# Optimal Issuance under Information Asymmetry and Accumulation of Cash Flows*

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**Abstract**

We study the optimal timing of security issuance to finance a new project when the firm's assets in place have unobservable quality. Stochastic cash flows generated by assets in place reveal information about their quality and simultaneously reduce the required outside funding. A high-quality firm optimally delays issuance unless its accumulated cash or the market belief about its quality is sufficiently high. A low-quality firm does the same and, additionally, issues if market belief and accumulated cash are sufficiently low. Under stated restrictions, the renegotiation-proof optimal security pays outside investors in full before paying anything to original shareholders.

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**Keywords**: dynamic information asymmetry, financing, cash flow accumulation, issuance delay, optimal security

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# 1 Introduction

A classic problem in corporate finance is how firms raise capital in the presence of asymmetric information. The seminal paper by Myers and Majluf (1984) begins by stating the following problem: "Consider a firm that has assets in place and also a valuable real investment opportunity. However, it has to issue common shares to raise part or all of the cash required to undertake the investment project. If it does not launch the project promptly, the opportunity will evaporate." A major finding of Myers and Majluf (1984) is that adverse selection can cause a financial market breakdown because the market cannot ascertain the quality of a firm's assets in place and will only offer a price that a high-quality firm finds too low to accept.

This market breakdown occurs in Myers and Majluf (1984) largely because the firm's manager has a single opportunity to raise capital and make the investment. In many practical applications, however, investment opportunities do not "evaporate" if not undertaken immediately. A firm usually has the option to delay financing and investment if the market conditions are unfavorable. A natural question, therefore, is how the firm should optimally time its financing and investment decisions to minimize adverse selection cost.

A major contribution in this direction is made by Daley and Green (2012). They consider a dynamic lemons market in which a seller wishes to sell an indivisible asset with unobservable quality, and buyers observe "news", or signals about the asset's quality over time. As news arrives, the market updates its belief about the asset quality, and a sale ultimately takes place. At any point in time, if the market is sufficiently optimistic about the asset quality, both types of sellers sell at the average price. If the market is sufficiently pessimistic about asset quality, only the low-quality seller sells, at the low price. For market beliefs between these two thresholds, both types of sellers wait for news and delay sales.

An important assumption in Daley and Green (2012) is that information about the asset quality does not affect the gains from trade between buyers and sellers, and only affects the beliefs of the buyers. In financial markets, however, information about assets often comes in the form of cash flows generated by these assets. This seemingly minor detail has important and interesting economic consequences. A higher-than-expected cash flow simultaneously conveys positive information about the firm's assets in place and relaxes a firm's financial constraint. Both channels reduce a high-quality firm's adverse selection cost in raising outside capital.

To the best of our knowledge, this paper is the first that formally analyzes the dual role of cash flows—reducing asymmetric information and relaxing financial constraint—when a firm wishes to raise outside financing under asymmetric information about its assets in place. We solve for the optimal timing of issuance as well as for the optimal security being issued.

Our model and result work roughly as follows. Time is continuous, $t \in [0, \infty)$. An all-equity firm's assets in place could be one of two types, high or low. The firm's type is observable only to the firm's management and not to the market. The assets in place generate unit cash flows that arrive according to a Poisson process, with intensity $\mu_H$ for the high type and intensity $\mu_L$ for the low type. Cash flows are observable and verifiable. Moreover, we allow for accumulation of cash flows within the firm. At time 0, the firm also has a positive-NPV investment opportunity. The investment requires a cash outlay of $I$, which is a positive integer. Once undertaken, the new project immediately increases the rate of cash flow arrivals from $\mu_\theta$ to $\mu_\theta + k$, where $\theta \in \{H, L\}$ is the unobservable type and $k$ is a commonly known constant. We assume that the firm cannot spin off the new project and finance it separately; instead, the firm must issue securities on the entire firm (assets in place plus new project) to finance investment. The firm chooses the timing of investment and the security type from a broad class of securities that we specify later. The market is competitive and only seeks to break even in expectation. Everyone is risk neutral.

Our primary contribution is to characterize the optimal timing of issuance and investment when cash flows reveal information and relax financial constraint. For ease of discussion, let us take the optimal security as given for now and describe its properties shortly. (The analysis on the optimal issuance timing actually works for any given security.) The general shape of the equilibrium strategy of the two types of firms is illustrated in the Figure 1 below. In the figure, $p$ denotes the probability that the firm is of the high type and $n$ denotes the amount of cash already saved within the firm. In equilibrium, there are two belief thresholds, $\overline{p}$ and $\underline{p}$, partitioning the $(p, n)$ space into three regions: pooling, separating, and waiting.

In the pooling region, i.e., $p > \overline{p}$, both types of firm immediately issue at the average "pooling" price. The high type firm suffers some adverse selection cost, but this cost is dominated by the early realization of the positive NPV of the new project. Obviously, the low type firm also issues whenever the high type firm does. On one hand, for a fixed $n$, pooling is more likely if $p$ is higher because the high type firm suffers a lower adverse selection cost on the amounts to be raised, $I - n$. This information channel is similar to that of Daley and Green (2012). On other other hand, for a fixed $p$, pooling is more likely if $n$ is larger because the amount to be raised in the market, $I - n$, gets smaller. This financial constraint channel is new to our model.

In the waiting region, i.e., $p \in [\underline{p}, \overline{p}]$, both types delay. Again, the high type waits for

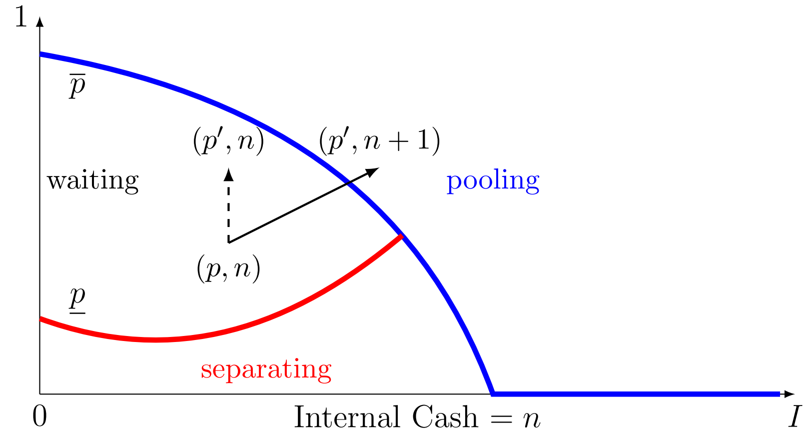



Figure 1: Equilibrium belief thresholds as functions of internal cash level.

a new cash flow because the cash flow will improve the market belief and relax the firm's financial constraint. By contrast, the low type firm waits only because he wishes to imitate the high type for a chance to reach the pooling region, where he would sell the security at an inflated price (relative to the low type's firm value). If the internal cash saving $n$ is sufficiently large, the waiting region actually degenerates into a point.

As illustrated in Figure 1, given current market belief $p$ and internal cash $n$ in the waiting region, the arrival of a new cash flow not only increases the market belief to $p' > p$ but also increases the cash reserve to $n + 1$. In this example, the combination of both channels triggers immediate issuance. By contrast, the information channel of Daley and Green (2012) corresponds to the evolution of $(p, n)$ to $(p', n)$.

The separating region ($p < \underline{p}$) obtains if the market belief and internal cash is sufficiently low. In this case the low type firm expects a very long waiting time to reach the pooling region, if ever. His optimal strategy is to reveal his type probabilistically and raise funds at the low "separating" price.

Now, let us turn to the optimal security design. Since it is a dynamic market, we look for securities that are renegotiation-proof after issuance. As it turns out, it is possible to characterize the renegotiation-proof optimal security under two assumptions/restrictions. First, recovery value upon default is zero. This assumption implies that the new security holders only get paid when cash flows arrive. Second, at issuance, the division of future cash flows between the original shareholders and new security holders is deterministic (i.e., not contingent upon events yet to realize). The optimal security stipulates that shareholders cannot be paid sooner than outside investors. The optimal security is illustrated in Figure

2 below. Specifically, the optimal security specifies an endogenous integer $J \geq 1$ such that cash flows 1, 2, ..., $J-1$ are paid exclusively to the new security holders (full solid fill), cash flow $J$ is possibly split between equity holders and new security holders (partial solid fill), and cash flows $J+1$, $J+2$, ... are paid exclusively to equity holders (no solid fill).

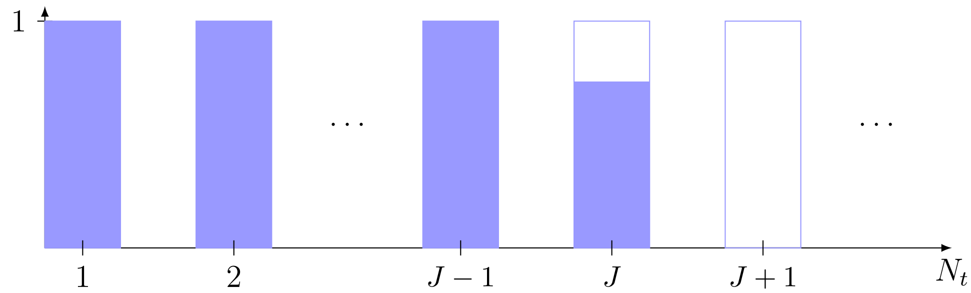



Figure 2: Payment schedule of the optimal security. For each cash flow (bar), the solid fill represents the payment to new security holders and white space represents the payment to original shareholders.

The front-loading of cash flows in the optimal security is intuitive. Just like the timing of issuance, the security choice is determined in equilibrium by the high type firm, whose objective is to minimize adverse selection costs. The more distant is a cash flow in the future, the more severe is adverse selection. Thus, the security that minimizes adverse section cost for the high type firm should front-load payments to new security holders as much as possible.

Our theory makes a few unique empirical predictions. First, issuance and investment can happen after lower-than-expected cash flows. In the model, this is driven by the separation of the low type firm. For example, Korajczyk, Lucas, and McDonald (1990) find that while most equity issuances take place after large positive abnormal returns, 18% of issuances occur after share price declines relative to the market. Second, if issuance follows lower-than-expected cash flows, the post-issue market value of the firm immediately jumps down because issuance in this case is by separation of the low type, a negative signal of the quality of assets in place. By contrast, if issuance follows higher-than-expected cash flows, the post-issue market value of the firm remains a martingale because issuance in this case is by pooling. Third, conditional on issuance after lower-than-expected cash flows, the size of downward drop in market value of the firm is generally non-monotone in the level of internal cash reserves. This is because the separating belief threshold $\underline{p}$ is generally non-monotone in internal cash (see Figure 1). Thus, it is possible to see a larger drop in firm value even if the firm's cash reserve is higher. Finally, the high type firm accumulates more cash prior to investment than the low type does, because the high type only issues in the pooling region but the low type issues in both the pooling and separating regions. Thus, firms making investment using more internal cash would have higher subsequent returns.

From a methodology perspective, our solution technique of the optimal timing of issuance may also be useful in handling other dynamic adverse selection problems with two-dimensional state variables. A key step in our solution method is to discretize the level of internal cash, which allows us to define the value functions with cash level $n$ recursively by the value function with cash level $n + 1$.

### Related Literature

The classic problem of issuance and investment under asymmetric information has been studied by many papers since Myers and Majluf (1984). The main theme of this literature is how a high type firm reduces its adverse selection cost. Our results contribute to two ways of doing so: delayed issuance and security design.

On delayed issuance, the most related paper is Daley and Green (2012). As discussed above, in Daley and Green (2012) new information only gradually reveals the type of the seller but does not affect the seller's financial constraint or willingness to sell. In our model, not only do cash flows gradually reveal the quality of the assets in place, they also relax the firm's financial constraint and reduce the amount of required outside funding. The incentive to delay issuance and investment becomes stronger. Moreover, we also go beyond Daley and Green (2012) in solving the renegotiation-proof optimal security.

Investment delay in our model (and that of Daley and Green (2012)) has an economic underpinning that is distinct from those in the real option literature, where investment delays are caused by the optimal timing of real option exercise. For instance, Morellec and Schürhoff (2011) consider a real option setting in which the firm chooses the optimal investment timing as well as the type of security to issue. They model the firm's revenue from the new project as the firm's unobservable type multiplied by a publicly observable cash flow process. The observable cash flow shocks in their model are independent of firm type and hence cannot reveal information about the firm's type over time. In our model, the market learns about the firm's type by observing its cash flows; the firm delays investment because learning reduces adverse selection and because accumulated cash reduces required outside funding.

Some earlier models feature short-lived information asymmetry. For instance, Lucas and McDonald (1990) consider the impact of information asymmetry on equity financing decisions. In their model, information asymmetry is reset at fixed time intervals, so undervalued firms do not issue equity until the complete resolution of information asymmetry. The immediate issuance by overvalued firms and the delayed issuance by undervalued firms jointly predict an increase in share prices prior to issuance. More recently, Hennessy, Livdan, and Miranda (2010) develop a dynamic signaling model of investment and financing. In their paper, the firm's manager has superior information relative to the market, but this informational advantage is short-lived. Markovian evolution of the firm's type together with short lived private information generate a time-invariant level of information asymmetry in their model. In contrast to both Lucas and McDonald (1990) and Hennessy, Livdan, and Miranda (2010), our model features persistent, time-varying information asymmetry, stemming from persistent firm types and gradual information revelation.

The second way of reducing adverse selection cost is by security design. Models in this category tend to be static. Nachman and Noe (1994) show that if the probability distribution of the firm's value satisfies the "conditional stochastic dominance," debt is the optimal security. Although the optimal security in our model is not debt in the conventional sense (fixed payments at fixed times), the economic underlying mechanism is quite closely linked to Nachman and Noe (1994), as we discuss in Section 3.

When conditional stochastic dominance is violated, there are no one-size-fits-all designs of the optimal security. For example, Fulghieri, Garcia, and Hackbarth (2015) show that, depending on model parameters and the existing capital structure, the optimal security could be straight debt, convertible debt, or warrants.

Chakraborty and Yilmaz (2011) show that in some situations the adverse selection problem can be costlessly solved by the issuance of properly structured convertible debt. Their result requires that (i) information asymmetry should be sufficiently low at the time of maturity of the convertible debt, and (ii) managers cannot benefit from the assets in place or the growth option before the debt matures. In our setup, the availability of cash flows prior to the resolution of market uncertainty limits the benefits of using convertible bonds. In this respect, our results complement those of Chakraborty and Yilmaz (2011).

Our model does not analyze costly signaling by creatively using cash reserves. Bond and Zhong (2016) consider a multi-period Myers and Majluf (1984) in which the firm may repurchase shares using a small amount of cash before raising a larger amount later. While desirable for the sake of generality, a model extension combining the dual role of cash flows and stock repurchases quickly becomes intractable. Nonetheless, our economic insights and Bond and Zhong's do complement each other.

# 2 Model

In this section, we develop a dynamic model of financing and investment decisions under asymmetric information. For ease of reference, Appendix A provides a glossary of the main model variables.

## 2.1 The Firm and the Market

**Assets in Place.** Consider an all-equity firm with pledgable assets in place that belong to one of two types $\theta$, $\theta \in \{H, L\}$. The type is the private information of the firm's management, which we simply refer to as the "firm." As in Myers and Majluf (1984), the existing shareholders are passive, and there is no conflict of interest between existing shareholders and management. All parameters other than the firm's type are common knowledge and the prior probability that $\theta = H$ is $p_0 \in (0, 1)$. The cumulative cash flows generated by the assets in place of the type $\theta$ firm at time $t$ is given by $N_t^\theta$, which is a Poisson process with intensity of arrival $\mu_\theta$ with $\mu_H > \mu_L > 0$. That is, after the previous cash flow arrival, one unit of new cash flow arrives at an exponentially distributed random time with expected time of waiting $1/\mu_\theta$. These cash flows are commonly observable and verifiable. Without loss of generality, at time $t = 0$ the firm has zero cash reserve. Future payoffs are discounted at a rate $r > 0$. Thus, the net present value (NPV) of the assets in place of type $\theta$ firm is

$$\mathbf{E}\left[\int_0^\infty e^{-rt} dN_t^\theta \,\middle|\, \theta\right] = \frac{\mu_\theta}{r}. \tag{1}$$

In addition to its assets in place, the firm has a growth option, which consists of a monopoly access to a new investment opportunity. The firm can invest at any time.[1] The firm is unable to spin-off the new project and finance it independently of the assets in place.[2] If undertaken, the new project immediately incurs a one-off cost of $I$, where $I$ is a positive integer, and increases the rate of free cash flow arrival from $\mu_\theta$ to $\mu_\theta + k$, where $k$ is a commonly known positive constant that is independent of $\theta$. A type-independent new project provides a clear benchmark and allows us to focus on the asymmetric information of

[1] If the investment opportunity expires with some intensity, then the firm effectively has a higher discount rate than $r$, but the qualitative nature of our results still applies.

[2] This would be the case if the new project is inherently nonseparable from the assets in place. For example, they may use the same property, machinery, and key personnel.

assets in place.[3] The NPV of the new project is thus $\frac{k}{r} - I$, which we assume to be positive:

$$\frac{k}{r} > I. \tag{2}$$

The total cost of investment $I$ can be funded with a combination of accumulated internal cash and capital raised from outside investors. Prior to raising external funds and investing, the cash flows generated by the assets in place are assumed to be saved within the firm and used to (partially) finance the new project. One way to interpret this blunt-appearing assumption is that the firm has no credible way to prove its use of accumulated cash other than undertake the investment. For example, Bond and Zhong (2016) consider repurchase of underpriced shares from the market. In their model, repurchases happen partly because it is profitable for undervalued firms and partly because it improves terms for subsequent issuance. In our setting, because shareholders and managers have no conflict of interest, the repurchase of shares is merely a transfer from shareholders to themselves, and hence cannot be a costly, credible signal. For a similar reason, paying dividends to shareholders cannot be a credible signal of firm quality, either. Besides these motivations, the assumption that accumulated cash is automatically saved within the firm is primarily made for tractability. Finally, almost without loss of generality, we assume that cash savings within the firm earn zero return.[4]

**Security Issuance.** The firm can raise funds from outside investors at any time $t \in [0, \infty)$ by issuing a security $S$. Every security $S$ issued at time $t$ can be characterized by a stochastic process $(S_{t+s})_{s \geq 0}$, where $S_{t+s}$ is the *cumulative payment*, without discounting, to investors up to time $t + s$. We will focus attention to the following class of securities, denoted $\mathcal{S}$, that the firm can issue. We call $\mathcal{S}$ the feasible class of securities.

**Definition 1** *A security $S$ issued at time $t$ belongs to class $\mathcal{S}$ if its cumulative payment*

[3] One could overlay asymmetric information about the new project by adjusting our model, but such an extension does not change our main results. For example, we can incorporate type-dependent $k$ as follows. Denote by $k_\theta$ the increase in a type-$\theta$ firm's cash flow arrival rate after investment. As long as $\mu_H + k_H > \mu_L + k_L$ the same solution procedure carries through and the qualitative nature of the equilibrium does not change.

[4] This assumption allows us to solve a complicated two dimensional problem by discretizing the state space along one of the dimensions. One way to incorporate interest-bearing internal cash savings is to consider a cash-flow generating process $\Delta_N \cdot N_t$ for a arbitrary small $\Delta_N > 0$ and allow discrete internal cash appreciation. Qualitatively, a positive rate of return on internal cash holdings increases incentives of the high type firm to delay investment, by reducing the implied costs of waiting.

*process can be decomposed as follows:*

$$dS_{t+s} = dS^d_{t+s} + \lambda_{N_{t+s}-N_t} dN_{t+s}, \tag{3}$$

*where*

*(i)* $S^d_{t+s}$ *is a deterministic non-decreasing function of* $s$*, given public information observed up to time* $t$*;*

*(ii)* $(\lambda_n)_{n=1}^{\infty}$ *is a deterministic sequence with* $0 \leq \lambda_n \leq 1$ *for all* $n$*, given public information observed up to time* $t$*; and*

*(iii)* $N_{t+s} - N_t$ *is the cumulative cash flow of the firm generated in the time interval* $(t, t+s]$.

The class $\mathcal{S}$ of securities is broad enough to include many commonly used securities, including:

- equity (with $S^d_t \equiv 0$ for all $t$ and $\lambda_n \equiv \alpha < 1$ for some $\alpha$ for all $n$);
- fixed maturity zero-coupon debt (with $dS^d_T > 0$ for some fixed $T$ and $\lambda_n \equiv 0$ for all $n$);
- perpetuity debt (with $dS^d_t = cdt$ for some $c > 0$ and $\lambda_n \equiv 0$ for all $n$); and
- any combination of those instruments.

The main restrictive feature of class $\mathcal{S}$ is that the division of upcoming cash flows between the firm and the market is determined at issuance, rather than contingent on information that is potentially revealed later. For instance, a security issued at time 0 that says "the holder of the security receives 100% of the first cash flow if the first cash flow arrives within one month, and nothing otherwise" is not in class $\mathcal{S}$, because the division of the first cash flow is contingent on events that are yet to realize at time of issuance. That said, our solution technique for the optimal *timing* of issuance actually does not hinge upon the security itself. As long as the issued security does not entirely eliminate the adverse selection problem, the intuition behind our results is applicable for securities beyond class $\mathcal{S}$.

**Market.** There is a group of competitive risk-neutral outside investors, called the "market," who do not observe $\theta$. The market has a prior belief $p_0$ at time 0 that the firm is of type $H$. The market participants are risk-neutral, discount future payoffs at a rate $r > 0$, and update their beliefs about the firm based on the history of cash flows, timing of issuance, and the type of security chosen.

**Post-Issuance.** Once a security is issued and the new project is undertaken, the cash flows may be used to pay dividend to existing shareholders, pay the obligations to the holders of the new security, or be kept within the firm. Denote by $D_t$ the cumulative dividend paid to shareholders up to time $t$. The amount of cash $C_t$ the firm keeps for future payments is given by

$$C_t = N_t - S_t - D_t. \tag{4}$$

If at any time $\tau$, the firm is scheduled to pay the security holder ($dS_\tau > 0$) but runs out of cash ($C_\tau = 0$), it defaults. For simplicity, we make the following assumption about default.

**Assumption 1** *The firm has zero recovery value upon default.*

Although the zero-recovery-value assumption seems strong, it is not unreasonable for firms in either one of the following situations. First, the firm's assets are valuable only if combined with the human capital of key personnel, who are unlikely to stay if the firm is in bankruptcy. Second, the firm structure is so complicated that liquidation is long and painful, and the security holders would be willing to accept a much lower market value for their claims. For instance, right after Lehman Brothers filed bankruptcy in 2008, the CDS-auction-determined recovery value of Lehman senior debt was below 9 cents on the dollar (see Du and Zhu (2016)). As of early 2015, Lehman's estimated recovery value is more than 32 cents on the dollar.[5]

We allow the firm to renegotiate future payments to the security holders at any time after issuance but prior to default. Renegotiation is an important feature of a dynamic model, in which "the game is not over" after issuance. The renegotiation protocol is specified as follows.

**Assumption 2** *At any time $t$ after issuance:*

1. *Shareholders may propose a take-it-or-leave-it new schedule of payments $(\tilde{S}_u)_{u\geq t}$.*

2. *The security holders either accept or reject.*

3. *If the security holders accept, the new security $(\tilde{S}_u)_{u\geq t}$ replaces the old one immediately; otherwise, the initial security $(S_u)_{u\geq t}$ remains in place.*

[5]See "Lehman to Pay Out Additional \$7.6 Billion to Creditors," by Patrick Fitzgerald, *Wall Street Journal*, March 26, 2015.

For simplicity, we view the new security holders as one entity that is not subject to free-riding or hold-out problems.

As we elaborate shortly, the combination of zero recovery value and renegotiation implies that there are no defaults on equilibrium path. If a pending payment $dS_t > 0$ is due in the next instant and the firm has zero cash, the shareholders can always propose to postpone the payment $dS_t$ to some later date, and the security holders will agree, for the failure of renegotiation implies zero recovery value.

Because promised payments from securities that can be renegotiated will not be creditable anyway, we will focus on securities that are renegotiation-proof.

**Definition 2** *A security $S$ is called **renegotiation-proof**, written as $S \in \mathcal{S}_{RP}$, if no type of firm can strictly gain by successfully renegotiating its payments at any point in time after issuance.*

## 2.2 Strategies

Now, we turn to the firm's and the market's strategies.

Instead of defining strategies of the market explicitly, we model the market as a collection of price processes $M(S) = (M_t(S))_{t \geq 0}$ for all $S \in \mathcal{S}$, where $M_t(S)$ is the price that the market is willing to pay to the issuer of security $S$ at time $t$. This modeling approach is without loss of generality. In order to allow for mixed strategies of the market, we require that

$$\text{for all } S \in \mathcal{S}, \text{ the process } M(S) \text{ is adapted to } \mathbb{F}^N \times \mathbb{F}^{R_M},$$

where $\mathbb{F}^N = (\mathcal{F}_t^N)_{t \geq 0}$ is the filtration generated by firm's cash flow process $N_t$ and $\mathbb{F}^{R_M}$ is a sufficiently rich[6] filtration independent of $\mathbb{F}^N$.[7] Intuitively, $\mathbb{F}^N$ allows the security prices to depend on the market beliefs about the firm's type, and $\mathbb{F}^{R_M}$ allows for additional randomization conditional on market belief.

Heuristically, an issuing strategy of the firm is a choice of the security and the probability of issuance at any time $t$. Formally, we define a *mixed strategy* of a type $\theta$ firm to be a pair of processes $(S^\theta, \pi^\theta)$. The process $S^\theta$ is adapted to the filtration $\mathbb{F}^M$ of observed market prices $M(S)$, for all $S \in \mathcal{S}$. At every point in time $t$, $S_t^\theta$ is the security chosen by type $\theta$. $\pi_t^\theta$

[6] It is sufficient for $\mathbb{F}^{R_M}$ to contain a countable number of independent uniform random variables.

[7] Throughout the paper we assume that all filtrations satisfy standard conditions, i.e., that they are right continuous and contain all **P**-null sets. All processes are assumed to be cádlág (right continuous with existing left limits).

is the cumulative probability of issuance by the type $\theta$ firm up to time $t$. This definition of strategies accommodates randomized issuance conditional on observed price $M_t(S^\theta)$.

As an example, consider the following "pure" strategy: issue security $\hat{S}$ as soon as its price is above some threshold $\hat{M}$. The corresponding processes $(S^\theta, \pi^\theta)$ are given by

$$S_t^\theta \equiv \hat{S}, \qquad \pi_t^\theta = \begin{cases} 0, & \text{if } \sup_{s\le t} M_t(S_t^\theta) < \hat{M}, \\ 1, & \text{if } \sup_{s\le t} M_t(S_t^\theta) \ge \hat{M}. \end{cases} \tag{5}$$

Our general definition of the mixed strategies allows the firm to mix every instance between issuing security $S_t^\theta$ and waiting with "probabilities" $\frac{d\pi_t^\theta}{1-\pi_{t-}^\theta}$ and $1 - \frac{d\pi_t^\theta}{1-\pi_{t-}^\theta}$, respectively. The firm may choose not to issue at all, that is, $\mathbf{P}(\lim_{t\to\infty} \pi_t^\theta = 1)$ could be less that 1.

After issuing a renegotiation-proof security $S \in \mathcal{S}_{RP}$ at time $t$ the firm decides on the dividend policy. Define the dividend strategy of the type $\theta$ firm to be a process $D^\theta = (D_{t+s}^\theta)_{s\ge 0}$ adapted to $\mathbb{F}^N$ with $D_{t+s}^\theta$ being the cumulative dividend paid up to time $t+s$ by type $\theta$ firm.

## 2.3 Equilibrium

For every renegotiation-proof security $S \in \mathcal{S}_{RP}$ issued at time $t$, let $D^*$ be the optimal dividend policy that maximizes the shareholders' value:

$$F_E^\theta(S) \equiv \sup_D \mathbf{E}\left[\int_t^\tau e^{-ry} dD_{t+y} \,\Big|\, \theta\right], \tag{6}$$

where $\tau = \inf\{y > 0 : C_{t+y} = N_{t+y} - N_t - D_{t+y} - S_{t+y} < 0\}$ is the time of default. Then, $F_E^\theta(S)$ is the resulting supremum of shareholders' value given security $S$ and type $\theta$.

Let $F_M^\theta(S)$ denote the implied market value of the security $S$ conditional on the type $\theta$:

$$F_M^\theta(S) = \mathbf{E}\left[\int_t^{\tau^*} e^{-ry} dS_{t+y} \,\Big|\, \theta\right], \tag{7}$$

where $\tau^* = \inf\{y > 0 : C_{t+y} = N_{t+y} - N_t - D_{t+y}^* - S_{t+y} < 0\}$ and $D^*$ solves (6).

We are now ready to define the equilibrium in our model.

**Definition 3** *Strategies of the firm $(S^\theta, \pi^\theta), \theta \in \{H, L\}$, market prices $M(S), S \in \mathcal{S}$, and market belief $p$ constitute an equilibrium if:*

1. ***Optimality:*** *For each type $\theta$, the strategy $(S^\theta, \pi^\theta)$ maximizes the value of the firm's original shareholders, given the market prices $M(S), S \in \mathcal{S}$:*

$$(S^\theta, \pi^\theta) \in \text{argmax}_{S,\pi} \, \mathbf{E}\left[\int_0^\infty e^{-rt} \, F_E^\theta(S_t) \, d\pi_t\right]. \tag{8}$$

2. ***Zero Profit:*** *Conditional on the firm's investment at time $t$, the market earns zero profits. That is, if a high type firm and a low type firm issue the same security $S$ in equilibrium, then*

$$M_t(S) = \mathbf{E}\left[F_M^\theta(S) \,\middle|\, p_t, \; \pi^H, \pi^L\right]. \tag{9}$$

*Otherwise (i.e., $S_t^H \neq S_t^L$),*

$$M_t(S_t^\theta) = F_M^\theta(S_t^\theta). \tag{10}$$

3. ***Consistency:*** *Market belief $p_t$ is consistent with the history of cash flows and strategies of the firm, i.e., it satisfied Bayes rule along the equilibrium path.*

Optimality and consistency are standard features of any equilibrium. Firm optimality is required by Part 1 of Definition 3, while Parts 2 and 3 of Definition 3 guarantee market optimality. Since our market is perfectly competitive, investors earn zero profits (Part 2) any time the firm issues any security.

# 3 Optimal Security Design Conditional on Issuance

The general problem of dynamic financing can be decomposed into the optimal timing of issuance and the security choice conditional on issuance. In this section we solve for the optimal security conditional on the firm deciding to issue and invest. Of course, the decision to issue and invest is consistent with the optimal security being issued. The optimal timing of issuing the optimal security will be solved in the Section 4.

Conditional on issuance, the security choice is determined by only two factors: (a) market belief $p$ that the firm is of the high type and (b) the amount of capital raised from the market $I - n$, where $n$ is accumulated cash within the firm. Because the cash flow arrival process is Poisson, the security choice problem is stationary. Therefore, without loss of generality, we solve for the optimal security conditional on issuing at time $t = 0$, with an arbitrary level of accumulated cash $n$.

We proceed in two steps. The first step is to show that the renegotiation-proof requirement implies that payments to new security holders must be made on cash-flow arrival times.

The second step is to solve for the division schedules $\{\lambda_j\}$, where $\lambda_j$ is the fraction of the $j$-th cash flow paid to the security holders, counting from the time of issuance.

Recall that any security in $\mathcal{S}$ has two kinds of payments: (i) those that happen on deterministic dates and (ii) those that happen on (stochastic) dates of cash flow arrivals. If the deterministic part of a renegotiation-proof security payments $S^d$ were not zero, then there would be a positive probability that the firm will not be able to make its promised payment after a sufficiently long time without any cash flow arrival. Because of zero recovery value in bankruptcy, an instant before the payment is due, the firm that has zero cash flow would be able to successfully renegotiate the future payments by offering the market a security with an arbitrary small value $\varepsilon > 0$ and avoid bankruptcy. Thus, any renegotiation-proof security must pay only at the times of firm's cash flow arrivals. The following lemma summarizes this result.

**Lemma 1** *Security $S \in \mathcal{S}$ is renegotiation-proof only if $S^d \equiv 0$.*

Although Lemma 1 does not fully characterize the set of renegotiation-proof securities $\mathcal{S}_{RP}$, it shows that it is sufficient to consider securities that make payments only when the firm receive cash flows (i.e., at times when $dN_t > 0$). For all such securities the firm is better off paying out all excess cash as dividends after the investment. Therefore, for every renegotiation-proof security $S \in \mathcal{S}_{RP}$ the value of shareholders immediately after the issuance is given by

$$F_E^\theta(S) = \mathbf{E}\left[\int_0^\infty e^{-rt}(1-\lambda_{N_t})dN_t \;\middle|\; \theta\right] = \frac{\mu_\theta + k}{r} - F_M^\theta(S), \tag{11}$$

with

$$F_M^\theta(S) = \mathbf{E}\left[\int_0^\infty e^{-rt}\lambda_{N_t}dN_t \;\middle|\; \theta\right]. \tag{12}$$

The next lemma shows that a high type firm cannot use renegotiation-proof securities to avoid pooling in equilibrium.

**Lemma 2** *No security $S \in \mathcal{S}_{RP}$ can be used as a separating device.*

Intuitively, any security that pays only at the times of cash flow arrivals is cheaper for the low type firm due to a higher discounting of future payoffs. Since the average cost of issuing a pooling security for both types is exactly $I$, the low type's cost is strictly below $I$. By issuing any other security different from the high type's, the low type would reveal herself and have to pay exactly $I$ in present value terms. Thus, if the high type issues some security $S \in \mathcal{S}_{RP}$ at any point in time, the low type mimics.

Given the previous lemma, any security $S \in \mathcal{S}_{RP}$ issued by the high type induces pooling in equilibrium. Since a fraction $\lambda_j$ of the $j$-th cash flow is paid to new security holders, the expected value of the security $S$ is

$$pF_M^H(S) + (1-p)F_M^L(S) = p\sum_{j=1}^{\infty}\lambda_j\left(\frac{\mu_H + k}{\mu_H + k + r}\right)^j + (1-p)\sum_{j=1}^{\infty}\lambda_j\left(\frac{\mu_L + k}{\mu_L + k + r}\right)^j, \quad (13)$$

where $p$ is the probability of a high type firm and

$$\left(\frac{\mu_\theta + k}{\mu_\theta + k + r}\right)^j \equiv q_j(\theta) \quad (14)$$

is the present value of the $j$-th cash flow conditional on a type $\theta$ firm.

The following proposition characterizes a high type firm's optimal security, simply referred to as the "optimal security." This security incurs the smallest adverse selection cost to a high type firm. Because the low type will imitate the high type, this security will be the one issued in a pooling equilibrium.

**Proposition 1** *Suppose that the firm wishes to raise $I - n$ from the market, whose belief that the firm is of the high type is $p$. The security $S^*(p, I-n)$ that maximizes payoff of the original shareholders of a high type firm has the following structure. There exists a unique positive integer $J(p, I-n)$ such that*

$$\lambda_j = \begin{cases} 1, & \text{if } j < J(p, I-n), \\ 0, & \text{if } j > J(p, I-n), \end{cases} \quad (15)$$

*and $J(p, I-n)$ and $\lambda_{J(p,I-n)} \in (0, 1]$ satisfy:*

$$I - n = p\left[\sum_{j=1}^{J(p,I-n)-1} q_j(H) + \lambda_{J(p,I-n)}q_{J(p,I-n)}(H)\right] + (1-p)\left[\sum_{j=1}^{J(p,I-n)-1} q_j(L) + \lambda_{J(p,I-n)}q_{J(p,I-n)}(L)\right]. \quad (16)$$

Obviously, the security $S^*(p, I-n)$ of Proposition 1 is renegotiation-proof because the firm's cash flow arrivals are verifiable (and hence, the payments can be enforced in court). Figure 3 illustrates the optimal security. Payments to outside security holders are front-

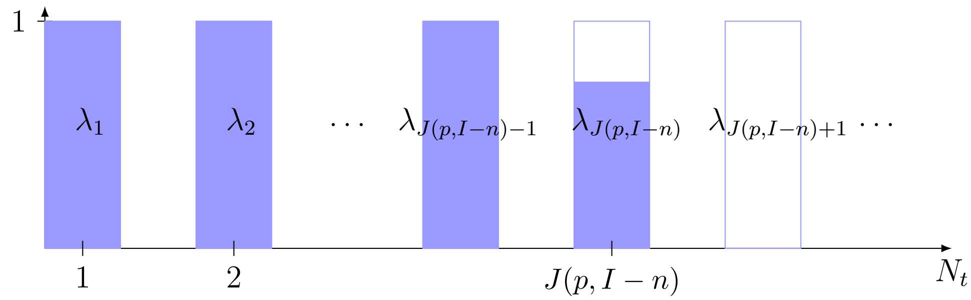



Figure 3: Payment schedule of the optimal security $S^*(p, I-n)$.

loaded as much as possible, in the sense that the firm's original shareholders never get paid sooner than outside security holders. Intuitively, the ratio of the present values of the $j$-th cash flow generated by the high type firm and the low type firm,

$$\left(\frac{\mu_H + k}{\mu_H + k + r}\right)^j \Bigg/ \left(\frac{\mu_L + k}{\mu_L + k + r}\right)^j = \frac{q_j(H)}{q_j(L)}, \tag{17}$$

is increasing in $j$, since $\mu_H > \mu_L$. Therefore, later cash flows incur a higher adverse selection cost for the high type firm, hence the desire to front-load payments. The fraction $\lambda_j$, $j < J(p, I-n)$, is capped at 100% since the firm cannot credibly pledge to pay more than 100% of early cash flows. The total number of cash flows to outside security holders, $J(p, I-n)$, and the fraction of the last one, $\lambda_{J(p,I-n)}$, are determined by the break-even condition of the market.

Using Proposition 1, we can define the *pooling value* of the original shareholders of a type $\theta$ firm that has accumulated $n$ units of cash internally and issues security $S^*(p, I-n)$ to finance the project as

$$\begin{aligned}\Phi_\theta(p, n) = F_E^\theta(S^*(p, I-n)) &= \frac{\mu_\theta + k}{r} - F_M^\theta(S^*(p, I-n)) \\ &= \frac{\mu_\theta + k}{r} - \sum_{j=1}^{J(p,I-n)-1} q_j(\theta) - \lambda_{J(p,I-n)} q_{J(p,I-n)}(\theta).\end{aligned} \tag{18}$$

**Remark.** The result of Proposition 1 is deeply connected to the findings of Nachman and Noe (1994). To see this, write the value of the security $S$, conditional on type $\theta$, as

$$F_E^\theta(S) = \mathbf{E}\left[\int_0^\infty e^{-rt} \lambda_{N_t} dN_t \;\Big|\; \theta\right] = \sum_{j=1}^\infty \lambda_j \left(\frac{\mu_\theta + k}{\mu_\theta + k + r}\right)^j = \sum_{j=1}^\infty \lambda_j q_j(\theta). \tag{19}$$

One can view $Q(\theta) = \{q_j(\theta)\}_{j=1}^{\infty}$ as a measure on $\mathbb{N}$. Although $Q(\theta)$ is not a probability measure, it inherits several useful properties from its probabilistic counterpart, the geometric distribution. For the sake of this exercise, it is important that the ratio $q_j(H)/q_j(L)$ is increasing in $j$, which implies strict conditional stochastic dominance for probability measures. Nachman and Noe (1994) and Fulghieri, Garcia, and Hackbarth (2015) show that under conditional stochastic dominance, debt minimizes the cost of adverse selection and is hence the optimal security. The security $S^*(p, I-n)$ has properties similar to static debt: in "low-cash-flow" states outside investors are paid first, and original shareholders receive the residual cash flows.

**Uniqueness.** One can show that with additional restrictions on market beliefs, namely (D1), pooling on security $S^*(p, I-n)$ is the unique equilibrium of the issuance game at $t = 0$. The proof follows Theorem 5 of Nachman and Noe (1994) and we omit it for brevity.

# 4 Optimal Timing of Issuing the Optimal Security

In the previous section we have solved the optimal security conditional on the firm's issuing decision. In this section we solve the optimal timing of issuing the optimal security. The unique feature of the dynamic model with cash flow accumulation is that the arrival of cash flows changes market belief and internal cash reserve simultaneously.

## 4.1 Market Belief

To explore the evolution of market belief, note that the market revises its belief based on two sources of information: publicly available cash flows and equilibrium investment strategies. The former gives rise to the non-strategic component of the belief process, while the latter gives rise to the strategic (signaling) one. To disentangle these two components, as well as to simplify exposition, we introduce an auxiliary belief process $\hat{p}_t$, defined as:

$$\hat{p}_t = \mathbf{P}(\theta = H \mid \mathcal{F}_t^N). \tag{20}$$

The process $\hat{p}_t$ is the probability of facing a high type firm, conditional *only* on the observed cash flows up to time $t$. In other words, $\hat{p}_t$ represents the non-strategic component of the belief process.

It is convenient to work in the log-likelihood space $\hat{z}_t = \ln\left(\frac{\hat{p}_t}{1-\hat{p}_t}\right)$. Because the number of cash flows that have arrived by time $t$ for the type $\theta$ firm is Poisson distributed with parameter $\mu_\theta t$, we have

$$\hat{z}_t = \ln\left(\frac{p_0}{1-p_0}\frac{\frac{1}{N_t}\exp(-\mu_H t)(\mu_H t)^{N_t}}{\frac{1}{N_t}\exp(-\mu_L t)(\mu_L t)^{N_t}}\right) = \ln\left(\frac{p_0}{1-p_0}\right) + N_t \ln(\mu_H/\mu_L) - (\mu_H - \mu_L)t. \quad (21)$$

Letting

$$\Delta = \ln(\mu_H/\mu_L) > 0, \quad (22)$$

we see that $\hat{z}_t$ evolves according to

$$d\hat{z}_t = -(\mu_H - \mu_L)dt + \Delta dN_t. \quad (23)$$

That is, the non-strategic component of the market belief drifts down at a deterministic rate and jumps up whenever a cash flow arrives. A sample path of non-strategic belief component $\hat{z}_t$ is shown in Figure 4. Since there is a one-to-one, monotone increasing mapping between $\hat{p}$ and $\hat{z}$, we refer to both of them as "belief."

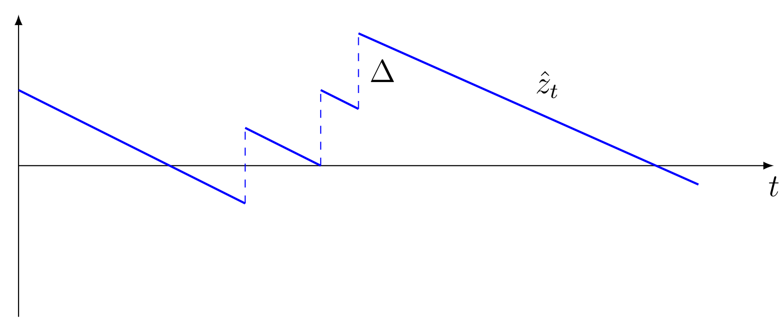



Figure 4: Sample path of the non-strategic belief component $\hat{z}_t$.

To demonstrate the effect of the second, strategic, source of information, define $z_t$ as $z_t = \ln\left(\frac{p_t}{1-p_t}\right)$. Then, applying the Bayes rule we can decompose belief $z_t$ into: [8]

$$z_t = \hat{z}_t + \ln\left(\frac{1-\pi^H_{t-}}{1-\pi^L_{t-}}\right). \quad (24)$$

The second term in (24) captures the pure signaling effect on the market's belief. For example, if $\pi^H_{t-} < \pi^L_{t-}$, then the absence of issuance before time $t$ is a positive signal of the

[8] We denote by $\pi^\theta_{t-}$ the right limit of $\pi^\theta$ at $t$, i.e., $\pi^\theta_{t-} = \lim_{u\uparrow t}\pi^\theta_u$.

firm's quality.

The non-strategic component of market belief process, $\hat{z}_t$, does not depend on the firm's type. The firm, however, knows its type, and the manager observes the "true" belief process $\hat{z}_t$, denoted $\hat{z}_t^\theta$. Separating the martingale component in (23), we can write:

$$d\hat{z}_t^H = \mu_H \Delta dt - (\mu_H - \mu_L)dt + \Delta(dN_t - \mu_H dt), \tag{25}$$

$$d\hat{z}_t^L = \mu_L \Delta dt - (\mu_H - \mu_L)dt + \Delta(dN_t - \mu_L dt), \tag{26}$$

Path by path, $\hat{z}_t$ coincides with $\hat{z}_t^H$ if the firm is of the high type, and it coincides with $\hat{z}_t^L$ if the firm is of the low type. At any instance $s$, the market's conditional distribution of future realizations of $\hat{z}_t$, $t > s$, is given by (23), the high type's corresponding distribution of $\hat{z}_t^H$ is given by (25), and the low type's conditional distribution of $\hat{z}_t^L$ is given by (26). This difference in the assessment of future conditional distributions is the essential source of *dynamic* information asymmetry.

## 4.2 Preliminary Steps

In this subsection we lay down a few preliminary steps towards a full analysis of the optimal timing of issuance. We begin by highlighting the properties that any equilibrium of the dynamic game possesses.

**Lemma 3** *In any equilibrium in which the high type plays a pure strategy, investment happens only after one of the following events:*

1. *(Pooling) The firm raises $I - N_t$ by issuing $S^*(p_t, I - N_t)$;*
2. *(Separating) The firm raises $I - N_t$ by issuing a security different from $S^*(p_t, I - N_t)$;*
3. *(Investment with cash) The firm accumulates $N_t = I$ units of cash and invests only using cash.*

This lemma summaries the "skimming" property of any equilibrium: along any equilibrium path that involves issuance of securities the low type firm either separates by issuing a security at a low price on an early date in expectation, or pools with the high type firm on a later date in expectation.

The intuition behind Lemma 3 relies on the fact that waiting is more costly for the low type firm than for the high type. The cost differential stems from two sources: information arrival and cash accumulation, which are linked in our model. Over time, the firm accumulates internal cash and the need for external financing is reduced. Thus, the low type firm's expected benefits of issuing overpriced securities go down. Moreover, since $\mu_H > \mu_L$, the low type expects the future market beliefs (and corresponding security prices) to be below the expectations of the high type, and as a result the low type is more likely to issue earlier.

More subtly, the restriction that the high type plays a pure strategy pins down the pooling security to be $S^*(p_t, I - N_t)$, where $p_t$ is the current market belief. Indeed, in the unique equilibrium that satisfies a set of refinements discussed in Section 4.4, the high type plays a pure strategy, given explicitly in the next subsection.

Given Lemma 3, we conjecture that the equilibrium consists of the following regions in terms of the state variables $(z_t, N_t)$: the separating region $\Omega_M$, the pooling region $\Omega_P$, and the waiting region $\Omega_W$. The first two of these regions are characterized by the following conditions:[9]

$$V_H(t) = \Phi_H(z_t, N_t) \qquad (z_t, N_t) \in \Omega_P, \tag{27}$$

$$V_L(t) = \frac{\mu_L + k}{r} - I \qquad (z_t, N_t) \in \Omega_S. \tag{28}$$

In other words, the high type is willing to pool only if the continuation value from of waiting $V_H(t)$ is exactly equal to the expected value from pooling $\Phi_H(z_t, N_t)$. Similarly, if the low type firm were to issue a security at a price revealing its type, it has to be indifferent between doing so and receiving the continuation value.

In the waiting region (if it is not degenerate) the value functions $V_\theta(t)$ are function of two state variables, market belief $z$ and accumulated cash flows $n$, and satisfy the following Hamilton-Jacobi-Bellman (HJB) equations for $(z, n) \in \Omega_W$:

$$\begin{cases} (r + \mu_H)V_H(z, n) = \mu_H V_H(z + \Delta, n + 1) - (\mu_H - \mu_L)\frac{\partial}{\partial z}V_H(z, n), \\ (r + \mu_L)V_L(z, n) = \mu_L V_L(z + \Delta, n + 1) - (\mu_H - \mu_L)\frac{\partial}{\partial z}V_L(z, n). \end{cases} \tag{29}$$

The HJB equation reflects the fact that in the next $dt$ interval of time the next unit of cash flow arrives with intensity $\mu_\theta$, in which case the market belief jumps up by $\Delta$ and the firm's payoff changes by $V_\theta(z + \Delta, n + 1) - V_\theta(z, n)$. If the cash flow does not arrive, the waiting payoff is discounted by $-rV_\theta(z, n)$ and further decreases due to a downward drift of market belief by $-(\mu_H - \mu_L)\frac{\partial}{\partial z}V_\theta(z, n)$.

[9] We convert beliefs into the loglikelihood space and with a slight abuse of notation write $\Phi_\theta(z, n)$.

In order to characterize the equilibrium, we solve equations (29) with endogenous boundary conditions (27) and (28).

## 4.3 Dynamic Equilibrium

For any market belief $z_t$ and internal cash $N_t$, we define the *pooling price* to be

$$\frac{e^{z_t}}{e^{z_t}+1} F_M^H(S^*(z_t, I-N_t)) + \frac{1}{e^{z_t}+1} F_M^L(S^*(z_t, I-N_t)), \tag{30}$$

and we let the *separating price* be $F_M^L(S^*(-\infty, I-N_t))$. The next two propositions characterize the equilibrium issuance strategies of the firm and market prices. Proposition 2 (Proposition 3) applies to situations with relatively low (high) levels of cash accumulation within the firm.

**Proposition 2 (Two-threshold case)** *The equilibrium is characterized by a series of thresholds $\{(\underline{z}(n), \overline{z}(n))\}_{n=0}^{I}$ and a cutoff $n^*$. Let $n$ be the amount of cash that the firm has accumulated up to time $t$. Then, for $n \leq n^*$, the equilibrium has the following shape:*

1. *The thresholds $\underline{z}(n)$ and $\overline{z}(n)$ are given by equations (49) and (52) in Appendix B.4. Moreover, $\underline{z}(n) < \overline{z}(n)$.*

2. *The market offers the separating price if $z < \overline{z}(n)$ and the pooling price if $z \geq \overline{z}(n)$. If $z_0 = \overline{z}(0)$, the probability that the market makes a pooling offer at $t = 0$ is arbitrary.*

3. *A high type firm issues at the pooling price whenever $z \geq \overline{z}(n)$ and a pooling price is offered. It does not issue otherwise. The high type firm's value function $V_H(z, n)$ is given by equation (53) in Appendix B.4.*

4. *A low type firm issues for sure if any pooling price is offered. If only the separating price is offered and $z < \underline{z}(n)$, it mixes at the separating price with the acceptance probability $1 - e^{z - \underline{z}(n)}$. If only the separating price is offered and $z = \underline{z}(n)$, it mixes at the separating price with the rate of acceptance $\mu_H - \mu_L$. The low type firm's value function $V_L(z, n)$ is given by equation (55) in Appendix B.4.*

The equilibrium described in Proposition 2 applies to situations with relatively low level of internal cash, since it requires $n \leq n^*$. Figure 5 shows a sample path of market belief $z$ with corresponding lower and upper equilibrium thresholds $(\underline{z}(n), \overline{z}(n))$.

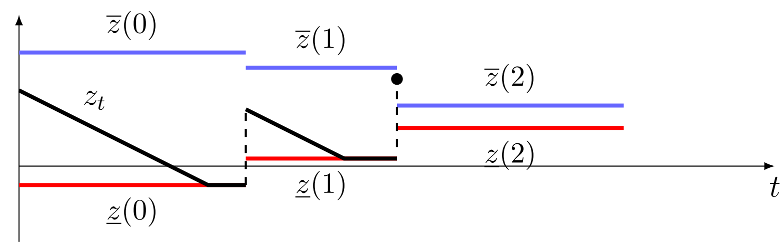



Figure 5: Sample path of market belief process $z_t$ with equilibrium thresholds $(\underline{z}(n), \overline{z}(n))_{n=0}^2$.

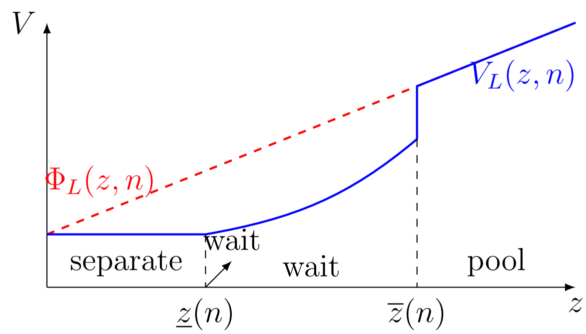
When the market's belief reaches the upper threshold $\overline{z}(n)$ (i.e., when outside investors become sufficiently optimistic about the firm's quality), both types invest by issuing $S^*(z, I - n)$ at a pooling price. The upper threshold $\overline{z}(n)$ is essentially chosen by a high type firm that trades off a higher current adverse selection cost against the time value of delaying investment in a positive NPV project.

When the market's belief reaches $\underline{z}(n)$, only a low type firm issues. By part 4 of Proposition 2, it does so probabilistically. The lower threshold is chosen so that a low type firm is indifferent between investing now (and thus revealing its type) and postponing investing with the hope that positive cash flow shocks would lead the market's belief to hit the upper threshold $\overline{z}$ in the future. The equilibrium rate of mixing by a low type firm at the lower threshold forces the beliefs to be *reflecting*. That is, conditional on not observing issuance at $\underline{z}(n)$, the market's belief immediately adjusts upwards, because a high type firm would never invest at the low threshold. The exact rate of separation by the low type firm exactly offsets the downward drift in the market beliefs and results in a belief process that never falls below $\underline{z}(n)$, i.e. conditional on no issuance:

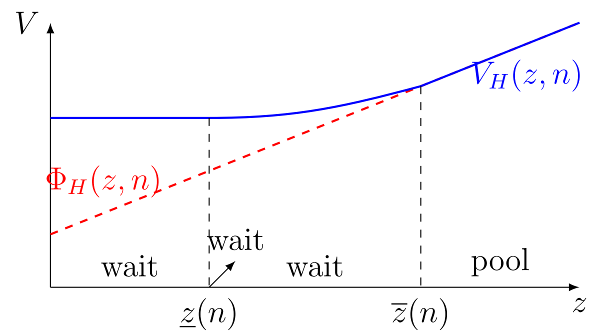
$$dz_t = \begin{cases} -(\mu_H - \mu_L)dt + \Delta dN_t, & \text{if } z_t > \underline{z}(N_t), \\ \Delta dN_t, & \text{if } z_t = \underline{z}(N_t), \\ \underline{z}(N_t) - z_t, & \text{if } z_t < \underline{z}(N_t). \end{cases} \tag{31}$$

For all the market belief levels between the two thresholds, $\overline{z}(n)$ and $\underline{z}(n)$, there is a region of optimal inaction in which both firm types postpone their decisions. A high type firm expects more cash flow arrivals, which reduce both underpricing per unit of capital raised and the amount external capital needed. A low type firm "speculates" on positive shocks to its cash flows and higher overpricing of its (yet to be issued) securities, even though new cash flow arrivals will reduce the amount of external financing. The dynamic lemons model of Daley and Green (2012) only captures the channel that cash flows convey information, but our model additionally captures the channel that cash flows also reduce the required outside financing.

Figure 6 shows the shapes of the continuation values $V_H(z,n)$ and $V_L(z,n)$ as functions of $z$. If the market belief is below the threshold $\underline{z}(n)$, the mixing strategy of the low type guarantees that the belief jumps immediately to $\underline{z}(n)$ and stays there until a new cash flow arrives. This implies that $V_H(z,n)$ and $V_L(z,n)$ do not change below $\underline{z}(n)$, hence the flat parts of the graphs. Given the low type's mixed strategies, the lower threshold $\underline{z}(n)$ is also a "waiting point" where the firm spends a positive amount of time in expectation until a new cash flow arrives.




(a) Low type's value function $V_L(z,n)$




(b) High type's value function $V_L(z,n)$

Figure 6: Equilibrium value functions for $n$ such that $\overline{z}(n) > \underline{z}(n)$

In the waiting region, the value functions increase in $z$ because more favorable market beliefs imply higher market prices of securities issued in the future. At the upper threshold $\overline{z}(n)$ the continuation value $V_L(z,n)$ is generally discontinuous. This discontinuity comes from the model feature that cash flows are lumpy. Intuitively, just above $\overline{z}(n)$ a low type firm issues at the pooling price; just below $\overline{z}(n)$ the best a low type firm can do is to wait until a new cash flow arrives, but waiting for the new cash flow incurs the cost of time value of money and reduces the expected profit from selling overpriced securities. Hence the discrete jump of $V_L(z,n)$ as $z$ crosses $\overline{z}(n)$. The high type's continuation value $V_H(z,n)$ is continuous at $\overline{z}(n)$ because it is the high type's incentive conditions that determines this upper threshold in equilibrium.

The next proposition shows that if internal cash is sufficiently high, the waiting region collapses to a single point, i.e. $\underline{z}(n) = \overline{z}(n)$.

**Proposition 3 (One-threshold case)** *The equilibrium is characterized by a series of thresholds* $\{(\underline{z}(n), \overline{z}(n))\}_{n=0}^{I}$ *and a cutoff* $n^*$. *Let* $n$ *be the amount of cash that the firm has accu-*

*mulated up to time $t$. Then, for $n > n^*$, the equilibrium has the following shape:*

1. $\underline{z}(n) = \overline{z}(n)$.

2. *The market offers the separating price if $z < \overline{z}(n)$, offers the pooling price if $z > \overline{z}(n)$, and mixes between the separating price and pooling price if $z = \overline{z}(n)$ such that at any time $t > 0$ the pooling price arrives with intensity $\delta(n)$ that solves*

$$\frac{\mu_L + k}{r} - n = \frac{\delta(n)}{\mu_L + \delta(n) + r}\Phi_L(\overline{z}(n), n) + \frac{\mu_L}{\mu_L + \delta(n) + r}V_L(\overline{z}(n) + \Delta, n+1), \quad (32)$$

*where $V_L$ is given by Part 4 below. If $z_0 = \overline{z}(0)$, the probability that the market makes a pooling offer at $t = 0$ is arbitrary.*

3. *A high type firm issues at the pooling price whenever $z \geq \overline{z}(n)$ and a pooling price is offered. It does not issue otherwise. The high type firm's value function $V_H(z, n)$ is given by*

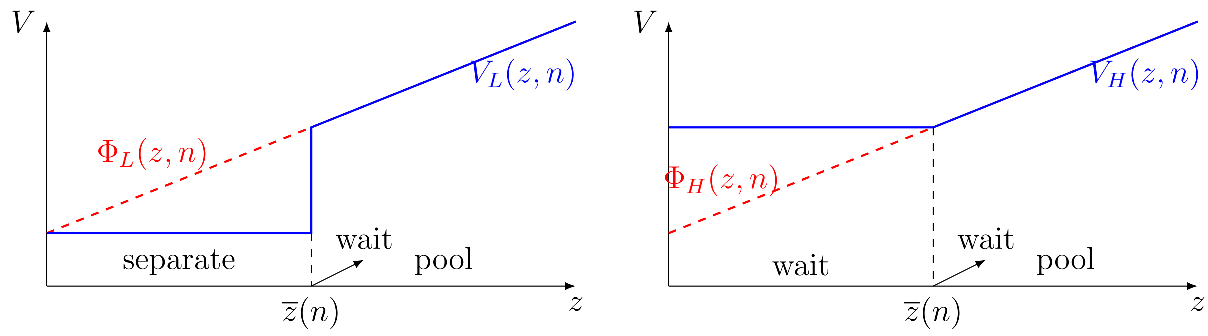
$$V_H(z,n) = \begin{cases} \Phi_H(z,n), & z > \overline{z}(n) \\ \Phi_H(\overline{z}(n), n), & z \leq \overline{z}(n) \end{cases}. \quad (33)$$

4. *A low type firm issues for sure if any pooling price is offered. If only the separating price is offered and $z < \overline{z}(n)$, it mixes at the separating price with the acceptance probability $1 - e^{z - \overline{z}(n)}$. If only the separating price is offered and $z = \overline{z}(n)$, it mixes at the separating price with the rate of acceptance $\mu_H - \mu_L$. The low type firm's value function $V_L(z, n)$ is given by*

$$V_L(z,n) = \begin{cases} \Phi_L(z,n), & z > \overline{z}(n) \\ \frac{\mu_L + k}{r} - n, & z \leq \overline{z}(n) \end{cases}. \quad (34)$$

Figure 7 illustrates the shapes of the equilibrium value functions $V_L(z, n)$ and $V_H(z, n)$ as functions of market belief $z$ for the one-threshold case of Proposition 3. The equilibrium has a few notable properties. First, similar to the two-threshold case, partial revelation of the low type firm implies that $V_H(z, n)$ and $V_L(z, n)$ do not change below $\overline{z}(n)$, hence the flat parts of the graphs.

Second, due to the mixed strategy played by the market at $\overline{z}(n)$, there is strictly positive waiting time at $\overline{z}(n)$ before a pooling price is offered. If the waiting time at $\overline{z}(n)$ were zero, then the low type would reject the separating offer for sure below $\overline{z}(n)$; this implies that the




(a) Low type's value function $V_L(z, n)$

(b) High type's value function $V_L(z, n)$

Figure 7: Equilibrium value functions for $n$ such that $\overline{z}(n) = \underline{z}(n)$

market belief cannot jump up, which would defeat the purpose of rejecting separating offers below $\overline{z}(n)$ to start with. This waiting point is highlighted in Figure 7.

Third, the low-type firm's value function is generally discontinuous at $\overline{z}(n)$. Such discontinuity is incentive-compatible in equilibrium because the waiting time at $\overline{z}(n)$ is strictly positive with probability 1. Similarly, while the high-type firm's value function is continuous at $\overline{z}(n)$, it has a kink at $\overline{z}(n)$, and this kink is incentive-compatible also because of a positive waiting time at $\overline{z}$. In diffusion-based models such as Daley and Green (2012), waiting time at any particular belief is zero with probability 1; hence, in these models value functions must be continuous and have continuous first derivatives.

**Example.** To further illustrate the intuition, Figure 8 plots the numerical solutions of the belief thresholds for $I = 6$. In this plot we have converted all log-likelihood ("$z$") variables to probabilities ("$p$") using $p = e^z/(e^z + 1)$. The horizontal axis is the cash already accumulated, $n$, and the vertical axis is the belief variables. In this example, if $n \in \{0, 1\}$, the two-threshold case of Proposition 2 applies. In this case $\underline{p}(n) < \overline{p}(n)$. If $n \geq 2$, the single threshold case of Proposition 3 applies, and $\overline{p} = \underline{p}$.

In both cases (two thresholds and one threshold), $\overline{p}$ is decreasing in $n$, implying that the high-type firm is more willing to pool because the expected adverse selection cost is smaller. Simultaneously, the low type faces two countervailing incentives. On one hand, waiting becomes more attractive for the low type because the high type is more willing to pool. On the other hand, more internal cash also means a lower benefit of pooling for the low type, so the low type firm is more willing to separate. These two effects generate a lower threshold $\underline{p}$ that is generally non-monotone in $n$.

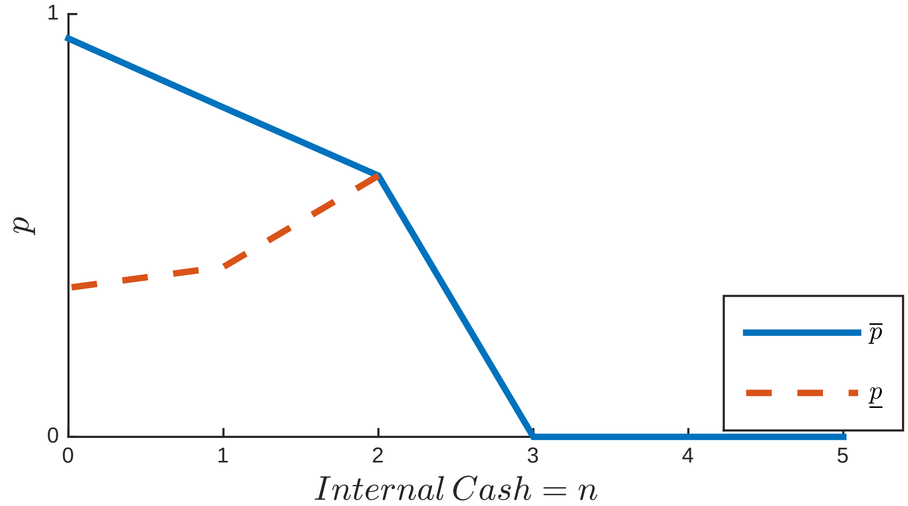



Figure 8: Equilibrium belief thresholds as functions of internal cash level. We convert log-likelihood variables to probabilities by $p = e^z/(e^z + 1)$. Parameters: $I = 6$, $\mu_H = 3.5$, $\mu_L = 1.5$, $r = 0.07$, and $k = 0.45$.

## 4.4 A Brief Discussion of Equilibrium Selection

Dynamic games typically have multiple equilibria. In our model, this multiplicity problem is amplified by the presence of asymmetric information. Therefore, characterizing all possible perfect equilibria is infeasible. Instead, we show that the equilibrium of Propositions 2 and 3 is unique under the following three restrictions. We will briefly describe their intuition and leave formal discussions to the appendix.

First, due to the lack of mathematical apparatus to deal with arbitrary path-dependent strategies, we restrict attention to *stationary* strategies only, i.e., the strategies do not depend explicitly on $t$.[10] In the current setting, this restriction is natural because the dynamics of our model are driven by the time-homogeneous Poisson cash flow process $\{N_t\}_{t\geq 0}$.

Second, we require beliefs to be *monotone non-decreasing*, which means that $z_t - \hat{z}_t$ is a non-decreasing process in $t$, where recall $z_t$ is the market belief and $\hat{z}_t$ is the component of $z_t$ that only depends on the cash flows. This refinement is a natural extension of the Divinity refinement of Banks and Sobel (1987) to the continuous-time setting and has been used by Daley and Green (2012) and Gul and Pesendorfer (2012). This restriction allows us

[10] Although we confine the strategies used in the construction of equilibrium to be stationary, we place no restrictions on the set of deviations. In particular, equilibria derived in Propositions 2 and 3 survive against arbitrary non-stationary deviations.

to rule out potential unreasonable equilibria in which the market forces a particular timing of issuance by "threatening" that a failure to issue will always be attributed to the low type.

Finally, we focus only on equilibria in which the value functions of the two types of firm are *monotone non-decreasing* in market belief and satisfy the "*No Deals*" condition. Monotonicity rules out equilibria in which the low type firm has incentives to "sabotage", i.e., decrease the market belief and still receive a higher payoff. The No Deals condition was first introduced by Daley and Green (2012) to capture in a reduced-form "private offers" of Swinkels (1999). (See Definition 4 in Appendix for details.) The No Deals condition precludes the existence of off-equilibrium prices at which the firm would like to issue and the market would earn positive profits.

The next proposition provides sufficient conditions for our equilibrium to be the unique one.

**Proposition 4 (Equilibrium Uniqueness)** *For a sufficiently small $\mu_L$, the equilibrium of Propositions 2 and 3 is the unique equilibrium that has: (i) stationary strategies, (ii) monotone non-decreasing beliefs, and (iii) non-decreasing value functions satisfying the No Deals condition.*

Instead of proving Proposition 4 directly, we refer readers to Daley and Green (2012)'s Theorem 5.1. Using essentially the same technique, one could use backward induction on the amount of internal cash $n$ to show that the uniqueness of the equilibrium in their model implies the uniqueness of the equilibrium in Propositions 2 and 3. The additional difficulty in our analysis arises because the Static Lemons condition of Daley and Green (2012) eventually and inevitably stops binding as the firm accumulates more cash in equilibrium. As a result, additional care needs to be taken to make sure that the No Deals condition is satisfied for all $n$. While the uniqueness proof is similar, we stress that our model goes beyond Daley and Green (2012) in allowing cash flow accumulation and characterizing the optimal security (under certain restrictions).

# 5 Empirical Implications

Our results generate a number of empirical implications.

First, our model predicts that issuance and investment can take place after a series of lower-than-expected cash flows. Such behavior arises because of the separation by the low type firm at the lower reflecting threshold $\underline{z}(n)$.

Second, related to the previous prediction, if issuance happens after low realized cash flows, the new issuance immediately leads to a drop on firm value because the firm is revealed to be of the low type. By contrast, if issuance happens after high realized cash flows, the firm value remains a martingale because this kind of issuance happens in a pooling equilibrium.

Third, unconditionally, the high type firm accumulates more cash prior to investment than the low type does. This is because the high type firm only invests in the pooling region, whereas the low type firm invests in both pooling and separation regions. That is, if a higher fraction of a firm's investment is financed by internal cash (retained earnings), the firm's subsequent return is higher.

Fourth, conditional on issuance happening after low realized cash flows, the magnitude of the drop in firm value can be non-monotone in the level of internal cash reserve. This is because the lower threshold $\underline{z}(n)$ is generally non-monotone in $n$. Thus, it is possible that a firm with a higher internal cash reserve experiences a larger drop in value.

# 6 Concluding Remarks

When a firm is financially constrained, asymmetric information is an impediment to outside financing and hence investment. This paper characterizes equilibrium behavior of the firm when the firm's assets in place generate cash flows that both reveal the firm's quality and relax its financial constraint. We show conditions under which the firm optimally delays issuance and investment. Moreover, under stated assumptions, the renegotiation-proof optimal security (of the high-quality firm) front-loads cash flows to outside investors in order to minimize adverse selection cost.

One possible extension for future work is to broaden the class of possible securities. For securities considered in our model, the split of cash flows between outside investors and original shareholders are determined upfront and cannot depend on future events that are yet to realize. Relaxing this assumption is likely to further reduce the adverse selection cost, but we do not expect it to eliminate the adverse selection problem altogether. For example, one can consider a credit line with face value $I$ that grows at rate $r$ and can be paid back by the firm at any point in the future.[11] A credit line is not in the class of securities we consider in this paper because the repayment schedule is not specified upfront but is determined by the firm in real time. For example, if cash flows arrive later than expected, then it would take longer to repay the growing balance. In Appendix C we show that the credit line does

[11] We thank Peter DeMarzo for suggesting this security.

not eliminate the adverse selection problem. Thus, our solution method of optimal issuance timing can also be applied to the credit line, and we expect the results to be qualitatively similar to that in Propositions 2 and 3. It remains an interesting question whether the credit line is the optimal security design under a broader class of securities, and if not, what is the optimal security. We have not found a tractable method of solving this problem.

Another possible extension of our model is to allow more flexible use of accumulated cash. Share repurchases analyzed by Bond and Zhong (2016) is one example. Other examples include paying dividends, burning cash, and buying and selling other derivatives contracts in the open markets as signals of firm quality. Since there are virtually unlimited number of such choices, analyzing flexible use of cash is an intractable problem for solution methods we are aware of.

# Appendix

## A A Glossary of Main Model Variables

This appendix summarizes the main model variables used in this paper.

| Variable | Explanation |
|---|---|
| $\mu_\theta$ | Type $\theta$ firm's mean cash flow arrival rate is $\mu_\theta$ |
| $N_t$ | The number of cash flows that arrive by time $t$ |
| $k$ | Incremental arrival rate of cash flows brought by the new project |
| $r$ | Discount rate |
| $I$ | Required amount of funds to undertake the investment |
| $\mathcal{S}$ | The class of securities we consider |
| $\mathcal{S}_{RP}$ | Renegation-proof securities; a subset of $\mathcal{S}$ |
| $S$ | A generic security, whose cumulative payment up to time $t$ is denoted $S_t$ |
| $\lambda_m$ | The fraction of the $m$-th cash flow paid to outside investors |
| $\pi_t^\theta$ | The cumulative probability of issuance by type $\theta$ up to time $t$ |
| $D_t$ | The cumulative dividend paid to shareholders up to time $t$ |
| $C_t$ | The amount of cash saved within the firm by time $t$ |
| $F_E^\theta(S)$ | The original shareholders' value given security $S$ and type $\theta$ |
| $F_M^\theta(S)$ | The market value of security $S$ conditional on type $\theta$ |
| $M(S)$ | The market price of the security $S$, $M(S) \in \{\mathbf{E}[F_M^\theta(S)], F_M^L(S)\}$ |
| $p$ | The market belief that the firm is of the high type |
| $z$ | Likelihood ratio, $z = \ln(p/1-p)$ |
| $\hat{p}$, $\hat{z}$ | Market belief based only on cash flows and not on past issuance decisions |
| $V_H(t) = V_H(z_t, N_t)$ | The high type firm's value function at time $t$ |
| $V_L(t) = V_L(z_t, N_t)$ | The low type firm's value function at time $t$ |
| $J(p, I-n)$ | In the optimal security, original shareholders start to get paid from the $J(p, I-n)$-th cash flow |
| $S^*(p, I-n)$ | The optimal security issued if market belief is $p$ and internal cash is $n$ |
| $\Phi_\theta(p,n)$, $\Phi_\theta(z,n)$ | Value of pooling immediately for type $\theta$ if belief is $p$ or $z$, and internal cash is $n$ |
| $\Delta$ | $\ln(\mu_H/\mu_L)$, the upward jump in belief $z$ after a cash flow arrives |
| $\underline{z}(n), \overline{z}(n)$ | The lower and upper belief thresholds, if internal cash is $n$ |
| $\delta(n)$ | The intensity that the market proposes the pooling price, if internal cash is $n$ |

# B Proofs

## B.1 Proof of Lemma 2

**Proof.** Suppose that a high type firm issues some security $S$. The payoff to a low type firm from mimicking the high type and issuing the same security $S$ is

$$F_E^L(S) = \frac{\mu_\theta + k}{r} - F_M^L(S). \tag{35}$$

Because security $S$ is priced by a competitive market that breaks even in expectation, we have

$$pF_M^H(S) + (1-p)F_M^L(S) = I \quad \Rightarrow \quad F_M^H(S) > I > F_M^L(S). \tag{36}$$

Hence a low type firm's payoff from pooling,

$$F_E^L(S) = \frac{\mu_\theta + k}{r} - F_M^L(S) > \frac{\mu_\theta + k}{r} - I = \frac{\mu_\theta + k}{r} - F_M^L(\tilde{S}), \tag{37}$$

is higher than the payoff from issuing a security $\tilde{S}$ that separates a low type firm. ■

## B.2 Proof of Proposition 1

**Proof.** We look for security $S^*(p, I-n)$ that solves

$$S^*(p, I-n) = \text{argmax}_{S \in \mathcal{S}_{RP}} F_E^H(S), \quad s.t. \quad pF_M^H(S) + (1-p)F_M^L(S) = I. \tag{38}$$

The main argument of the proof is to establish that the optimal security should front-load payments as much as possible, in order to reduce adverse-selection costs.

Suppose, for simplicity, that the firm wishes to raise a small quantity of cash $\varepsilon$ from the market. Consider a security $S^1$ such that $\lambda_j^1 = 0$ for all $j \neq 1$ but $\lambda_1^1 \neq 0$. The first payment $\lambda_1^1$ solves

$$\lambda_1^1 \left( p\frac{\mu_H + k}{\mu_H + k + r} + (1-p)\frac{\mu_L + k}{\mu_L + k + r} \right) = \varepsilon. \tag{39}$$

The corresponding equity value of a high type firm is

$$\begin{aligned}
F_E^H(S^1) &= \frac{\mu_H + k}{r} - F_M^H(S^1) \\
&= \frac{\mu_H + k}{r} - \frac{\mu_H + k}{\mu_H + k + r}\lambda_1^1 \\
&= \frac{\mu_H + k}{r} - \frac{\varepsilon}{p + (1-p)\cdot \frac{\mu_L + k}{\mu_L + k + r}\cdot \frac{\mu_H + k + r}{\mu_H + k}}.
\end{aligned}$$

Alternatively, consider a security $S^2$ such that $\lambda_j^2 = 0$ for all $j \neq 2$ but $\lambda_2^2 \neq 0$. Analogous calculations reveal that

$$F_E^H(S^2) = \frac{\mu_H + k}{r} - \frac{\varepsilon}{p + (1-p)\cdot \left(\frac{\mu_L + k}{\mu_L + k + r}\cdot \frac{\mu_H + k + r}{\mu_H + k}\right)^2}. \tag{40}$$

Since $\frac{\mu_L + k}{\mu_L + k + r}\cdot \frac{\mu_H + k + r}{\mu_H + k} < 1$, we have

$$F_E^H(S^1) > F_E^H(S^2). \tag{41}$$

That is, a high type firm prefers $S^1$ to $S^2$.

The above argument implies that if $\lambda_1 < 1$ and $\lambda_2 > 0$ for any security, a high type firm can strictly profit by moving $\epsilon > 0$ promised payment from the second cash flow time to the first.

Repeating this argument for subsequent cash flow times, we can see that the security cannot be optimal if $\lambda_m < 1$ but $\lambda_n > 0$ for $n > m$. Thus, the unique optimal security has absolute priority in allocating the cash flows: the original shareholders receive nothing until the outside security holder receives all their promised payments. This optimal security is precisely the one stated in the proposition.

Explicit calculations show that

$$J(p, I-n) = \min\left\{ J \,:\, \sum_{j=1}^{J}\left[p\left(\frac{\mu_H + k}{\mu_H + k + r}\right)^j + (1-p)\left(\frac{\mu_L + k}{\mu_L + k + r}\right)^j\right] \geq I \right\}, \tag{42}$$

and

$$\lambda_{J(p,I-n)} = \frac{I - \sum_{j=1}^{J(p,I-n)-1}\left[p\left(\frac{\mu_H + k}{\mu_H + k + r}\right)^j + (1-p)\left(\frac{\mu_L + k}{\mu_L + k + r}\right)^j\right]}{p\left(\frac{\mu_H + k}{\mu_H + k + r}\right)^{J(p,I-n)} + (1-p)\left(\frac{\mu_L + k}{\mu_L + k + r}\right)^{J(p,I-n)}}. \tag{43}$$

■

## B.3 Proof of Lemma 3

**Proof.** Denote by $V_\theta(t)$ continuation value of the type $\theta$ at time $t$. Notice that $V_\theta(t)$ is an $\mathbb{F}^N$-adapted process, so it can be written as $V_\theta(z_t, N_t)$. It is clear that if there exists a security at time $t$, $S_t$, and a market price $V_t(S_t)$ such that $(\mu_\theta + k)/r - F_S^\theta(S_t) > V_\theta(z_t, N_t)$, it is optimal for the firm to issue and invest rather than wait and receive $V_\theta(z_t, N_t)$. That it, each firm follows a threshold strategy.

Next we show that if the high type firm is willing to issue a security $S$ at time $t$, then so does the low type firm, i.e.

$$\text{if} \quad F_E^H(S) > V_H(z_t, N_t) \quad \text{then} \quad F_E^L(S) > V_L(z_t, N_t)$$

Suppose that in the continuation game the high type will issue a security $\tilde{S}$ at time $\tau$, then

$$\begin{aligned}
F_E^L(S) &= F_E^L(S) + F_E^H(S) - F_E^H(S) > F_E^L(S) + \mathbf{E}^H[e^{-r\tau}F_E^H(\tilde{S})] - F_E^H(S) \\
&= F_E^L(S) - \mathbf{E}^L[e^{-r\tau}F_E^L(S)] + \mathbf{E}^L[e^{-r\tau}(F_E^L(S) - F_E^L(\tilde{S}))] \\
&- F_E^H(S) + \mathbf{E}^H[e^{-r\tau}F_E^H(S)] - \mathbf{E}^H[e^{-r\tau}(F_E^H(S) - F_E^H(\tilde{S}))] + \mathbf{E}^L[e^{-r\tau}F_E^L(\tilde{S})] \\
&> F_E^L(S) - \mathbf{E}^L[e^{-r\tau}F_E^L(S)] \\
&- F_E^H(S) + \mathbf{E}^H[e^{-r\tau}F_E^H(S)] + \mathbf{E}^L[e^{-r\tau}F_E^L(\tilde{S})] \\
&> \mathbf{E}^L[e^{-r\tau}F_E^L(\tilde{S})].
\end{aligned}$$

In the above derivation, the first inequality is due to the assumption that the high type would rather issue $S$ now than issue $\tilde{S}$ at $\tau$. The second inequality is due to the fact that a security $\tilde{S} - S$ is more valuable if issued by the high type. And the final inequality holds since the high type's cash flows arrive faster; thus, he loses less by waiting until $\tau$ to issue $S$. Therefore, the low type would also prefer to issue $S$ immediately. ■

## B.4 Proof of Propositions 2 and 3

**Preliminary steps.**

Consider an auxiliary function $W_H(z,n)$, defined by:

$$W_H(z,n) = \Phi_H(z,n) - \frac{\mu_H}{\mu_H + r}\Phi_H(z+\Delta, n+1), \tag{44}$$

$$z_H(n) \equiv \sup\{z : W_H(z,n) < 0\} \quad \text{with } \sup\{\emptyset\} = -\infty. \tag{45}$$

The second term on the right-hand side of the definition of $W_H(z,n)$ is a high type firm's expected discounted value of issuing the optimal security after a new cash flow, conditional on the market belief staying at $z$ in the mean time. Intuitively, if $W_H(z,n) < 0$, then the high type prefers to raise funds $I-n$ and invest immediately rather than wait for the arrival of the next cash flow. For sufficiently high $z$, the function $W_H(z,n)$ is always positive. Therefore, $z_H(n)$ is the upper bound on the size of the waiting region.

Similarly, define an auxiliary function $W_L(z,n)$ and its root $z_L(n)$ by:

$$W_L(z,n) = \frac{\mu_L + k}{r} - (I-n) - \frac{\mu_L}{\mu_L + k}\Phi_L(z+\Delta, n+1), \tag{46}$$

$$z_L(n) = \inf\{z : W_L(z,n) < 0\} \quad \text{with } \inf\{\emptyset\} = \infty. \tag{47}$$

If $W_L(z,n) > 0$, then a low type firm would reveal its type immediately and capture the positive NPV of the project, rather than wait in hopes of pooling in the future. Since in any equilibrium the payoff of a low type firm is no greater than the pooling value $\Phi_L(z,n)$, the threshold $z_L(n)$ is the lower bound of the waiting region.

The following lemma characterizes the behavior of the auxiliary thresholds $z_H(n)$ and $z_L(n)$ for various level of internal cash $n$.

**Lemma 4** *$z_H(n)$ is decreasing in $n$ and $z_L(n)$ is increasing in $n$.*

**Proof.** First, notice that $\Phi_\theta(z,n)$ is an increasing function of $z$ regardless of $\theta$ and $n$. Thus, $W_L(z,n)$ is decreasing in $z$. Second, notice that $\Phi_L(z,n)$ is decreasing in $n$. Hence,

$$W_L(z,n) - W_L(z,n-1) = 1 + \frac{\mu_L}{\mu_L + r}\left[\Phi_L(z+\Delta,n) - \Phi_L(z+\Delta,n+1)\right] > 0. \tag{48}$$

Since $W_L(z,n)$ is increasing in $n$, the unique root $z_L(n)$ is also increasing in $n$.

Now consider $W_H(z,n)$:

$$\begin{aligned}
W_H(z,n-1) - W_H(z,n) &= \Phi_H(z,n-1) - \frac{\mu_H}{\mu_H+r}\Phi_H(z+\Delta,n) \\
&- \Phi_H(z,n) + \frac{\mu_H}{\mu_H+r}\Phi_H(z+\Delta,n+1) \\
&= \Phi_H(z,n-1) - \Phi_H(z,n) + \frac{\mu_H}{\mu_H+r}\left[\Phi_H(z+\Delta,n+1) - \Phi_H(z+\Delta,n)\right] \\
&> \Phi_H(z,n) - \Phi_H(z,n+1) + \frac{\mu_H}{\mu_H+r}\left[\Phi_H(z+\Delta,n+1) - \Phi_H(z+\Delta,n)\right] \\
&> \Phi_H(z+\Delta,n) - \Phi_H(z+\Delta,n+1) \\
&+ \frac{\mu_H}{\mu_H+r}\left[\Phi_H(z+\Delta,n+1) - \Phi_H(z+\Delta,n)\right] \\
&= \frac{r}{\mu_H+r}\Big[\Phi_H(z+\Delta,n) - \Phi_H(z+\Delta,n+1)\Big] \\
&> 0.
\end{aligned}$$

Here, the first inequality holds because $\Phi_H(z,n-1)-\Phi_H(z,n)$ is decreasing in $n$, the second inequality holds because $\Phi_H(z,n) - \Phi_H(z,n+1)$ is decreasing in $z$, and the last inequality holds because $\Phi_H(z,n)$ is increasing in $n$. Thus, the function $W_H(z,n)$ is increasing in $n$ and the right-most root $z_H(n)$ is decreasing in $n$. ■

**Proof of Propositions 2 and 3.**

**Case $n = I-1$.** First, notice that $z_L(I-1) = +\infty$ since after the arrival of the final cash flow the project will be financed internally and the low type firm can not benefit from issuing overpriced security. Thus, $\overline{z}(I-1) = \underline{z}(I-1) = z_H(I-1)$. We now verify the strategies given the belief threshold $\overline{z}(I-1)$ and mixing intensity $\delta(I-1)$ at $\overline{z}(I-1)$. As usual, when verifying each player's strategy, we hold the other players' strategies fixed.

1. High-type firm's strategy. At the belief threshold $\overline{z}(I-1)$, the high type firm is indifferent between pooling immediately and waiting. Below $\overline{z}(I-1)$, rejecting the separating offers is optimal since belief jumps up to $\overline{z}(I-1)$ immediately. Above $\overline{z}(I-1)$ accepting the pooling is optimal, since beliefs have a downward drift $\mu_H - \mu_L$; thus, the continuation value of rejecting a pooling offer is strictly below $\frac{\mu_H}{\mu_H+r}\Phi_H(z+\Delta,I)$, which in turn is less than $\Phi_H(z,I-1)$.

2. Low-type firm's strategy. It is clear that the low type firm should accept a pooling price immediately. But the low type firm cannot accept a separating offer with probability

1, for otherwise the market belief $z_t$ would jump to $\infty$ immediately afterwards, which means the low type firm would wish to delay instead.

Nor could the low type firm reject the separating offer with probability 1, for otherwise the market belief would deteriorate over time, which defeats the purpose of rejecting the separating offer. Thus, the low-type firm always mixes at the separating price. The mixing probability must be such that the resulting market belief is $\overline{z}(I-1)$, for the following reason. If the resulting market belief were strictly below $\overline{z}(I-1)$, the market would not offer a pooling price. If the resulting market belief were strictly above $\overline{z}(I-1)$, then immediate pooling would follow, which implies that the low type firm would reject the separating offer for sure.

The above logic implies that the market belief must stay at $\overline{z}(I-1)$ without a new cash flow. If $z_t < \overline{z}(I-1)$, the probability that the low-type firm accepts the separating offer must be $1 - e^{z_t - \overline{z}(I-1)}$, for this implies that the new market belief after rejecting the separating offer is

$$z_t + \log\left(\frac{1}{e^{z_t - \overline{z}(I-1)}}\right) = \overline{z}(I-1).$$

If $z_t = \overline{z}(I-1)$, the mixing probability must be such that the resulting belief remains at $\overline{z}(I-1)$. A mixing strategy that involves accepting a separating offer at rate $\mu_H - \mu_L$ implies that the signaling component of market belief has a drift of

$$\log \frac{\mathbf{P}(\text{delay by } dt \mid \text{high type})}{\mathbf{P}(\text{delay by } dt \mid \text{low type})} = \log \frac{1}{1-(\mu_H-\mu_L)dt} \approx (\mu_H - \mu_L)dt.$$

This approximation is exact in continuous time and exactly offsets the downward drift of market belief at rate $\mu_H - \mu_L$ without a new cash flow.

**Case $n < I - 1$.** Next we construct the equilibrium by induction. If $n < I-1$ but $z_H(n) \leq z_L(n)$, the equilibrium with a single belief threshold can be constructed in a similar way as in the $n = I-1$ case above.

For the rest of the proof we focus on $n$'s such that $z_H(n) > z_L(n)$. Since $z_H(n)$ is decreasing in $n$ and $z_L(n)$ is increasing, there exists an $n^*$ such that for all $n < n^*$, $z_H(n) > z_L(n)$. That is, the ordering between $z_H(n)$ and $z_L(n)$ never changes as we decrease $n$ from $n^*$.

For $n < n^*$, we conjecture the following equilibrium structure: there exist two distinct thresholds $\underline{z}(n) < \overline{z}(n)$ and continuation values $V_H(z,n)$ and $V_L(z,n)$ such that

$$z \geq \overline{z}(n) \Rightarrow V_H(z,n) = \Phi_H(z,n), \quad V_L(z,n) = \Phi_L(z,n), \tag{Pooling}$$
$$z \in (\underline{z}(n), \overline{z}(n)) \Rightarrow \quad V_L(z,n) < \Phi_L(z,n), \tag{Waiting}$$
$$z \leq \underline{z}(n) \Rightarrow V_H(z,n) = V_H(\underline{z}(n),n), \quad V_L(z,n) = \frac{\mu_L + k}{r} - (I-n). \tag{Separating}$$

We now proceed with the induction step of going from $n$ to $n-1$, with $V_H(\cdot,n)$ and $V_L(\cdot,n)$ already constructed from step $n$.

Define $\underline{z}(n-1)$ to be the solution of

$$\frac{\mu_L + k}{r} - (n-1) = \frac{\mu_L}{\mu_L + r} V_L(\underline{z}(n-1) + \Delta, n). \tag{49}$$

Note that $\underline{z}(n-1) + \Delta > \underline{z}(n)$ because

$$\frac{\mu_L + k}{r} - (I-(n-1)) > \frac{\mu_L}{\mu_L + r}\left(\frac{\mu_L + k}{r} - (I-n)\right) = \frac{\mu_L}{\mu_L + r} V_L(\underline{z}(n), n),$$

i.e., the low type prefers separating now to separating after one cash flow by the time value of money.

This leaves us with two cases: $\underline{z}(n-1) + \Delta \in (\underline{z}(n), \overline{z}(n)]$ or $\underline{z}(n-1) + \Delta > \overline{z}(n)$. We show below that at $\underline{z}(n-1)$ the high type prefers to wait, since at $\underline{z}(n-1)$ only the low price (corresponding to belief $p = 0$ or $z = -\infty$) is offered. This boils down to showing that

$$\Phi_H(-\infty, n-1) \leq \frac{\mu_H}{\mu_H + r} V_H(\underline{z}(n-1) + \Delta, n).$$

This is true due to the skimming property. Suppose the contrary, i.e. that the high type strictly prefers to take the low price, then

$$\Phi_H(-\infty, n-1) > \frac{\mu_H}{\mu_H + r} V_H(\underline{z}(n-1) + \Delta, n),$$

which implies

$$\Phi_L(-\infty, n-1) > \frac{\mu_H}{\mu_H + r} V_L(\underline{z}(n-1) + \Delta, n).$$

Thus,

$$\frac{\mu_L + k}{r} - (n-1) = \Phi_L(-\infty, n-1) > \frac{\mu_L}{\mu_L + r} V_L(\underline{z}(n-1) + \Delta, n).$$

The last inequality contradicts the definition of $\underline{z}(n-1)$. Hence, the initial conjecture that the high type strictly prefers to accept the low price is false. Overall, the high type prefers to wait at $\underline{z}(n-1)$.

Having constructed $\underline{z}(n-1)$, we construct $\overline{z}(n-1)$. Define an auxiliary function $H(z)$ by:

$$H(z) = \mathbf{E}\left[e^{-r\tau} V_H(z_\tau + \Delta, n) \mid z_0 = z\right], \tag{50}$$

where $\tau$ is the arrival time of a new cash flow from a Poisson process with intensity $\mu_H$ and $z_t$ evolves as

$$dz_t = \begin{cases} -(\mu_H - \mu_L)dt, & \text{if } z_t \geq \underline{z}(n-1), \\ 0, & \text{if } z_t = \underline{z}(n-1). \end{cases} \tag{51}$$

Since $V_H(\cdot, n)$ has been constructed recursively already (induction step), $H$ can be evaluated by integration. We define $\overline{z}(n-1)$ to be the maximal solution of

$$\Phi_H(\overline{z}(n-1), n-1) = H(\overline{z}(n-1)). \tag{52}$$

If the equation above has no solution, then put $\overline{z}(n-1) = +\infty$. Thus, the high type's value function $V_H(z, n-1)$ can be constructed as:

$$V_H(z, n-1) = \begin{cases} \Phi_H(z, n-1), & z \geq \overline{z}(n-1) \\ H(z), & z \in (\underline{z}(n-1), \overline{z}(n-1)), \\ \frac{\mu_H}{\mu_H + r} V_H(\underline{z}(n-1) + \Delta, n), & z \leq \underline{z}(n-1). \end{cases} \tag{53}$$

In the last case of $z \leq \underline{z}(n-1)$, note that even if $z < \underline{z}(n-1)$, the market belief jumps to $\underline{z}(n-1)$ immediately conditional on no investment. Thus, the high type's value is equal to $\frac{\mu_H}{\mu_H + r} V_H(\underline{z}(n-1) + \Delta, n)$.

The low type's value can be constructed similarly. Let $L(z)$ be

$$L(z) = \mathbf{E}\left[e^{-r\tau} V_L(z_\tau + \Delta, n) \mid z_0 = z\right], \tag{54}$$

where $\tau$ is the arrival time of a new cash flow from a Poisson process with intensity $\mu_L$ and $z_t$ evolves according to (51). Again, since $V_L(\cdot, n)$ has been constructed already, $L(z)$ can be evaluated by the induction step. The low type's value function $V_L(z, n-1)$ is given by

$$V_L(z, n-1) = \begin{cases} \Phi_L(z, n-1), & z \geq \overline{z}(n-1) \\ L(z), & z \in (\underline{z}(n-1), \overline{z}(n-1)), \\ \frac{\mu_L + k}{r} - (I - (n-1)), & z \leq \underline{z}(n-1). \end{cases} \tag{55}$$

■

## B.5 Proof of Proposition 4

**Definition 4** *The Equilibrium value functions of high type and low type firms, $V_H$ and $V_L$, satisfy the* ***No Deals*** *condition if*

$$V_H(t) \geq \frac{\mu_H + k}{r} - \min_S\{F_M^H(S) : p_t F_M^H(S) + (1 - p_t) F_M^L(S) = I - N_t\}, \tag{$ND_H$}$$

$$V_L(t) \geq \frac{\mu_L + k}{r} - (I - N_t). \tag{$ND_L$}$$

A sufficient condition that grantees the uniqueness of our equilibrium is formalized by Assumption 3.

**Assumption 3** *If for some $n$, $z_L(n) < z_H(n)$, then $W_H(z, n) < 0$ for all $z \in (z_L(n), z_H(n))$.*

In other words, if $z_L(n) < z_H(n)$, then the high type would rather wait if the optimal strategy after a cash flow arrival is pooling, at the belief $z + \Delta$.

The following lemma establishes a simple sufficient condition on the primitive model parameters for Assumption 3 to hold.[12]

**Lemma 5** *If $\mu_L$ is sufficiently close to 0, then Assumption 3 is satisfied.*

**Proof.** Step 1. For a sufficiently high $n$ and a sufficiently low $\mu_L$, $W_H(z, n)$ has only one root.

First, define $\hat{z}_H(n)$ by:

$$\hat{z}_H(n) = \sup\left\{z : \Phi_H(z, n) \leq \frac{\mu_H}{r}\right\}, \quad \sup\{\emptyset\} = -\infty. \tag{56}$$

[12] If Assumption 3 fails, our equilibrium remains a valid equilibrium, although the No Deals condition may fail, implying potential multiple equilibria. The equilibrium that satisfies the No Deals condition involves a lower pooling region, but that equilibrium fails the "non-decreasing value function" condition.

Since immediate pooling at low beliefs is impossible when $n = 0$ $\hat{z}_H(0) > -\infty$, thus, there exists a maximal $n < I$ such that $\hat{z}_H(n) > -\infty$; denote it by $n^*$.

Notice that when $z < \hat{z}_H(n)$, then $W_H(z, n)$ is negative:

$$\begin{aligned} W_H(z,n) \quad & vs. \quad 0 \\ \frac{\mu_H + k}{r} - \Phi_H(z,n) \quad & vs. \quad \frac{\mu_H}{\mu_H + r}\left(\frac{\mu_H + k}{r} - \Phi_H(z+\Delta, n+1)\right) \\ \frac{k}{r} - \Phi_H(z,n) \quad & vs. \quad \frac{\mu_H}{\mu_H + r}\left(\frac{k}{r} - 1 - \Phi_H(z+\Delta, n+1)\right) \end{aligned}$$

Since $z < \hat{z}_H(n)$ the left hand side is negative. Additionally,

$$\Phi_H(z,n) > 1 + \Phi_H(z, n+1),$$

hence the right hand side of the comparison above is either positive, or negative but larger that the LHS.

Denote by $z^* = \hat{z}_H(n^*)$ at the original value of $\mu_L$. As we take $\mu_L$ to 0 the threshold $\hat{z}_H(n^*)$ only increases, so $W_H(z, n^*)$ remains negative for all $z < z^*$. Moreover, since $W_H(z, n)$ is increasing in $n$, $W_H(z, n) < 0$ for all $z < z^*$ and all $n \le n^*$

Additionally, for all $z > z^*$ and $n < n^*$, $W_H(z, n)$ and its derivative $\frac{\partial}{\partial z} W_H(z, n)$ can be approximated (uniformly in $z$ and $n$) by

$$W_H(z,n) \approx \Phi_H(z,n) - \frac{\mu_H}{\mu_r + r}\Phi_H(+\infty, n+1), \quad \frac{\partial}{\partial z} W_H(z,n) \approx \frac{\partial}{\partial z}\Phi_H(z,n) > 0. \tag{57}$$

Thus, for sufficiently small $\mu_L$, $W_H(z, n)$ is increasing in $z$ for all $z \ge z^*$ and $n \le n^*$.

Step 2. For sufficiently low $\mu_L$, $z_L(n^*) = +\infty$.

If $z_L(n^*) = +\infty$ at the new (smaller) $\mu_L$ obtained in Step 1, then proceed to Step 3. Otherwise, notice that as $\mu_L \to 0$, $W_L(z, n)$ converges (uniformly in $z$ and $n$) to

$$W_L(z,n) = \frac{\mu_L + k}{r} - n - \frac{\mu_L}{\mu_L + r}\left(\frac{\mu_L + k}{r} - V_L(z+\Delta, n+1)\right) \to \frac{k}{r} > 0. \tag{58}$$

Thus, for sufficiently small $\mu_L$, $z_L(n^*) = +\infty$.

Step 3.

Recall that $z_L(n)$ is an increasing sequence, implying that $z_L(n) = +\infty$ for all $n > n^*$. For these $n$, Assumption 3 is satisfied automatically.

For $n < n^*$, $W_H(z, n)$ has only one root. Since $W_H(+\infty, n) > 0$ it has to be that for $z < z_H(n)$, $W_H(z,n) < 0$ and for $z > z_H(n)$, $W_H(z,n) > 0$. Hence, if for some $n$ $z_L(n) < z_H(n)$ then $W_H(z,n) < 0$ for all $z \in (z_L(n), z_H(n))$ and the Assumption 3 is satisfied for these $n$ as well.

■

**Proof of Proposition 4.** Equilibrium value functions of Proposition 2 satisfy Non-Decreasing Value Functions and the second part of the No Deals condition by construction regardless whether Assumption 3 holds.

We need to verify only the first part of the No Deals condition.

By construction $V_H(z, I-1) \geq \Phi_H(z, I-1)$ for all $z$. Suppose that $V_H(z,n) \geq \Phi_H(z,n)$ has been shown and consider the case of $n-1$.

Clearly, $\underline{z}(n-1) + \Delta > \underline{z}(n)$ since otherwise low type would strictly prefer to revel itself at $\underline{z}(n-1)$. This leaves us with two cases: $\underline{z}(n-1)+\Delta \in (\underline{z}(n), \overline{z}(n)]$ or $\underline{z}(n-1)+\Delta > \overline{z}(n)$.

If $\underline{z}(n-1) + \Delta \in (\underline{z}(n), \overline{z}(n)]$, then

$$\underline{z}(n-1) + \Delta < \overline{z}(n) < z_H(n) < z_H(n-1), \tag{59}$$

hence

$$\begin{aligned}
\Phi_H(\underline{z}(n-1), n-1) &< \frac{\mu_H}{\mu_H + r}\Phi_H(\underline{z}(n-1)+\Delta, n) \\
&\leq \frac{\mu_H}{\mu_H + r}V_H(\underline{z}(n-1)+\Delta, n) \\
&= V_H(\underline{z}(n-1), n-1).
\end{aligned}$$

If $\underline{z}(n-1) + \Delta > \overline{z}(n)$, then $\underline{z}(n-1) = z_L(n-1) < z_H(n-1)$. Thus,

$$\begin{aligned}
\Phi_H(\underline{z}(n-1), n-1) &< \frac{\mu_H}{\mu_H + r}\Phi_H(\underline{z}(n-1)+\Delta, n) \\
&= V_H(\underline{z}(n-1), n-1).
\end{aligned}$$

Overall, $\Phi_H(z, n-1) < V_H(z, n-1)$ for all $z \leq \underline{z}(n-1)$. For all $z > \underline{z}(n-1)$ this inequality holds by construction given the value of $V_H(z, n-1)$. ■

# C A Credit Line

This appendix shows that a credit line does not eliminate the adverse selection problem.

Without loss of generality, suppose that the credit line is opened at time 0, with a balance of $X_0 > 0$ (i.e., the firm borrowed $X_0$). The credit line balance grows at rate $r$ and can be repaid at any time. The best a high type firm can do is to repay the credit line as soon as possible, i.e., after each cash flow arrival. Thus, $X_t$ evolves according to:

$$dX_t = rX_t dt - \min(X_t, 1) dN_t, \tag{60}$$

where $N_t$ is the cumulative cash flow process generated by the firm. The credit line is repaid in full at time

$$\tau = \inf\{t > 0 : X_t \leq 0\}. \tag{61}$$

Denote by $f_\theta(x)$ the market value of such a credit line conditional on type $\theta$, i.e.,

$$f_\theta(x) = \mathrm{E}\Big[\int_0^\tau e^{-rt} \min(X_{t-}, 1) dN_t \,\Big|\, \theta\Big]. \tag{62}$$

The function $f_\theta(x)$ solves:

$$r f_\theta(x) = \begin{cases} (\mu_\theta + k)(x - f_\theta(x)) + rx f_\theta'(x), & \text{if } x \leq 1 \\ (\mu_\theta + k)(1 + f_\theta(x-1) - f_\theta(x)) + rx f_\theta'(x), & \text{if } x > 1 \end{cases} \tag{63}$$

with boundary conditions

$$f_\theta(0) = 0 \quad f_\theta(+\infty) = \frac{\mu_\theta + k}{r}. \tag{64}$$

**Lemma 6** *The market value of the credit line is always below the face value, i.e. $f_\theta(x) < x$ for all $x > 0$.*

**Proof.** Clearly $f_\theta(x) \leq x$. Now suppose that for some $\hat{x} > 0$ $f_\theta(\hat{x}) = \hat{x}$. With the additional boundary condition $f_\theta(0) = 0$, the ODE $f$ is uniquely pinned down as

$$f_\theta(x) \equiv x. \tag{65}$$

However, this function does not satisfy the other boundary condition $f_\theta(+\infty) = \frac{\mu_\theta + k}{r}$. Thus, the market value of the credit line is always below its face value. ■

Intuitively, $f_\theta(x) < x$ because for each time $T$, there is a positive probability that the firm's realized cash flows will be lower than the outstanding balance by time $T$. Given the above lemma, and the fact that $f_\theta(x)$ is strictly increasing in $\mu_\theta$, such a credit line would always incur the high type firm a positive adverse selection cost.

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