Disclosure, Competition, and Learning from Asset Prices

Liyan Yang*

December 2018

Abstract

This paper studies information sharing in a duopoly product market in which firms learn information from asset prices. By disclosing information, a firm incurs a proprietary cost of losing competitive advantage to its rival firm but benefits from learning from a more informative asset market. Learning from asset prices can dramatically change firms' disclosure behaviors: without learning from prices, firms do not disclose at all; but with learning from prices, firms can fully disclose their information. Firms' disclosure decisions can exhibit strategic complementarity, which leads to both a disclosure equilibrium and a nondisclosure equilibrium.

Keywords: Disclosure, product market competition, proprietary cost, feedback effect, complementarity and multiplicity.

JEL Classifications: D61; G14; M41

---

*Yang: liyan.yang@rotman.utoronto.ca, Rotman School of Management, University of Toronto, 105 St. George Street, Toronto, M5S3E6, ON, Canada. I thank Ayan Bhattacharya, Wen Chen, Theirry Foucault, Zhenyu Gao, Eitan Goldman, Itay Goldstein, Chong Huang, Shiyang Huang, Eric Hughson, Xuewen Liu, Lin Peng, Ioanid Rosu, Francesco Sangiorgi, Jan Schneemeier, James Thompson, Charles Trzcinka, Laura Veldkamp, Xavier Vives, Pengfei Wang, Yajun Wang, Wei Xiong, Xiaoyun Yu, seminar participants at CUNY Baruch, HEC Paris, HKUST, Indian Kelley, Temple Fox, UNC Charlotte, and University of Geneva, and conference participants at 2018 Bank of Canada - Laurier Market Structure Conference, 2018 China International Conference in Finance (CICF), 2018 China International Forum on Finance and Policy (CIFFP), 2018 HKUST Finance Symposium, 2018 North American Summer Meeting of the Econometric Society, 2018 Northern Finance Association (NFA) Annual Meeting, and 2018 PHBS Workshop in Macroeconomics and Finance. I thank Social Sciences and Humanities Research Council of Canada (SSHRC) for financial support.

1. Introduction

I examine the incentives for information sharing among oligopoly firms when they can learn information from a financial market. My analysis builds on the classic information-sharing duopoly setting with demand uncertainty and Cournot competition (e.g., Vives, 1984; Gal-Or, 1985; Darrough, 1993). In such a setting, disclosure incurs a cost, which is often labeled as "proprietary cost" (Darrough, 1993) or "competitive disadvantage cost"/"loss of competitive advantage" (Bhattacharya and Ritter, 1983; Foster, 1986): disclosure reveals strategic information to competitors and reduces the disclosing firm's competitive advantage.[1] For instance, high demand of the disclosing firm may be indicative of high demand for competitors (i.e., "a rising tide lifts all boats"), which encourages competitors to expand their production, eroding the disclosing firm's profits. The literature shows that proprietary cost concerns make oligopoly firms choose to withhold information in equilibrium (see the excellent survey paper by Vives (2008)).

The new feature of my analysis is that firms learn new information from a financial market and use this information to guide their production decisions. Going back at least to Hayek (1945), researchers argue that asset prices are a useful source of information for real decisions. Asset prices aggregate myriad pieces of information from various traders who trade in financial markets;[2] in turn, firms have an incentive to use this price information, in addition to other sources of information, in making their production decisions. This effect is known as the "feedback effect" from finan-

[1] Survey evidence indicates that proprietary cost is indeed one major barrier to voluntary disclosure of real companies (Graham, Harvey, and Rajgopal, 2005).

[2] The archetypal examples of financial markets include the stock market and the commodity futures market. For instance, Fama and Miller (1972, p. 335) note: "at any point in time market prices of securities provide accurate signals for resource allocation; that is, firms can make production-investment decisions...." Black (1976, p. 174–176) writes: "futures prices provide a wealth of valuable information for those who produce, store, and use commodities. Looking at futures prices for various transaction months, participants in this market can decide on the best times to plant, harvest, buy for storage, sell from storage, or process the commodity...The big benefit from futures markets is the side effect: the fact that participants in the futures markets can make production, storage, and processing decisions by looking at the pattern of futures prices, even if they don't take positions in that market."

cial markets to the real economy, and has received extensive empirical support.[3] I incorporate this feedback effect by introducing a futures market. The futures contract is on the commodity produced by the two competing firms. Financial speculators, such as hedge funds or commodity index traders, trade the futures contracts (against liquidity traders) based on their private information about the later product demand, and their trading injects new information into the futures price.

One might be tempted to conjecture that adding the element of learning from asset prices does not change the nondisclosure equilibrium identified in the information-sharing literature (e.g., Vives (2008)). Intuitively, the asset price is effectively a public signal shared by both firms and so its main role may be simply changing the firms' prior distribution about the product demand, which should not affect firms' incentives to share their private information. This conjecture is only partially correct. What it misses is that the informational content of the asset price is endogenous and that firms can employ disclosure to affect the informativeness of the price. This creates a benefit for firms to share their private information.

Specifically, in my setting with a feedback effect, firms face the following trade-off in deciding on their disclosure policies. The negative effect of disclosure is the proprietary cost identified in the literature (e.g., Vives, 1984; Gal-Or, 1985; Darrough, 1993). The positive effect of disclosure comes from a more informative asset price that improves firms' learning quality. Specifically, the payoff on the futures contract is driven by different pieces of demand shocks, which are observed respectively by the two firms and financial speculators. So, publicly releasing the private information of firms reduces the uncertainty faced by financial speculators.[4] This encourages

[3]See Bond, Edmans, and Goldstein (2012) for a survey on the feedback effect. For empirical evidence, see, for example, Luo (2005), Chen, Goldstein, and Jiang (2007), Bakke and Whited (2010), Foucault and Frésard (2012). In particular, Ozoguz and Rebello (2013), Foucault and Frésard (2014), and Dessaint, Foucault, Frésard, and Matray (2018) provide evidence on firms learning from the stock price of their product-market peers.

[4]There exists evidence suggesting that company managers indeed use disclosure to reduce uncertainty faced by investors. For instance, Bochkay, Chychyla, and Nanda (2016) show that "new CEOs use disclosure to cut uncertainty and boost their careers" (Columbia Law School Blue Sky Blog, August 29, 2016). Graham, Harvey, and Rajgopal (2005) provide survey evidence that managers

risk-averse speculators to trade more aggressively on their private information. In consequence, the futures price will aggregate more of speculators' private information, benefiting firms' learning from the asset price. Each firm weighs this benefit of improved learning from the asset price against the proprietary cost to determine its optimal disclosure policy.

There are three types of equilibrium: a nondisclosure equilibrium, in which firms do not disclose any information; a full disclosure equilibrium, in which firms disclose their private information perfectly; and a partial disclosure equilibrium, in which firms voluntarily disclose their private information with added noises. This result runs in sharp contrast to the information-sharing literature which shows that firms never disclose their private information about market demand in Cournot settings (e.g., Gal-Or, 1985; Darrough, 1993). In my setting, the nondisclosure equilibrium is more likely to prevail as the unique equilibrium only when financial speculators know less information and when the financial market features less noise trading. This is because under both conditions, firms have a weaker incentive to learn from the financial market. First, when speculators know little information, firms do not have much to learn from speculators via the asset price. Second, when there is little noise trading in the financial market, the asset price has already aggregated speculators' information very well and thus, the scope to improve price informativeness via disclosure is small.

I show that firms' disclosure decisions can be a strategic complement. Complementarity arises when there is a lot noise trading in the financial market. If this complementarity is sufficiently strong, both a partial/full disclosure equilibrium and a nondisclosure equilibrium can be supported. This multiplicity result also runs in sharp contrast to the information-sharing literature which shows that there always exists a unique equilibrium. When multiplicity arises, both firms are better off on the disclosure equilibrium than on the nondisclosure equilibrium for two reasons. First, disclosure of each firm directly benefits its rival by releasing new information about

make voluntary disclosures to reduce information risk faced by investors.

product demand. Second, disclosure of both firms reduces the uncertainty faced by speculators who in turn trade more aggressively on their information. This makes the asset price more informative, thereby benefiting both firms. Taken together, it is in the firms' interests to coordinate on the disclosure equilibrium.

My analysis reveals that adding a feedback effect can dramatically change firms' equilibrium disclosure behavior. Specifically, as mentioned before, in a standard setting without learning from prices, firms do not disclose at all. That is, the equilibrium level of disclosure precision is zero in a setting in which the size of noise trading is infinity. Now consider a setting with learning from prices and suppose that there is a lot noise trading so that multiple equilibria arise. As argued above, firms prefer to coordinate on a partial/full disclosure equilibrium. It can be shown that as the size of noise trading diverges to infinity, firms' disclosure precision level also diverges to infinity. Thus, there is a discontinuity of disclosure policy at infinitely large noise trading. Intuitively, when the noise trading at the financial market is infinity, firms cannot at all learn from the asset price and so the benefit of disclosure disappears, leading to the nondisclosure equilibrium. However, when the noise trading is finite (although large) so that firms can learn from the asset price, they coordinate on a very aggressive disclosure equilibrium to improve the informativeness of asset prices, which is beneficial for both firms.

At a broad level, my analysis is aligned with the recent evidence that voluntary disclosure is an important tool in companies' arsenal to shape their information environments. For instance, Graham, Harvey, and Rajgopal (2005, p. 4) provide survey evidence that "managers make voluntary disclosures to reduce information risk and boost stock price." Anantharaman and Zhang (2011, p. 1851) show that "managers increase the volume of public financial guidance in response to decreases in analyst coverage of their firms" to "recoup analysts." Balakrishnan, Billings, Kelly, and Ljungqvist (2014, p. 2237) find that "(f)irms respond to an exogenous loss of public information by providing more timely and informative earnings guidance" to "improve liquidity." In my setting, in response to an increase in noise trading in the futures market, firms can voluntarily disclose more information to improve asset price informativeness that benefits firms' real investment decisions.

1.1. Related Literature

This paper contributes to two different strands of research. First, it advances the classic literature on information sharing of firms in oligopoly settings (e.g., Gal-Or, 1986; Darrough, 1993; Raith, 1996; Vives, 1984, 2008; Bagnoli and Watts, 2015; Arya, Mittendorf, and Yoon, 2016). This literature shows whether firms want to voluntarily disclose information depends upon the nature of competition (Cournot or Bertrand) and the nature of information (common value or private value). Common-value information represents shocks affecting all firms (e.g., a common demand shock), while private-value information represents shocks affecting each firm separately (e.g., idiosyncratic cost shocks). The literature finds that firms choose to withhold information in settings of Cournot/common-value and Bertrand/private-value, while they choose to share information completely in settings of Cournot/private-value and Bertrand/common-value.

My paper builds on a Cournot/common-value setting which features the proprietary cost. My analysis extends the canon of existing studies to include the realistic feature that firms learn information from asset prices. This extension generates two novel insights. First, firms either choose not to disclose information at all, or to disclose information to the public fully or partially. This differs from the literature which finds that firms do not disclose in a Cournot/common-value setting. Second, in the presence of learning from asset prices, firms' disclosure decisions can be a strategic complement, which gives rise to multiple equilibria. This also differs from the unique nondisclosure equilibrium identified in the standard setting.

The second related strand of literature is the literature on the feedback effect of a financial market, as reviewed by Bond, Edmans, and Goldstein (2012). A few recent papers study the effect of disclosure in contexts that feature a feedback effect. In Gao and Liang (2013), disclosure crowds out private-information production, which reduces price informativeness and harms managers' learning and investments. In Banerjee, Davis, and Gondhi (2018), public information can lower price efficiency by encouraging traders choose to acquire non-fundamental information exclusively. In Han, Tang, and Yang (2016), disclosure attracts noise trading that harms managers' learning quality. In Amador and Weill (2010), disclosure about monetary and/or productivity shocks can reduce welfare through reducing the informational efficiency of the good price system. In Goldstein and Yang (2018), disclosure can be either good or bad, depending on whether disclosure is about the dimension about which the firm already knows. In contrast, in my paper, disclosure benefits rather than harms firms via the feedback effect, and the cost of disclosure is endogenously generated from losing a competitive advantage that is unique to the oligopoly setting.

The positive effect of disclosure in my paper is related to the "residual risk effect" in Bond and Goldstein (2015) and the "uncertainty reduction effect" in Goldstein and Yang (2015). Releasing information about shocks that are unknown to traders reduces the uncertainty faced by traders. Since traders are risk averse, the reduction in risk incentivizes them to trade more on their information. In consequence, the price will aggregate more of traders' private information, benefiting the firms' learning from the asset price.

In a contemporaneous paper, Schneemeier (2018) also studies firms' optimal disclosure policies in the presence of a feedback effect, albeit in a very different framework. The two papers explore very different channels that can be relevant to different scenarios. In Schneemeier's setting, the key trade-off of disclosure is a combination of Gao and Liang (2013) and Dow, Goldstein, and Guembel (2017): on the one hand, as in Gao and Liang (2013), disclosure crowds out speculators' information production because it reduces the speculators' information advantage; on the other hand, disclosure can attract information production if it can credibly convey to the market when the firm makes information-sensitive investment, which raises the profitability of information acquisition (in a similar spirit as Dow, Goldstein, and Guembel (2017)). In contrast, in my analysis, the cost of disclosure arises from the proprietary cost of leaking information to competing companies. Disclosure in my setting crowds in, instead of crowds out, speculators' information in the price, because information disclosed by firms reduces the risk perceived by risk-averse speculators. The different trade-offs lead to different theory insights; for instance, in my setting, disclosure decisions can exhibit complementarity, leading to multiple equilibria.

2. The Model

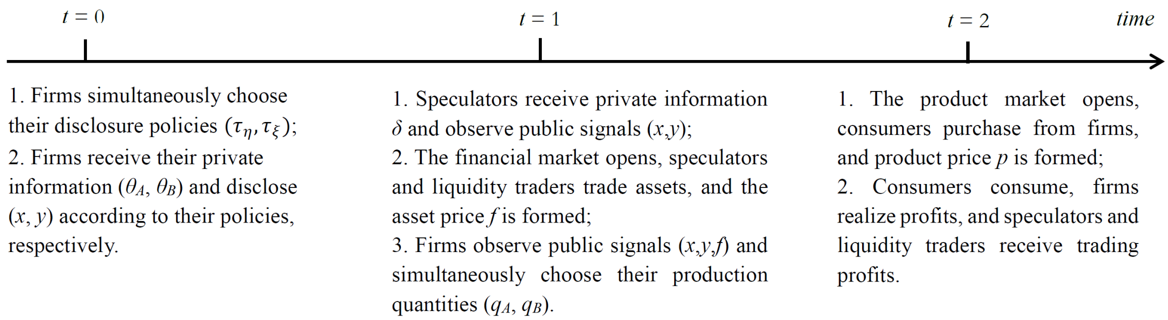
I consider a standard information-sharing duopoly setting (e.g., Vives, 1984; Gal-Or, 1985; Darrough, 1993), which is extended with a financial market, or more specifically, with a futures market on the commodity produced by two competitive firms. There are three dates, $t = 0, 1$, and 2. The order of events is described in Figure 1. On date 0, two competing firms, firm A and firm B, simultaneously decide on their disclosure policies. On date 1, financial speculators and liquidity traders trade commodity futures. Financial speculators are endowed with private information about the later demand for the firms' products, which is aggregated into the equilibrium futures price. Firms make inference on this information from the futures price to guide their production decisions (the feedback effect). On date 2, the product market opens and the product price is determined.

2.1. Demand for Products

The date-2 demand for firms' products is generated by a representative consumer who maximizes consumer surplus,

$$\mathcal{C}\left(Q, \theta_A, \theta_B, \delta, \varepsilon\right) = \mathcal{U}\left(Q, \theta_A, \theta_B, \delta, \varepsilon\right) - pQ, \tag{1}$$

Figure 1: Model Timeline



where Q is the amount of products purchased from the firms and p is the product price. In equation (1), $\mathcal{U}(Q, \theta_A, \theta_B, \delta, \varepsilon)$ captures the consumer's intrinsic utility from consuming the products, while the term pQ is the cost of purchasing the products. Following the literature (e.g., Singh and Vives, 1984), I specify a quasi-linear intrinsic utility function as follows:

$$\mathcal{U}(Q, \theta_A, \theta_B, \delta, \varepsilon) = (m + \theta_A + \theta_B + \delta + \varepsilon) Q - \frac{Q^2}{2}. \tag{2}$$

Parameter m is a positive constant that captures the size of the product market. Variables $\theta_A, \theta_B, \delta$, and ε are mutually independent demand shocks that are normally distributed; that is, $\theta_A \sim N\left(0, \tau_\theta^{-1}\right)$, $\theta_B \sim N\left(0, \tau_\theta^{-1}\right)$, $\delta \sim N\left(0, \tau_\delta^{-1}\right)$, and $\varepsilon \sim N\left(0, \tau_\varepsilon^{-1}\right)$ (with $\tau_\theta > 0$, $\tau_\delta > 0$, and $\tau_\varepsilon > 0$). The demand shocks $(\theta_A, \theta_B, \delta)$ are observed by firm A, firm B, and financial speculators, respectively, while the demand shock ε reflects the residual uncertainty that is hard to predict by firms and financial speculators.

The representative consumer knows her preference shocks and chooses product quantity Q to maximize her preference (1) taking the product price p as given. This maximization problem leads to the following standard linear inverse demand function for firms' products:

$$p = (m + \theta_A + \theta_B + \delta + \varepsilon) - Q. \tag{3}$$

For the sake of simplicity, I have assumed that both firms produce identical products.

Alternatively, I can assume that firms produce differentiated products and the results do not change under this alternative assumption.

2.2. Information Disclosure and Commodity Production

The two firms make two decisions in the economy, a disclosure-policy decision on date 0 and a commodity-production decision on date 1. Their production decisions determine the supply of products in the product market. Following Darrough (1993), I assume that on date 0, firms A and B respectively observe demand shocks θ_A and θ_B. Firms precommit themselves in advance to a particular disclosure policy ex ante before they receive their private information. Such a commitment may be coordinated and enforced by trade associations or regulatory agencies such as the FASB or the SEC. Firm A discloses a noisier version of θ_A to the public in the form of

$$x = \theta_A + \eta,$$

where $\eta \sim N\left(0, \tau_\eta^{-1}\right)$ (with $\tau_\eta \in [0, \infty]$) and η is independent of all other shocks. Similarly, firm B discloses a nosier version of θ_B in the form of

$$y = \theta_B + \xi,$$

where $\xi \sim N\left(0, \tau_\xi^{-1}\right)$ (with $\tau_\xi \in [0, \infty]$) and ξ is independent of all other shocks.

The random variables η and ξ are the noises added respectively by the two firms in their disclosed signals. The precision levels τ_η and τ_ξ are chosen by the firms at the beginning of date 0 to maximize their unconditional expected profits. A higher value of τ_η and τ_ξ signifies that x and y are more informative about θ_A and θ_B, respectively, which can be achieved by making more frequent announcements (e.g., through press releases, conference calls, monthly newsletters) and/or by releasing more accurate data (e.g., by adding an extra line in financial statements to separate core from non-core items).[5] In particular, I allow τ_η and τ_ξ to take values of 0 and ∞, which correspond respectively to the case in which the firms do not disclose (i.e.,

[5] For instance, on November 1, 2018, Apple announced that it will stop reporting unit sales figures for its three most recognizable brands, the iPhone, iPad and Mac, in the future reports starting from the next quarter. This corresponds to a decrease in τ_η.

disclose with infinite noise) and to the case in which the firms disclose their private information perfectly (i.e., disclose without noise). In the literature, these two values are the only possible equilibrium choices (see the survey by Vives (2008)). As I will show shortly, in the presence of learning from asset prices, firms can choose to disclose their information imperfectly (i.e., $\tau_\eta \in (0, \infty)$ and $\tau_\xi \in (0, \infty)$).

On date 1, firms make production decisions to maximize profits based on private and public information. Firm A's private information is θ_A and firm B's private information is θ_B. There are three pieces of public information: public disclosure x released by firm A, public disclosure y released by firm B, and the price f of a financial asset. The innovation of this paper is that firms extract information from the asset price f to guide their production decisions, which is the feedback effect.

I normalize the marginal cost of production as 0. As known in the literature, this normalization does not affect the results. Under this normalization, firm i's profit is

$$\Pi_i \left(q_i, q_j, \theta_A, \theta_B, \delta, \varepsilon\right) = pq_i = \left(m + \theta_A + \theta_B + \delta + \varepsilon - q_j\right) q_i - q_i^2, \tag{4}$$

for $i, j \in \{A, B\}$ and $i \neq j$. Variables q_i and q_j are respectively the amount of products produced by the firm i and firm j. The second equality in (4) follows from the inverse demand function (3) and $Q = q_A + q_B$. The optimal date-1 production q_i^* of firm i is determined by

$$\max_{q_i} E\left[\Pi_i\left(q_i, q_j^*, \theta_A, \theta_B, \delta, \varepsilon\right) \middle| \theta_i, x, y, f\right],$$

where $E\left[\cdot \middle| \theta_i, x, y, f\right]$ is the conditional expectation operator and q_j^* refers to firm j's optimal production, which is taken as given in firm i's production decision problem.

The optimal date-0 disclosure decision τ_η^* of firm A is determined by

$$\max_{\tau_\eta} E\left[\Pi_A\left(q_A^*, q_B^*, \theta_A, \theta_B, \delta, \varepsilon\right)\right].$$

Similarly, the optimal date-0 disclosure decision τ_ξ^* of firm B is determined by

$$\max_{\tau_\xi} E\left[\Pi_B\left(q_A^*, q_B^*, \theta_A, \theta_B, \delta, \varepsilon\right)\right].$$

When making the disclosure policy choice, each firm takes the other firm's disclosure policy as given and also takes into account how its own disclosure affects the optimal production decisions of both firms in the product market.

2.3. Financial Market

The financial market opens on date 1. There are two tradable assets: a futures contract and a risk-free asset. I normalize the net risk-free rate as 0. The payoff on the futures contract is the date-2 product spot price p. Each unit of futures contract is traded at an endogenous price f. The total supply of futures contracts is 0.

There are two groups of market participants: financial speculators and liquidity traders. Liquidity traders represent random transient demands in the futures market and they as a group demand u units of the commodity futures, where $u \sim N\left(0, \tau_u^{-1}\right)$ with $\tau_u \in (0, \infty)$. As usual, liquidity traders, also known as "noise traders," provide the randomness (noise) necessary to make the rational expectations equilibrium partially revealing. I do not endogenize the behavior of liquidity traders; rather, I view them as individuals who are trading to invest new cash flows or to liquidate assets to meet unexpected consumption needs.

There is a continuum $[0, 1]$ of financial speculators who derive expected utility only from their date-2 wealth. They have constant absolute risk aversion (CARA) utility functions with a common coefficient of risk aversion $\gamma > 0$. Speculators are endowed with cash only, and for simplicity I suppose that their endowment is 0. These traders can be interpreted as hedge funds or commodity index traders.[6] Financial speculators privately observe demand shock δ and thus their trading injects this information into the futures price f.

Remark 1. (Comments on Assumptions) *First, for the sake of tractability, I have assumed that speculators' private information δ is independent of firms' private information θ_A and θ_B. This assumption is not crucial for driving the qualitative results. What matters is that speculators as a group own some information which is new to firms, so that firms learn some information from the asset price (which is the key feature in the literature on feedback effects). Second, I have assumed that speculators*

[6] According to Cheng and Xiong (2014, p. 424), "(o)ver the past decade, there has been a large inflow of investment capital from a class of investors, so-called commodity index traders (CITs), also known as index speculators."

observe identical information. A more realistic view is that they own disperse information (potentially very coarse) which is aggregated into the price, leading to a very valuable signal to firms (e.g., Hayek (1945)). I do not take this alternative view to keep the model tractable;[7] *the current setup is sufficient for modeling the feature that firms learn from asset prices. Third, firms do not participate in the futures market, which allows me to isolate the informational role of asset prices in driving the results.*

3. Equilibrium Characterization

Definition 1. *An equilibrium consists of date-0 disclosure policies of firms* $\left(\tau_\eta^*, \tau_\xi^*\right)$, *date-1 production policies of firms* $q_A(\theta_A, x, y, f)$ *and* $q_B(\theta_B, x, y, f)$, *a date-1 trading strategy of speculators* $D(\delta, x, y, f)$, *a date-1 futures price function* $f(\delta, x, y, u)$, *and a date-2 spot price function* $p(\theta_A, \theta_B, \delta, x, y, f, \varepsilon)$, *such that:*

(a) Disclosure policies $\left(\tau_\eta^*, \tau_\xi^*\right)$ *form a Nash equilibrium, i.e.,*

$$\begin{aligned}\tau_\eta^* &= \arg\max_{\tau_\eta} E\left[\Pi_A\left(q_A(\theta_A, x, y, f), q_B(\theta_B, x, y, f), \theta_A, \theta_B, \delta, \varepsilon\right)\right],\\ \tau_\xi^* &= \arg\max_{\tau_\xi} E\left[\Pi_B\left(q_A(\theta_A, x, y, f), q_B(\theta_B, x, y, f), \theta_A, \theta_B, \delta, \varepsilon\right)\right];\end{aligned}$$

(b) Trading strategy $D(\delta, x, y, f)$ *and futures price function* $f(\delta, x, y, u)$ *form a noisy rational expectations equilibrium (noisy-REE) in the financial market, i.e.,*

$$\begin{aligned}D(\delta, x, y, f) &= \arg\max_D E\left[-e^{-\gamma D[p(\theta_A,\theta_B,\delta,x,y,f,\varepsilon)-f(\delta,x,y,u)]}\middle|\, \delta, x, y, f\right],\\ D(\delta, x, y, f) + u &= 0;\end{aligned}$$

(c) Production policies $q_A(\theta_A, x, y, f)$ *and* $q_B(\theta_B, x, y, f)$ *form a Bayesian-Nash equilibrium in the product market, i.e.,*

$$\begin{aligned}q_A(\theta_A, x, y, f) &= \arg\max_{q_A} E\left[\Pi_A\left(q_A, q_B(\theta_B, x, y, f), \theta_A, \theta_B, \delta, \varepsilon\right)\middle|\, \theta_A, x, y, f\right],\\ q_B(\theta_B, x, y, f) &= \arg\max_{q_B} E\left[\Pi_B\left(q_A(\theta_A, x, y, f), q_B, \theta_A, \theta_B, \delta, \varepsilon\right)\middle|\, \theta_B, x, y, f\right];\ \textit{and}\end{aligned}$$

[7] Specifically, the current setup allows me to first analytically compute the product market equilibrium, which is then inserted into the speculators' demand function and the market-clearing condition to compute the financial market equilibrium. By contrast, in a setting with diverse signals, I have to simultaneously solve the product market equilibrium and the financial market equilibrium.

(d) The spot price $p(\theta_A, \theta_B, \delta, x, y, f, \varepsilon)$ clears the product market, i.e.,

$$q_A(\theta_A, x, y, f) + q_B(\theta_B, x, y, f) = (m + \theta_A + \theta_B + \delta + \varepsilon) - p(\theta_A, \theta_B, \delta, x, y, f, \varepsilon).$$

A linear equilibrium is an equilibrium in which policy functions and price functions are linear.

Following the literature, I consider symmetric equilibrium in which both firms choose the same disclosure policy (i.e., $\tau_\eta^* = \tau_\xi^*$).[8] There are three types of symmetric equilibrium as defined below.

Definition 2. *If $\tau_\eta^* = \tau_\xi^* = 0$, then the equilibrium is referred to as the "nondisclosure equilibrium." If $\tau_\eta^* = \tau_\xi^* = \infty$, then the equilibrium is referred to as the "full disclosure equilibrium." If $\tau_\eta^* = \tau_\xi^* \in (0, \infty)$, then the equilibrium is referred to as a "partial disclosure equilibrium." Either the full disclosure equilibrium or a partial disclosure equilibrium is referred to as a disclosure equilibrium.*

Before formally characterizing the equilibrium, I first analyze a benchmark setting in which firms do not learn from a financial market.

3.1. A Benchmark Setting without Feedback Effects

If I shut down the feature that firms learn information from the asset price f, the model degenerates to a standard information-sharing setting with demand shocks and Cournot competition. As well-known in the literature (e.g., Gal-Or, 1985; Darrough, 1993), concealing information is a dominant strategy, so that both firms choose not to disclose information in equilibrium. This is because disclosure reveals strategic information to competitors, thereby reducing the disclosing firm's competitive advantage. I summarize the equilibrium of this benchmark setting in the following proposition, where I label variables with superscript "$\varnothing$" to indicate that in this setting, firms do not extract information from an asset price. The proof is standard and hence omitted.

[8] As Gal-Or (1985, p. 330) argued, "(s)ymmetric equilibrium is a reasonable solution concept for this model since all firms face the same technology and observe signals of the same precision."

Proposition 1. (No Learning from Asset Prices) *In a setting where firms do not learn information from a financial market, there exists a unique linear Bayesian-Nash equilibrium in the product market for given disclosure policies* (τ_η, τ_ξ)*, in which*

$$\begin{aligned} q_A^{\varnothing} &= \frac{m}{3} + \frac{1}{2}\theta_A - \frac{\tau_\eta}{6(\tau_\theta + \tau_\eta)}x + \frac{\tau_\xi}{3(\tau_\theta + \tau_\xi)}y, \\ q_B^{\varnothing} &= \frac{m}{3} + \frac{1}{2}\theta_B + \frac{\tau_\eta}{3(\tau_\theta + \tau_\eta)}x - \frac{\tau_\xi}{6(\tau_\theta + \tau_\xi)}y, \end{aligned}$$

and on date 0, no firm chooses to disclose information, i.e., $\tau_\eta^{\varnothing} = \tau_\xi^{\varnothing} = 0$*.*

In the following two subsections, I will derive the equilibrium in a setting where firms learn information from the financial market. There will be two main results that differ from Proposition 1. First, firms may choose to disclose information on date 0, i.e., $\tau_\eta^* = \tau_\xi^* > 0$ for some parameters. Second, there may exist multiple equilibria due to the coordination motives across firms, that is, it is possible that both $\tau_\eta^* = \tau_\xi^* = 0$ and $\tau_\eta^* = \tau_\xi^* > 0$ can be supported as an equilibrium.

3.2. Product Market Equilibrium and Financial Market Equilibrium

Following the literature (e.g., Gal-Or, 1985; Darrough, 1993), I consider linear Bayesian-Nash equilibria in the product market. That is, the production policies of firms A and B are linear in their information variables as follows:

$$q_A^* = a_0 + a_\theta \theta_A + a_x x + a_y y + a_f f, \tag{5}$$

$$q_B^* = b_0 + b_\theta \theta_B + b_x x + b_y y + b_f f, \tag{6}$$

where the a-coefficients and the b-coefficients are endogenous constants.

The optimal productions q_A^* and q_B^* are determined respectively by the following first-order conditions (FOCs) of the profit-maximization problems in Part (c) of Definition 1 (the second-order conditions (SOCs) are always satisfied):

$$q_A^* = \frac{1}{2}E\left(m + \theta_A + \theta_B + \delta + \varepsilon - q_B^* | \theta_A, x, y, f\right), \tag{7}$$

$$q_B^* = \frac{1}{2}E\left(m + \theta_A + \theta_B + \delta + \varepsilon - q_A^* | \theta_B, x, y, f\right). \tag{8}$$

A Bayesian-Nash equilibrium requires that the above implied policy functions (7) and (8) agree with the conjectured policy functions (5) and (6). In doing so, one needs to express out the conditional moments in (7) and (8), namely to figure out how each firm uses both private and public information (in particular, the asset price f) to forecast later demand shocks and its opponent's production.

Take firm A as an example. Inserting the conjectured production policy (6) of firm B into the FOC (7) of firm A's profit-maximization problem yields

$$q_A^* = \frac{1}{2}\left[\begin{array}{c} m + \theta_A - (b_0 + b_x x + b_y y + b_f f) \\ +E(\delta|\theta_A, x, y, f) + (1 - b_\theta) E(\theta_B|\theta_A, x, y, f) \end{array}\right]. \tag{9}$$

So, firm A needs to forecast two variables, θ_B and δ. The idea is that the public signal y disclosed by firm B is useful for predicting θ_B, while the asset price f, together with public disclosure x and y, is useful for predicting δ, because the trading of speculators injects information δ into the futures price f. I now turn to the futures market to figure out how firms extract information from the asset price f.

Solving the speculators' utility-maximization problem in Part (b) of Definition 1 gives rise to their demand function under CARA preference,

$$D(\delta, x, y, f) = \frac{E(p|\delta, x, y, f) - f}{\gamma Var(p|\delta, x, y, f)}, \tag{10}$$

where $E(\cdot|\,\delta, x, y, f)$ and $Var(\cdot|\,\delta, x, y, f)$ are the conditional expectation and variance, respectively. Inserting the conjectured policy functions (5) and (6) into the market-clearing condition of product market in Part (d) of Definition 1 yields

$$\begin{aligned} p \;&=\; (1 - a_\theta)\,\theta_A + (1 - b_\theta)\,\theta_B + \varepsilon \\ &\quad + (m - a_0 - b_0) + \delta - (a_x + b_x)\,x - (a_y + b_y)\,y - (a_f + b_f)\,f. \end{aligned} \tag{11}$$

Since speculators observe $\{\delta, x, y, f\}$, they only need to forecast $(1 - a_\theta)\,\theta_A + (1 - b_\theta)\,\theta_B$ in the above expression of p. In doing so, speculators use public information x to predict θ_A and public information y to predict θ_B. Applying Bayes' rule to compute $E(p|\delta, x, y, f)$ and $Var(p|\delta, x, y, f)$, which are in turn inserted into demand function (10) and the market-clearing condition of the futures market, $D(\delta, x, y, f) + u = 0$, I derive the futures price function as follows:

$$\begin{aligned} f &= \frac{m-a_0-b_0}{a_f+b_f+1}+\frac{\delta}{a_f+b_f+1} \\ &+\frac{\frac{(1-a_\theta)\tau_\eta}{\tau_\theta+\tau_\eta}-(a_x+b_x)}{a_f+b_f+1}x+\frac{\frac{(1-b_\theta)\tau_\xi}{\tau_\theta+\tau_\xi}-(a_y+b_y)}{a_f+b_f+1}y \\ &+\frac{\gamma\left[\frac{(1-a_\theta)^2}{\tau_\theta+\tau_\eta}+\frac{(1-b_\theta)^2}{\tau_\theta+\tau_\xi}+\frac{1}{\tau_\varepsilon}\right]}{a_f+b_f+1}u. \end{aligned} \tag{12}$$

Thus, to firm A, the futures price f is equivalent to the following signal in predicting demand shock δ:

$$\begin{aligned} s &\equiv (a_f+b_f+1)f-(m-a_0-b_0) \\ &-\left[\frac{(1-a_\theta)\tau_\eta}{\tau_\theta+\tau_\eta}-(a_x+b_x)\right]x-\left[\frac{(1-b_\theta)\tau_\xi}{\tau_\theta+\tau_\xi}-(a_y+b_y)\right]y \\ &= \delta+\gamma\left[\frac{(1-a_\theta)^2}{\tau_\theta+\tau_\eta}+\frac{(1-b_\theta)^2}{\tau_\theta+\tau_\xi}+\frac{1}{\tau_\varepsilon}\right]u, \end{aligned} \tag{13}$$

which has an endogenous precision level of

$$\tau_s=\frac{\tau_u}{\gamma^2\left[\frac{(1-a_\theta)^2}{\tau_\theta+\tau_\eta}+\frac{(1-b_\theta)^2}{\tau_\theta+\tau_\xi}+\frac{1}{\tau_\varepsilon}\right]^2}. \tag{14}$$

The signal s formalizes the fact that firms learn information about δ from the asset price f, and its precision τ_s captures the informational content in the asset price. I follow the literature and refer to variable τ_s as "price informativeness."

Firm A's information set $\{\theta_A, x, y, f\}$ is equivalent to $\{\theta_A, x, y, s\}$, among which y and s are respectively useful for predicting demand shocks θ_B and δ. Applying Bayes' rule to compute $E(\delta|\theta_A, x, y, f) = E(\delta|s)$ and $E(\theta_B|\theta_A, x, y, f) = E(\theta_B|y)$ and combining with the expression of s in (13), I can express q_A^* in (9) as a function of (θ_A, x, y, f). Comparing this expression with the conjectured policy in (5), I can form five conditions in terms of the unknown a-coefficients and b-coefficients. Conducting a similar analysis for firm B leads to another five conditions in terms of a's and b's. Solving this system of ten equations yields the values of a's and b's. Finally, inserting the values of a's and b's into equations (11) and (12) gives rise to the spot price function and the futures price function, respectively.

Proposition 2. (Product and Futures Markets) *For any disclosure polices* (τ_η, τ_ξ)*, there exists a unique linear Bayesian-Nash equilibrium in the product market, in which*

$$\begin{aligned} q_A^* &= a_0 + a_\theta \theta_A + a_x x + a_y y + a_f f, \\ q_B^* &= b_0 + b_\theta \theta_B + b_x x + b_y y + b_f f, \end{aligned}$$

where

$$\begin{aligned} a_0 &= b_0 = \frac{\tau_\delta}{\tau_s + 3\tau_\delta} m, a_\theta = b_\theta = \frac{1}{2}, \\ a_x &= -\frac{\tau_s + \tau_\delta}{2(\tau_s + 3\tau_\delta)} \frac{\tau_\eta}{\tau_\theta + \tau_\eta}, b_x = \frac{\tau_\delta}{\tau_s + 3\tau_\delta} \frac{\tau_\eta}{\tau_\theta + \tau_\eta}, \\ a_y &= \frac{\tau_\delta}{\tau_s + 3\tau_\delta} \frac{\tau_\xi}{\tau_\theta + \tau_\xi}, b_y = -\frac{\tau_s + \tau_\delta}{2(\tau_s + 3\tau_\delta)} \frac{\tau_\xi}{\tau_\theta + \tau_\xi}, \\ a_f &= b_f = \frac{\tau_s}{\tau_s + 3\tau_\delta}, \end{aligned}$$

and

$$\tau_s = \frac{\tau_u}{\gamma^2 \left[\frac{1}{4(\tau_\theta + \tau_\eta)} + \frac{1}{4(\tau_\theta + \tau_\xi)} + \frac{1}{\tau_\varepsilon} \right]^2}.$$

The date-2 spot price function is

$$\begin{aligned} p = {} & \frac{\tau_s + \tau_\delta}{\tau_s + 3\tau_\delta} m + \frac{1}{2}\theta_A + \frac{1}{2}\theta_B + \delta + \varepsilon \\ & - \frac{\tau_\delta - \tau_s}{2(\tau_s + 3\tau_\delta)} \frac{\tau_\eta}{\tau_\theta + \tau_\eta} x - \frac{\tau_\delta - \tau_s}{2(\tau_s + 3\tau_\delta)} \frac{\tau_\xi}{(\tau_\theta + \tau_\xi)} y - \frac{2\tau_s}{\tau_s + 3\tau_\delta} f. \end{aligned}$$

The date-1 futures price function is

$$f = \frac{1}{3} m + \frac{\tau_s + 3\tau_\delta}{3(\tau_s + \tau_\delta)} \delta + \frac{\tau_\eta}{3(\tau_\theta + \tau_\eta)} x + \frac{\tau_\xi}{3(\tau_\theta + \tau_\xi)} y + \frac{\tau_s + 3\tau_\delta}{3(\tau_s + \tau_\delta)} \sqrt{\frac{\tau_u}{\tau_s}} u.$$

By the expression of τ_s in Proposition 2, disclosing information improves firms' learning quality from the asset price. Intuitively, demand shocks θ_A and θ_B in the spot price p in (11) are the uncertainty exposed to speculators when they trade futures contracts. Releasing information about these two shocks reduces the uncertainty faced by speculators. Being risk averse, speculators then trade more aggressively on their own private information δ, thereby injecting more information on δ into the futures price f. This effect shares a similar flavor as the "residual risk effect" in Bond and Goldstein (2015) and the "uncertainty reduction effect" in Goldstein and Yang (2015).

Corollary 1. (Price Informativeness) *Disclosure of firms improves the informational content of the asset price. That is,* $\frac{\partial \tau_s}{\partial \tau_\eta} > 0$ *and* $\frac{\partial \tau_s}{\partial \tau_\xi} > 0$.

3.3. Equilibrium Disclosure Policy

3.3.1. Profit Function

At the beginning of date 0, firms choose disclosure policies to maximize unconditional expected profits. Again, take firm A as an example. Using the FOC of firm A's profit-maximization problem in Part (c) of Definition 1 and the equilibrium production policy in Proposition 2, I can compute firm A's expected profit as follows:

$$E\Pi_A(\tau_\eta, \tau_\xi) = \underbrace{\frac{m^2}{9}}_{\text{market size}} + \underbrace{\frac{9\tau_\theta + 4\tau_\eta}{36\tau_\theta(\tau_\theta + \tau_\eta)}}_{\text{proprietary cost}} + \underbrace{\frac{\tau_\xi}{9\tau_\theta(\tau_\theta + \tau_\xi)}}_{\text{disclosure by firm } B} + \underbrace{\frac{\tau_s}{9\tau_\delta(\tau_s + \tau_\delta)}}_{\text{learning from prices}}. \tag{15}$$

Here, I explicitly express $E\Pi_A$ as a function of disclosure precision (τ_η, τ_ξ) to emphasize the dependence of expected profit on disclosure policies. Firm A chooses its optimal disclosure policy τ_η^* to maximize $E\Pi_A\left(\tau_\eta, \tau_\xi^*\right)$, taking as given the optimal disclosure τ_ξ^* of firm B.

There are four terms that go into firm A's expected profit in (15). The first term $\frac{m^2}{9}$ is simply the size of the product market. Disclosure has no effect on this term. The second term $\frac{9\tau_\theta+4\tau_\eta}{36\tau_\theta(\tau_\theta+\tau_\eta)}$ captures the "proprietary cost" (Darrough, 1993) or "competitive disadvantage cost" (Foster, 1986), whereby disclosing private information reduces the disclosing firm's competitive advantage. Disclosure harms firm A's profits via this second term; that is, $\frac{\partial}{\partial \tau_\eta}\frac{9\tau_\theta+4\tau_\eta}{36\tau_\theta(\tau_\theta+\tau_\eta)} < 0$. The third term $\frac{\tau_\xi}{9\tau_\theta\left(\tau_\theta+\tau_\xi\right)}$ captures the benefit from observing the public signal disclosed by the competing firm B, which is determined by firm B's disclosure precision τ_ξ and so independent of firm A's disclosure precision τ_η. The last term $\frac{\tau_s}{9\tau_\delta(\tau_s+\tau_\delta)}$ represents the benefit from learning from the asset price f. Disclosure benefits firm A via this last term. That is, $\frac{\partial}{\partial \tau_\eta}\frac{\tau_s}{9\tau_\delta(\tau_s+\tau_\delta)} = \frac{1}{9(\tau_s+\tau_\delta)^2}\frac{\partial \tau_s}{\partial \tau_\eta} > 0$, since $\frac{\partial \tau_s}{\partial \tau_\eta} > 0$ by Corollary 1.

In sum, the trade-off faced by firm A in the disclosure choice can be captured by the following FOC:

$$\frac{\partial E\Pi_A\left(\tau_\eta,\tau_\xi\right)}{\partial\tau_\eta}=\underbrace{-\frac{5}{36\left(\tau_\theta+\tau_\eta\right)^2}}_{<0,\text{ proprietary cost}}+\underbrace{\frac{1}{9\left(\tau_s+\tau_\delta\right)^2}\frac{\partial\tau_s}{\partial\tau_\eta}}_{>0,\text{ learning from prices}}. \tag{16}$$

That is, disclosing private information harms firm A via the proprietary cost but benefits firm A via improving price informativeness.

3.3.2. Disclosure Policy Characterization

The equilibrium disclosure policies $\left(\tau_\eta^*,\tau_\xi^*\right)$ form a Nash equilibrium. That is,

$$\tau_\eta^*=\arg\max_{\tau_\eta}E\Pi_A\left(\tau_\eta,\tau_\xi^*\right)\text{ and }\tau_\xi^*=\arg\max_{\tau_\xi}E\Pi_B\left(\tau_\eta^*,\tau_\xi\right),$$

where firm A's profit function $E\Pi_A\left(\tau_\eta,\tau_\xi\right)$ is given by equation (15) and firm B's profit function $E\Pi_B\left(\tau_\eta,\tau_\xi\right)$ is defined similarly. There are three types of disclosure policies in a symmetric equilibrium: (1) a "nondisclosure equilibrium," where both firms do not disclose information (i.e., $\tau_\eta^*=\tau_\xi^*=0$); (2) a "full disclosure equilibrium," where both firms disclose all of their information perfectly (i.e., $\tau_\eta^*=\tau_\xi^*=\infty$); and (2) a "partial disclosure equilibrium," where both firms disclose information with noise (i.e., $\tau_\eta^*=\tau_\xi^*\in(0,\infty)$). The following three theorems respectively characterize these three types of equilibrium.

Theorem 1. (Nondisclosure Equilibrium) *A nondisclosure equilibrium ($\tau_\eta^*=\tau_\xi^*=0$) exists if and only if one of the following two sets of conditions holds:*

$$(a)\begin{cases}16\gamma^2\tau_u\tau_\theta^3\tau_\varepsilon^3\left(3\tau_\varepsilon+8\tau_\theta\right)\\ \le 5\begin{pmatrix}\gamma^2\tau_\delta\tau_\varepsilon^2+4\tau_u\tau_\theta^2\tau_\varepsilon^2\\+4\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta^2\tau_\delta\end{pmatrix}\begin{pmatrix}\gamma^2\tau_\delta\tau_\varepsilon^2+16\tau_u\tau_\theta^2\tau_\varepsilon^2\\+8\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+16\gamma^2\tau_\theta^2\tau_\delta\end{pmatrix},\\ 4\gamma^2\tau_u\tau_\theta^3\tau_\varepsilon^3\left(7\tau_\varepsilon+16\tau_\theta\right)\\ \le 5\begin{pmatrix}\gamma^2\tau_\delta\tau_\varepsilon^2+4\tau_u\tau_\theta^2\tau_\varepsilon^2\\+4\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta^2\tau_\delta\end{pmatrix}\begin{pmatrix}\gamma^2\tau_\delta\tau_\varepsilon^2+8\tau_u\tau_\theta^2\tau_\varepsilon^2\\+6\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+8\gamma^2\tau_\theta^2\tau_\delta\end{pmatrix},\\ 16\gamma^2\tau_u\tau_\theta^3\tau_\varepsilon^3\left(\tau_\varepsilon+2\tau_\theta\right)\le 5\left(\gamma^2\tau_\delta\tau_\varepsilon^2+4\tau_u\tau_\theta^2\tau_\varepsilon^2+4\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta^2\tau_\delta\right)^2;\end{cases}$$

or

$$(b)\begin{cases} 16\gamma^2\tau_u\tau_\theta^3\tau_\varepsilon^3\left(3\tau_\varepsilon+8\tau_\theta\right) \\ <5\begin{pmatrix} \gamma^2\tau_\delta\tau_\varepsilon^2+4\tau_u\tau_\theta^2\tau_\varepsilon^2 \\ +4\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta^2\tau_\delta \end{pmatrix}\begin{pmatrix} \gamma^2\tau_\delta\tau_\varepsilon^2+16\tau_u\tau_\theta^2\tau_\varepsilon^2 \\ +8\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+16\gamma^2\tau_\theta^2\tau_\delta \end{pmatrix}, \\ 4\gamma^2\tau_u\tau_\theta^3\tau_\varepsilon^3\left(7\tau_\varepsilon+16\tau_\theta\right) \\ >5\begin{pmatrix} \gamma^2\tau_\delta\tau_\varepsilon^2+4\tau_u\tau_\theta^2\tau_\varepsilon^2 \\ +4\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta^2\tau_\delta \end{pmatrix}\begin{pmatrix} \gamma^2\tau_\delta\tau_\varepsilon^2+8\tau_u\tau_\theta^2\tau_\varepsilon^2 \\ +6\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+8\gamma^2\tau_\theta^2\tau_\delta \end{pmatrix}, \\ 2\gamma^2\tau_\theta\tau_\varepsilon\begin{pmatrix} 5\gamma^2\tau_\delta^2\tau_\varepsilon^3+2\tau_u\tau_\theta^3\tau_\varepsilon^3+20\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^3 \\ +30\gamma^2\tau_\theta\tau_\delta^2\tau_\varepsilon^2+40\tau_u\tau_\theta^3\tau_\delta\tau_\varepsilon^2+60\gamma^2\tau_\theta^2\tau_\delta^2\tau_\varepsilon+40\gamma^2\tau_\delta^2\tau_\theta^3 \end{pmatrix} \\ \leq 25\tau_\delta\left(\gamma^2\tau_\delta\tau_\varepsilon^2+4\tau_u\tau_\theta^2\tau_\varepsilon^2+4\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta^2\tau_\delta\right)^2. \end{cases}$$

Theorem 2. (Full Disclosure Equilibrium) *A full disclosure equilibrium ($\tau_\eta^*=\tau_\xi^*=\infty$) exists if and only if one of the following two sets of conditions holds:*

$$(a)\begin{cases} 5\left(\tau_u\tau_\varepsilon^2+\gamma^2\tau_\delta\right)^2\leq 2\gamma^2\tau_u\tau_\varepsilon^3, \\ 10\left(\tau_u\tau_\varepsilon^2+\gamma^2\tau_\delta\right)\left(4\tau_u\tau_\theta\tau_\varepsilon^2+\gamma^2\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta\tau_\delta\right)\leq\gamma^2\tau_u\tau_\varepsilon^3\left(\tau_\varepsilon+16\tau_\theta\right), \\ 5\left(\tau_u\tau_\varepsilon^2+\gamma^2\tau_\delta\right)\begin{pmatrix} \gamma^2\tau_\delta\tau_\varepsilon^2+16\tau_u\tau_\theta^2\tau_\varepsilon^2 \\ +8\gamma^2\tau_\theta\tau_\delta\tau_\varepsilon+16\gamma^2\tau_\theta^2\tau_\delta \end{pmatrix}\leq 4\gamma^2\tau_u\tau_\theta\tau_\varepsilon^3\left(\tau_\varepsilon+8\tau_\theta\right); \end{cases}$$

or

$$(b)\begin{cases} 5\left(\tau_u\tau_\varepsilon^2+\gamma^2\tau_\delta\right)^2<2\gamma^2\tau_u\tau_\varepsilon^3, \\ 10\left(\tau_u\tau_\varepsilon^2+\gamma^2\tau_\delta\right)\left(4\tau_u\tau_\theta\tau_\varepsilon^2+\gamma^2\tau_\delta\tau_\varepsilon+4\gamma^2\tau_\theta\tau_\delta\right)>\gamma^2\tau_u\tau_\varepsilon^3\left(\tau_\varepsilon+16\tau_\theta\right), \\ \gamma^2\tau_\varepsilon\left(\tau_u\tau_\varepsilon^3+20\tau_u\tau_\delta\tau_\varepsilon^2+20\gamma^2\tau_\delta^2\right)\leq 100\tau_\delta\left(\tau_u\tau_\varepsilon^2+\gamma^2\tau_\delta\right)^2. \end{cases}$$

Theorem 3. (Partial Disclosure Equilibrium) *A partial disclosure equilibrium $\left(\tau_\eta^*,\tau_\xi^*\right)\in\mathbb{R}_{++}^2$ (with $\tau_\xi^*=\tau_\eta^*$) is characterized by the following three conditions:*

(a) (FOC) $\tau_\eta^*>0$ is a solution to the fourth order polynomial,

$$F\left(\tau_\eta^*\right)\equiv F_4\tau_\eta^{*4}+F_3\tau_\eta^{*3}+F_2\tau_\eta^{*2}+F_1\tau_\eta^*+F_0=0;$$

(b) (SOC) τ_η^* satisfies the second-order condition,

$$S\left(\tau_\eta^*\right)\equiv S_6\tau_\eta^{*6}+S_5\tau_\eta^{*5}+S_4\tau_\eta^{*4}+S_3\tau_\eta^{*3}+S_2\tau_\eta^{*2}+S_1\tau_\eta^*+S_0\leq 0;$$

(c) (Global maximum) τ_η^* is a global maximum of $E\Pi_A\left(\tau_\eta,\tau_\xi^*\right)$, that is,

$$E\Pi_A\left(\tau_\eta^*,\tau_\xi^*\right)\geq E\Pi_A\left(\tau_\eta,\tau_\xi^*\right),\ \textit{for}\ \tau_\eta\in\left\{0,\infty,\hat{\tau}_\eta\right\},$$

where $\hat{\tau}_\eta$ is the positive roots of the fourth order polynomial:

$$G\left(\tau_\eta^*\right) \equiv G_4\hat{\tau}_\eta^4 + G_3\hat{\tau}_\eta^3 + G_2\hat{\tau}_\eta^2 + G_1\hat{\tau}_\eta + G_0 = 0.$$

The F-coefficients, S-coefficients, and G-coefficients are given in Online Appendix.

Theorems 1 and 2 respectively characterize the conditions that support the nondisclosure equilibrium and the full disclosure equilibrium. Theorem 3 characterizes a partial disclosure equilibrium in three conditions in the form of polynomials of the disclosure policy τ_η. The first two conditions respectively correspond to the first and second order conditions, while the last condition ensures that the optimal disclosure maximizes ex ante expected profits globally, rather than only locally.

Theorems 1–3 suggest the following five-step algorithm to compute all the linear symmetric equilibria:

Step 1: Employ Theorem 1 to check whether the nondisclosure equilibrium is supported.

Step 2: Employ Theorem 2 to check whether the full disclosure equilibrium is supported.

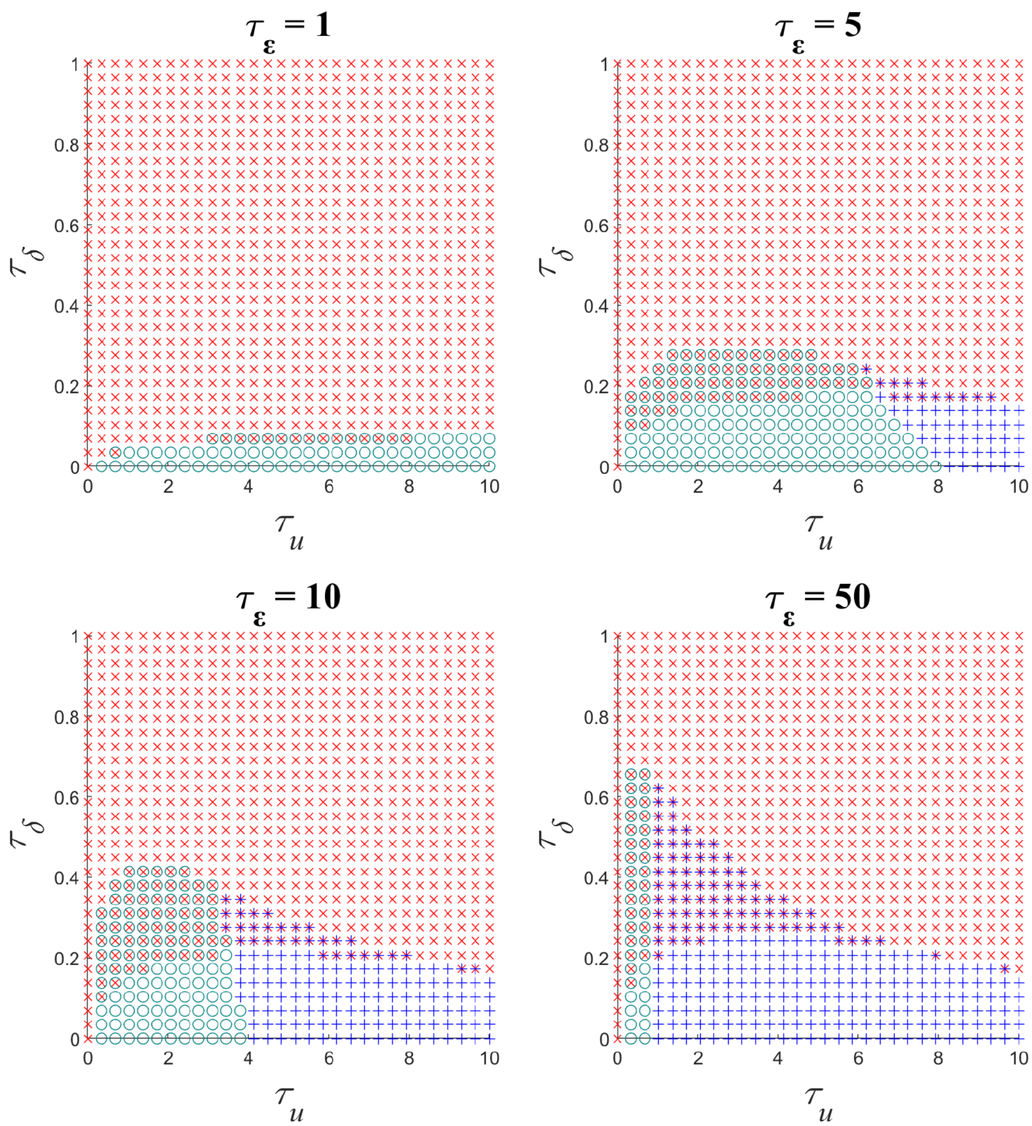
Step 3: Compute all the positive roots τ_η^* of the fourth order polynomial in Part (a) of Theorem 3 to serve as candidates of partial disclosure equilibria.

Step 4: For each root τ_η^* computed in Step 3, check whether the SOC in Part (b) of Theorem 3 is satisfied. Retain those roots that satisfy the SOC.

Step 5: For each value retained in Step 4, check whether the condition in Part (c) of Theorem 3 is satisfied. If yes, then it is a partial disclosure equilibrium; otherwise, it is not.

Figure 2 plots the regimes of equilibrium types in the parameter space of (τ_u, τ_δ) when $\tau_\theta = 1$, $\gamma = 10$, and $\tau_\varepsilon \in \{1, 5, 10, 50\}$. I use "x" to indicate the nondisclosure equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* = 0$), "o" to indicate the full disclosure equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* = \infty$), and "+" to indicate a partial disclosure equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* \in (0, \infty)$).

Figure 2: Parameter Space for Equilibrium Types



This figure plots the regions of equilibrium types in the parameter space of (τ_u, τ_δ). Parameter τ_u denotes the precision of noise trading in the financial market, and parameter τ_δ is the precision of financial specualtors' information. Parameter τ_ε is the precision of residual uncertainty in commodity demand. The other parameter values are: $\tau_\theta = 1$ and $\gamma = 10$. I use "x" to indicate the nondisclosure equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* = 0$), "o" to indicate the full disclosure equilbrium (i.e., $\tau_\eta^* = \tau_\xi^* = \infty$), and "+" to indicate a partial disclosure equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* \in (0, \infty)$).

Two observations emerge from Figure 2, both of which are unique to a setting with learning from asset prices. First, unlike a standard setting with demand uncertainty and Cournot competition in which nondisclosure forms a dominant strategy for firms (e.g., Gal-Or, 1985, 1986; Darrough, 1993; Vives, 1984, 2008), introducing learning from asset prices causes firms to disclose information in some cases and not to disclose in other cases. Firms are more likely to withhold information only when τ_δ or τ_u are sufficiently high. When τ_δ is high (i.e., $Var(\delta)$ is low), speculators know little new information so that the value of learning from asset prices is low and hence firms choose not to disclose because of the proprietary-cost concern as in the standard setting. When τ_u is high (i.e., $Var(u)$ is low), there is little noise trading in the financial market and thus, the market is already very efficient in communicating speculators' information to firms. Again, in this case, the value of learning from prices is low and the only equilibrium is the nondisclosure equilibrium.

Proposition 3. (Nondisclosure) *When τ_u or τ_δ is sufficiently high, the nondisclosure equilibrium prevails as the unique linear symmetric equilibrium.*

The second observation emerging from Figure 2 is that multiple equilibria can be supported. That is, when τ_u and τ_δ are relatively small, both the nondisclosure equilibrium and a full/partial disclosure equilibrium can be supported. This is also different from the standard setting where the nondisclosure equilibrium prevails as the unique equilibrium. The multiplicity of equilibrium is generated by the coordination motivates among firms, which are explored in detail in the next section.

4. Disclosure in a Noisy Financial Market

In this section, I illustrate two points. First, I examine the multiplicity result to understand its driving forces. Second, I show that firms can use disclosure as an effective device to shape the informativeness of financial markets and improve real decisions. I achieve my illustration in two steps. In Subsections 4.1 and 4.2, I remove residual uncertainty ε in the commodity demand and examine the limit of increasing noise trading in the financial market ($\tau_u \to 0$). These simplifications enhance tractability and make the analysis transparent. In Subsection 4.3, I conduct numerical analyses to examine the robustness/interpretation of results.

4.1. Complementarity and Multiplicity

By removing residual uncertainty in commodity demand, I can show that the full disclosure equilibrium is not supported. That is, $\tau_\eta^* = \tau_\xi^* < \infty$ when $\tau_\varepsilon = \infty$ (see Theorem 2). As a result, the possible equilibria are either nondisclosure or partial disclosure. The following theorem characterizes the equilibrium when there is a lot of noise trading in the financial market (i.e., τ_u is low and so $Var(u)$ is high).

Theorem 4. (Multiplicity) *In an economy without residual uncertainty, suppose there is a lot of noise trading in the financial market (i.e.,* $\tau_\varepsilon = \infty$ *and* τ_u *is sufficiently low). Then:*

(a) If $4\tau_\theta \geq 5\tau_\delta$*, there are two symmetric linear equilibria:*

$$\tau_\eta^* = \tau_\xi^* = 0 \text{ and } \tau_\eta^* = \tau_\xi^* = \frac{\gamma^2}{5\tau_u} + o(1),$$

where $o(1)$ *is a term that converges to zero as* $\tau_u \to 0$*.*

(b) If $4\tau_\theta < 5\tau_\delta$*, there exists a unique symmetric linear equilibrium, which is the nondisclosure equilibrium.*

Theorem 4 suggests that multiplicity arises in the limiting economy if and only if

$$4\tau_\theta \geq 5\tau_\delta \Longleftrightarrow \frac{Var(\delta)}{Var(\theta_A + \theta_B + \delta)} \geq 38.46\%.$$

That is, multiple equilibria are supported if and only if the financial market knows more than 38.46% of the total demand shock. This condition sounds likely to hold in reality, given that the market aggregates information from a large number of market participants (although many of them are noise traders).

On the qualitative side, Theorem 4 says that multiplicity is more likely to arise when speculators know more information that is useful to firms (i.e., $Var(\delta)$ is relatively large). This multiplicity is driven by a strategic complementarity in the disclosure decisions of firms. Specifically, recall that in the profit expression (15), the benefit of disclosing information comes from the fact that firms learn from the asset price. When there is a lot of noise trading in the market, the scope to improve price informativeness via disclosure is large; it is particularly helpful for both firms to disclose information to reduce the uncertainty faced by speculators, which in turn encourages speculators to trade more aggressively on their private information δ. When this complementarity is sufficiently strong, both disclosure and nondisclosure equilibria are supported.

Proposition 4. (Complementarity) *In an economy without residual uncertainty, when there is a lot of noise trading in the financial market, there is strategic complementarity in disclosure decisions. That is,* $\frac{\partial^2 E\Pi_A}{\partial \tau_\eta \partial \tau_\xi} > 0$ *and* $\frac{\partial^2 E\Pi_B}{\partial \tau_\eta \partial \tau_\xi} > 0$ *when* $\tau_\varepsilon = \infty$ *and* τ_u *is sufficiently low.*

Remark 2. (Complementarity and Multiplicity) *I make two remarks about the result on complementarity and multiplicity. First, Corollary 1 can be viewed as complementarity between firm disclosure and speculative trading: more disclosure encourages more informed trading. However, this firm-speculator complementarity alone does not lead to the multiplicity result in Theorem 4. In a variation setting in which* θ_B *is always set at its mean* 0 *so that the complementarity between firms is removed, there exists a unique disclosure equilibrium of firm A. Second, I have assumed that firms and speculators are endowed with information. If instead, speculators can determine which information—*θ_A, θ_B*, or* δ*—to acquire, the complementarity and multiplicity results can be strengthened. For instance, the following two types of equilibrium can be simultaneously supported. In one equilibrium, firms disclose* θ*-information and speculators acquire* δ*. This is because firms' disclosure weakens trading profits based on* θ *and at the same time speculators' acquisition about* δ *encourages firms to disclose. In another equilibrium, speculators acquire information* θ_A*, firm A does not disclose, and firm B discloses. Intuitively, when speculators acquire information* θ_A*,* *firm A has no benefit of learning from the asset price, and so it does not disclose; firm B may disclose to encourage speculators to trade more aggressively, making the price more informative about θ_A; speculators want to acquire θ_A not θ_B since firm B's disclosure lowers the trading profits on θ_B. The results will depend on the variance of information and the structure of information-acquisition cost.*

4.2. Shaping Price Informativeness by Coordinated Disclosure

When the size of noise trading is infinitely large, both firms choose not to disclose in equilibrium. That is, $\tau_\eta^* = \tau_\xi^* = 0$ when $\tau_u = 0$. This is because firms do not learn from asset prices when the financial market is populated with infinitely many liquidity traders (and thus the economy degenerates to the standard setting without learning from asset prices).

Now suppose that τ_u is small but positive. According to Part (a) of Theorem 4, a partial disclosure equilibrium is supported provided $4\tau_\theta \geq 5\tau_\delta$. In addition, as $\tau_u \to 0$, the optimal disclosure precision τ_η^* diverges to infinity on the partial disclosure equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* = \frac{\gamma^2}{5\tau_u} + o(1) \to \infty$ as $\tau_u \to 0$). In addition, this disclosure equilibrium is a preferred equilibrium from the perspective of firms: both firms are better off on the partial disclosure equilibrium than on the nondisclosure equilibrium. This is because on the disclosure equilibrium firms make more informed decisions after equipped with more public information (the additional public information disclosed by both firms and the more informative asset price). In this sense, the disclosure equilibrium is more likely to be selected by firms. Under this selection criterion, adding learning from prices dramatically changes the firms' disclosure behavior: without learning from prices, firms do not disclose information at all; in contrast, with learning from prices, firms may disclose their information almost perfectly.

On the disclosure equilibrium, firms choose to disclose more information when there is more noise trading (i.e., τ_η^* and τ_ξ^* increase as τ_u decreases). Hence, firms effectively coordinate to disclose information to offset the negative effect of added noise trading on price informativeness τ_s^*. Formally, by the expression of τ_s^* in equation (14), decreasing τ_u has two effects on τ_s^*. The direct effect is negative: other things being equal, more noise trading clouds the speculators' information in the order flow, which reduces price informativeness. The indirect effect is positive: more noise trading encourages more disclosure from firms, which in turn reduces the uncertainty faced by speculators and so they trade more aggressively on their private information δ, making the price more informative. Overall, the positive indirect effect dominates, so that a decrease in τ_u leads to an increase in price informativeness τ_s^*.

The improved price informativeness has real consequences on firms' production activities through firms' learning from the futures price. First, firms' production policies rely more on asset prices, i.e., both a_f and b_f increase with τ_s^*. Second, the products of both firms comove more strongly; that is, a decrease in τ_u raises $Cov\left(q_A^*, q_B^*\right)$. This is because the products of both firms are driven more by public information than by private information: as τ_u decreases, both firms release more public information (τ_η^* and τ_ξ^* increase), and at the same time, the futures price f becomes more informative. Third, the volatility of firms' products also increases, i.e., $Var\left(q_A^*\right)$ and $Var\left(q_B^*\right)$ increase as τ_u decreases. Intuitively, as firms learn more information from disclosure x and y and from the price f, they adjust their production better to accommodate the later commodity demand. This increased flexibility of production raises product volatility and also firms' profits (formally, $E\Pi_i^* = Var\left(q_i^*\right) + m^2/9$). Finally, a decrease in τ_u increases the volatility $Var\left(Q^*\right)$ of total product Q^*, since $Var\left(Q^*\right) = Var\left(q_A^*\right) + Var\left(q_B^*\right) + 2Cov\left(q_A^*, q_B^*\right)$, where all the three terms, $Var\left(q_A^*\right)$, $Var\left(q_B^*\right)$, and $Cov\left(q_A^*, q_B^*\right)$, increase as τ_u decreases.

Proposition 5. (Coordinated Disclosure, Price Informativeness, and Real Effects) *Suppose that there is no residual uncertainty (i.e., $\tau_\varepsilon = \infty$). Then:*

(a) When $\tau_u = 0$, the unique symmetric linear equilibrium is the nondisclosure equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* = 0$). When $\tau_u \to 0$ and when $4\tau_\theta \geq 5\tau_\delta$, there are two

symmetric linear equilibria: $\tau_\eta^* = \tau_\xi^* = 0$ *and* $\tau_\eta^* = \tau_\xi^* = \frac{\gamma^2}{5\tau_u} + o(1)$*; and firms are better off on the partial disclosure equilibrium than on the nondisclosure equilibrium.*
(b) Suppose $4\tau_\theta \geq 5\tau_\delta$*. On the partial disclosure equilibrium, as* τ_u *decreases toward* 0*, all of the following variables increase: disclosure precision levels* τ_η^* *and* τ_ξ^**, the informativeness of futures price* τ_s^**, investment-price sensitivities* a_f *and* b_f*, product variances and covariance* $Var(q_A^*)$*,* $Var(q_B^*)$*,* $Var(Q^*)$*, and* $Cov(q_A^*, q_B^*)$*.*

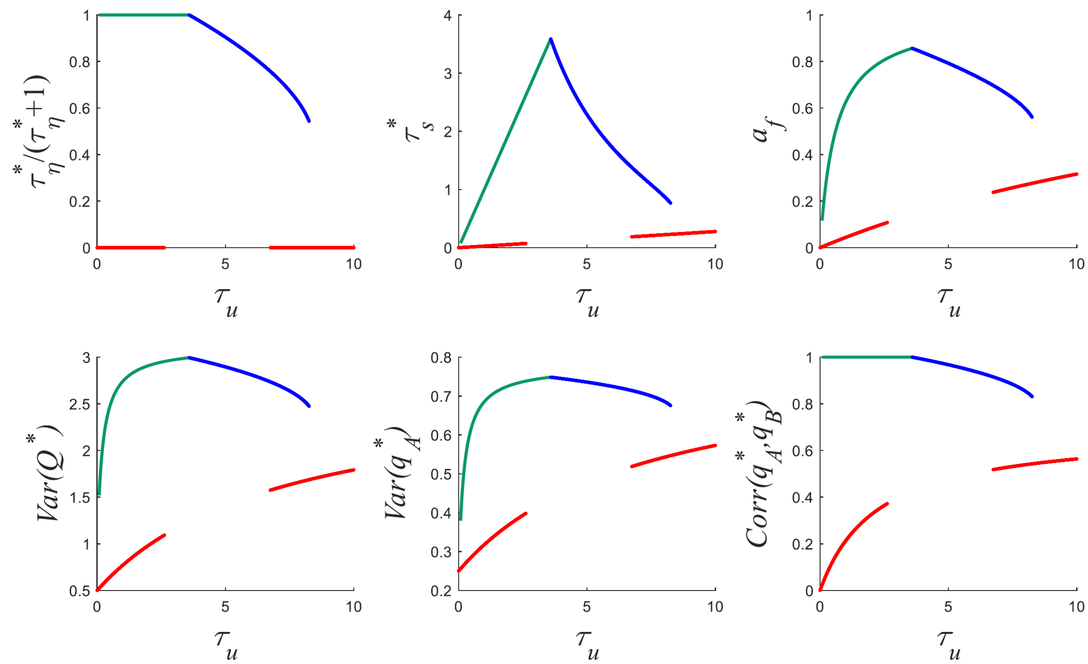
4.3. Numerical Analysis with Residual Uncertainty

I now add back residual uncertainty ε to the commodity demand to examine the robustness/interpretation of Proposition 5. The complexity of the setting with residual uncertainty precludes an analytical characterization, and so I instead rely on numerical analysis. In Figure 3, I plot disclosure policy, price informativeness, and product features for the parameter configuration $\tau_\theta = 1$, $\tau_\delta = 0.2$, $\tau_\varepsilon = 10$, and $\gamma = 10$. The general patterns are robust to parameter choices.

To facilitate the drawing of the full disclosure equilibrium, the first panel depicts a monotonic transformation $\frac{\tau_\eta^*}{\tau_\eta^*+1}$ of disclosure precision τ_η^* against noise trading precision τ_u. The variable $\frac{\tau_\eta^*}{\tau_\eta^*+1}$ takes values on $[0,1]$: $\frac{\tau_\eta^*}{\tau_\eta^*+1} = 1$ on the full disclosure equilibrium; $\frac{\tau_\eta^*}{\tau_\eta^*+1} = 0$ on the nondisclosure equilibrium; and $\frac{\tau_\eta^*}{\tau_\eta^*+1} \in (0,1)$ on a partial disclosure equilibrium. At $\tau_u = 0$, there is a unique equilibrium, which is the nondisclosure equilibrium $\tau_\eta^* = 0$. When τ_u is small, there are two equilibria: one is the nondisclosure equilibrium $\tau_\eta^* = 0$, and the other is the full disclosure equilibrium $\tau_\eta^* = \infty$. This is broadly consistent with Part (a) of Proposition 5.

Unlike Part (b) of Proposition 5, some variables, such as price informativeness τ_s^* and investment-price sensitivity a_f, exhibit non-monotone relation with τ_u. For instance, as τ_u decreases from 8 toward 0, τ_s^* first increases and then decreases on the partial/full disclosure equilibrium. This is driven by the switch between disclosure equilibria. Specifically, as τ_u starts to decrease from 8, the disclosure equilibrium is a partial disclosure equilibrium, and the disclosure precision τ_η^* increases as τ_u

Figure 3: Disclosure, Multiplicity, and Real Effects



This figure plots the disclosure policies (τ^*_η), price informativeness (τ^*_s), investment-price sensitivity (a_f), and variances and covariance of firms' product quantities ($Var\,(Q^*)\,, Var\,(q^*_A)\,, Corr\,(q^*_A, q^*_B)$) against the precision τ_u of noise trading in the financial market. The nondisclosure equilibrium is plotted in red, the full disclosure is plotted in green, and the partial disclosure equilbrium is plotted in in blue. The other parameters are: $\tau_\theta = 1, \tau_\delta = 0.2, \tau_\varepsilon = 10$, and $\gamma = 10$.

decreases. On this regime, as what Part (b) of Proposition 5 predicts, τ^*_s increases as well because of the increase in τ^*_η in response to the decrease in τ_u. However, as τ_u continues to decrease, the disclosure equilibrium switches to the full disclosure equilibrium $\tau^*_\eta = \infty$, and as a result, τ^*_η no longer increases as τ_u decreases. Now, on this regime, the indirect positive effect on τ^*_s vanishes and thus, τ^*_s has to decrease with more noise trading. The non-monotone patterns for other variables a_f, $Var\,(q^*_A)$, and $Var\,(Q^*)$ can be explained in a similar manner.

Nonetheless, if one focuses only on the partial disclosure equilibrium, Part (b) of Proposition 5 continues to hold. That is, on the partial disclosure equilibrium, all of the six variables—τ^*_η, τ^*_s, a_f, $Var\,(q^*_A)$, $Var\,(Q^*)$, and $Cov\,(q^*_A, q^*_B)$—increase, as

τ_u decreases. This makes sense, since only on the partial disclosure equilibrium, can firms have the flexibility to disclose more information in response to an increase in noise trading, which in turn makes the indirect effect active.

Remark 3. (IPO Waves) *The literature on initial public offerings (IPOs) has identified a hot-issue market phenomenon characterized by the clustering of IPOs in some periods and industries (see Derrien (2010) for a survey on the IPO literature). A popular explanation is a sentiment-based behavioral theory: when the market is too optimistic about an industry, companies in this industry take advantage of this mispricing by selling overvalued stocks to the market. The first panel of Figure 3 suggests an alternative theory connecting sentiment with IPO waves to the extent that noise trading is partially driven by sentiment. Intuitively, firms face the following trade-off in making the decision to go public: on the one hand, going public offers an additional signal, the price on the firm share, which is useful for real investment decisions; on the other hand, going public is associated with more disclosure requirement, which can result in releasing confidential information to competitors.*[9] *This trade-off is the same as the main model in Section 2 and thus, one can associate IPO with the partial/full disclosure equilibrium. As the first panel of Figure 3 suggests, when the financial market becomes more sentiment-driven (i.e., τ_u becomes smaller), both firms are more likely to go public (i.e., $\tau_\eta^* > 0$).*

5. Conclusion

I study the classic information-sharing problem in a duopoly setting augmented with a financial market. Disclosure improves price informativeness via reducing the uncertainty faced by financial speculators. When making disclosure decisions, firms face a trade-off between incurring the proprietary cost and improving learning quality from

[9] In Bhattacharya and Ritter (1983) and Maksimovic and Pichler (2001), the costs of going public also comes from releasing confidential information to competitors at the time of IPO, but the benefit arises from raising capital at a cheaper rate in the public equity markets.

asset prices. In equilibrium, firms may optimally choose to disclose information in a setting with learning from asset prices, which differs from the standard setting where firms always withhold information. In addition, firms' disclosure decisions can be a strategic complement. When this complementarity is sufficiently strong, both a disclosure equilibrium and a nondisclosure equilibrium are supported. Overall, my analysis highlights the importance of incorporating the feature of learning from asset prices in understanding firms' disclosure behavior.

Appendix: Proofs

Proof of Proposition 2

After expressing q_A^* in (9) as functions of (θ_A, x, y, f) and comparing with the conjectured policy in (5), I obtain the following five conditions in terms of the unknown a coefficients and b coefficients:

$$\begin{aligned}
2a_0 &= m - \frac{\tau_s}{\tau_\delta + \tau_s}(m - a_0 - b_0) - b_0, \\
2a_\theta &= 1, \\
2a_x &= -\frac{\tau_s}{\tau_\delta + \tau_s}\left[\frac{(1-a_\theta)\tau_\eta}{\tau_\theta + \tau_\eta} - (a_x + b_x)\right] - b_x, \\
2a_y &= -\frac{\tau_s}{\tau_\delta + \tau_s}\left[\frac{(1-b_\theta)\tau_\xi}{\tau_\theta + \tau_\xi} - (a_y + b_y)\right] + \frac{(1-b_\theta)\tau_\xi}{\tau_\theta + \tau_\xi} - b_y, \\
2a_f &= \frac{\tau_s}{\tau_\delta + \tau_s}(a_f + b_f + 1) - b_f.
\end{aligned}$$

Conducting a similar analysis for firm B leads to the following additional five equations:

$$\begin{aligned}
2b_0 &= m - \frac{\tau_s}{\tau_\delta + \tau_s}(m - a_0 - b_0) - a_0, \\
2b_\theta &= 1, \\
2b_x &= -\frac{\tau_s}{\tau_\delta + \tau_s}\left[\frac{(1-a_\theta)\tau_\eta}{\tau_\theta + \tau_\eta} - (a_x + b_x)\right] + \frac{(1-a_\theta)\tau_\eta}{\tau_\theta + \tau_\eta} - a_x, \\
2b_y &= -\frac{\tau_s}{\tau_\delta + \tau_s}\left[\frac{(1-b_\theta)\tau_\xi}{\tau_\theta + \tau_\xi} - (a_y + b_y)\right] - a_y, \\
2b_f &= \frac{\tau_s}{\tau_\delta + \tau_s}(a_f + b_f + 1) - a_f.
\end{aligned}$$

Solving the above system yields the expressions of a's and b's in Proposition 2. The expressions of τ_s, p, and f in Proposition 2 are obtained by plugging a's and b's respectively into equations (14), (11), and (12).

Proof of Corollary 1

By the expression of τ_s in Proposition 2, we can directly compute the partial derivatives and show that $\frac{\partial \tau_s}{\partial \tau_\eta} > 0$ and $\frac{\partial \tau_s}{\partial \tau_\xi} > 0$.

Proof of Theorem 1

Nondisclosure is an equilibrium if and only if $\tau_\eta^* = 0$ is the best response to $\tau_\xi^* = 0$, i.e., if and only if

$$E\Pi_A(0,0) \geq \max_{\tau_\eta} E\Pi_A(\tau_\eta, 0).$$

By the expression of τ_s in Proposition 2 and the expression of expected profit $E\Pi_A(\tau_\eta, \tau_\xi)$ in (15), direct computations show that

$$E\Pi_A(0,0) - E\Pi_A(\tau_\eta, 0) \geq 0 \Longleftrightarrow H(\tau_\eta) \equiv H_2\tau_\eta^2 + H_1\tau_\eta + H_0 \leq 0,$$

where

$$\begin{aligned} H_2 = {} & \left[48\gamma^2\tau_u\tau_\theta^3 - 5\left(\gamma^2\tau_\delta + 4\tau_u\tau_\theta^2\right)\left(\gamma^2\tau_\delta + 16\tau_u\tau_\theta^2\right)\right]\tau_\varepsilon^4 \\ & -4\gamma^2\tau_\theta\left(15\gamma^2\tau_\delta^2 - 32\tau_u\tau_\theta^3 + 120\tau_u\tau_\theta^2\tau_\delta\right)\tau_\varepsilon^3 - 20\gamma^2\tau_\theta^2\tau_\delta\left(13\gamma^2\tau_\delta + 32\tau_u\tau_\theta^2\right)\tau_\varepsilon^2 \\ & -480\gamma^4\tau_\theta^3\tau_\delta^2\tau_\varepsilon - 320\gamma^4\tau_\theta^4\tau_\delta^2, \end{aligned}$$

$$H_1 = 4\tau_\theta \left[\begin{array}{c} \left[28\gamma^2\tau_u\tau_\theta^3 - 5\left(\gamma^2\tau_\delta + 4\tau_u\tau_\theta^2\right)\left(\gamma^2\tau_\delta + 8\tau_u\tau_\theta^2\right)\right]\tau_\varepsilon^4 \\ -2\gamma^2\tau_\theta\left(25\gamma^2\tau_\delta^2 - 32\tau_u\tau_\theta^3 + 140\tau_u\tau_\theta^2\tau_\delta\right)\tau_\varepsilon^3 \\ -20\gamma^2\tau_\theta^2\tau_\delta\left(9\gamma^2\tau_\delta + 16\tau_u\tau_\theta^2\right)\tau_\varepsilon^2 - 280\gamma^4\tau_\theta^3\tau_\delta^2\tau_\varepsilon - 160\gamma^4\tau_\theta^4\tau_\delta^2 \end{array}\right], \text{ and}$$

$$H_0 = 4\tau_\theta^2 \left[\begin{array}{c} \left(16\gamma^2\tau_u\tau_\theta^3 - 5\left(\gamma^2\tau_\delta + 4\tau_u\tau_\theta^2\right)^2\right)\tau_\varepsilon^4 - 8\gamma^2\tau_\theta\left(5\gamma^2\tau_\delta^2 - 4\tau_u\tau_\theta^3 + 20\tau_u\tau_\theta^2\tau_\delta\right)\tau_\varepsilon^3 \\ -40\gamma^2\tau_\theta^2\tau_\delta\left(3\gamma^2\tau_\delta + 4\tau_u\tau_\theta^2\right)\tau_\varepsilon^2 - 160\gamma^4\tau_\theta^3\tau_\delta^2\tau_\varepsilon - 80\gamma^4\tau_\theta^4\tau_\delta^2 \end{array}\right].$$

Thus, nondisclosure is an equilibrium if and only if

$$H(\tau_\eta) \leq 0, \forall \tau_\eta \geq 0. \tag{A1}$$

Clearly, a necessary condition for (A1) to hold is $H_0 \leq 0$. Now suppose $H_0 \leq 0$ and discuss the possible values of H_2 and H_1 to check when condition (A1) holds.

If $H_2 > 0$, then $H(\tau_\eta) > 0$ for sufficiently large τ_η, so that condition (A1) is violated. If $H_2 = 0$, then $H(\tau_\eta)$ becomes linear, and condition (A1) holds if and only if $H_1 \leq 0$.

Suppose $H_2 < 0$. If in addition, $H_1 \leq 0$, then the range of $\tau_\eta > 0$ lies on the right branch of $H(\tau_\eta)$ and thus condition (A1) holds. If $H_1 > 0$, then condition (A1) holds if and only if the discriminant of $H(\tau_\eta)$ is nonpositive (i.e., if and only if $H_1^2 - 4H_2H_0 \leq 0$).

To summarize, (A1) holds if and only if one of the following two sets of conditions holds:

$$\{H_2 \leq 0, H_1 \leq 0, H_0 \leq 0\} \text{ or } \left\{H_2 < 0, H_1 > 0, H_1^2 - 4H_2H_0 \leq 0\right\},$$

which are respectively the two sets of conditions in Theorem 1.

Proof of Theorem 2

The proof of Theorem 2 follows the same logic as the proof of Theorem 1. A full disclosure equilibrium exists if and only if

$$E\Pi_A(\infty,\infty) \geq \max_{\tau_\eta} E\Pi_A(\tau_\eta,\infty).$$

By the expression of τ_s in Proposition 2 and the expression of expected profit $E\Pi_A(\tau_\eta,\tau_\xi)$ in (15), we can compute

$$E\Pi_A(\infty,\infty) - E\Pi_A(\tau_\eta,\infty) \geq 0 \Longleftrightarrow K(\tau_\eta) \equiv K_2\tau_\eta^2 + K_1\tau_\eta + K_0 \leq 0,$$

where $K_2 = 16\left(5\gamma^4\tau_\delta^2 + 5\tau_u^2\tau_\varepsilon^4 - 2\gamma^2\tau_u\tau_\varepsilon^3 + 10\gamma^2\tau_u\tau_\delta\tau_\varepsilon^2\right)$,

$K_1 = 40\gamma^4\tau_\delta^2\tau_\varepsilon + 160\tau_\theta\gamma^4\tau_\delta^2 + 40\gamma^2\tau_u\tau_\delta\tau_\varepsilon^3 + 320\tau_\theta\gamma^2\tau_u\tau_\delta\tau_\varepsilon^2 - 4\gamma^2\tau_u\tau_\varepsilon^4 - 64\tau_\theta\gamma^2\tau_u\tau_\varepsilon^3 + 160\tau_\theta\tau_u^2\tau_\varepsilon^4$,

$K_0 = 80\gamma^4\tau_\theta^2\tau_\delta^2 + 5\gamma^4\tau_\delta^2\tau_\varepsilon^2 + 80\tau_u^2\tau_\theta^2\tau_\varepsilon^4 - 32\gamma^2\tau_u\tau_\theta^2\tau_\varepsilon^3 - 4\gamma^2\tau_u\tau_\theta\tau_\varepsilon^4 + 5\gamma^2\tau_u\tau_\delta\tau_\varepsilon^4 + 40\gamma^4\tau_\theta\tau_\delta^2\tau_\varepsilon + 160\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^2 + 40\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^3$.

Thus, full disclosure is an equilibrium if and only if

$$K(\tau_\eta) \leq 0, \forall \tau_\eta \geq 0. \tag{A2}$$

Then, following the same logic as the proof of Theorem 1, (A2) holds if and only if one of the following two sets of conditions holds:

$$\{K_2 \leq 0, K_1 \leq 0, K_0 \leq 0\} \text{ or } \left\{K_2 < 0, K_1 > 0, K_1^2 - 4K_2K_0 \leq 0\right\},$$

which are respectively the two sets of conditions in Theorem 2.

Proof of Theorem 3

A symmetric disclosure equilibrium requires that $\tau_\eta^* > 0$ is the best response to $\tau_\xi^* = \tau_\eta^* > 0$. That is,

$$\tau_\eta^* = \arg\max_{\tau_\eta} E\Pi_A\left(\tau_\eta, \tau_\eta^*\right).$$

I characterize the value of τ_η^* in three steps. First, I use the FOC to find the candidates for τ_η^*. Second, I use the SOC to ensure that τ_η^* is a local maximum of $E\Pi_A\left(\tau_\eta, \tau_\eta^*\right)$.

Third, I compare $E\Pi_A\left(\tau_\eta^*, \tau_\eta^*\right)$ with the other extreme values of $E\Pi_A\left(\tau_\eta, \tau_\eta^*\right)$ to ensure that τ_η^* is a global maximum of $E\Pi_A\left(\tau_\eta, \tau_\eta^*\right)$.

For the FOC, direct computations show

$$\left.\frac{\partial E\Pi_A\left(\tau_\eta, \tau_\xi\right)}{\partial \tau_\eta}\right|_{\tau_\xi=\tau_\eta} = 0 \Longleftrightarrow F\left(\tau_\eta\right) \equiv F_4\tau_\eta^4 + F_3\tau_\eta^3 + F_2\tau_\eta^2 + F_1\tau_\eta + F_0 = 0,$$

where the expressions of the F-coefficients are provided in the Online Appendix. Any candidate disclosure policy $\tau_\eta^* > 0$ must satisfy $F\left(\tau_\eta^*\right) = 0$.

For the SOC, direct computations show

$$\left.\frac{\partial^2 E\Pi_A\left(\tau_\eta, \tau_\xi\right)}{\partial \tau_\eta^2}\right|_{\tau_\xi=\tau_\eta} \leq 0 \Longleftrightarrow S\left(\tau_\eta\right) \equiv S_6\tau_\eta^6 + S_5\tau_\eta^5 + S_4\tau_\eta^4 + S_3\tau_\eta^3 + S_2\tau_\eta^2 + S_1\tau_\eta + S_0 \leq 0,$$

where the S-coefficients are given in the Online Appendix. Any candidate disclosure policy $\tau_\eta^* > 0$ must satisfy $S\left(\tau_\eta^*\right) \leq 0$.

Finally, fixing $\tau_\xi = \tau_\eta^*$, I can find the interior extreme values of $E\Pi_A\left(\tau_\eta, \tau_\xi\right)$ by setting its FOC at zero, that is,

$$\frac{\partial E\Pi_A\left(\tau_\eta, \tau_\xi\right)}{\partial \tau_\eta} = 0 \Longleftrightarrow G\left(\tau_\eta\right) \equiv G_4\tau_\eta^4 + G_3\tau_\eta^3 + G_2\tau_\eta^2 + G_1\tau_\eta + G_0 = 0,$$

where the G-coefficients are given in the Online Appendix. The extreme values of $E\Pi_A\left(\tau_\eta, \tau_\xi\right)$ include (1) the positive roots of $G\left(\tau_\eta\right) = 0$; and (2) the two boundaries $\tau_\eta = 0$ and $\tau_\eta = \infty$.

Proof of Proposition 3

Fix the other parameters and let $\tau_u \to \infty$. Condition (a) in Theorem 1 is satisfied and thus nondisclosure is an equilibrium.

For sufficiently large values of τ_u, we have $5\left(\tau_u\tau_\varepsilon^2 + \gamma^2\tau_\delta\right)^2 > 2\gamma^2\tau_u\tau_\varepsilon^3$. Thus, both condition (a) and condition (b) in Theorem 2 is violated. In consequence, full disclosure is not an equilibrium.

Condition (a) in Theorem 3 is violated, because all the F coefficients are negative for sufficiently large values of τ_u, which implies $F\left(\tau_\eta\right) < 0$ for all $\tau_\eta > 0$. So, there are no disclosure equilibria.

The proof for large values of τ_δ is identical to the proof for large values of τ_u and thus is omitted.

Proof of Theorem 4

To remove residual uncertainty, we let $\tau_\varepsilon \to \infty$ for a fixed $(\gamma, \tau_\theta, \tau_\delta, \tau_u)$. As $\tau_\varepsilon \to \infty$, we have $5\left(\tau_u \tau_\varepsilon^2 + \gamma^2 \tau_\delta\right)^2 > 2\gamma^2 \tau_u \tau_\varepsilon^3$; by Theorem 2, the full disclosure equilibrium is not supported. We then consider the process of $\tau_u \to 0$. Condition (a) of Theorem 1 is satisfied for small values of τ_u and thus the nondisclosure equilibrium is supported.

The key is to characterize the partial disclosure equilibrium. I conduct this characterization in four steps. First, I use the FOC in Part (a) of Theorem 3 to compute all the candidates for a partial disclosure equilibrium. It turns out that there are two possible values of disclosure policy τ_η^*, which I label as τ_η^{large} and τ_η^{small}, respectively. Second, I employ the SOC in Part (b) of Theorem 3 to rule out candidate τ_η^{small} and retain the other candidate τ_η^{large}. Third, I compare $E\Pi_A\left(0, \tau_\eta^{large}\right)$ with $E\Pi_A\left(\tau_\eta^{large}, \tau_\eta^{large}\right)$ to show that under condition $4\tau_\theta < 5\tau_\delta$, the unique equilibrium is the nondisclosure equilibrium (i.e., Part (b) of Theorem 4). Lastly, I show that if $4\tau_\theta \geq 5\tau_\delta$, then $\tau_\eta^* = \tau_\xi^* = \tau_\eta^{large}$ is supported as a partial disclosure equilibrium (i.e., Part (a) of Theorem 4).

Compute disclosure equilibrium candidates

A partial disclosure equilibrium requires $F(\tau_\eta^*) = 0$ in Part (a) of Theorem 3. I can rewrite this equation as follows:

$$-80\left(\tau_\theta + \tau_\eta^*\right)^4 \tau_u^2 + 8\gamma^2 \left(\tau_\theta + \tau_\eta\right)^2 \left(2\tau_\theta - 5\tau_\delta + 2\tau_\eta^*\right)\tau_u = 5\gamma^4 \tau_\delta^2. \tag{A3}$$

Now consider the process of $\tau_u \to 0$ and examine the order of τ_η^*. Clearly, τ_η^* must diverge to ∞ as $\tau_u \to 0$, because if τ_η^* converges to a finite value, then the left-hand-side (LHS) of equation (A3) converges to 0, which cannot maintain equation (A3).

The highest order of the LHS of equation (A3) is $-80\tau_\eta^{*4}\tau_u^2 + 16\gamma^2\tau_\eta^{*3}\tau_u$. Thus, by equation (A3),

$$-80\tau_\eta^{*4}\tau_u^2 + 16\gamma^2\tau_\eta^{*3}\tau_u \propto 5\gamma^4\tau_\delta^2, \tag{A4}$$

where $\propto$ means that the LHS has the same order as the right-hand-side (RHS). Equation (A4) determines the order of τ_η^*.

Given that the RHS of (A4) is positive and that only the term $16\gamma^2\tau_\eta^{*3}\tau_u$ in the LHS of (A4) is positive, there are two possibilities. First, $-80\tau_\eta^{*4}\tau_u^2$ has a lower order than $16\gamma^2\tau_\eta^{*3}\tau_u$, i.e., $-80\tau_\eta^{*4}\tau_u^2 = o\left(16\gamma^2\tau_\eta^{*3}\tau_u\right)$, where the notation $X_2 = o\left(X_1\right)$ means $\lim_{\tau_u\to 0}\frac{X_2}{X_1} = 0$. Second, $-80\tau_\eta^{*4}\tau_u^2$ has the same order as $16\gamma^2\tau_\eta^{*3}\tau_u$, i.e., $-80\tau_\eta^{*4}\tau_u^2 = O\left(16\gamma^2\tau_\eta^{*3}\tau_u\right)$, where the notation $X_2 = O\left(X_1\right)$ means $\frac{X_2}{X_1}$ converges to a finite constant as $\tau_u \to 0$.

Case 1. $-80\tau_\eta^{*4}\tau_u^2 = o\left(16\gamma^2\tau_\eta^{*3}\tau_u\right)$

By equation (A4),

$$16\gamma^2\tau_\eta^{*3}\tau_u = 5\gamma^4\tau_\delta^2 + o\left(1\right) \Rightarrow \tau_\eta^* = \sqrt[3]{\frac{5\gamma^2\tau_\delta^2}{16}\frac{1}{\tau_u}} + o\left(\sqrt[3]{\frac{1}{\tau_u}}\right).$$

I denote this candidate disclosure policy as τ_η^{small}.

Case 2. $-80\tau_\eta^{*4}\tau_u^2 = O\left(16\gamma^2\tau_\eta^{*3}\tau_u\right)$

In this case, τ_η^* diverges at the order of $\frac{1}{\tau_u}$, that is, $\tau_u\tau_\eta^*$ converges to a finite value as $\tau_u \to 0$. By equation (A4),

$$\begin{aligned} -80\tau_\eta^{*4}\tau_u^2 + 16\gamma^2\tau_\eta^{*3}\tau_u &= 5\gamma^4\tau_\delta^2 = O\left(1\right) \Rightarrow \\ 16\tau_u\tau_\eta^*\left(\gamma^2 - 5\tau_u\tau_\eta^*\right)\tau_\eta^{*2} &= O\left(1\right). \end{aligned}$$

Note that $16\tau_u\tau_\eta^* = O\left(1\right)$ and $\tau_\eta^{*2} = O\left(\frac{1}{\tau_u^2}\right)$, and thus

$$\gamma^2 - 5\tau_u\tau_\eta^* = O\left(\frac{1}{\tau_\eta^{*2}}\right) \Rightarrow 5\tau_u\tau_\eta^* = \gamma^2 + O\left(\frac{1}{\tau_\eta^{*2}}\right) \Rightarrow \tau_\eta^* = \frac{\gamma^2}{5\tau_u} + O\left(\tau_u\right).$$

Hence, the other candidate is:

$$\tau_\eta^* = \frac{\gamma^2}{5\tau_u} + o\left(1\right),$$

which is labeled as τ_η^{large}, where the superscript "large" follows from $\frac{\gamma^2}{5\tau_u} > \sqrt[3]{\frac{5\gamma^2\tau_\delta^2}{16}\frac{1}{\tau_u}}$ for small values of τ_u.

Check the SOC

Inserting the candidate disclosure policy $\tau_\eta^{small} = \sqrt[3]{\frac{5\gamma^2\tau_\delta^2}{16}\frac{1}{\tau_u}} + o\left(\sqrt[3]{\frac{1}{\tau_u}}\right)$ into the SOC in Part (b) of Theorem 3 and keeping the highest order, I compute $S\left(\tau_\eta^{small}\right) \propto \frac{15}{4}\gamma^6\tau_\delta^3 > 0$. That is, the SOC is violated and thus τ_η^{small} cannot be supported as a partial disclosure equilibrium.

Similarly, for the other candidate policy $\tau_\eta^{large} = \frac{\gamma^2}{5\tau_u} + o\left(1\right)$, I can compute

$S\left(\tau_\eta^{large}\right) \propto -\frac{16}{3125}\frac{\gamma^{12}}{\tau_u^3} < 0$, which means that τ_η^{large} is a local maximum for function $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$.

In sum, the value of τ_η^{large} serves as the only candidate for a partial disclosure equilibrium.

Compare $E\Pi_A\left(\tau_\eta^{large}, \tau_\eta^{large}\right)$ ***with*** $E\Pi_A\left(0, \tau_\eta^{large}\right)$ ***(Proof of Part (b))***

By the profit expression (15) and using $\tau_\eta^{large} = \frac{\gamma^2}{5\tau_u} + o(1)$, I can show:

$$E\Pi_A\left(\tau_\eta^{large}, \tau_\eta^{large}\right) < E\Pi_A\left(0, \tau_\eta^{large}\right) \Longleftrightarrow$$

$$\left(-200\,000\tau_\theta^6\right)\tau_u^5 - 20\,000\gamma^2\tau_\theta^4\left(6\tau_\theta + 5\tau_\delta\right)\tau_u^4 - 500\gamma^4\tau_\theta^2\left(44\tau_\theta^2 + 25\tau_\delta^2 + 100\tau_\theta\tau_\delta\right)\tau_u^3$$
$$-100\gamma^6\tau_\theta\left(4\tau_\theta^2 + 25\tau_\delta^2 + 85\tau_\theta\tau_\delta\right)\tau_u^2 + 5\gamma^8\left(48\tau_\theta^2 - 25\tau_\delta^2 - 120\tau_\theta\tau_\delta\right)\tau_u + 4\gamma^{10}\left(4\tau_\theta - 5\tau_\delta\right) < 0.$$

For sufficiently small τ_u,

$$E\Pi_A\left(\tau_\eta^{large}, \tau_\eta^{large}\right) < E\Pi_A\left(0, \tau_\eta^{large}\right) \Longleftrightarrow 4\tau_\theta < 5\tau_\delta.$$

Thus, if $4\tau_\theta < 5\tau_\delta$, τ_η^{large} does not form a global maximum for function $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$, and hence τ_η^{large} cannot be supported as a partial disclosure equilibrium. Given that τ_η^{large} is the only partial disclosure equilibrium candidate, there is no partial disclosure equilibrium when $4\tau_\theta < 5\tau_\delta$ and τ_u is sufficiently small.

Proof of Part (a)

Now suppose $4\tau_\theta \geq 5\tau_\delta$, so that $E\Pi_A\left(\tau_\eta^{large}, \tau_\eta^{large}\right) > E\Pi_A\left(0, \tau_\eta^{large}\right)$ for sufficiently small τ_u. I then examine the shape of $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$ and show that τ_η^{large} forms a global maximum of $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$. Using Part (c) of Theorem 3 and the expression of $\tau_\eta^{large} = \frac{\gamma^2}{5\tau_u} + o(1)$, I can show that the FOC of $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$ has the same sign as

$$A(\tau_\eta) = A_4\tau_\eta^4 + A_3\tau_\eta^3 + A_2\tau_\eta^2 + A_1\tau_\eta + A_0,$$

where

$$\begin{aligned}
A_4 &= -1280\tau_u^2, A_3 = 128\tau_u\left(\gamma^2 - 40\tau_u\tau_\theta\right), \\
A_2 &= 32\tau_u\left(12\gamma^2\tau_\theta - 5\gamma^2\tau_\delta - 240\tau_u\tau_\theta^2\right), \\
A_1 &= 64\tau_u\tau_\theta\left(6\gamma^2\tau_\theta - 5\gamma^2\tau_\delta - 80\tau_u\tau_\theta^2\right), \\
A_0 &= -\left(5\gamma^4\tau_\delta^2 + 1280\tau_u^2\tau_\theta^4 - 128\gamma^2\tau_u\tau_\theta^3 + 160\gamma^2\tau_u\tau_\theta^2\tau_\delta\right).
\end{aligned}$$

Thus, for sufficiently small τ_u, if $4\tau_\theta \geq 5\tau_\delta$, then $A_4 < 0, A_3 > 0, A_2 > 0, A_1 > 0$, and $A_0 < 0$.

Taking derivative of $A(\tau_\eta)$ yields:

$$A'(\tau_\eta) = 4A_4\tau_\eta^3 + 3A_3\tau_\eta^2 + 2A_2\tau_\eta + A_1.$$

Given $4A_4 < 0, 3A_3 > 0, 2A_2 > 0$, and $A_1 > 0$, it must be the case that $A'(0) > 0$ and $A'(\infty) < 0$ and that $A'(\tau_\eta)$ changes signs only once (by Descartes' "rule of signs"). Hence, $A(\tau_\eta)$ first increases and then decreases. Given that $A(\tau_\eta)$ is negative at small and large values of τ_η and that τ_η^{large} is a local maximum for function $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$ (i.e., $A\left(\tau_\eta^{large} - \epsilon\right) > 0$ for sufficiently small ϵ), $A(\tau_\eta)$ crosses zero twice, which corresponds to two local extreme values of τ_η. Recall that $A(\tau_\eta)$ has the same sign as the FOC of $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$, function $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$ must first decrease, then increase, and finally decrease. Thus, the two local maximum values are 0 and τ_η^{large}. Given that $E\Pi_A\left(\tau_\eta^{large}, \tau_\eta^{large}\right) > E\Pi_A\left(0, \tau_\eta^{large}\right)$ (under the condition $4\tau_\theta \geq 5\tau_\delta$), it is clear that τ_η^{large} forms a global maximum of $E\Pi_A\left(\cdot, \tau_\eta^{large}\right)$, which implies that τ_η^{large} is supported as a partial disclosure equilibrium.

Proof of Proposition 4

Let $\tau_\varepsilon = \infty$. By the FOC (16) in firm A's disclosure decision problem,

$$\frac{\partial^2 E\Pi_A}{\partial \tau_\eta \partial \tau_\xi} = \frac{\partial}{\partial \tau_\xi}\left[\frac{1}{9(\tau_s + \tau_\delta)^2}\frac{\partial \tau_s}{\partial \tau_\eta}\right].$$

Using the expression of τ_s in Proposition 2, I can show that

$$\frac{\partial}{\partial \tau_\xi}\left[\frac{1}{9(\tau_s + \tau_\delta)^2}\frac{\partial \tau_s}{\partial \tau_\eta}\right] \propto -16(\tau_\theta + \tau_\eta)^2(\tau_\theta + \tau_\xi)^2\tau_u + 3\gamma^2\tau_\delta(2\tau_\theta + \tau_\xi + \tau_\eta)^2.$$

Hence, when τ_u is sufficiently small, $\frac{\partial^2 E\Pi_A}{\partial \tau_\eta \partial \tau_\xi} > 0$. Given symmetry, $\frac{\partial^2 E\Pi_B}{\partial \tau_\eta \partial \tau_\xi} > 0$.

Proof of Proposition 5

Proof of Part (a)

When $\tau_u = 0$, price informativeness τ_s is equal to 0, and so the profit expression in equation (15) becomes

$$E\Pi_A(\tau_\eta, \tau_\xi)\big|_{\tau_u=0} = \frac{m^2}{9} + \frac{9\tau_\theta + 4\tau_\eta}{36\tau_\theta(\tau_\theta + \tau_\eta)} + \frac{\tau_\xi}{9\tau_\theta(\tau_\theta + \tau_\xi)}.$$

Taking derivatives shows $\frac{\partial E\Pi_A\left(\tau_\eta,\tau_\xi\right)\big|_{\tau_u=0}}{\partial \tau_\eta} < 0$. Thus, no disclosure is a dominant strategy, which implies that the nondisclosure equilibrium serves as the unique equilibrium (i.e., $\tau_\eta^* = \tau_\xi^* = 0$).

The multiplicity result follows immediately from Part (a) of Theorem 4.

Using the expression of $\tau_\eta^* = \tau_\xi^* = \frac{\gamma^2}{5\tau_u} + o\left(1\right)$ and the profit expression in equation (15), I can show that $E\Pi_A\left(\frac{\gamma^2}{5\tau_u}, \frac{\gamma^2}{5\tau_u}\right) - E\Pi_A\left(0,0\right)$ has the same sign as

$$\Delta\Pi\left(\tau_u\right) = -2000\tau_\theta^4\tau_u^3 - 1000\gamma^2\tau_\theta^2\tau_\delta\tau_u^2 + 5\gamma^4\left(32\tau_\theta^2 - 25\tau_\delta^2 - 40\tau_\theta\tau_\delta\right)\tau_u + 4\gamma^6\left(4\tau_\theta - 5\tau_\delta\right).$$

Thus, when τ_u is sufficiently small, $\Delta\Pi\left(\tau_u\right) > 0$ provided $4\tau_\theta > 5\tau_\delta$.

Proof of Part (b)

By $\tau_\eta^* = \tau_\xi^* = \frac{\gamma^2}{5\tau_u} + o\left(1\right)$, it is clear that τ_η^* decreases with τ_u and diverges to ∞ as $\tau_u \to 0$. By the expression of τ_s in Proposition 2, direct computation shows that $\tau_s^* = \frac{4}{25}\frac{\gamma^2}{\tau_u} + o\left(1\right)$. Thus, as τ_u decreases, τ_s^* increases. By the expression of a_f in Proposition 2, we know that a_f and τ_s^* change in the same direction.

Direct computation shows $\frac{\partial Var\left(q_A^*\right)}{\partial \tau_u} \propto -\frac{5}{9\gamma^2}$, and thus, $Var\left(q_A^*\right)$ increases as τ_u decreases. Finally, one can compute $Cov\left(q_A^*, q_B^*\right) = \frac{2}{9}\frac{\tau_\eta^*}{\left(\tau_\theta+\tau_\eta^*\right)\tau_\theta} + \frac{\tau_s^*}{9\tau_\delta\left(\tau_s^*+\tau_\delta\right)}$. Since both τ_η^* and τ_s^* increase as τ_u decreases, we know that $Cov\left(q_A^*, q_B^*\right)$ increases as well when τ_u decreases.

References

Amador, M., Weill, P. O., 2010, "Learning from prices: Public communication and welfare," *Journal of Political Economy* 118, 866–907.

Anantharaman, D., Zhang, Y., 2011, "Cover me: Managers' responses to changes in analyst coverage in the post-Regulation FD period," *Accounting Review* 86, 1851–1885.

Arya, A., Mittendorf, B., Yoon, D-H., 2016, "Public disclosures in the presence of suppliers and competitors," Working Paper.

Bagnoli, M, Watts, S., 2015, "Competitive intelligence and disclosure," *RAND Journal of Economics* 46, 709–729.

Bakke T. E., Whited, T. M., 2010, "Which firms follow the market? An analysis of corporate investment decisions," *Review of Financial Studies* 23, 1941–1980.

Balakrishnan, K., Billings, M. B., Kelly, B., Ljungqvist, A. 2014, "Shaping liquidity: On the causal effects of voluntary isclosure," *Journal of Finance* 69, 2237–2278.

Banerjee, S., Davis, J., Gondhi, N., 2018, "When transparency improves, must prices reflect fundamentals better?", *Review of Financial Studies* 31, 2377–2414.

Bhattacharya, S., Ritter, J. R., 1983, "Innovation and communication: Signaling with partial disclosure," *Review of Economic Studies* 50, 331–46.

Black, F., 1976, "The pricing of commodity contracts," *Journal of Financial Economics* 3, 167-179.

Bochkay, K., Chychyla, R., Nanda, D., 2016, "Dynamics of CEO communication style," Working paper.

Bond, P., Edmans, A., Goldstein, I., 2012, "The real effects of financial markets," *Annual Review of Financial Economics* 4, 339–360.

Bond, P., Goldstein, I., 2015, "Government intervention and information aggregation by prices," *Journal of Finance* 70, 2777–2811.

Chen, Q., Goldstein, I., Jiang, W., 2007, "Price informativeness and investment sensitivity to stock price," *Review of Financial Studies* 20, 619–650.

Cheng, I. H., Xiong, W., 2014, "The financialization of commodity markets," *Annual Review of Financial Economics* 6, 419–441.

Darrough, M., 1993, "Disclosure policy and competition: Cournot vs. Bertrand," *Accounting Review* 68, 534–561.

Derrien, F., 2010, "Initial public offerings," In B. Finance H. Kent Baker & J. Nofsinger (Eds.). Hoboken: Wiley.

Dessaint, O., Foucault, T., Frésard, L., Matray, A., 2018, "Noisy stock prices on corporate investment," *Review of Financial Studies*, Forthcoming.

Dow, J., Goldstein, I., Guembel, A., 2017, "Incentives for information production in markets where prices affect real investment," *Journal of the European Economic Association* 15, 877–909.

Easley, D., O'Hara, M., Yang, L., 2016, "Differential access to price information in financial markets," *Journal of Financial and Quantitative Analysis* 51, 1071–1110.

Fama, E. F., Miller, M., 1972, *The Theory of Finance.* Holt, Rinehart & Winston, New York.

Foster, G., 1986, Financial Statement Analysis. Englewood Cliffs, NJ: Prentice-Hall.

Foucault, T., Frésard, L., 2012, "Investment sensitivity to stock price and the learning hypothesis," *Review of Financial Studies* 25, 3305–3350.

Foucault, T., Frésard, L., 2014, "Learning from peers' stock prices and corporate investment," *Journal of Financial Economics* 111, 554–577.

Gal-Or, E., 1985, "Information sharing in oligopoly," *Econometrica* 53, 329–343.

Gal-Or, E., 1986, "Information transmission—Cournot and Bertrand equilibria," *Review of Economic Studies* 53, 85–92.

Gao, P., Liang, P. J., 2013, "Informational feedback, adverse selection, and optimal disclosure policy," *Journal of Accounting Research* 51, 1133–1158.

Goldstein, I., Yang, L., 2015, "Information diversity and complementarities in trading and information acquisition," *Journal of Finance* 70, 1723–1765.

Goldstein, I., Yang, L., 2017, "Information disclosure in financial markets," *Annual Review of Financial Economics* 9, 101–125.

Goldstein, I., Yang, L., 2018, "Good disclosure, bad disclosure," *Journal of Financial Economics*, Forthcoming.

Graham, J. R., Harvey, C. R., Rajgopal, S., 2005, "The economic implications of corporate financial reporting," *Journal of Accounting and Economics* 40, 3–73.

Grossman, S., Stiglitz, J., 1980, "On the impossibility of informationally efficient markets," *American Economic Review* 70, 393–408.

Han, B., Tang, Y., Yang, L., 2016, "Public information and uninformed trading: Implications for market liquidity and efficiency," *Journal of Economic Theory* 163, 604–643.

Hayek, F., 1945, "The use of knowledge in society," *American Economic Review* 35, 519–530.

Hirshleifer, J. 1971, "The private and social value of information and the reward to inventive activity," *American Economic Review* 61, 561–574.

Kurlat, P., Veldkamp, L., 2015, "Should we regulate financial information?", *Journal of Economic Theory* 158, 697–720.

Luo, Y., 2005, "Do insiders learn from outsiders? Evidence from mergers and acquisitions," *Journal of Finance* 60, 1951–1982.

Maksimovic, V., Pichler, P., 2001, “Technological innovation and Initial Public Offerings,” *Review of Financial Studies* 14, 459–494.

Ozoguz, A., Rebello, M. J., 2013, “Information, competition, and investment sensitivity to peer stock prices,” Working paper.

Raith, M., 1996, “A general model of information sharing in oligopoly,” *Journal of Economic Theory* 71, 260–288.

Schneemeier, J., 2018. “Optimal disclosure and fight for attention,” Working paper.

Singh, N., Vives, X., 1984, “Price and quantity competition in a differentiated duopoly,” *RAND Journal of Economics* 15, 546–554.

Vives, X., 1984, “Duopoly information equilibrium: Cournot and Bertrand,” *Journal of Economic Theory* 34, 71–94.

Vives, X., 2008, “Information sharing among firms,” *The New Palgrave Dictionary of Economics*, ed. by S. N. Durlauf and L. E. Blume, Basingstoke: Palgrave Macmillan.

Online Appendix (Not for Publication)

Expressions of $F's, S's$, and $G's$ in Theorem 3

The F-coefficients are:

$$F_4 = -16\left(5\tau_u^2\tau_\varepsilon^4 - 2\gamma^2\tau_u\tau_\varepsilon^3 + 10\gamma^2\tau_u\tau_\delta\tau_\varepsilon^2 + 5\gamma^4\tau_\delta^2\right),$$

$$F_3 = 16\left(\begin{array}{c} \gamma^2\tau_u\tau_\varepsilon^4 - 20\tau_u^2\tau_\theta\tau_\varepsilon^4 + 8\gamma^2\tau_u\tau_\theta\tau_\varepsilon^3 - 10\gamma^2\tau_u\tau_\delta\tau_\varepsilon^3 \\ -40\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^2 - 10\gamma^4\tau_\delta^2\tau_\varepsilon - 20\gamma^4\tau_\theta\tau_\delta^2 \end{array}\right),$$

$$F_2 = 8\left(\begin{array}{c} 6\gamma^2\tau_u\tau_\theta\tau_\varepsilon^4 - 5\gamma^2\tau_u\tau_\delta\tau_\varepsilon^4 - 60\tau_u^2\tau_\theta^2\tau_\varepsilon^4 \\ +24\gamma^2\tau_u\tau_\theta^2\tau_\varepsilon^3 - 60\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^3 \\ -120\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^2 - 15\gamma^4\tau_\delta^2\tau_\varepsilon^2 \\ -60\gamma^4\tau_\theta\tau_\delta^2\tau_\varepsilon - 60\gamma^4\tau_\theta^2\tau_\delta^2 \end{array}\right),$$

$$F_1 = 8\left(\begin{array}{c} 6\gamma^2\tau_u\tau_\theta^2\tau_\varepsilon^4 - 10\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^4 - 40\tau_u^2\tau_\theta^3\tau_\varepsilon^4 \\ +16\gamma^2\tau_u\tau_\theta^3\tau_\varepsilon^3 - 5\gamma^4\tau_\delta^2\tau_\varepsilon^3 - 60\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^3 \\ -30\gamma^4\tau_\theta\tau_\delta^2\tau_\varepsilon^2 - 80\gamma^2\tau_u\tau_\theta^3\tau_\delta\tau_\varepsilon^2 \\ -60\gamma^4\tau_\theta^2\tau_\delta^2\tau_\varepsilon - 40\gamma^4\tau_\theta^3\tau_\delta^2 \end{array}\right),$$

$$F_0 = -\left(\begin{array}{c} 5\gamma^4\tau_\delta^2\tau_\varepsilon^4 + 80\tau_u^2\tau_\theta^4\tau_\varepsilon^4 - 16\gamma^2\tau_u\tau_\theta^3\tau_\varepsilon^4 + 40\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^4 \\ -32\gamma^2\tau_u\tau_\theta^4\tau_\varepsilon^3 + 40\gamma^4\tau_\theta\tau_\delta^2\tau_\varepsilon^3 + 160\gamma^2\tau_u\tau_\theta^3\tau_\delta\tau_\varepsilon^3 \\ +120\gamma^4\tau_\theta^2\tau_\delta^2\tau_\varepsilon^2 + 160\gamma^2\tau_u\tau_\theta^4\tau_\delta\tau_\varepsilon^2 \\ +160\gamma^4\tau_\theta^3\tau_\delta^2\tau_\varepsilon + 80\gamma^4\tau_\theta^4\tau_\delta^2 \end{array}\right).$$

The S-coefficients are:

$$S_6 = 64\left(\tau_u\tau_\varepsilon^2 + \gamma^2\tau_\delta\right)\left(5\tau_u^2\tau_\varepsilon^4 - 2\gamma^2\tau_u\tau_\varepsilon^3 + 10\gamma^2\tau_u\tau_\delta\tau_\varepsilon^2 + 5\gamma^4\tau_\delta^2\right),$$

$$S_5 = 16\left(\begin{array}{c} 120\tau_u^3\tau_\theta\tau_\varepsilon^6 - 5\gamma^2\tau_u^2\tau_\varepsilon^6 + 60\gamma^2\tau_u^2\tau_\delta\tau_\varepsilon^5 - 48\gamma^2\tau_u^2\tau_\theta\tau_\varepsilon^5 \\ -9\gamma^4\tau_u\tau_\delta\tau_\varepsilon^4 + 360\gamma^2\tau_u^2\tau_\theta\tau_\delta\tau_\varepsilon^4 + 120\gamma^4\tau_u\tau_\delta^2\tau_\varepsilon^3 - 48\gamma^4\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^3 \\ +360\gamma^4\tau_u\tau_\theta\tau_\delta^2\tau_\varepsilon^2 + 60\gamma^6\tau_\delta^3\tau_\varepsilon + 120\gamma^6\tau_\theta\tau_\delta^3 \end{array}\right),$$

$$S_4 = 16\left(\begin{array}{c} 300\tau_u^3\tau_\theta^2\tau_\varepsilon^6 - 25\gamma^2\tau_u^2\tau_\theta\tau_\varepsilon^6 + 15\gamma^2\tau_u^2\tau_\delta\tau_\varepsilon^6 \\ +300\gamma^2\tau_u^2\tau_\theta\tau_\delta\tau_\varepsilon^5 - 120\gamma^2\tau_u^2\tau_\theta^2\tau_\varepsilon^5 - 3\gamma^4\tau_u\tau_\delta\tau_\varepsilon^5 \\ +900\gamma^2\tau_u^2\tau_\theta^2\tau_\delta\tau_\varepsilon^4 - 45\gamma^4\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^4 + 90\gamma^4\tau_u\tau_\delta^2\tau_\varepsilon^4 \\ +600\gamma^4\tau_u\tau_\theta\tau_\delta^2\tau_\varepsilon^3 - 120\gamma^4\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^3 \\ +75\gamma^6\tau_\delta^3\tau_\varepsilon^2 + 900\gamma^4\tau_u\tau_\theta^2\tau_\delta^2\tau_\varepsilon^2 + 300\gamma^6\tau_\theta\tau_\delta^3\tau_\varepsilon + 300\gamma^6\tau_\theta^2\tau_\delta^3 \end{array}\right),$$

$$S_3 = 4\left(\begin{array}{c} 1600\tau_u^3\tau_\theta^3\tau_\varepsilon^6 - 200\gamma^2\tau_u^2\tau_\theta^2\tau_\varepsilon^6 - \gamma^4\tau_u\tau_\delta\tau_\varepsilon^6 + 240\gamma^2\tau_u^2\tau_\theta\tau_\delta\tau_\varepsilon^6 \\ +2400\gamma^2\tau_u^2\tau_\theta^2\tau_\delta\tau_\varepsilon^5 + 120\gamma^4\tau_u\tau_\delta^2\tau_\varepsilon^5 - 640\gamma^2\tau_u^2\tau_\theta^3\tau_\varepsilon^5 - 48\gamma^4\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^5 \\ +1440\gamma^4\tau_u\tau_\theta\tau_\delta^2\tau_\varepsilon^4 - 360\gamma^4\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^4 + 4800\gamma^2\tau_u^2\tau_\theta^3\tau_\delta\tau_\varepsilon^4 \\ -640\gamma^4\tau_u\tau_\theta^3\tau_\delta\tau_\varepsilon^3 + 200\gamma^6\tau_\delta^3\tau_\varepsilon^3 + 4800\gamma^4\tau_u\tau_\theta^2\tau_\delta^2\tau_\varepsilon^3 \\ +4800\gamma^4\tau_u\tau_\theta^3\tau_\delta^2\tau_\varepsilon^2 + 1200\gamma^6\tau_\theta\tau_\delta^3\tau_\varepsilon^2 + 2400\gamma^6\tau_\theta^2\tau_\delta^3\tau_\varepsilon + 1600\gamma^6\tau_\theta^3\tau_\delta^3 \end{array}\right),$$

$$S_2 = 4\left(\begin{array}{c} 1200\tau_u^3\tau_\theta^4\tau_\varepsilon^6 + 15\gamma^4\tau_u\tau_\delta^2\tau_\varepsilon^6 - 200\gamma^2\tau_u^2\tau_\theta^3\tau_\varepsilon^6 + 360\gamma^2\tau_u^2\tau_\theta^2\tau_\delta\tau_\varepsilon^6 - 3\gamma^4\tau_u\tau_\theta\tau_\delta\tau_\varepsilon^6 \\ +360\gamma^4\tau_u\tau_\theta\tau_\delta^2\tau_\varepsilon^5 - 72\gamma^4\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^5 - 480\gamma^2\tau_u^2\tau_\theta^4\tau_\varepsilon^5 + 2400\gamma^2\tau_u^2\tau_\theta^3\tau_\delta\tau_\varepsilon^5 \\ +75\gamma^6\tau_\delta^3\tau_\varepsilon^4 - 360\gamma^4\tau_u\tau_\theta^3\tau_\delta\tau_\varepsilon^4 + 3600\gamma^2\tau_u^2\tau_\theta^4\tau_\delta\tau_\varepsilon^4 + 2160\gamma^4\tau_u\tau_\theta^2\tau_\delta^2\tau_\varepsilon^4 \\ +600\gamma^6\tau_\theta\tau_\delta^3\tau_\varepsilon^3 - 480\gamma^4\tau_u\tau_\theta^4\tau_\delta\tau_\varepsilon^3 + 4800\gamma^4\tau_u\tau_\theta^3\tau_\delta^2\tau_\varepsilon^3 \\ +1800\gamma^6\tau_\theta^2\tau_\delta^3\tau_\varepsilon^2 + 3600\gamma^4\tau_u\tau_\theta^4\tau_\delta^2\tau_\varepsilon^2 + 2400\gamma^6\tau_\theta^3\tau_\delta^3\tau_\varepsilon + 1200\gamma^6\tau_\theta^4\tau_\delta^3 \end{array}\right),$$

$$S_1 = 4\left(\begin{array}{c} 480\tau_u^3\tau_\theta^5\tau_\varepsilon^6 + 30\gamma^4\tau_u\tau_\theta\tau_\delta^2\tau_\varepsilon^6 - 3\gamma^4\tau_u\tau_\theta^2\tau_\delta\tau_\varepsilon^6 + 240\gamma^2\tau_u^2\tau_\theta^3\tau_\delta\tau_\varepsilon^6 - 100\gamma^2\tau_u^2\tau_\theta^4\tau_\varepsilon^6 \\ +15\gamma^6\tau_\delta^3\tau_\varepsilon^5 - 48\gamma^4\tau_u\tau_\theta^3\tau_\delta\tau_\varepsilon^5 + 1200\gamma^2\tau_u^2\tau_\theta^4\tau_\delta\tau_\varepsilon^5 + 360\gamma^4\tau_u\tau_\theta^2\tau_\delta^2\tau_\varepsilon^5 - 192\gamma^2\tau_u^2\tau_\theta^5\tau_\varepsilon^5 \\ +150\gamma^6\tau_\theta\tau_\delta^3\tau_\varepsilon^4 - 180\gamma^4\tau_u\tau_\theta^4\tau_\delta\tau_\varepsilon^4 + 1440\gamma^2\tau_u^2\tau_\theta^5\tau_\delta\tau_\varepsilon^4 + 1440\gamma^4\tau_u\tau_\theta^3\tau_\delta^2\tau_\varepsilon^4 \\ +600\gamma^6\tau_\theta^2\tau_\delta^3\tau_\varepsilon^3 + 2400\gamma^4\tau_u\tau_\theta^4\tau_\delta^2\tau_\varepsilon^3 - 192\gamma^4\tau_u\tau_\theta^5\tau_\delta\tau_\varepsilon^3 \\ +1200\gamma^6\tau_\theta^3\tau_\delta^3\tau_\varepsilon^2 + 1440\gamma^4\tau_u\tau_\theta^5\tau_\delta^2\tau_\varepsilon^2 + 1200\gamma^6\tau_\theta^4\tau_\delta^3\tau_\varepsilon + 480\gamma^6\tau_\theta^5\tau_\delta^3 \end{array}\right),$$

$$\begin{aligned} S_0 &= \left(5\gamma^6\tau_\delta^3 - 4\gamma^4\tau_u\tau_\theta^3\tau_\delta + 60\gamma^4\tau_u\tau_\theta^2\tau_\delta^2 - 80\gamma^2\tau_u^2\tau_\theta^5 + 240\gamma^2\tau_u^2\tau_\theta^4\tau_\delta + 320\tau_u^3\tau_\theta^6\right)\tau_\varepsilon^6 \\ &+ \left(60\gamma^6\tau_\theta\tau_\delta^3 - 48\gamma^4\tau_u\tau_\theta^4\tau_\delta + 480\gamma^4\tau_u\tau_\theta^3\tau_\delta^2 - 128\gamma^2\tau_u^2\tau_\theta^6 + 960\gamma^2\tau_u^2\tau_\theta^5\tau_\delta\right)\tau_\varepsilon^5 \\ &+ \left(300\gamma^6\tau_\theta^2\tau_\delta^3 - 144\gamma^4\tau_u\tau_\theta^5\tau_\delta + 1440\gamma^4\tau_u\tau_\theta^4\tau_\delta^2 + 960\gamma^2\tau_u^2\tau_\theta^6\tau_\delta\right)\tau_\varepsilon^4 \\ &+ \left(800\gamma^6\tau_\theta^3\tau_\delta^3 - 128\tau_u\gamma^4\tau_\theta^6\tau_\delta + 1920\tau_u\gamma^4\tau_\theta^5\tau_\delta^2\right)\tau_\varepsilon^3 \\ &+ \left(1200\gamma^6\tau_\theta^4\tau_\delta^3 + 960\tau_u\gamma^4\tau_\theta^6\tau_\delta^2\right)\tau_\varepsilon^2 + \left(960\gamma^6\tau_\theta^5\tau_\delta^3\right)\tau_\varepsilon + 320\gamma^6\tau_\theta^6\tau_\delta^3. \end{aligned}$$

For the G-coefficients, let us set $\tau_\xi = \tau_\eta^*$. Then, we have:

$$\begin{aligned} G_4 &= \left(\begin{array}{c} 5\gamma^4\tau_\delta^2 - 128\gamma^2\tau_u\tau_\theta^3 + 160\gamma^2\tau_u\tau_\theta^2\tau_\delta - 384\gamma^2\tau_u\tau_\theta^2\tau_\xi \\ +320\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\xi - 384\gamma^2\tau_u\tau_\theta\tau_\xi^2 + 160\gamma^2\tau_u\tau_\delta\tau_\xi^2 \\ -128\gamma^2\tau_u\tau_\xi^3 + 1280\tau_u^2\tau_\theta^4 + 5120\tau_u^2\tau_\theta^3\tau_\xi \\ +7680\tau_u^2\tau_\theta^2\tau_\xi^2 + 5120\tau_u^2\tau_\theta\tau_\xi^3 + 1280\tau_u^2\tau_\xi^4 \end{array}\right)\tau_\varepsilon^4 \\ &+ \left(\begin{array}{c} 80\gamma^4\tau_\theta\tau_\delta^2 + 80\gamma^4\tau_\delta^2\tau_\xi - 512\tau_u\gamma^2\tau_\theta^4 \\ +1280\tau_u\gamma^2\tau_\theta^3\tau_\delta - 2048\tau_u\gamma^2\tau_\theta^3\tau_\xi + 3840\tau_u\gamma^2\tau_\theta^2\tau_\delta\tau_\xi \\ -3072\tau_u\gamma^2\tau_\theta^2\tau_\xi^2 + 3840\tau_u\gamma^2\tau_\theta\tau_\delta\tau_\xi^2 - 2048\tau_u\gamma^2\tau_\theta\tau_\xi^3 \\ +1280\tau_u\gamma^2\tau_\delta\tau_\xi^3 - 512\tau_u\gamma^2\tau_\xi^4 \end{array}\right)\tau_\varepsilon^3 \\ &+ \left(\begin{array}{c} 480\gamma^4\tau_\theta^2\tau_\delta^2 + 960\gamma^4\tau_\theta\tau_\delta^2\tau_\xi + 480\gamma^4\tau_\delta^2\tau_\xi^2 \\ +2560\tau_u\gamma^2\tau_\theta^4\tau_\delta + 10\,240\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi \\ +15\,360\tau_u\gamma^2\tau_\theta^2\tau_\delta\tau_\xi^2 + 10\,240\tau_u\gamma^2\tau_\theta\tau_\delta\tau_\xi^3 + 2560\tau_u\gamma^2\tau_\delta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^2 \\ &+ \left(1280\gamma^4\tau_\theta^3\tau_\delta^2 + 3840\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi + 3840\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^2 + 1280\gamma^4\tau_\delta^2\tau_\xi^3\right)\tau_\varepsilon \\ &+ \left(\begin{array}{c} 1280\gamma^4\tau_\theta^4\tau_\delta^2 + 5120\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi + 7680\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^2 \\ +5120\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^3 + 1280\gamma^4\tau_\delta^2\tau_\xi^4 \end{array}\right), \end{aligned}$$

$$\begin{aligned} G_3 &= \left(\begin{array}{c} 40\gamma^4\tau_\theta\tau_\delta^2 + 20\gamma^4\tau_\delta^2\tau_\xi - 640\gamma^2\tau_u\tau_\theta^4 + 960\gamma^2\tau_u\tau_\theta^3\tau_\delta \\ -2048\gamma^2\tau_u\tau_\theta^3\tau_\xi + 2240\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\xi - 2304\gamma^2\tau_u\tau_\theta^2\tau_\xi^2 \\ +1600\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\xi^2 - 1024\gamma^2\tau_u\tau_\theta\tau_\xi^3 + 320\gamma^2\tau_u\tau_\delta\tau_\xi^3 \\ -128\gamma^2\tau_u\tau_\xi^4 + 5120\tau_u^2\tau_\theta^5 + 20\,480\tau_u^2\tau_\theta^4\tau_\xi \\ +30\,720\tau_u^2\tau_\theta^3\tau_\xi^2 + 20\,480\tau_u^2\tau_\theta^2\tau_\xi^3 + 5120\tau_u^2\tau_\theta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^4 \\ &+ \left(\begin{array}{c} 560\gamma^4\tau_\theta^2\tau_\delta^2 + 800\gamma^4\tau_\theta\tau_\delta^2\tau_\xi + 240\gamma^4\tau_\delta^2\tau_\xi^2 \\ -2048\tau_u\gamma^2\tau_\theta^5 + 6400\tau_u\gamma^2\tau_\theta^4\tau_\delta - 8192\tau_u\gamma^2\tau_\theta^4\tau_\xi \\ +20\,480\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi - 12\,288\tau_u\gamma^2\tau_\theta^3\tau_\xi^2 \\ +23\,040\tau_u\gamma^2\tau_\theta^2\tau_\delta\tau_\xi^2 - 8192\tau_u\gamma^2\tau_\theta^2\tau_\xi^3 \\ +10\,240\tau_u\gamma^2\tau_\theta\tau_\delta\tau_\xi^3 - 2048\tau_u\gamma^2\tau_\theta\tau_\xi^4 + 1280\tau_u\gamma^2\tau_\delta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^3 \\ &+ \left(\begin{array}{c} 2880\gamma^4\tau_\theta^3\tau_\delta^2 + 6720\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi + 4800\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^2 \\ +960\gamma^4\tau_\delta^2\tau_\xi^3 + 10\,240\tau_u\gamma^2\tau_\theta^5\tau_\delta + 40\,960\tau_u\gamma^2\tau_\theta^4\tau_\delta\tau_\xi \\ +61\,440\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi^2 + 40\,960\tau_u\gamma^2\tau_\theta^2\tau_\delta\tau_\xi^3 \\ +10\,240\tau_u\gamma^2\tau_\theta\tau_\delta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^2 \\ &+ \left(\begin{array}{c} 6400\gamma^4\tau_\theta^4\tau_\delta^2 + 20\,480\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi + 23\,040\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^2 \\ +10\,240\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^3 + 1280\gamma^4\tau_\delta^2\tau_\xi^4 \end{array}\right)\tau_\varepsilon \end{aligned}$$

$$
+\left(\begin{array}{c} 5120\gamma^4\tau_\theta^5\tau_\delta^2+20\,480\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi+30\,720\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^2 \\ +20\,480\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^3+5120\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^4 \end{array}\right),
$$

$$
G_2=\left(\begin{array}{c} 120\gamma^4\tau_\theta^2\tau_\delta^2+120\gamma^4\tau_\theta\tau_\delta^2\tau_\xi+30\gamma^4\tau_\delta^2\tau_\xi^2 \\ -1152\gamma^2\tau_u\tau_\theta^5+2080\gamma^2\tau_u\tau_\theta^4\tau_\delta-3840\gamma^2\tau_u\tau_\theta^4\tau_\xi \\ +5440\gamma^2\tau_u\tau_\theta^3\tau_\delta\tau_\xi-4608\gamma^2\tau_u\tau_\theta^3\tau_\xi^2+4800\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\xi^2 \\ -2304\gamma^2\tau_u\tau_\theta^2\tau_\xi^3+1600\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\xi^3-384\gamma^2\tau_u\tau_\theta\tau_\xi^4 \\ +160\gamma^2\tau_u\tau_\delta\tau_\xi^4+7680\tau_u^2\tau_\theta^6+30\,720\tau_u^2\tau_\theta^5\tau_\xi \\ +46\,080\tau_u^2\tau_\theta^4\tau_\xi^2+30\,720\tau_u^2\tau_\theta^3\tau_\xi^3+7680\tau_u^2\tau_\theta^2\tau_\xi^4 \end{array}\right)\tau_\varepsilon^4
$$

$$
+\left(\begin{array}{c} 1440\gamma^4\tau_\theta^3\tau_\delta^2+2640\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi+1440\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^2 \\ +240\gamma^4\tau_\delta^2\tau_\xi^3-3072\tau_u\gamma^2\tau_\theta^6+11\,520\tau_u\gamma^2\tau_\theta^5\tau_\delta \\ -12\,288\tau_u\gamma^2\tau_\theta^5\tau_\xi+38\,400\tau_u\gamma^2\tau_\theta^4\tau_\delta\tau_\xi-18\,432\tau_u\gamma^2\tau_\theta^4\tau_\xi^2 \\ +46\,080\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi^2-12\,288\tau_u\gamma^2\tau_\theta^3\tau_\xi^3 \\ +23\,040\tau_u\gamma^2\tau_\theta^2\tau_\delta\tau_\xi^3-3072\tau_u\gamma^2\tau_\theta^2\tau_\xi^4+3840\tau_u\gamma^2\tau_\theta\tau_\delta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^3
$$

$$
+\left(\begin{array}{c} 6240\gamma^4\tau_\theta^4\tau_\delta^2+16\,320\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi+14\,400\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^2 \\ +4800\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^3+480\gamma^4\tau_\delta^2\tau_\xi^4+15\,360\tau_u\gamma^2\tau_\theta^6\tau_\delta \\ +61\,440\tau_u\gamma^2\tau_\theta^5\tau_\delta\tau_\xi+92\,160\tau_u\gamma^2\tau_\theta^4\tau_\delta\tau_\xi^2 \\ +61\,440\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi^3+15\,360\tau_u\gamma^2\tau_\theta^2\tau_\delta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^2
$$

$$
+\left(\begin{array}{c} 11\,520\gamma^4\tau_\theta^5\tau_\delta^2+38\,400\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi+46\,080\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^2 \\ +23\,040\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^3+3840\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^4 \end{array}\right)\tau_\varepsilon
$$

$$
+\left(\begin{array}{c} 7680\gamma^4\tau_\theta^6\tau_\delta^2+30\,720\gamma^4\tau_\theta^5\tau_\delta^2\tau_\xi+46\,080\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi^2 \\ +30\,720\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^3+7680\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^4 \end{array}\right),
$$

$$
G_1=\left(\begin{array}{c} 160\gamma^4\tau_\theta^3\tau_\delta^2+240\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi+120\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^2 \\ +20\gamma^4\tau_\delta^2\tau_\xi^3-896\gamma^2\tau_u\tau_\theta^6+1920\gamma^2\tau_u\tau_\theta^5\tau_\delta \\ -3072\gamma^2\tau_u\tau_\theta^5\tau_\xi+5440\gamma^2\tau_u\tau_\theta^4\tau_\delta\tau_\xi-3840\gamma^2\tau_u\tau_\theta^4\tau_\xi^2 \\ +5440\gamma^2\tau_u\tau_\theta^3\tau_\delta\tau_\xi^2-2048\gamma^2\tau_u\tau_\theta^3\tau_\xi^3+2240\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\xi^3 \\ -384\gamma^2\tau_u\tau_\theta^2\tau_\xi^4+320\gamma^2\tau_u\tau_\theta\tau_\delta\tau_\xi^4+5120\tau_u^2\tau_\theta^7 \\ +20\,480\tau_u^2\tau_\theta^6\tau_\xi+30\,720\tau_u^2\tau_\theta^5\tau_\xi^2 \\ +20\,480\tau_u^2\tau_\theta^4\tau_\xi^3+5120\tau_u^2\tau_\theta^3\tau_\xi^4 \end{array}\right)\tau_\varepsilon^4
$$

$$
+\left(\begin{array}{c} 1600\gamma^4\tau_\theta^4\tau_\delta^2+3520\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi+2640\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^2 \\ +800\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^3+80\gamma^4\tau_\delta^2\tau_\xi^4-2048\tau_u\gamma^2\tau_\theta^7 \\ +8960\tau_u\gamma^2\tau_\theta^6\tau_\delta-8192\tau_u\gamma^2\tau_\theta^6\tau_\xi+30\,720\tau_u\gamma^2\tau_\theta^5\tau_\delta\tau_\xi \\ -12\,288\tau_u\gamma^2\tau_\theta^5\tau_\xi^2+38\,400\tau_u\gamma^2\tau_\theta^4\tau_\delta\tau_\xi^2-8192\tau_u\gamma^2\tau_\theta^4\tau_\xi^3 \\ +20\,480\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi^3-2048\tau_u\gamma^2\tau_\theta^3\tau_\xi^4+3840\tau_u\gamma^2\tau_\theta^2\tau_\delta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^3
$$

$$
+\left(\begin{array}{c} 5760\gamma^4\tau_\theta^5\tau_\delta^2+16\,320\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi+16\,320\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^2 \\ +6720\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^3+960\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^4+10\,240\tau_u\gamma^2\tau_\theta^7\tau_\delta \\ +40\,960\tau_u\gamma^2\tau_\theta^6\tau_\delta\tau_\xi+61\,440\tau_u\gamma^2\tau_\theta^5\tau_\delta\tau_\xi^2 \\ +40\,960\tau_u\gamma^2\tau_\theta^4\tau_\delta\tau_\xi^3+10\,240\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi^4 \end{array}\right)\tau_\varepsilon^2
$$

$$
+\left(\begin{array}{c} 8960\gamma^4\tau_\theta^6\tau_\delta^2+30\,720\gamma^4\tau_\theta^5\tau_\delta^2\tau_\xi+38\,400\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi^2 \\ +20\,480\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^3+3840\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^4 \end{array}\right)\tau_\varepsilon
$$

$$
+\left(\begin{array}{c} 5120\gamma^4\tau_\theta^7\tau_\delta^2+20\,480\gamma^4\tau_\theta^6\tau_\delta^2\tau_\xi+30\,720\gamma^4\tau_\theta^5\tau_\delta^2\tau_\xi^2 \\ +20\,480\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi^3+5120\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^4 \end{array}\right),
$$

$$G_0 = \begin{pmatrix} 80\gamma^4\tau_\theta^4\tau_\delta^2 + 160\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi + 120\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^2 \\ +40\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^3 + 5\gamma^4\tau_\delta^2\tau_\xi^4 - 256\gamma^2\tau_u\tau_\theta^7 + 640\gamma^2\tau_u\tau_\theta^6\tau_\delta \\ -896\gamma^2\tau_u\tau_\theta^6\tau_\xi + 1920\gamma^2\tau_u\tau_\theta^5\tau_\delta\tau_\xi - 1152\gamma^2\tau_u\tau_\theta^5\tau_\xi^2 \\ +2080\gamma^2\tau_u\tau_\theta^4\tau_\delta\tau_\xi^2 - 640\gamma^2\tau_u\tau_\theta^4\tau_\xi^3 + 960\gamma^2\tau_u\tau_\theta^3\tau_\delta\tau_\xi^3 \\ -128\gamma^2\tau_u\tau_\theta^3\tau_\xi^4 + 160\gamma^2\tau_u\tau_\theta^2\tau_\delta\tau_\xi^4 + 1280\tau_u^2\tau_\theta^8 \\ +5120\tau_u^2\tau_\theta^7\tau_\xi + 7680\tau_u^2\tau_\theta^6\tau_\xi^2 + 5120\tau_u^2\tau_\theta^5\tau_\xi^3 + 1280\tau_u^2\tau_\theta^4\tau_\xi^4 \end{pmatrix} \tau_\varepsilon^4$$

$$+ \begin{pmatrix} 640\gamma^4\tau_\theta^5\tau_\delta^2 + 1600\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi + 1440\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^2 \\ +560\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^3 + 80\gamma^4\tau_\theta\tau_\delta^2\tau_\xi^4 - 512\tau_u\gamma^2\tau_\theta^8 \\ +2560\tau_u\gamma^2\tau_\theta^7\tau_\delta - 2048\tau_u\gamma^2\tau_\theta^7\tau_\xi + 8960\tau_u\gamma^2\tau_\theta^6\tau_\delta\tau_\xi \\ -3072\tau_u\gamma^2\tau_\theta^6\tau_\xi^2 + 11\,520\tau_u\gamma^2\tau_\theta^5\tau_\delta\tau_\xi^2 - 2048\tau_u\gamma^2\tau_\theta^5\tau_\xi^3 \\ +6400\tau_u\gamma^2\tau_\theta^4\tau_\delta\tau_\xi^3 - 512\tau_u\gamma^2\tau_\theta^4\tau_\xi^4 + 1280\tau_u\gamma^2\tau_\theta^3\tau_\delta\tau_\xi^4 \end{pmatrix} \tau_\varepsilon^3$$

$$+ \begin{pmatrix} 1920\gamma^4\tau_\theta^6\tau_\delta^2 + 5760\gamma^4\tau_\theta^5\tau_\delta^2\tau_\xi + 6240\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi^2 \\ +2880\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^3 + 480\gamma^4\tau_\theta^2\tau_\delta^2\tau_\xi^4 + 2560\tau_u\gamma^2\tau_\theta^8\tau_\delta \\ +10\,240\tau_u\gamma^2\tau_\theta^7\tau_\delta\tau_\xi + 15\,360\tau_u\gamma^2\tau_\theta^6\tau_\delta\tau_\xi^2 \\ +10\,240\tau_u\gamma^2\tau_\theta^5\tau_\delta\tau_\xi^3 + 2560\tau_u\gamma^2\tau_\theta^4\tau_\delta\tau_\xi^4 \end{pmatrix} \tau_\varepsilon^2$$

$$+ \begin{pmatrix} 2560\gamma^4\tau_\theta^7\tau_\delta^2 + 8960\gamma^4\tau_\theta^6\tau_\delta^2\tau_\xi + 11\,520\gamma^4\tau_\theta^5\tau_\delta^2\tau_\xi^2 \\ +6400\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi^3 + 1280\gamma^4\tau_\theta^3\tau_\delta^2\tau_\xi^4 \end{pmatrix} \tau_\varepsilon$$

$$+ \begin{pmatrix} 1280\gamma^4\tau_\theta^8\tau_\delta^2 + 5120\gamma^4\tau_\theta^7\tau_\delta^2\tau_\xi + 7680\gamma^4\tau_\theta^6\tau_\delta^2\tau_\xi^2 \\ +5120\gamma^4\tau_\theta^5\tau_\delta^2\tau_\xi^3 + 1280\gamma^4\tau_\theta^4\tau_\delta^2\tau_\xi^4 \end{pmatrix}.$$

Expressions of Moment Variables in Section 4

$$Var\left(q_A^*\right) = \frac{9\tau_\theta + 4\tau_\eta}{36\tau_\theta\left(\tau_\theta + \tau_\eta\right)} + \frac{1}{9\left(\tau_\theta + \tau_\xi\right)}\frac{\tau_\xi}{\tau_\theta} + \frac{\tau_s}{9\tau_\delta\left(\tau_s + \tau_\delta\right)},$$

$$Cov\left(q_A^*, q_B^*\right) = \frac{\tau_\theta\tau_\xi + \tau_\theta\tau_\eta + 2\tau_\xi\tau_\eta}{9\tau_\theta\left(\tau_\theta + \tau_\eta\right)\left(\tau_\theta + \tau_\xi\right)} + \frac{\tau_s}{9\tau_\delta\left(\tau_s + \tau_\delta\right)},$$

$$Var\left(p - f\right) = \frac{2\tau_\theta + \tau_\xi + \tau_\eta}{4\left(\tau_\theta + \tau_\eta\right)\left(\tau_\theta + \tau_\xi\right)} + \frac{1}{\tau_s} + \frac{1}{\tau_\varepsilon},$$

$$Var\left(p\right) = \frac{9\tau_\theta + 4\tau_\eta}{36\tau_\theta\left(\tau_\theta + \tau_\eta\right)} + \frac{9\tau_\theta + 4\tau_\xi}{36\tau_\theta\left(\tau_\theta + \tau_\xi\right)} + \frac{\tau_s + 9\tau_\delta}{9\tau_\delta\left(\tau_s + \tau_\delta\right)} + \frac{1}{\tau_\varepsilon},$$

$$Var\left(f\right) = \frac{\tau_\eta}{9\left(\tau_\theta + \tau_\eta\right)\tau_\theta} + \frac{\tau_\xi}{9\left(\tau_\theta + \tau_\xi\right)\tau_\theta} + \frac{\left(\tau_s + 3\tau_\delta\right)^2}{9\tau_s\tau_\delta\left(\tau_s + \tau_\delta\right)}.$$