the subperiods 1999 to 2008 and 2009 to 2018.

Risk-minimizing currency demands for internationally diversified bond market investors are generally very small and not statistically significant. The Japanese yen is an exception. The optimal demand for the Japanese yen is negative and statistically significant at the 5% level. However, the yen exposure is economically small. It is almost zero in the first subperiod, and grows larger in the second subperiod. It is interesting to observe that the optimal exposure in Japanese yen was positive for the global stock investor and it is negative for the global bond investor, reflecting the fact that the Japanese yen tends to depreciate when the Japanese bond market declines.

Overall, our results imply that international bond investors should fully hedge the currency exposure implicit in their bond portfolios, with possibly a small short bias towards the Japanese yen. The results for the global bond investors are consistent with the existing literature which proposes full currency hedging for the international bond investors.

Risk Reduction from Currency Hedging

Table 7 reports the portfolio standard deviations that investors can achieve by combining their global stock and bond portfolios with risk-minimizing currency exposures. For each base currency, we report the annualized standard deviations of daily returns given several alternative currency hedging strategies, where hedging is performed at quarterly time intervals. The first three columns report volatilities for unhedged, half hedged, and fully hedged portfolios. Half hedging is a compromise strategy that is popular with some institutional investors. In the fourth column, we report the in-sample volatility of optimally hedged portfolio. The volatilities are independent of the base currency for the cases with full or unconditionally optimal hedging, but not for the cases with no hedging and half hedging. The right-hand part of the table reports F-statistics and p-values to test the statistical significance of the risk reductions achieved by unconditionally optimal currency hedging in comparison to full hedging and zero hedging. Full-sample results, in Table 7, show that the benefit of full currency hedging depends sensitively on an investor's base currency. It is particularly large for Japanese and Swiss investors because these investors have a risk-reducing base currency, so they gain by hedging back to that currency and out of foreign currencies. The volatility reduction from full currency hedging is particularly small for Australian and Canadian investors because the home currency for these investors is risky in the sense that it is positively correlated with their equity positions. In fact, full currency hedging actually increases risk for the Australian and the Canadian investor.

Optimal hedging, in-sample, reduces risk for all investors. An important point is that the benefit of optimal hedging is substantial, both economically and statistically. This difference is statistically highly significant, with p-values that are well below 1%. The gains from currency hedging are also substantial for global bond investors, but in this case almost all the gains can be achieved by full currency hedging.

The results presented in this section are completely in line with the ones reported in Campbell et al. (2010). However, one has to note that the optimal hedging (and the corresponding optimization) is performed in-sample. The variance-covariance structure is known to fluctuate over time and hence the prediction power becomes vital for the investors trying to manage the risk of currency exposure.

Optimal Currency Exposure for an Equally Weighted Global Bond Portfolio: Multiple Currency Case

	Currency						
Time Period	Australia	Canada	Switzerland	Eurozone	UK	Japan	USA
Panel A: Seven-Country Optimization							
Full Period	0.02	0.10^{*}	−0.09	−0.03	0.11^{*}	-0.10^{**}	−0.01
	(0.06)	(0.06)	(0.06)	(0.07)	(0.06)	(0.05)	(0.06)
Subperiod I	0.10	0.03	−0.13	−0.15	0.13	0.01	0.01
	(0.08)	(0.08)	(0.16)	(0.17)	(0.10)	(0.06)	(0.08)
Subperiod II	−0.06	0.14^{**}	−0.02	0.09	0.06	-0.17^{***}	−0.04
	(0.07)	(0.07)	(0.04)	(0.07)	(0.07)	(0.05)	(0.09)
Panel B: Five-Country Optimization							
Full Period	0.07		-0.12^{*}		0.11^{**}	-0.09^{*}	0.03
	(0.04)		(0.06)		(0.05)	(0.05)	(0.05)
Subperiod I	0.09		-0.25^{***}		0.09	0.03	0.05
	(0.06)		(0.06)		(0.09)	(0.06)	(0.06)
Subperiod II	0.04		0.02		0.10	-0.17^{***}	0.01
	(0.05)		(0.05)		(0.06)	(0.05)	(0.08)

Table 6 – This table studies an investor holding a portfolio composed of long-term bonds from all countries, with equal weights, who chooses a vector of positions in all available foreign currencies to minimize the variance of her portfolio. Panel A considers a case where all seven currencies are available, whereas Panel B excludes the Canadian dollar and the euro. Within each panel, rows indicate the time period over which the optimization is computed and columns the currencies used to manage risk. The full period runs from 1999 to 2018, the first subperiod covers the years 1999 through 2008, and the second subperiod covers the rest of the sample. Reported currency positions are the amounts of dollars invested in foreign currency per dollar in the portfolio. We run monthly regressions of overlapping 3-month returns. Standard errors are corrected for autocorrelation due to overlapping intervals using the Newey-West procedure. We mark with one, two, or three asterisks coefficients for which we reject the null of zero at a 10%, 5%, and 1% significance level, respectively.

Standard Deviations of Hedged Global Equity and Bond Portfolios

Base Country	No Hedge	Half Hedge	Full Hedge	Optimal Hedge	Optimal vs. Full Hedge F-Stat	Optimal vs. Full Hedge p-value	Optimal vs. No Hedge F-Stat	Optimal vs. No Hedge p-value
Equity								
Australia	11.88	11.67	13.12	11.52	1.30	0.00	1.06	0.04
Canada	12.65	12.43	13.12	11.52	1.30	0.00	1.21	0.00
Switzerland	16.31	14.39	13.12	11.52	1.30	0.00	2.00	0.00
Eurozone	13.97	13.28	13.12	11.52	1.30	0.00	1.47	0.00
UK	13.86	13.13	13.12	11.52	1.30	0.00	1.45	0.00
Japan	19.27	15.80	13.12	11.52	1.30	0.00	2.80	0.00
USA	15.79	14.17	13.12	11.52	1.30	0.00	1.88	0.00
Bonds								
Australia	10.36	6.23	3.34	3.17	1.11	0.00	10.64	0.00
Canada	8.31	5.34	3.34	3.17	1.11	0.00	6.86	0.00
Switzerland	7.00	4.23	3.34	3.17	1.11	0.00	4.87	0.00
Eurozone	6.16	4.16	3.34	3.17	1.11	0.00	3.77	0.00
UK	7.45	4.85	3.34	3.17	1.11	0.00	5.50	0.00
Japan	8.90	4.81	3.34	3.17	1.11	0.00	7.88	0.00
USA	7.25	4.69	3.34	3.17	1.11	0.00	5.23	0.00

Table 7 – This table reports the standard deviation of portfolios featuring different uses of currency exposure for risk management. We present results for equally weighted global portfolios of equity and bonds as described in Tables 4 and 6. Within each panel, rows represent base countries and columns represent the risk management strategy. "No Hedge" refers to the simple equity portfolio. "Half Hedge" refers to a portfolio in which half of the implicit currency risk is neutralized. "Full Hedge" refers to a portfolio in which all of the implicit currency risk is neutralized. "Optimal Hedge" refers to a portfolio in which the currency position is chosen optimally to minimize variance. Reported portfolio return standard deviations are annualized and measured in percentage points. All results presented are computed considering daily returns and hedging at a quarterly horizon.