

# Crises: Equilibrium Shifts and Large Shocks\*

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November 2016

## Abstract

We study the informational events that trigger equilibrium shifts in coordination games with incomplete information. Assuming that the distribution of the changes in fundamentals have fat tails, we show that majority play shifts *either* if fundamentals reach a critical threshold *or* if there are large common shocks, even before the threshold is reached. The fat-tail assumption matters because it implies that large shocks make players more unsure about whether their payoffs are higher than others. This feature is crucial for large shocks to matter.

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\*We are grateful for comments from the editor, three anonymous referees, George-Marios Angeletos, Andy Atkeson, Markus Brunnermeier, Tianjiao Dai, Ulrich Mueller, Alessandro Pavan, Elia Sartori, Hyun-Song Shin, Laura Veldkamp, Ming Yang, and participants at 2015 Time, Uncertainties and Strategies Workshop (PSE), Global Games Conference (Iowa State), 2017 Cowles Foundation Conference on Macroeconomics (Yale), 2017 Econometric Society North American Summer Meeting (St Louis), and seminars at MIT, the University of Chicago, UCLA, UCSD, and University of Wisconsin-Madison.

# 1 Introduction

On July 26th, 2012, Mario Draghi gave a speech in which he promised "...to do whatever it takes to preserve the euro. And believe me, it will be enough....". Many commentators have credited the "whatever it takes..." speech with shifting the Eurozone economy from a self-fulfilling "bad equilibrium," with high sovereign debt spreads and growing fiscal deficits mutually reinforcing each other, to a self-fulfilling "good equilibrium"—with low spreads and sustainable fiscal policy.<sup>1</sup> There are many other economic and social contexts where strategic complementarities are thought to give rise to the possibility of self-fulfilling equilibria in the form of currency crises, economic booms, financial panics and revolutions. In these cases, also, commentators seek to identify the informational events that trigger crises or recoveries that can be interpreted as shifts into (bad or good) self-fulfilling equilibria. Economists and game theorists have well developed explanations and models of the strategic complementarities giving rise to the self-fulfilling equilibria in these applications but have less well developed explanations of which informational events trigger the shift in self-fulfilling equilibria. This paper contributes to the latter question.

We analyze a canonical coordination game with a continuum of players making a binary choice, where each player's payoff to taking a "good" action, "invest," is increasing in the proportion of others investing and also increasing in a fundamental state; there is incomplete information about fundamentals. We ask: which informational events ensure that one action is uniquely rationalizable, and thus must be played independent of history or other context? We will say that a crisis is triggered if not investing becomes the unique rationalizable action for the players in the game. In our theory, two distinct kinds of informational events can trigger a crisis. First, fundamentals fall below some critical threshold. Second, fundamentals deteriorate sharply to a level where they are somewhat weak, but still better than the critical threshold guaranteeing a crisis. The first trigger is a *level* effect: independent of whether fundamentals are worse than expected, or by how much, sufficiently bad fundamentals trigger a crisis. The second trigger is a *change* effect and corresponds to the main large shock result of the paper. Over a wide range of levels of fundamentals, it is the size of the negative shocks that moved the economy to that fundamental that determines whether a crisis is triggered.

Before explaining the mechanism behind the large shock result, we first describe a key assumption about incomplete information and a key statistic implication of that assumption.

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<sup>1</sup>See Brunnermeier, James, and Landau (2016) for one discussion.

Players cannot identify the common component of fundamentals, observing only a private return that is the sum of the common component and an idiosyncratic noise term. The common component of fundamentals is assumed to have fat tails (the density of the tail exceeds a power law distribution), while the idiosyncratic noise has thinner tails. Under this assumption, a player who observes a large shock will be convinced that the shock is due to the common component of fundamentals. In particular, nothing about the level of his shock will tell a player anything about how large is the idiosyncratic component of his return. Thus a player who observes a large shock will not know anything about whether he is more optimistic about fundamentals than other players. He will therefore have "uniform rank beliefs": if asked what his rank in the population is with respect to his optimism about fundamentals, any rank between 0 and 1 is equally likely. Thus large shocks create diffuse beliefs about what other people are thinking, our key statistical observation.

Uniform rank beliefs pin down strategic behavior. For a player who has not observed a large shock and whose rank belief is not uniform, invest can be rationalized for a wide range of fundamentals by the belief that others are more optimistic than him. But consider a player who has observed a large negative shock, and thus has uniform rank beliefs and believes that players with larger negative shocks also have uniform rank beliefs. Suppose that this player was just indifferent between investing and not investing, and believed that other players are investing only if they have a higher return than him. This player would have a uniform belief on the proportion of other players who are investing. We say that an action is "risk dominant" if it is a best response for a player with such a uniform belief over the proportion of his opponents choosing each action. We conclude that invest can only be rationalizable after observing a large negative shock if it is risk dominant. Thus if invest is not risk dominant and there is a large negative shock, not invest is uniquely rationalizable and there is a crisis. This is the large shock result.

Our large shock result uses the key argument from the "global games" literature (Carlsson and Van Damme (1993)) but in a novel context, and it is useful to contrast the results. A classic benchmark result in the global game literature is the following. Suppose that players observe the payoffs of a game drawn according to a smooth prior, with a small amount of idiosyncratic noise. Look at the equilibria of the sequence of games as the amount of noise goes to zero. In the limit as the noise goes to zero, there is "global uniqueness": each player has a unique rationalizable action whatever signal she observes. In a binary action

symmetric payoff game, the unique rationalizable action is the risk dominant action. The global uniqueness and the selection of the risk dominant action are a consequence of the fact that, as the noise goes to zero, rank beliefs always become uniform and thus there is common certainty of uniform rank beliefs.<sup>2</sup>

In this paper, we do not study the case where idiosyncratic noise goes to zero, and we therefore do not have common certainty of uniform rank beliefs and we do not have global uniqueness. We identify some situations where there is unique rationalizable behavior but in other cases there are multiple rationalizable actions. We instead identify a novel set of conditions under which uniform rank beliefs arise: after a large shock when the common component of fundamentals has fat tails. A player who observes a large shock has uniform rank beliefs and knows that all players observing larger shocks also have uniform rank beliefs. This allows us to establish a "local uniqueness" result: after observing a large shock, it is uniquely rationalizable to play the risk dominant action. We believe that this more selective use of the global game reasoning is best able to generate insights about which informational events trigger equilibrium shifts.

Our analysis will be presented in two parts. We first characterize rationalizable individual play in the static coordination game. This represents the analytic core of the paper. We show that invest is uniquely rationalizable for a player if (i) his private return exceeds a critical level or (ii) invest is risk dominant for him and there was a large shock in his return. Symmetrically, not invest is uniquely rationalizable if his private return is below another critical level or not invest is risk dominant and there was a large negative shock. When the coordination motives are sufficiently strong, both actions are rationalizable in other cases, leading to a characterization. In the second part of the paper, we discuss implications of the theory for observations about aggregate play and dynamics. First, when the fundamentals exceeds the critical level in (i), a majority of returns also exceed the critical level. In that case, a majority of players invest under any rationalizable solution. Likewise, when there is a large shock to the fundamentals, then a majority of players experience a large shock in their private returns, leading them to invest under all rationalizable solutions. In a dynamic

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<sup>2</sup>Carlsson and Van Damme (1993) showed limit global uniqueness and risk dominant selection in two player two action games. Frankel, Morris, and Pauzner (2003) show limit global uniqueness in a class of supermodular global games and risk dominant selection in binary action symmetric payoffs games (see also Morris and Shin (2003)). Morris, Shin, and Yildiz (2016) formalize the idea that global uniqueness and risk dominant selection follow from common certainty of uniform rank beliefs.

setup this leads to equilibrium shifts. When the fundamentals exceed a critical threshold or invest is risk dominant and there was a large shock to the fundamentals, a majority of players invest. Thus, the above events trigger a shift to majority investing if they were not doing so in the previous period. Likewise, if the fundamentals go below a critical threshold or not invest is risk dominant and there was a large shock to the fundamentals, a majority of players stop investing, triggering a crisis.

We must emphasize that our results rely on our assumption that the common shock has fat tails while idiosyncratic shocks have thinner tails. For example, global games papers typically assume that both the common shock and the idiosyncratic shocks are normally distributed. In that case, large shocks will not play any remarkable role, and, when the coordination motives are strong, the critical level in (i) will be near the threshold at which invest becomes dominant. The rationalizable behavior will be unremarkable. Now, suppose that the variance of the common shock is unknown and distributed with inverse  $\chi^2$ -distribution. This leads to a  $t$ -distribution for the common shock, satisfying all of our distributional assumptions, exhibiting the rationalizable behavior established in our paper.

Our fat-tail assumption is in line with a long-standing empirical literature that establishes that changes in key economic variables have fat tailed distributions, going back to Pareto's (1897) observation about income distribution (see surveys by Benhabib and Bisin (2016), Gabaix (2009) and Ibragimov and Prokharov (2016)). For example, changes in GDP, prices, asset returns and foreign exchange rates all have fat-tailed distributions (see pioneering works of Mandelbrot (1963) and Fama (1963), as well as contemporary studies such as Cont (2001), Gabaix, Gopikrishnan, Plerou, and Stanley (2006), and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017)). Moreover, many commonly used theoretical models, such as GARCH models and models with stochastic volatility, naturally lead to fat tailed changes in the fundamental, as in the example of  $t$ -distribution above. In our model, individual returns are more volatile than the fundamentals as they contain additional idiosyncratic shocks. Motivated with model uncertainty, we assumed that idiosyncratic shocks have thinner tails,<sup>3</sup> so that the tails of the changes in payoff types are as thick as the tails of common shocks. This is similar to the fact that the empirical tail indices of stock and market returns are both approximately 3 (Gabaix (2009)). This assumption is plausible especially when the players

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<sup>3</sup>We allow idiosyncratic shocks to have a larger variance, consistent with cross-sectional data exhibiting larger variance than the aggregate data.

learn the distribution of the shocks from the past realizations; there is a large cross-sectional data about the individual shocks while there is only a single time series about the common shocks. Idiosyncratic variation can also be interpreted as variation in players' signals of the fundamental; under this interpretation, thinner idiosyncratic tails correspond to a well understood (if noisy) observation technology.

In the next section, we introduce our model. In Section 3, we define and characterize the rank belief functions that will drive our results, and give our basic characterization of equilibria and rationalizable behavior. In that section, we also illustrate our key results graphically assuming the shocks are normally distributed and the variance of common shock may not be known. Our main results are reported in Section 4. We study the implications for aggregate behavior in Section 5, a dynamic extension of the model in Section 6. In Section 7, we review what happens if we relax the assumption that common shocks have a fat-tailed distribution. We discuss our broader contribution to the global games literature in Section 8. Some proofs are relegated to the Appendix.

## 2 Model

We study the following Bayesian game, parametrized by real numbers  $y$  and  $\sigma > 0$ . There is a continuum of players  $i \in N = [0, 1]$ . Simultaneously, each player  $i$  chooses between actions invest and not invest; the chosen action is denoted by  $a_i$ . The payoff from not invest is normalized to zero. The payoff from invest depends on a *type*  $z_i \in \mathbb{R}$  and the fraction  $A$  of individuals who invest:

$$u(A, z_i) = y + \sigma z_i + A - 1. \quad (1)$$

The type  $z_i$  has two components:

$$z_i = \eta + \varepsilon_i, \quad (2)$$

a *common shock*  $\eta$  that affects all players' payoffs, and an *idiosyncratic shock*  $\varepsilon_i$  that affects only the payoff of player  $i$ . Player  $i$  (privately) knows the sum  $z_i$ , but not its components.

We write

$$x_i = y + \sigma z_i \quad (3)$$

for the private *return* from investment for type  $z_i$ . The shock  $z_i$  will have zero mean. Hence, the ex-ante expectation of the return is  $y$ , which we call *prior mean*. The sensitivity of the return to shock  $z_i$  is  $\sigma$ , which we call *shock sensitivity*. Note that the the coordination

motives are inversely proportional to the shock sensitivity (i.e.  $\frac{\partial u}{\partial A} / \frac{\partial u}{\partial z_i} = 1/\sigma$ ). We will pay a special attention to the case of small  $\sigma$ , when the coordination motives are large.

We assume that  $\varepsilon_i$  and  $\eta$  are independently drawn—across the players—from distributions  $F$  and  $G$ , respectively, with positive continuous densities  $f$  and  $g$  everywhere on real line. We will assume that these distributions are symmetric around zero, i.e.,  $f(\varepsilon) = f(-\varepsilon)$  and  $f(\eta) = f(-\eta)$ , and that  $f$  and  $g$  are weakly decreasing on  $(0, \infty)$ . By symmetry, the idiosyncratic shock  $\varepsilon_i$  has zero mean and  $F(\varepsilon) = 1 - F(-\varepsilon)$ . Likewise, the common shock  $\eta$  has zero mean and  $G(\eta) = 1 - G(-\eta)$ .

Our key distributional assumptions are:

1. the distribution of idiosyncratic shocks is log-concave (i.e.,  $\log f$  is concave), and
2. the distribution of common shocks has regularly-varying tails, i.e.,

$$\lim_{\lambda \rightarrow \infty} \frac{g(\lambda\eta)}{g(\lambda\eta')} \in (0, \infty) \text{ for all } \eta, \eta' \in \mathbb{R}_+. \quad (4)$$

The log-concavity of  $f$  implies that the idiosyncratic shocks have subexponential (thin) tails (i.e.  $\int e^{c|\varepsilon|} f(\varepsilon) d\varepsilon$  is finite for some  $c > 0$ ), and common distributions with subexponential tails, such as normal and exponential distributions, are log-concave. In contrast, the second part states that  $g$  has regularly-varying (i.e. fat) tails, as in Pareto and  $t$ -distributions. In that case,  $g(\eta)$  is approximately proportional to  $\eta^{-\alpha}$  for some  $\alpha > 1$  when  $\eta$  is large, and the tails are thicker than the exponential function. Altogether, we assume that the common shock has thicker tails than the idiosyncratic shocks, reflecting our assumption that there is more tail uncertainty about the common shock. The log-concavity of  $f$  also ensures that each player's belief about other players' types is increasing in his own type in the sense of the first-order stochastic dominance, making our game monotone supermodular. While such monotonicity and the tail properties are important in our analysis, log-concavity is assumed for exposition. We will maintain these assumptions throughout the paper. We will return in Section 7 to see what happens to rank beliefs and thus our results when we relax these assumptions.

We now introduce some useful terminology. Each player  $i$  has a strictly dominant strategy to invest if  $x_i$  is strictly more than one, has a strictly dominant strategy to not invest if  $x_i < 0$  and otherwise no action is strictly dominated. We will therefore refer to  $[0, 1]$  as *the undominated region* and 0 and 1 as *the dominance triggers*. Under complete information, there are

multiple equilibria in the region  $[0, 1]$ : all invest and all not invest. Game theoretic analysis suggests *refinements* to select among equilibria. An action is said to be *risk-dominant* if it is a best response when each action is equally likely to be played by other players. Invest is the risk-dominant action when  $x_i > 1/2$ ; and not invest is the risk-dominant action when  $x_i < 1/2$ . More generally, we say that an action is *p-dominant* if it is a best response when a player's expectation of the proportion of others taking the same action is at least  $p$ . Now invest is strictly dominant if it is 0-dominant; it is an equilibrium action under complete information if it is 1-dominant, and it is risk dominant if it is  $\frac{1}{2}$ -dominant.

In our model, we made a host of simplifying assumptions, such as a continuum of players and independence of idiosyncratic shocks. We do not have a theoretical foundation for these assumptions. Our motivation is rather pragmatic. For example, the independence assumption ensures that the common shock is the only source of correlation among returns. Likewise, the continuum and independence assumptions together allow us to obtain a deterministic aggregate behavior as a function of the average returns.

### 3 Rank Beliefs and Equilibrium Structure

In this section, we present the main ingredients of our analysis. We formally introduce the rank beliefs and identify their key properties for our analysis. Rank beliefs are key to our analysis as they determine how a player thinks his return relates to others'. We then describe the structure of equilibria and rationalizable strategies: rationalizable strategies are bounded by symmetric equilibria in cutoff strategies, and the return is equal to the rank belief at the equilibrium cutoffs. Finally, we illustrate these results on a canonical example in which common and idiosyncratic shocks have  $t$  and normal distributions, respectively.

#### 3.1 Rank Beliefs

We define the *rank belief* of player  $i$  as the probability he assigns to the event that another player's type  $z_j$  is lower than his own:

$$R(z) = \Pr(z_j \leq z_i | z_i = z) = \frac{\int F(\varepsilon) f(\varepsilon) g(z - \varepsilon) d\varepsilon}{\int f(\varepsilon) g(z - \varepsilon) d\varepsilon}. \quad (5)$$



We refer to the function  $R$  as the *rank-belief function*. The following properties of rank belief functions will be important for us.

**Symmetry** Rank belief function is said to be *symmetric* if

$$R(-z) = 1 - R(z).$$

That is,  $R$  is symmetric around  $1/2$  for positive and negative values. In that case, we have  $R(0) = 1/2$ .

**Single-Crossing Property** Rank belief function is said to satisfy *single-crossing property* if  $R(z) > 1/2 > R(-z)$  whenever  $z > 0$ . That is,  $R$  takes the value of  $1/2$  at  $z = 0$  and remains above  $1/2$  for positive  $z$ , and symmetrically remains below  $1/2$  for negative  $z$ .

**Uniform Limit Rank Beliefs** We say that *limit rank beliefs are uniform* if

$$R(z) \rightarrow \frac{1}{2} \text{ as } z \rightarrow \infty.$$

That is, as  $z \rightarrow \infty$ , the rank belief converges back to  $1/2$ . Uniformity of limit rank beliefs implies immediately some further properties. The rank belief  $R$  is bounded away from 0 and 1. We write  $\bar{R} < 1$  for the upper bound. And the rank belief is decreasing over some interval.

Rank beliefs exhibit these properties in our model:

**Lemma 1** *The function  $R$  is differentiable, symmetric, and satisfies single-crossing and uniform limit rank belief properties.*

Here, differentiability follows from having density, and symmetry and single crossing properties follow from the symmetry of these densities. Uniformity of limit rank beliefs is special. It is a consequence of our assumption that the common shock has fat tails, and the idiosyncratic shocks have thinner tails.

We plot a typical rank belief function  $R$  in Figure 1 as a function of shock  $z$ , along with the return  $x = y + \sigma z$ . The rank belief function is non-linear. At  $z = 0$ , by symmetry, the rank belief is  $1/2$ . As  $z$  increases,  $R$  first gets larger by single-crossing property, and finally it goes back to  $1/2$  by uniformity of limit rank beliefs. By single crossing property,  $R$  behaves symmetrically for negative shocks.

### 3.2 Structure of Equilibria and Rationalizable Behavior

A (*Bayesian Nash*) *equilibrium* is defined as usual by requiring each type to play a best response. We first characterize a class of symmetric "threshold" equilibria. Suppose that each player invested only if his type  $z_i$  were greater than a critical threshold  $\hat{z}$ . Consider a player whose type was that critical threshold  $\hat{z}$ . His payoff to investing would be

$$\underbrace{y + \sigma\hat{z}}_{\text{own return}} + \underbrace{(1 - R(\hat{z}))}_{\text{expected proportion of others' investing}} - 1.$$

The threshold type  $\hat{z}$  will be indifferent only if this payoff is equal to 0, i.e.,

$$R(\hat{z}) = y + \sigma\hat{z}. \quad (6)$$

This is thus a necessary condition for there to be a  $\hat{z}$ -threshold equilibrium. But this is also sufficient for equilibrium. Suppose that a player anticipated that all other players were going to play a  $\hat{z}$ -threshold strategy, and was therefore indifferent between investing and not investing when his type was  $\hat{z}$ . If his type was  $z_i > \hat{z}$ , he would have higher incentive to invest since both his return from investment would be higher and his expectation of the proportion of others' investing would be higher (by log-concavity of  $f$ ).

The largest and smallest threshold strategy equilibria will play a key role in our analysis. Write  $z^*$  and  $z^{**}$  for the smallest and the largest solutions to (6), respectively; see Figure 1 for an illustration. We write  $x^* = \sigma z^* + y$  and  $x^{**} = \sigma z^{**} + y$  for the corresponding returns. Define symmetric strategies  $s^*$  and  $s^{**}$  associated with these cutoffs by

$$s_i^*(z_i) = \begin{cases} \text{Invest} & \text{if } z_i \geq z^* \\ \text{Not Invest} & \text{otherwise} \end{cases}$$

$$s_i^{**}(z_i) = \begin{cases} \text{Invest} & \text{if } z_i \geq z^{**} \\ \text{Not Invest} & \text{otherwise.} \end{cases}$$

Our next result establishes that  $s^*$  and  $s^{**}$  are Bayesian Nash equilibria and they bound all rationalizable (and hence equilibrium) strategies. In particular,  $s^*$  is the equilibrium with the most investment, while  $s^{**}$  is the equilibrium with the least investment.

**Lemma 2**  *$s^*$  and  $s^{**}$  are Bayesian Nash equilibria. Moreover, invest is uniquely rationalizable whenever  $z_i > z^{**}$ , and not invest is uniquely rationalizable whenever  $z_i < z^*$ .*

Thus the set of rationalizable actions is as follows. When  $z^* \leq z_i \leq z^{**}$ , both actions are rationalizable, and there is a unique rationalizable action otherwise. The unique rationalizable action is invest when  $z_i > z^{**}$ , and it is not invest when  $z_i < z^*$ . In the appendix, we prove this result by checking that our game is monotone supermodular (Van Zandt and Vives (2007)), and thus the Bayesian Nash equilibria and the rationalizable strategies are bounded by monotone Bayesian Nash equilibria. The key step in the proof is to show that under log-concave  $f$ , the beliefs about the common shock are increasing in  $z_i$  (i.e.  $\Pr(\eta \leq \bar{\eta}|z_i)$  is decreasing in  $z_i$  for any  $\bar{\eta}$ ).

### 3.3 Example: $t$ Distribution

We now graphically illustrate the key properties of rank beliefs and the equilibrium structure. We assume that the idiosyncratic shocks have the standard normal distribution. We assume that the common shock is also normally distributed but its variance is not known: the reciprocal of its variance has a  $\chi^2$  distribution. Such variance uncertainty leads to the  $t$ -distribution, satisfying all of our distributional assumptions. We can interpret this as model uncertainty: the player does not know what is the true data generating function for a parameter that affects everybody.

There will be two effects of an increase in a player's shock on this rank belief. First, there will be the *reversion to mean* effect: a player will attribute some of the shock to his return to the common shock and some of it to his own idiosyncratic shock. Because of the last attribution, a player's expectation of the common shock will be further from his own shock as his own shock increases. This effect increases the rank belief as a player's shock increases. But second, there will be a *learning* effect. When the variance of the common shock is unknown, a high shock will lead a player to conclude that the variance of the common shock is higher, and he will attribute an increasing portion of his payoff shock to the common shock. This effect will tend to decrease rank beliefs. The shape of the rank belief function will then depend on which of these two effects predominates. Figure 1 plots the rank belief function for this example: when a player's shock becomes large, he attributes it almost entirely to the common shock; the learning effect will predominate and the rank belief will approach 1/2.

The extremal cutoffs are plotted in Figure 1. They correspond to the extremal intersection of non-monotone rank belief function  $R$  and the line that represents the private return  $y + \sigma z$

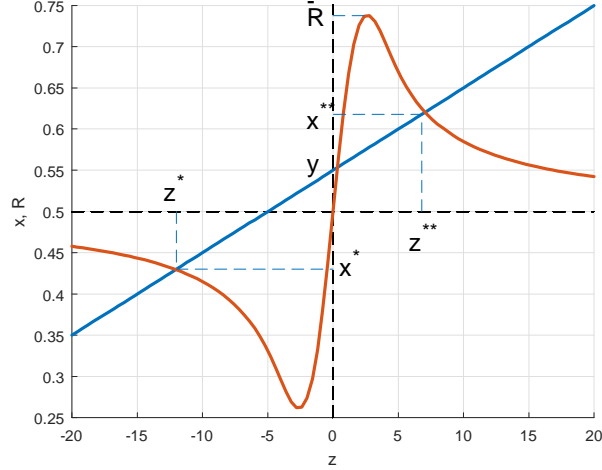


Figure 1: Extremal cutoffs when the variance of the common shock is unknown

as a function of shock  $z$ . The private returns  $x^*$  and  $x^{**}$  at the extremal cutoffs are in between  $1 - \bar{R}$  and  $\bar{R}$ , where  $\bar{R}$  is the maximum possible rank belief; this is marked on Figure 1 and is approximately 0.738. Hence, invest will become uniquely rationalizable for every shock when the return is more than  $\bar{R}$ . This is the *level* effect we discussed in the introduction. Second, as we decrease  $y$ , the return  $x^{**}$  at the upper cutoff approaches  $1/2$ . Hence, for any  $x_i > 1/2$ , invest is uniquely rationalizable (i.e.  $x_i > x^{**}$ ) whenever the prior mean is sufficiently low, or equivalently whenever there was a sufficiently large positive shock. This is the *change* effect we discussed in the introduction. We will next establish these generally, as our main results.

## 4 Rationalizable Behavior and the Role of Shocks

Suppose that a player has a private return  $x$ , having received a shock  $z$  and thus having a prior mean  $y = x - \sigma z$ . Which actions are rationalizable? In particular, are both actions rationalizable or is invest or not invest the uniquely rationalizable action? Since the model is entirely symmetric between the two actions, we report formal necessary and sufficient conditions for invest to be the uniquely rationalizable action and the rest of the characterization will follow by symmetry. These characterizations will follow easily from our characterization of the rank belief function in Lemma 1 and rationalizable behavior in Lemma 2 to characterize rationalizable behavior. In particular, we will be able to explain

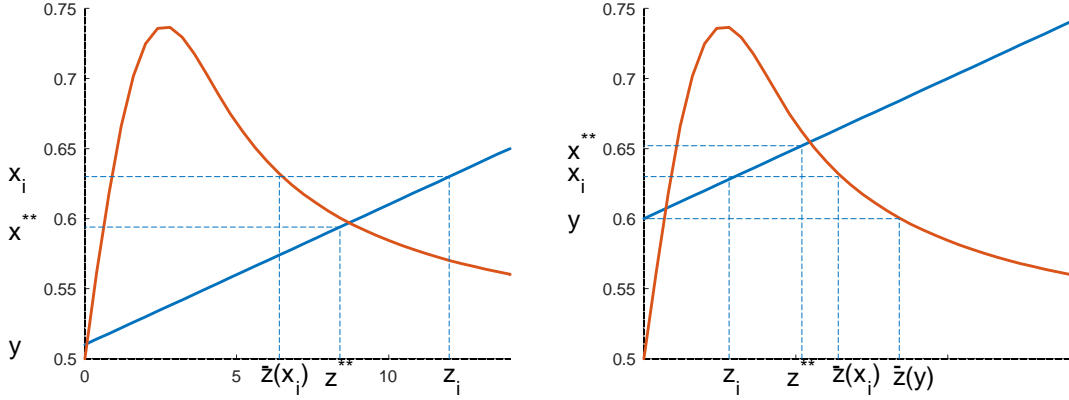


Figure 2: Effect of shocks; shock size is above the critical value on the left panel and below the critical value on the right panel.

the results by appeal to the simple geometry of Figure 1.

We first observe that when a player's private return exceeds the maximum rank belief  $\bar{R}$ , or equivalently when invest is  $(1 - \bar{R})$ -dominant, invest will be uniquely rationalizable independent of what his shock was.

**Proposition 1** *Invest is uniquely rationalizable if it is minimum rank belief dominant (i.e.  $x_i > \bar{R}$ ).*

**Proof.** Observe that, for any  $\sigma$  and  $y$ ,

$$x^{**} = R(z^{**}) \leq \bar{R},$$

where the equality is by definition of  $x^{**}$  and the inequality is by definition of  $\bar{R}$ . Therefore, whenever  $x_i > \bar{R}$ , we have  $x_i > x^{**}$  and invest is uniquely rationalizable. ■

But we also see from Figure 1 that even if a player's return is less than  $\bar{R}$ , invest will be uniquely rationalizable if the return is larger than the rank belief given the shock that gave rise to that return. In particular, invest is uniquely rationalizable whenever  $x_i > 1/2$  and  $z_i$  exceeds some threshold  $\bar{z}$  where  $\bar{z}$  is a function of  $x_i$ .

For each  $x_i > 1/2$ , at which invest is risk-dominant, define the cutoff

$$\bar{z}(x_i) = \max R^{-1}(x_i). \quad (7)$$

The cutoff  $\bar{z}(x_i)$  is illustrated in Figure 2, where we only show the part of Figure 1 where invest is risk-dominant and  $z_i \geq 0$ . As seen in the figure, for  $x_i \leq \bar{R}$ ,  $\bar{z}(x_i)$  is the maximum

level of shock under which a player's rank belief is  $x_i$ . (For  $x_i > \bar{R}$ ,  $\bar{z}(x_i) = -\infty$  by the convention that maximum of empty set is  $-\infty$ .)

It turns out that the cutoff  $\bar{z}(x_i)$  is the critical threshold for a shock to be effective in making the risk-dominant action uniquely rationalizable. This is formally established in our next result—the main result of our paper.

**Proposition 2** *Invest is uniquely rationalizable if it is risk-dominant (i.e.  $x_i > 1/2$ ) and the shock is sufficiently large, i.e.,*

$$z_i > \bar{z}(x_i). \quad (8)$$

**Proof.** The special case of  $x_i > \bar{R}$  is covered in Proposition 1. Hence, assume that  $\bar{R} \geq x_i > 1/2$  and  $z_i > \bar{z}(x_i)$ —as in the left panel of Figure 2. Then, for all  $z \geq z_i$  we have

$$R(z) < x_i = y + \sigma z_i \leq y + \sigma z,$$

where the strict inequality is by definition of  $\bar{z}(x_i)$  and  $z > \bar{z}(x_i)$ . Hence,  $z^{**} < z_i$ . Therefore, invest is uniquely rationalizable at  $z_i$ . ■

Proposition 2 provides sufficient conditions for invest being uniquely rationalizable: it is risk-dominant (i.e.,  $x_i > 1/2$ ) and there was a large positive shock with size more than critical level  $\bar{z}(x_i)$ . By symmetry, this also establishes that not invest is uniquely rationalizable if it is risk-dominant (i.e.  $x_i < 1/2$ ) and there is a large negative shock—with size more than  $\bar{z}(1 - x_i)$ . We will refer to  $\bar{z}(x_i)$  as the *critical shock size*. The critical shock size is independent of  $\sigma$ , rendering the critical shock size on returns,  $\sigma \bar{z}(x_i)$ , proportional to  $\sigma$ . Hence, the latter threshold can be arbitrarily small for small  $\sigma$  and large for large  $\sigma$ . For example, when coordination motives are strong, a very small positive jump in his return will lead a player to invest if invest is risk-dominant. Likewise, a very small drop in his return will lead a player not to invest if not investing is risk-dominant. Such behavior also arises in highly stable environments where one does not expect large shifts in returns. In the remainder of the paper, by a "large shock", we mean a shock of size that exceeds a critical shock size.

The proof of Proposition 2 is as illustrated on the left panel of Figure 2. Here,  $x_i = 0.63$  and hence invest is risk-dominant. Moreover, the shock  $z_i$  exceeds the critical shock size  $\bar{z}(x_i)$ . Now, for any shock level  $z \geq z_i$ , since  $z$  is strictly greater than  $\bar{z}(x_i)$ , the rank belief  $R(z)$  is strictly below  $x_i$  (by definition of  $\bar{z}(x_i)$ ). But clearly for any such  $z$ , the return  $y + \sigma z$

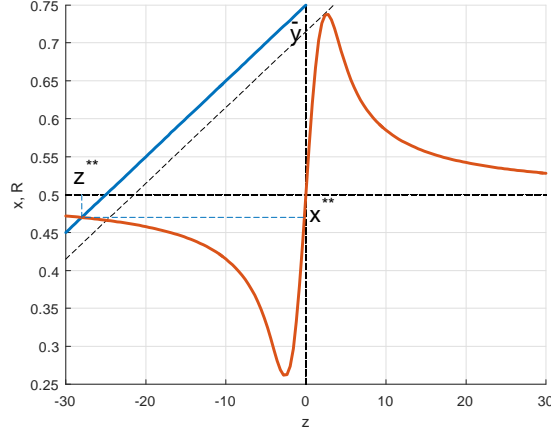


Figure 3: Equilibrium cutoffs when prior mean is above  $\bar{y}$ .

is above  $x_i$ . Hence, the returns remain strictly above the rank beliefs for all  $z \geq z_i$ . Thus, the maximal equilibrium cutoff  $z^{**}$  is strictly smaller than  $z_i$ . Therefore, invest is uniquely rationalizable at  $z_i$ . In contrast, the case on the right panel illustrates that invest may not be uniquely rationalizable without a large shock. Here, the return is still as in the left panel ( $x_i = 0.63$ ), but the shock  $z_i$  is now smaller than the critical level  $\bar{z}(x_i)$ . In that case, the equilibrium cutoff  $z^{**}$  is above  $z_i$ , and thus the cutoff  $x^{**}$  is above  $x_i$ , leading to multiplicity at  $x_i$ .

In Figure 1, the rank belief function  $R(\cdot)$  crossed the return  $y + \sigma z$  in the positive orthant. But if the mean  $y$  was high enough, the return would exceed the rank belief function for all positive shocks. This is illustrated in Figure 3. We can define cutoff  $\bar{y}$  as the largest  $y$  for which there exists  $z > 0$  such that

$$R(z) \geq \sigma z + y. \quad (9)$$

We will refer to  $\bar{y}$  as *the prior investment threshold*.<sup>4</sup> Define also cutoff  $\underline{y} = 1 - \bar{y}$ . These cutoffs play a prominent role in the remainder of the paper. When the prior mean is above the cutoff  $\bar{y}$ , the return remains strictly above the rank belief for non-negative shocks and hence the cutoff  $z^{**}$  is negative and  $x^{**} < 1/2$ . Therefore, invest is uniquely rationalizable whenever it is risk dominant, regardless of the size of the shock, and it can be uniquely

<sup>4</sup>The cutoff  $\bar{y}$  lies between  $1/2$  and  $\bar{R}$ . It is  $\bar{R}$  in the limit  $\sigma \rightarrow 0$ , and it decreases towards  $1/2$  as  $\sigma$  increases. When  $\sigma < \sup_z R(z)/z$ , the cutoff  $\bar{y}$  is determined by the tangency of the line  $\sigma z + y$  to  $R$  and is strictly above  $1/2$ . In contrast,  $\bar{y} = 1/2$  when  $\sigma > \sup_z R'(z)$ .

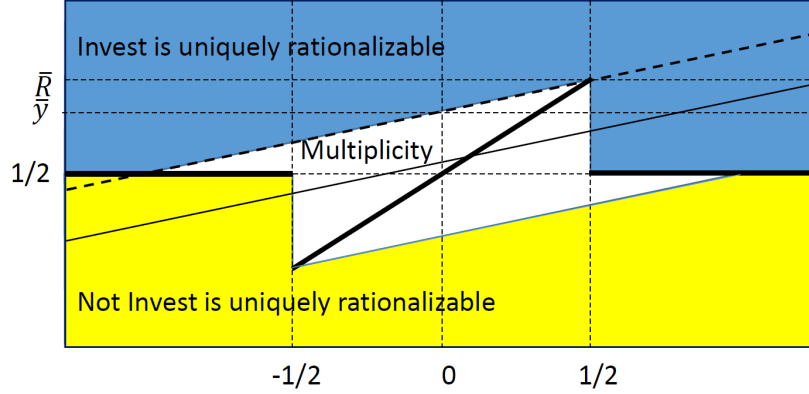


Figure 4: Rank beliefs and rationalizable behavior in Example 1. (Horizontal axis: shock  $z_i$ ; vertical axis:  $R(z_i)$  and return  $x_i$ .)

rationalizable even when it is not risk dominant and there is a negative shock.

**Proposition 3** *Invest is uniquely rationalizable if it is risk-dominant (i.e.  $x_i > 1/2$ ) and the prior mean exceeds the prior investment threshold (i.e.,  $y > \bar{y}$ ).*

**Proof.** For any  $y > \bar{y}$ , by definition of  $\bar{y}$ , the return exceeds the rank beliefs for all positive shocks:  $\sigma z + y > R(z)$  for all  $z > 0$ . Hence,  $z^{**} < 0$  as it is the largest  $z$  with  $R(z) = \sigma z + y$ . Thus,

$$x^{**} = R(z^{**}) < 1/2,$$

where the equality is by definition of  $x^{**}$  and the inequality is by the single-crossing property of rank beliefs in Lemma 1. Therefore, invest is uniquely rationalizable at each  $x_i > 1/2$ . ■

We have thus shown three sufficient conditions for invest to be uniquely rationalizable: (i) it is minimum rank belief dominant; (ii) it is risk-dominant and there was a large shock, and (iii) it is risk dominant and the prior mean was above  $\bar{y}$ . Symmetrically, not invest is uniquely rationalizable when (i) it is minimum rank belief dominant ( $x_i < 1 - \bar{R}$ ); (ii) it is risk-dominant ( $x_i < 1/2$ ) and there was a large negative shock, and (iii) it is risk dominant and the prior mean was below  $\bar{y}$ . We next illustrate our results using a parametric example. This stark example does not satisfy some of our technical assumptions such as continuity and positive density everywhere, but it clearly demonstrates the logic of our results and the role of model uncertainty.



**Example 1** *Idiosyncratic shocks are drawn from uniform distribution on  $[-1/2, 1/2]$ . There is uncertainty about the distribution of the common shock: with probability  $\pi$ , there is no common shock (i.e.  $\eta = 0$ ), but with complementarity probability  $1 - \pi$ , the common shock takes any value on the real line with improper uniform distribution. The rank belief function is plotted in Figure 4, along with the description of the rationalizable behavior for each shock-return pair  $(z_i, x_i)$  where  $\sigma < \pi/2$  is fixed while  $y$  varies. The rank belief function is given by<sup>5</sup>*

$$R(z_i) = \begin{cases} \frac{1}{2} + \pi z_i & \text{if } z_i \in [-1/2, 1/2] \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

*As in the figure, it is non-monotone: it starts at  $1/2$  for  $z_i < -1/2$ , drops to  $(1 - \pi)/2$  at  $z_i = -1/2$ , increases to*

$$\bar{R} = \frac{1}{2} + \frac{\pi}{2}$$

*until  $z_i = 1/2$  after which it drops back to  $1/2$ . The critical shock size is*

$$\bar{z} = 1/2$$

*for each return  $x_i \in (1/2, \bar{R})$ . Rationalizable behavior is as in the figure. First, as an illustration of condition (i) above, invest is uniquely rationalizable whenever the return is above  $\bar{R}$ , regardless of what shock is. Second, as an illustration of condition (ii), invest is uniquely rationalizable when the shock exceeds the critical level  $\bar{z}$  and return is above  $1/2$ —the area north-east of  $(1/2, 1/2)$  in the figure. Finally, as an illustration of condition (iii), invest is uniquely rationalizable when  $y$  is above*

$$\bar{y} = \frac{1}{2} + \frac{\pi - \sigma}{2}$$

*and the return is above  $1/2$ ; in the figure, this is the area above the dashed line  $x_i = \bar{y} + \sigma z_i$  and above the horizontal line  $x_i = 1/2$ . As in the figure, not invest is also uniquely rationalizable under symmetrical conditions. In this figure, these conditions characterize the rationalizable behavior. In all the remaining cases (i.e., in the unshaded area), both actions*

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<sup>5</sup>Intuitively, when  $z_i \notin [-1/2, 1/2]$ , player  $i$  learns that there is a common component and thus  $\eta$  is uniformly distributed on real line. Then, the rank belief is  $1/2$  as in standard global games. When  $z_i \in [-1/2, 1/2]$ , a player does not learn anything about whether there is a common component. With probability  $1 - \pi$ , there is a common component and rank beliefs are  $1/2$ . With probability  $\pi$ , there is no common component (i.e.  $\eta = 0$ ), and  $z_j$  is uniformly distributed on  $[-1/2, 1/2]$  independent of  $z_i$ , yielding the rank belief  $z_i + 1/2$ . His rank belief,  $\frac{1}{2} + \pi z_i$ , is the weighted average of  $z_i + 1/2$  and  $1/2$ .

are rationalizable, and we have multiple equilibrium behavior. In particular, for any prior mean  $y$  between  $1 - \bar{y}$  and  $\bar{y}$ , both actions are rationalizable whenever the shock size falls below the critical level. This is illustrated in Figure 4 by returns plotted in thin solid lines where  $z^* = -1/2$  and  $z^{**} = 1/2$ .

This characterization holds as long as the shock sensitivity,  $\sigma$ , is below  $\pi/2$ . The rationalizable behavior is similar if  $\sigma$  is in between  $\pi/2$  and  $\pi$ . When  $\sigma > \pi$ , the rationalizable behavior is quite different: the game is dominance solvable, and the unique rationalizable action is invest when the return is above a cutoff  $x^*$  and not invest when the return is below the cutoff  $x^*$ . Depending on  $y$ , the cutoff  $x^*$  can take any value between  $1 - \bar{R}$  and  $\bar{R}$ . Thus, unique rationalizable action is not determined by risk dominance or shock size.

The example illustrates the fact that when the shock sensitivity is small, the three sufficient conditions characterize unique rationalizability. In general, however, we see that there is a gap. As in this example, in general, when shock sensitivity is high, invest can be uniquely rationalizable without being risk dominant. In particular, when  $\sigma > \sup_z R'(z)$ , the rationalizable action is unique, and it depends only on whether the return is above or below the rank belief, independent of the size of the shock and the risk dominant action. We next rule out such scenarios (by requiring that  $\sigma \leq (\bar{R} - y) / \bar{z}(\bar{R})$ ), and obtain a characterization:

**Proposition 4** *Assume  $R$  is single peaked on  $\mathbb{R}_+$  and  $y \leq \bar{R} - \sigma \bar{z}(\bar{R})$ . Then, invest is uniquely rationalizable if and only if it is risk-dominant and  $z_i > \bar{z}(x_i)$ .*

That is, invest is uniquely rationalizable if and only if it is risk dominant and there was a large positive shock—as in Proposition 2. This also includes the case in Proposition 1 because when the return is above the maximum rank belief  $\bar{R}$ , the critical shock size is  $-\infty$ , and all shocks are considered large. In Proposition 3, invest was uniquely rationalizable whenever it was risk dominant and the prior mean was above the cutoff  $\bar{y}$ , even if there was a negative shock and the return was below the maximum rank belief. This case is ruled out by the condition that  $y \leq \bar{R} - \sigma \bar{z}(\bar{R})$  in the hypothesis. Once that case is ruled out, for any given return level  $x_i > 1/2$ , the rationalizable behavior is a monotone function of shock  $z_i$ : both actions are rationalizable when  $z_i \leq \bar{z}(x_i)$  and invest is uniquely rationalizable when  $z_i > \bar{z}(x_i)$ .

We must note that the condition  $y \leq \bar{R} - \sigma \bar{z}(\bar{R})$  is crucial for this characterization. In general, rationalizable behavior is non-monotone in shock  $z_i$  for any fixed return level

$x_i > 1/2$ . As one can glean from Figure 4, for any fixed  $x_i \in (1/2, \bar{y})$ , invest is uniquely rationalizable when  $z_i < (\bar{y} - x_i)/\sigma$ ; both actions are rationalizable when  $(\bar{y} - x_i)/\sigma \leq z_i \leq \bar{z}(x_i)$ , and invest is uniquely rationalizable once again when  $z_i > \bar{z}(x_i)$ . As the shock sensitivity gets smaller, the lower cutoff gets smaller, making our characterization more relevant. In the limit  $\sigma \rightarrow 0$ , the lower cutoff approaches  $-\infty$ , and the rationalizable behavior is as in our characterization.

This characterization is obtained by establishing a converse to our main result under the additional conditions in the hypothesis. The proof of the converse is depicted in the right panel of Figure 2. In this figure, invest is risk dominant, but the shock  $z_i$  is smaller than the critical level  $\bar{z}(x_i)$ . The additional conditions for the converse are also met in this example:  $R$  is single-peaked, and  $y \leq \bar{R} - \sigma\bar{z}(\bar{R})$ , so that  $R$  is decreasing at the cutoff  $z^{**}$ , where the line  $y + \sigma z$  cuts  $R$ . Then, as in the figure, the equilibrium cutoff  $z^{**}$  must be at least as large as  $z_i$ , and thus  $x^{**}$  must be above  $x_i$ .<sup>6</sup> Therefore, not invest is rationalizable.

We next focus on the case of small shocks to the returns. In that case, rationalizable actions depend only on the position of prior mean relative to the cutoffs  $\bar{y}$  and  $\underline{y}$ .

**Proposition 5** *For any  $\sigma < \sup_z (R(z) - 1/2)/z$ , there exists  $\Delta > 0$  such that whenever  $|x_i - y| \leq \Delta$ , invest is uniquely rationalizable if and only if  $y > \bar{y}$  and not invest is uniquely rationalizable if and only if  $y < \underline{y}$ .*

Proposition 5 establishes that, without a large shock, invest is uniquely rationalizable when  $y > \bar{y}$  and not invest is uniquely rationalizable when  $y < \underline{y}$ . But equilibrium play will depend on equilibrium selection when  $y$  is in the intermediate range  $[\underline{y}, \bar{y}]$ , as the action choice depends on which equilibrium is played. Using the extreme example, we have already illustrated the uniqueness when the prior mean is outside the range  $[\underline{y}, \bar{y}]$ . When the prior mean is within this range, by definition,  $z^* < 0 < z^{**}$  and hence both actions are rationalizable when the shock is sufficiently small. The proposition provides a uniform bound that guarantees multiplicity: the bound  $\Delta$  is independent of  $y$  although it may depend on  $\sigma$  and the rank belief function.

Figure 5 illustrates the qualitative properties we have established so far. In this figure, we plot equilibrium cutoffs and regions in which invest and not invest are uniquely rationalizable

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<sup>6</sup>Indeed, suppose  $z^{**} < \bar{z}$ . Then, since the straight line has positive slope,  $x^{**}$  would have been strictly below  $x_i$ , and this would be a contradiction:  $R$  would be decreasing from  $x^{**}$  at  $z^{**}$  to the larger value  $x_i$  at  $\bar{z}(x_i)$ .

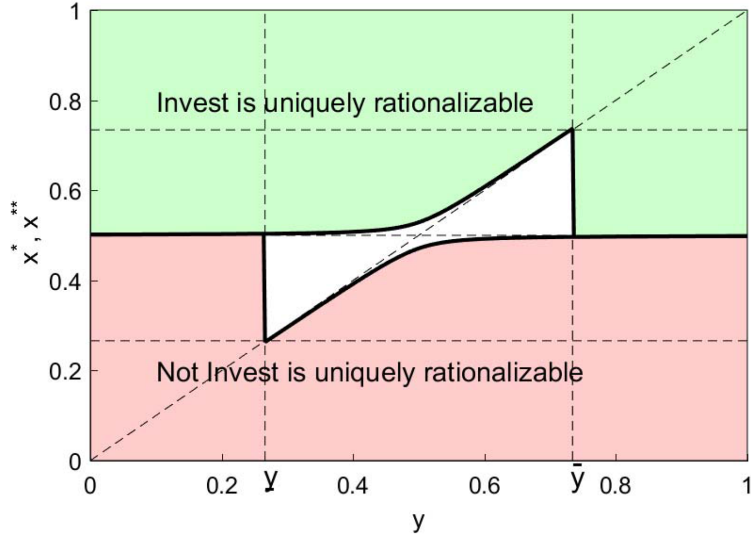


Figure 5: Equilibrium cutoffs and rationalizability as a function of prior mean  $y$  in  $t$ -distribution example ( $\sigma = 0.01$ ).

for the  $t$ -distribution example, assuming  $\sigma < \sup_z (R(z) - 1/2) / z$ . When  $y \in [\underline{y}, \bar{y}]$ , the upper equilibrium cutoff  $x^{**}$  is above  $\max\{y, 1/2\}$  and approaches  $\max\{y, 1/2\}$  as  $\sigma \rightarrow 0$  by Proposition 2. Hence, when  $y > 1/2$ , a large shock makes invest uniquely rationalizable. Since  $x^{**} > y$ , if the positive shock is sufficiently small, then both actions are rationalizable and can be played in equilibrium. Likewise, the lower cutoff  $x^*$  is below  $\min\{y, 1/2\}$  and approaches  $\min\{y, 1/2\}$  as  $\sigma \rightarrow 0$ . Once again, large negative shocks make not invest uniquely rationalizable when  $y < 1/2$ , while both actions are rationalizable under smaller shocks. Note that when  $y \in [\underline{y}, \bar{y}]$ , regions with uniquely rationalizable actions are confined to different sides of cutoff  $x_i = 1/2$ , and only a risk-dominant action can be uniquely rationalizable. Outside of  $[\underline{y}, \bar{y}]$ , a non-risk-dominant action can be uniquely rationalizable. For example, when  $y > \bar{y}$ , the cutoff  $x^{**}(y)$  is slightly below  $1/2$ —approaching  $1/2$  by Proposition 2 as  $\sigma \rightarrow 0$ . In the limit  $\sigma \rightarrow 0$ , risk-dominance is necessary for being uniquely rationalizable.

## 5 Aggregate Implications

In this section we will focus on the implications of our result on aggregate behavior. We will study how a common shock affects the aggregate investment, showing that there will be a shift in aggregate investment when the size of the common shock exceeds the critical level.

We define the *fundamental state* (or fundamentals) as

$$\theta = y + \sigma\eta,$$

which is the average return—as a function of the common shock. Since we assume a continuum of players with independently distributed idiosyncratic shocks, there is no aggregate uncertainty conditional on the fundamental state. In particular, the fraction of players with returns below a given threshold  $z$  is  $F(z - \eta)$ . Thus, the effect of shock on individuals' behavior directly translates as an effect of common shock on majority behavior:<sup>7</sup>

**Corollary 1** *Invest is uniquely rationalizable for a majority if it is risk-dominant under fundamentals (i.e.  $\theta > 1/2$ ) and, in addition, one of the following is true: (1) invest is maximum rank belief dominant (i.e.  $\theta > \bar{R}$ ); (2) there is a large common shock (i.e.,  $\eta > \bar{z}(\theta)$ ); or (3) the prior mean exceeds the prior investment threshold ( $y > \bar{y}$ ).*

**Proof.** Clearly, invest is uniquely rationalizable for a majority if and only if it is uniquely rationalizable for the player with the median return, for which  $x_i = \theta$  and  $z_i = \eta$ . We obtain our corollary by substituting these equalities in Propositions 1, 2, and 3. ■

That is, under rationalizability majority must invest if doing so is risk dominant under the fundamental state and there was a large positive common shock. When  $\sigma$  is small, nearly all players take the same action, and hence the aggregate investment is near 1 when investing is risk dominant and there was a large common shock.

It is straightforward to extend this result for an arbitrary percentile of players. For any  $p \in (0, 1)$  and for any  $x_i \in (1/2 + \sigma F^{-1}(p), \bar{R}]$ , invest is uniquely rationalizable for a fraction  $p$  of the players if the common shock  $\eta$  exceeds a critical shock size

$$\bar{z}_{p,\sigma}(\theta) = \bar{z}(\theta - \sigma F^{-1}(p)) + F^{-1}(p).$$

As  $\sigma \rightarrow 0$ , the critical shock size  $\bar{z}_{p,\sigma}(\theta)$  decreases to

$$\bar{z}_{p,0}(\theta) = \bar{z}(\theta) + F^{-1}(p),$$

a translation of the critical shock size for the majority, which is independent of  $\sigma$ .

We next consider the impact of common shocks on the level of aggregate investment under rationalizability. As a function of fundamental state  $\theta$  and the common shock  $\eta$

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<sup>7</sup>By a *majority* we mean a set of players whose Lebesgue measure is more than 1/2.

(where  $y = \theta - \sigma\eta$ ), the upper and lower bounds for aggregate investment are given by  $s^*$  and  $s^{**}$ , respectively, as follows:

$$\begin{aligned}\alpha(\theta, \eta | s^*) &= F(\eta - z^*(\theta - \sigma\eta)), \\ \alpha(\theta, \eta | s^{**}) &= F(\eta - z^{**}(\theta - \sigma\eta)).\end{aligned}$$

In the limit  $\sigma \rightarrow 0$ , for any  $\theta > 1/2$ , the upper bound  $\alpha(\theta, \eta | s^*)$  is 1. The lower bound is

$$\lim_{\sigma \rightarrow 0} \alpha(\theta, \eta | s^*) = F(\eta - \bar{z}(\theta)).$$

The lower bound remains near zero until the common shock approaches the critical shock size  $\bar{z}(\theta)$ , reaches 1/2 at  $\eta = \bar{z}(\theta)$ , and keeps rising towards 1 as the common shock rises further. For a given  $\sigma$ , the lower bound behaves similarly except for  $\eta < (\theta - \bar{y})/\sigma$ , when the lower bound is also near one as  $y > \bar{y}$ .

## 6 Dynamics

Our motivation for studying this problem comes from thinking about a dynamic model. In this section, we describe a dynamic model that can be solved using our results for the static model. This analysis provides an interpretation of and motivation for our earlier results.

For a fixed  $\sigma$ , the static game that we have analyzed can be parameterized by the prior mean  $y$ , and we will denote that game as  $G(y)$ . We will now consider the following dynamic game. At the beginning of each period  $t \geq 1$ , there is an expected return  $y_t$ . In each period  $t$ , the static game  $G(y_t)$  is played by a continuum of players. That is, a common shock  $\eta_t$  and idiosyncratic shocks  $\varepsilon_{it}$  are independently drawn across players, and players with types  $z_{it} = \eta_t + \varepsilon_{it}$  make investment choices as in the game  $G(y_t)$ . The expected return  $y_{t+1}$  at period  $t + 1$  is a function of the fundamental state  $\theta_t = y_t + \sigma\eta_t$  at  $t$ :

$$y_{t+1} = Y(\theta_t)$$

for some known increasing function  $Y : \mathbb{R} \rightarrow \mathbb{R}$ . At the beginning of  $t$ , the current expected return  $y_t$  and the previous aggregate investment  $A_{t-1}$ —the fraction of players who invest in the previous period—are publicly observable. Our interpretation is that  $y_t$  is the expected productivity in the economy. In each period, there is a common shock to productivity. The shock is persistent, but there may be a reversion to a mean productivity, as in the example below.

We now identify equilibrium shifts—when does equilibrium play switch from investment to non-investment and back—and the role the shocks play in such shifts. We will focus on the *hysteresis equilibrium*: each player  $i$  invests at any period  $t$  if and only if  $z_{it} > \hat{z}_t$  where  $\hat{z}_t = z^*(y_t)$  if  $t = 0$  or  $A_{t-1} \geq 1/2$  and  $\hat{z}_t = z^{**}(y_t)$  otherwise. The cutoff  $\hat{z}_t$  is a function of the current expected return and the previous aggregate investment. If a majority invested in the previous period, each player invests as long as investing is rationalizable for him, using the lowest equilibrium cutoff  $z^*(y_t)$  in the static game. Likewise, if majority did not invest in the previous period, he does not invest unless investing is the only rationalizable option for him. This leads to inertia in majority behavior: majority behavior changes if and only if the previous action taken by majority is no longer rationalizable for a majority. Combined with this simple characterization, our previous results lead to the following description of equilibrium shifts under hysteresis. (We say that there is *majority investment* at  $t - 1$  if  $A_{t-1} > 1/2$  and there is *minority investment* if  $A_{t-1} < 1/2$ .)

**Corollary 2** *Under the hysteresis equilibrium, at any  $t > 0$ , if there was minority investment in the previous period (i.e.,  $A_{t-1} < 1/2$ ), equilibrium shifts to majority investment (i.e.,  $A_t > 1/2$ ) whenever invest is risk dominant and, in addition, one of the following conditions hold (1) invest is minimum rank belief dominant (i.e.  $\theta_t > \bar{R}$ ); (2) there is a large common shock (i.e.,  $\eta_t > \bar{z}(\theta_t)$ ); or (3) the prior mean exceeds the prior investment threshold ( $y_t > \bar{y}$ ). If  $R$  is single peaked on  $\mathbb{R}_+$ , then equilibrium shifts to majority investment can occur only if (1) invest is maximum rank belief dominant (i.e.  $\theta_t > \bar{R}$ ); (2) invest is risk dominant and there is a large common shock (i.e.,  $\eta_t > \bar{z}(\theta_t)$ ); or (3) the prior mean exceeds  $\bar{R} - \sigma \bar{z}(\bar{R})$ .*

**Proof.** Equilibrium shifts to majority investment if and only if invest is uniquely rationalizable for the median type  $z_{it} = \eta_t$ , i.e.,  $\eta_t > z^{**}(y_t)$ . Then, the corollary immediately follows from Propositions 1-5. ■

Under hysteresis, Corollary 2 provides nearly a characterization of when equilibrium shifts to majority investment occur: invest is risk-dominant and either the expected return is above  $\bar{y}$  or there was a large positive shock at  $t$ . The converse rules out an equilibrium shift for all but a few remaining cases—negligible when  $\sigma$  is small—discussed in Section 4.

The dynamics under the hysteresis equilibrium is illustrated in Figure 6.<sup>8</sup> Time is on

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<sup>8</sup>In this example, we take  $y_{t+1} = \frac{1}{2} + \kappa(\theta_t - \frac{1}{2})$  with parameter  $\kappa = 0.99$ , so that fundamentals follow an AR(1) process around  $1/2$ :  $\theta_{t+1} - \frac{1}{2} = \kappa(\theta_t - \frac{1}{2}) + \sigma\eta_{t+1}$ . The distributions of shocks are as in the  $t$ -distribution example.

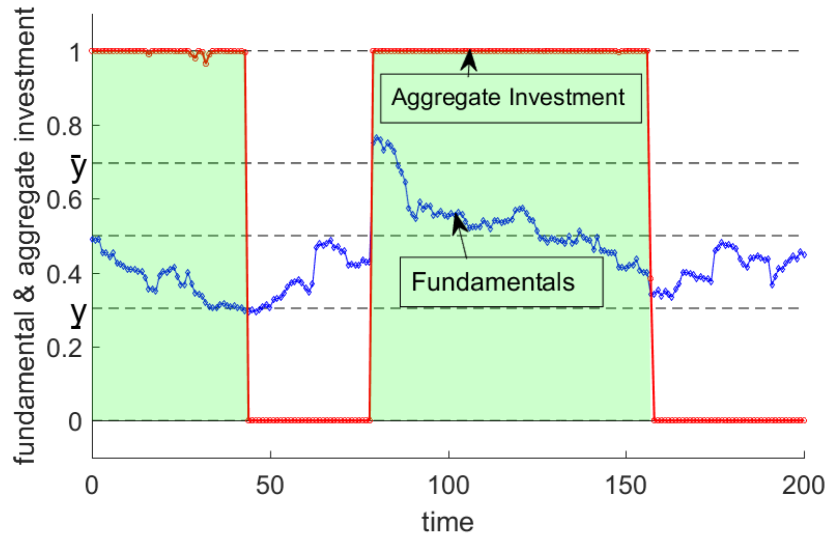


Figure 6: Equilibrium shifts on a typical sample path.

the horizontal axis. A sample path of fundamentals and aggregate investment are plotted on the vertical axis. But aggregate investment is always close to one or close to zero, so majority and minority investment corresponds to almost all investing and almost all not investing, respectively. There are two periods of majority investment—the shaded areas—interspersed with minority investment. At the beginning, investment is risk-dominant, and there is majority investment with aggregate investment nearly 1. As the fundamental drifts below  $1/2$ , the majority keeps investing due to hysteresis, although investing ceases to be risk-dominant. The equilibrium shifts when the fundamental drifts below  $y$ : the majority stop investing and aggregate investment drops near zero. This is the end of first period of majority investment. After that the fundamental fluctuates, but aggregate investment remains near zero. In particular, in this no investment period, a large positive shock has no discernible impact on aggregate investment as not invest remains risk dominant. Later, arrival of a major large positive shock makes investing risk dominant and shifts equilibrium back to majority investment. Thereafter, fundamentals drift down with occasional negative shocks, and a large negative shock ends the second investment period as it arrives when not invest is risk dominant.

Our result implies that it is preferable to avoid large unexpected negative shocks in good times in order to avoid crises, and preferable to have large unexpected positive shocks in the aftermath of a crisis especially after a substantial improvement of fundamentals in order



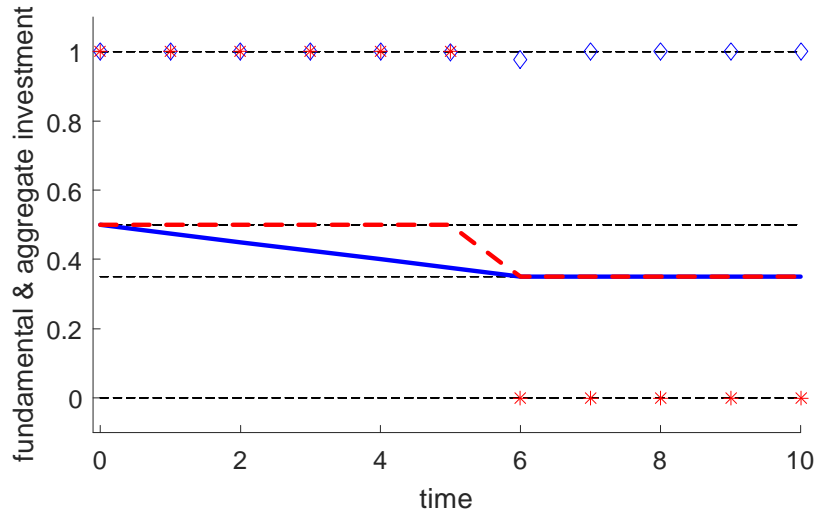


Figure 7: Aggregate investment under soft-landing (solid,  $\diamond$ ) and sudden drop (dashed,  $*$ ).

to hasten the economic recovery. This is illustrated in Figure 7, where we compare two alternative hypothetical paths in our  $t$ -distribution example for  $\sigma = 0.01$  and  $y_t = \theta_{t-1}$ . On both paths, the fundamental state starts at 0.5 and drops to 0.35. The paths differ in terms of how that change happens. On one path (in solid lines), fundamentals drop smoothly—as in soft landing of a bubble. In that case, the aggregate investment remains nearly one throughout (marked with  $\diamond$ ). On the other path (dashed lines), fundamentals drop suddenly after remaining high for a long while. In that case, the negative shock triggers a long-lasting crisis, dropping aggregate investment near zero (marked with  $*$ ).

It is useful to compare the dynamics here to two usual solution concepts. First, consider the hysteresis equilibrium under complete information (as in Cooper (1994)) where (all) players switch their action only when the previous action becomes inconsistent with equilibrium, switching to all invest when  $\theta$  goes above 1 and switching to nobody investing when  $\theta$  goes below 0. Under this equilibrium, the players keep investing throughout because the fundamental never drops below 0. There are more equilibrium shifts in our model in general because equilibrium shifts even before the fundamental reaches the cutoffs 0 and 1. Second, consider the classic global games solution in which all players play the risk dominant action. The equilibrium shifts as the fundamental crosses  $1/2$ , resulting in frequent equilibrium shifts when the fundamental is near  $1/2$  and no shift away from the cutoff. Our equilibrium is not sensitive the cutoff  $1/2$  per se, but the outcomes correlate because shocks

revert to the solution under risk dominance if they happen to be in the right direction.<sup>9</sup>

Hysteresis as a selection device is often assumed as a modelling device, see Krugman (1991) and Cooper (1994) among others. Romero (2015) has tested hysteresis in the laboratory, confirming its existence in a setting with evolving complete information payoffs. The switches occur before dominance regions are reached, consistent with our results. Chamley (1999) develops a dynamic model of global games in which hysteresis arises as a unique equilibrium. In his model, players can learn about the previous fundamentals when the fundamentals reach near dominance regions, when the equilibrium shifts occur. In another dynamic model with small amount of hysteresis in players' actions, Burdzy, Frankel, and Pauzner (2001) obtain risk dominant selection as the unique equilibrium. Finally, Angelatos, Hellwig, and Pavan (2007) study a dynamic model of global games with regime change. In their model, fundamentals do not change over time, but players learn about them as they observe the outcomes of the past play; learning leads to multiple equilibria and interesting dynamics.

## 7 Rank Beliefs Revisited

In our baseline model, we assume that the density  $f$  of idiosyncratic shocks is log-concave and the density  $g$  of the common shock has fat tails. We now consider the implications of dropping these assumptions. The key implications of these assumptions are that (i)  $f$  has thinner tails than  $g$  and (ii) the beliefs are increasing in types. The latter feature is crucial for our analysis and we will maintain it throughout this section:

**Assumption 1**  $\Pr(\eta \leq \bar{\eta} | z_i)$  is decreasing in  $z_i$  for any  $\bar{\eta}$ .

The fat-tail assumption (i) was used in establishing limit uniformity of rank beliefs, i.e., the property that  $R(z) \rightarrow \frac{1}{2}$  as  $z \rightarrow \infty$ . In fact, limit uniformity was the only property of rank beliefs we used in our analysis (i.e., our results hold even without fat tails if limit uniformity continues to hold). But limit uniformity does not hold in general. We first

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<sup>9</sup>In our model, by Proposition 2, when  $y > 1/2$ , equilibrium shifts to majority investing whenever  $z_i > \bar{z}(y)$ . Hence, in the limit  $\sigma \rightarrow 0$ , equilibrium shifts occur near the cutoff  $1/2$  almost surely. However, as in Figure 6, even for  $\sigma = 0.01$ , equilibrium shifts typically occur away from the cutoff  $1/2$  (because the odds of getting large shocks  $z > \bar{z}(1/2 + \varepsilon)$  are also very small for small  $\varepsilon$ ).

generalize our main result to arbitrary rank belief functions, identifying when shocks lead to equilibrium shifts in general. (Here, the cutoff  $\bar{z}(x_i) = \max R^{-1}(y)$  is defined for  $x_i > R_\infty$ .)

**Proposition 6** *Under Assumption 1, assume that*

$$R_\infty \equiv \lim_{z \rightarrow \infty} R(z)$$

*exists. Then, invest is uniquely rationalizable if  $x_i > R_\infty$  and*

$$z_i > \bar{z}(x_i).$$

*Conversely, assuming  $R$  is single peaked on  $\mathbb{R}_+$  with  $R_\infty < \bar{R}$  and  $y \leq \bar{R} - \sigma \bar{z}(\bar{R})$ , invest is not uniquely rationalizable if  $x_i < R_\infty$  or  $z_i \leq \bar{z}(x_i)$ .*

Proposition 6 extends Propositions 2 and 4 to arbitrary rank beliefs by simply replacing risk-dominance with  $p$ -dominance with  $p = R_\infty$ ; i.e., one replaces  $1/2$  in the definition of risk-dominance with  $R_\infty$ . One simply replaces  $1/2$  with  $R_\infty$  in the proofs of Propositions 2 and 4 to prove this result. Using Proposition 6 in place of Propositions 2 and 4, one can easily extend our qualitative results to arbitrary rank beliefs with the new  $p$ -dominance threshold. Note that, under general rank belief functions, there is a middle region  $[1 - R_\infty, R_\infty]$  on which no action is  $p$ -dominant and large shocks do not play a role. Under uniform limit rank beliefs, this is the degenerate case of  $x_i = 1/2$ .

To motivate our definitions below, in Figure 8, we plot several interesting cases where limit rank beliefs are not uniform. In the baseline case, both idiosyncratic and common shocks are normally distributed (solid lines on the left). The rank belief function is monotone and approaches 1, so that  $\bar{R} = \sup_z R(z) = 1$ . In an alternative case, the idiosyncratic shocks are normally distributed but the common shocks are (double) exponentially distributed instead (dashed lines on the left). Now the common shocks have thicker tails than the idiosyncratic shocks without having a fat tail. The rank belief function is still monotone but it is bounded away from 1 by  $\bar{R} \cong 0.76$ . In the middle panel, both idiosyncratic and common shocks have  $t$  distributions. Now common shocks have fat tails but their tails are not thicker than the tails of idiosyncratic shocks. As in the main model, the rank belief function is not monotone, but the limit rank beliefs are no longer uniform,  $R(z)$  approaching 0.75 in the limit.<sup>10</sup> In

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<sup>10</sup>The limit rank belief will always be  $3/4$  if common shocks and idiosyncratic shocks are drawn from the same fat-tailed distribution. This follows from the fact that the large shocks will be attributed to either to the common shock or to the idiosyncratic shock with equal probability. (See Feller (1968) exercise 27 on page 288 for a similar observation. We are grateful to Ulrich Mueller for this reference.)

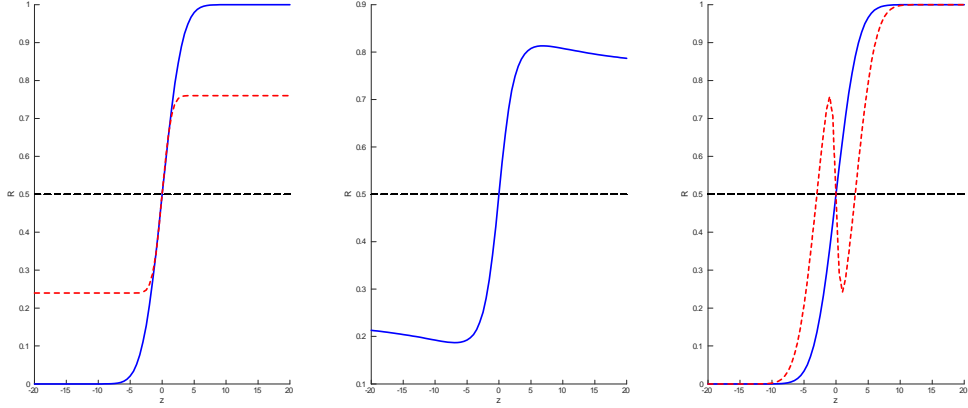


Figure 8: Rank belief function under various specifications. Left panel: both  $f$  and  $g$  are normal (solid);  $f$  normal,  $g$  double exponential (dashed). Middle panel: both  $f$  and  $g$  are  $t$  distributions. Right panel:  $f$  is normal, and  $g$  is normal with unknown mean  $\mu$ , where the distribution of  $\mu$  is either normal (solid) or binomial (dashed).

the right panel, both idiosyncratic and common shocks are normally distributed but the mean  $\mu$  of the common shock is unknown. When  $\mu$  is normally distributed, the common shocks are still normally distributed—with higher variance, and the rank belief function is qualitatively as in the baseline case (solid). When  $\mu$  has binomial distribution,  $g$  is no longer single-peaked, and the rank belief function is qualitatively different: it has no longer single-crossing property. Nonetheless such mean uncertainty does not affect the tails, and the limit rank beliefs are as in the baseline case once again. This motivates the following definitions.

**Definition 1** Rank beliefs are monotone if  $R(z)$  is increasing in  $z$ .

**Definition 2** Rank beliefs are bounded if  $\bar{R} = \sup_z R(z) < 1$ .

**Definition 3** Rank beliefs are limit certain if  $R_\infty = \lim_{z \rightarrow \infty} R(z) = 1$ .

Note that rank beliefs are monotone in both exponential and normal examples; they are bounded under exponential distribution while limit certain under normal distribution. The rank beliefs are limit certain in the examples with mean uncertainty.

When rank beliefs are monotone, shocks per se do not lead to equilibrium shifts:

**Proposition 7** *Under Assumption 1 and monotone rank beliefs, if invest is uniquely rationalizable at  $x_i$  under  $y$ , then invest is uniquely rationalizable at  $x_i$  under any  $y'$  with  $x_i - y' \leq x_i - y$ .*

Proposition 7 establishes that the shocks do not matter under monotone rank beliefs, as in the case of normal and exponential distributions. That is, if a positive shock makes investment uniquely rationalizable, then invest would have been uniquely rationalizable even with a smaller positive shock that brings the fundamental to the same level (starting from a higher prior mean).

We next focus on the limit case  $\sigma \rightarrow 0$  to discuss implications of above properties further. In this limiting case, under Assumption 1, for any  $y \in (1 - R_\infty, R_\infty)$ , the returns at the extremal equilibrium cutoffs take a simple form:

$$\lim_{\sigma \rightarrow 0} x^* = 1 - R_\infty \text{ and } \lim_{\sigma \rightarrow 0} x^{**} = R_\infty.$$

Hence, invest is uniquely rationalizable when the return is above  $R_\infty$ ; not invest is uniquely rationalizable when the return is below  $1 - R_\infty$ , and both actions can be played in equilibrium when the return is in between. When the rank beliefs are monotone, since  $R_\infty = \bar{R}$ , this yields a sharp characterization based on levels: invest is uniquely rationalizable if  $x_i > \bar{R}$ ; not invest is uniquely rationalizable if  $x_i < 1 - \bar{R}$ , and there are multiple equilibrium actions otherwise. Likewise, if the rank beliefs are limit certain, then the equilibrium cutoffs converge to the dominance triggers, so that the equilibrium shifts only when the fundamental enters a region in which an action is dominant, as in the complete information case.

Our focus on fat tails is motivated by model uncertainty. Model uncertainty plays a central role in our work when it leads to non-monotone behavior as in Proposition 6. Sometimes, model uncertainty may be too weak to affect the tails, as in the case of mean uncertainty plotted in the right panel of Figure 8. The behavior will converge to the behavior under complete information as  $\sigma \rightarrow 0$  in that case.

Model uncertainty also plays an important role in some other models. Chen and Suen (2016) study coordinated attack problem in which players are uncertain about how easy it is to change a regime. An unexpectedly successful attack by the previous cohort dramatically increases the probability that changing the regime is easy, enticing players to attack. Hence, successful attacks lead to further attacks by other players. Acemoglu, Chernozhukov, and Yildiz (2016) study learning and asymptotic agreement when players do not know the

conditional distribution of signals. Such a model uncertainty leads to asymptotic belief disagreement and possibly non-monotone beliefs (as in our paper). Such model uncertainty is also central to Liang (2016), who studies robustness of solution concepts to uncertainty about the statistical rules players use to learn the fundamentals. Kozłowski, Veldkamp, and Venkateswaran (2017) study a macroeconomic model in which the players do not know the distribution of shocks and update their belief by using a normal kernel estimation method. When they observe large unexpected shocks, they update their beliefs about tail probability drastically. Large shocks have large and long-lasting impact on economy as a result.

## 8 Discussion

In an economic environment with multiple equilibria, what explains which equilibrium is played? There are two versions of this question. In a static setting, how can we explain which equilibrium is played? In a dynamic setting, how can we explain switches among equilibria?

One response to the static question is to observe that the multiplicity may be an artifact of the assumption of complete information, or common certainty of the game's payoffs. A first generation of *global game* models (Carlsson and Van Damme (1993), Morris and Shin (1998) and Morris and Shin (2003)) argued that if the common certainty assumption were relaxed in a natural way, there is a unique equilibrium selection—the risk dominant one in two player two action games. The natural relaxation is to allow players to observe very accurate noisy signals of the state of the world.<sup>11</sup> Morris, Shin, and Yildiz (2016) formalize the idea that this information structure gives rise to (common certainty of) uniform rank beliefs, and this is what drives the results. Note that in this literature, the focus is on *global* uniqueness: there is a unique prediction of play for any signal that a player might observe.

Two basic criticisms of this first generation of global models are the following. First, with respect to assumptions, common knowledge of uniform rank beliefs will not hold even approximately in many environments (for example, when there are very accurate public

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<sup>11</sup>Weinstein and Yildiz (2007) pointed out that it mattered exactly how common certainty assumptions were relaxed: any rationalizable action in the underlying complete information game is a uniquely rationalizable action for a type of a player that is "close" to the complete information type, where closeness is in the product topology in the universal belief space of Mertens and Zamir (1985).

signals).<sup>12</sup> Second, with respect to predictions, as long as rank beliefs are approximately uniform throughout a model, outcomes will be largely determined by fundamentals. Thus in a dynamic model, the prediction would be that equilibrium play would always be switching when fundamentals crossed a threshold (which we call the risk-dominance threshold). Both predictions are counter-factual.

In this paper, we made an intermediate set of assumptions, relative to complete information and first generation global games. Like the first generation global games literature, we relax complete information and use the vital insight that properties of rank beliefs sometimes lead to unique predictions.<sup>13</sup> Like the complete information literature, we allowed for the possibility that information alone does not determine behavior and that some other factor or factors determine equilibrium choice—our focus was on hysteresis as that factor.

This approach generated three novel and intuitive predictions. First, if we look at the relationship between fundamentals and outcomes, play must shift when once fundamentals cross a fundamental threshold that arises before an action becomes dominant. Second, large shocks can trigger a shift before that threshold is reached. And third, those shifts can only occur once an action is risk-dominant—i.e., the best response to uniform rank beliefs and thus the first generation global game prediction.

We conclude by contrasting our explanation and modelling with the conventional account that equilibrium shifts that they are triggered by the arrival of public signals: even though the financial system has been coming under continuing pressure, a public event (such as the collapse of Lehman) triggers the shift to a bad equilibrium (a financial crisis); even though European fiscal and sovereign debt positions had been improving for some years, it was a public event (Draghi’s speech) that triggered the shift to the good equilibrium. Such explanations are common in a wide variety of settings—see Chwe (2013) for many examples across the social sciences. To model this, one can consider the case where players observe

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<sup>12</sup>Angeletos and Werning (2006) give a price revelation foundation for the assumption that idiosyncratic shocks should have no less variation than common shocks.

<sup>13</sup>We allow for a qualitatively richer class of rank beliefs than the first generation literature. The existing literature exclusively focusses on two cases (see Morris and Shin (2003) for a discussion of both cases). First, the case where we fix the distribution of common shocks / public signals and let the noise in idiosyncratic shocks / private signals go to zero; in this case there is common certainty of uniform rank beliefs in the limit. Second, the case where both common shocks / public signals and idiosyncratic shocks / private signals are normally distributed. As we noted in Section 7, rank beliefs are monotonic in this case and our large shock results cannot arise.

public signals which are sufficiently accurate to break uniform rank beliefs—and thus global selection of the risk dominant equilibrium—but not sufficiently accurate to break globally unique equilibrium. In such models, large shocks / public signals will play a disproportionate role in selecting the unique equilibrium (see Morris and Shin (2003), Morris and Shin (2004)). But is it really the case that there is more common knowledge after a large shock? Surely people are more uncertain what other people are thinking after a large shock? We offer the alternative explanation that a large shock gives rise to *less* common knowledge in the sense of uncertainty about others' relative optimism, i.e., *more* uniform rank beliefs, and it is this that triggers a shift to a new equilibrium. While both explanations appeal to large shocks, the mechanisms are opposite in terms of the properties of rank beliefs generating the results.

## A Omitted Proofs

### A.1 Properties of Beliefs and Equilibria

In this section, we present couple of basic properties of beliefs and prove Lemma 2, showing that the extremal equilibria  $s^*$  and  $s^{**}$  bound all rationalizable strategies. We write  $F(\eta, z_{-i}|z_i)$  for the cumulative distribution function of  $(\eta, z_{-i})$  conditional  $z_i$ , representing the interim beliefs of type  $z_i$  about the common shock and the other players' types.

**Lemma 3** *The interim beliefs are increasing in types in the sense of first-order stochastic dominance:*

$$F(\eta, z_{-i}|z_i) \text{ is decreasing in } z_i. \quad (10)$$

Moreover,  $f$  has thinner tails than  $g$ :

$$\lim_{\lambda \rightarrow \infty} \frac{f(\lambda z)}{g(\lambda z')} = 0 \quad (\forall z, z' \in \mathbb{R} \setminus \{0\}). \quad (11)$$

**Proof.** (Part 1) Since  $z_j = \eta + \sigma \varepsilon_j$  where  $\varepsilon_j$  is independent of  $\varepsilon_i$  and  $\eta$  for each  $j \neq i$ , it suffices to show that  $F(\eta|z_i)$  is decreasing in  $z_i$ , where  $F(\eta|z_i)$  is the conditional distribution of the common shock. To do this, it suffices to show that  $\eta$  and  $z_i$  are affiliated, i.e., the joint density  $h$  of  $(\eta, z_i)$  is log-supermodular. But since  $h(\eta, z_i) = g(\eta) f(z_i - \eta)$ ,  $\log h$  is supermodular:

$$\log h(\eta, z_i) = \log g(\eta) + \log f(z_i - \eta).$$

Here,  $\log g(\eta)$  is trivially supermodular, and  $\log f(z_i - \eta)$  is supermodular because  $\log f$  is concave.



(Part 2) Since  $f$  is log-concave, it is well known that  $f$  has subexponential tails, i.e.,

$$\lim_{z \rightarrow \infty} \frac{f(z)}{\exp(-cz)} = 0$$

for some  $c > 0$ . Thus, for any non-zero  $z$  and  $z'$ ,

$$\lim_{\lambda \rightarrow \infty} \frac{f(\lambda z)}{g(\lambda z')} = \lim_{\lambda \rightarrow \infty} \frac{f(\lambda z)}{\exp(-c\lambda z)} \frac{\exp(-c\lambda z)}{g(\lambda z)} \frac{g(\lambda z)}{g(\lambda z')} = 0.$$

(Since  $g$  has regularly varying tails,  $\lim_{\lambda \rightarrow \infty} \exp(-cz)/g(\lambda z) = 0$  and  $\lim_{\lambda \rightarrow \infty} g(\lambda z)/g(\lambda z') \in \mathbb{R}$ .)

■

The first part of the lemma is the main step in the proof of Lemma 2.

**Proof of Lemma 2.** It suffices to verify that our game is monotone supermodular, as in Van Zandt and Vives (2007). (Van Zandt and Vives (2007) also assume that the set of player is finite but their proof also applies to our game.) It is straightforward to verify the continuity and compactness assumptions as well as supermodularity of the payoff functions. Lemma 3 further establishes that the beliefs are monotone, and this fact immediately implies that  $s^*$  and  $s^{**}$  are Bayesian Nash equilibria. Since the game is monotone supermodular, all rationalizable strategies are bounded by  $s_i^*$  and  $s_i^{**}$ . In particular, all rationalizable strategies coincide whenever  $s_i^*(z_i) = s_i^{**}(z_i)$ . ■

## A.2 Properties of Rank Beliefs

We start with some useful notation. For any two functions  $h_1$  and  $h_2$  from reals to reals, we define *convolution*  $h_1 * h_2$  of  $h_1$  and  $h_2$  by

$$h_1 * h_2(z) = \int h_1(\varepsilon) h_2(z - \varepsilon) d\varepsilon. \quad (12)$$

Observe that

$$R(z) = \frac{Ff * g(z)}{f * g(z)}. \quad (13)$$

Since  $F(-\varepsilon) = 1 - F(\varepsilon)$  and  $f$  and  $g$  are even functions, we have the following useful properties:

$$f * g(z) = f * g(-z); \quad (14)$$

$$R(-z) = \frac{(1 - F)f * g(z)}{f * g(z)}, \quad (15)$$

where  $1 - F$  is the complementary cdf. The first property states that  $f * g$  is even, and the second property states that  $R(-z)$  is simply computed by using the complementary cdf. Hence,

$$\begin{aligned} R(z) - R(-z) &= \frac{(2F - 1)f * g(z)}{f * g(z)} = \frac{\int_{-\infty}^{\infty} (2F(\varepsilon) - 1)f(\varepsilon)g(z - \varepsilon) d\varepsilon}{f * g(z)} \\ &= \frac{\int_0^{\infty} (2F(\varepsilon) - 1)f(\varepsilon)(g(z - \varepsilon) - g(z + \varepsilon)) d\varepsilon}{f * g(z)} \end{aligned} \quad (16)$$

where the first equality is by (13), (14) and (15); the second equality is by definition of convolution, and the last property is by the fact that  $2F - 1$  is an odd function while  $f$  is even.

**Proof of Lemma 1. (Symmetry)** By (15),

$$R(-z) = \frac{(1-F)f * g(z)}{f * g(z)} = \frac{f * g(z) - Ff * g(z)}{f * g(z)} = 1 - R(z).$$

**(Single Crossing)** For any  $z > 0$ , observe that  $g(z - \varepsilon) - g(z + \varepsilon) \geq 0$  and the inequality is strict with positive probability; equality holds only if  $g$  is constant over the relevant range. Hence, by (16),  $R(z) - R(-z) > 0$ . Since  $R(-z) = 1 - R(z)$ , this also implies that  $R(z) > 1/2 > R(-z)$ .

**(Uniform Limit Rank Beliefs)** Fix any  $\varepsilon \in (0, 1)$ . Since  $g$  has regularly varying tails (4), there exist  $\beta > 0$  and  $\eta_0$  such that for all  $\eta' > \eta \geq \eta_0$ ,

$$\frac{g(\eta)}{g(\eta')} \leq (1 + \varepsilon/2) (\eta/\eta')^{-\beta}. \quad (17)$$

Fix also  $\gamma > 0$  such that

$$(1 + \varepsilon/2) \left( \frac{1 - \gamma}{1 + \gamma} \right)^{-\beta} < 1 + \varepsilon. \quad (18)$$

Now, by definition, for any  $z > 0$ ,

$$R(z) \leq (I_1 + I_2) / I_3$$

where

$$\begin{aligned} I_1 &= \int_{-\gamma z}^{\gamma z} f(\varepsilon) F(\varepsilon) g(z - \varepsilon) d\varepsilon \leq \frac{1}{2} (F(\gamma z) - F(-\gamma z)) g(z - \gamma z), \\ I_2 &= \int_{\varepsilon \notin (-\gamma z, \gamma z)} f(\varepsilon) F(\varepsilon) g(z - \varepsilon) d\varepsilon \leq f(\gamma z), \\ I_3 &= \int_{-\gamma z}^{\gamma z} f(\varepsilon) g(z - \varepsilon) d\varepsilon \geq (F(\gamma z) - F(-\gamma z)) g(z + \gamma z). \end{aligned}$$

Combining the above inequalities, we conclude that

$$R(z) \leq \frac{1}{2} \frac{g(z - \gamma z)}{g(z + \gamma z)} + \frac{f(\gamma z)}{(F(\gamma z) - F(-\gamma z)) g(z + \gamma z)}. \quad (19)$$

Now, by (17) and (18),

$$\frac{1}{2} \frac{g(z - \gamma z)}{g(z + \gamma z)} \leq \frac{1}{2} (1 + \varepsilon/2) \left( \frac{1 - \gamma}{1 + \gamma} \right)^{-\beta} < 1/2 + \varepsilon/2$$

for any  $z > \eta_0 / (1 - \gamma)$ . Moreover, by (11), there exists  $\bar{z} > \eta_0 / (1 - \gamma)$  such that for all  $z > \bar{z}$ ,

$$\frac{f(\gamma z)}{(F(\gamma z) - F(-\gamma z)) g(z + \gamma z)} < \varepsilon/2.$$

Substituting the two displayed inequalities in (19), we obtain  $R(z) < 1/2 + \varepsilon$  for all  $z > \bar{z}$ , as desired. ■

### A.3 Omitted Proofs of Main Results

We next prove Propositions 4, 5 and 7.

**Proof of Proposition 4.** By Proposition 2, it suffices to prove the necessity. Take any  $y \leq \bar{R} - \sigma \bar{z}(\bar{R})$ . Since  $y \leq \bar{R} - \sigma \bar{z}(\bar{R})$ ,

$$R(\bar{z}(\bar{R})) = \bar{R} \geq y + \sigma \bar{z}(\bar{R}).$$

Since  $R(z) < y + \sigma z$  for large values of  $z$ , by the intermediate-value theorem, this implies that  $z^{**} \geq \bar{z}(\bar{R}) > 0$ . Thus,

$$x^{**} > \max\{y, 1/2\}.$$

(Clearly,  $x^{**} = y + \sigma z^{**} > y$  and  $x^{**} = R(z^{**}) > 1/2$ .) Hence, if invest is not risk-dominant (i.e.  $x_i \leq 1/2$ ), then  $x^{**} > x_i$ , and therefore invest is not uniquely rationalizable. Now, assume that invest is risk-dominant (i.e.  $x_i > 1/2$ ) but inequality (8) does not hold—as in the right panel of Figure 2:

$$z_i \leq \bar{z}(x_i). \quad (20)$$

We **claim** that, if in addition  $R$  is single peaked, then (20) implies that  $x^{**} \geq x_i$ , and therefore invest is not uniquely rationalizable. To prove the **claim** that  $x^{**} \geq x_i$ , suppose  $x^{**} < x_i$  and equivalently

$$z^{**} < z_i. \quad (21)$$

Now, since  $z^{**} \geq \bar{z}(\bar{R})$ , by (20) and (21), we have

$$\bar{z}(\bar{R}) \leq z^{**} < z_i \leq \bar{z}(x_i).$$

However, since  $R$  is single-peaked with a peak at  $\bar{z}(\bar{R})$ , this implies that

$$x^{**} = R(z^{**}) > R(\bar{z}(x_i)) = x_i,$$

contradicting that  $x^{**} < x_i$ . ■

The following lemma will be useful in the proof of Proposition 5.

**Lemma 4** *Assume  $\sigma < \sup_z R(z)/z$ . Then,*

$$\begin{aligned} x^*(y) &> 1/2 > y \quad \text{for all } y < \underline{y} \equiv 1 - \bar{y}; \\ x^{**}(y) &> y > x^*(y) \quad \text{for all } y \in [\underline{y}, \bar{y}]; \\ x^{**}(y) &< 1/2 < y \quad \text{for all } y > \bar{y}. \end{aligned}$$

**Proof.** See our earlier working paper for a straight-forward proof. ■

**Proof of Proposition 5.** Set

$$\Delta = \min \{x^{**}(\bar{y}) - \bar{y}, \bar{y} - 1/2\}. \quad (22)$$

Observe that

$$\min_{y \geq \bar{y}} (y - x^*(y)) = \underline{y} - x^*(\underline{y}) = x^{**}(\bar{y}) - \bar{y} = \min_{y \leq \bar{y}} (x^{**}(y) - y) > 0, \quad (23)$$

where the first and the last equalities are by the fact that  $y - x^*(y)$  is increasing while  $x^{**}(y) - y$  is decreasing, and the middle equality is by symmetry. By Lemma 4,  $x^{**}(\bar{y}) - \bar{y} > 0$ .

Consider any  $y > \bar{y}$ . Clearly, by definition (22),

$$y > \bar{y} \geq \Delta + 1/2.$$

Hence, for any  $x_i$  with  $|x_i - y| \leq \Delta$ , we have

$$x_i \geq y - \Delta > 1/2 > x^{**}(y),$$

showing that invest is uniquely rationalizable at  $x_i$  under  $y$ . (Here, the last inequality is by Lemma 4.)

Now consider any  $y \leq \bar{y}$ . By (22) and (23),

$$x^{**}(y) - y \geq x^{**}(\bar{y}) - \bar{y} \geq \Delta.$$

Hence, for any  $x_i$  with  $|x_i - y| \leq \Delta$ , we have

$$x_i \leq y + \Delta \leq x^{**}(y),$$

showing that invest is not uniquely rationalizable at  $x_i$  under  $y$ .

The second statement in the proposition follows from the first one by symmetry. ■

**Proof of Proposition 7.** It immediately follows from Lemma 5 below. ■

**Lemma 5** *Extremal cutoff functions  $z^*$  and  $z^{**}$  are decreasing in  $y$ . Under monotone rank beliefs,  $x^*$  and  $x^{**}$  are also decreasing in  $y$ .*

**Proof.** Consider any  $y$  and  $y'$  with  $y > y'$ . Observe that

$$R(z^{**}(y)) = y + \sigma z^{**}(y) > y' + \sigma z^{**}(y').$$

Moreover, since  $R$  is bounded and  $y' + \sigma z$  goes to infinity, there exists  $\tilde{z} > z^{**}(y)$  such that  $R(\tilde{z}) < y' + \sigma \tilde{z}$ . Since  $R$  is continuous, by the intermediate value theorem, this implies that there exists  $z > z^{**}(y)$  with  $R(z) = y' + \sigma z$ , showing that  $z^{**}(y') > z^{**}(y)$ . One can similarly prove that  $z^*(y') > z^*(y)$ . ■

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