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FIRE-SALE EXTERNALITIES

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ABSTRACT

This paper characterizes the efficiency properties of competitive economies with financial constraints and fire sales. We show that two distinct pecuniary externalities occur in such settings: distributive externalities that arise from incomplete insurance markets and can take any sign; and collateral externalities that arise from price-dependent financial constraints and are conducive to over-borrowing. For both types of externalities, we identify three sufficient statistics that determine optimal taxes on financing and investment decisions to implement constrained efficient allocations. We illustrate how to employ our framework in a number of applications. We highlight how small changes in parameters may cause the sufficient statistics that drive distributive externalities to flip sign, leading to either under- or over-borrowing. We also show that financial amplification is neither necessary nor sufficient to generate inefficient fire-sale externalities.

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1 Introduction

Modern economies have experienced recurrent financial crises involving sharp drops in asset prices and amplification effects. Policy discussions in the aftermath of the 2008/09 Global Financial Crisis have understandably focused on the possibility that such "fire sales" may lead to inefficient externalities that call for regulatory intervention – as exemplified by the speech by Stein (2013). Understanding whether fire sales, i.e. asset sales at dislocated prices by financially constrained agents, provide a rationale for policy intervention is thus crucial to redesigning our financial regulatory framework.

In the existing literature, the seminal papers of Gromb and Vayanos (2002) and Lorenzoni (2008) describe how asset sales by financially constrained agents can generate pecuniary externalities that lead to constrained inefficient allocations.[1] Some policymakers and commentators have interpreted this as implying that sharp changes in prices always involve inefficient externalities. However, the efficiency properties of economies with financially constrained agents are less obvious than commonly understood, and a general description of the resulting externalities has been missing.

This paper seeks to fill this gap by developing a general framework to characterize the pecuniary externalities that arise in environments with financially constrained agents and fire sales. Our first main result characterizes constrained efficient allocations and optimal corrective policies with borrowers who are subject to financial constraints. We describe the optimal corrective policies for financing and investment decisions as a function of sufficient statistics that are invariant to the precise nature of the underlying financial frictions, e.g., uncontingent bonds, limited commitment, market segmentation, etc.

We show that two distinct types of pecuniary externalities arise in such environments. We refer to the first type as *distributive externalities*. Distributive externalities arise when marginal rates of substitution (MRS) between dates/states differ across agents, and a planner can improve on the allocation by affecting the relative prices at which agents trade. Potential reasons why the MRS are not equalized include, for instance, that the set of traded assets does not span all possible states of nature, or binding collateral constraints. Intuitively, when MRS are not equal, a planner can modify allocations to induce price changes that improve the terms of the transactions of those agents with relatively higher marginal utility in a given date/state. For example, a planner may internalize that reducing fire sales raises the price received by the sellers, who may greatly value having resources in those states as reflected by a high MRS.

[1] Whenever some agents are financially constrained, the market outcome is clearly not first-best: removing the frictions that underlie the financial constraints increases efficiency. However, in practice, policymakers frequently must take such frictions as given, which leads to question of whether decentralized equilibrium allocations are *constrained efficient*. In other words, can a policymaker subject to the same constraints as private agents improve on the market outcome?

We refer to the second type as *collateral externalities*. Collateral externalities arise when financial constraints depend on aggregate state variables, for example via market prices. The most common example, on which we focus in our main analysis, is when capital assets serve as collateral that limits borrowing. Intuitively, when agents are subject to a binding constraint that depends on aggregate variables, a planner internalizes that she can modify allocations to relax financial constraints. For example, the planner may reduce fire sales to raise the value of capital assets that serve as collateral, which increases the borrowing capacity of constrained agents.

The existing literature has found it remarkably difficult to provide general results on the direction of inefficiency – except in tightly-defined special cases. Our second main result explains why and delineates under what conditions the pecuniary externalities can be signed unambiguously and when they can go in either direction. The sign and magnitude of distributive externalities are determined by the product of three sufficient statistics: the difference in MRS of agents, the net trading positions (net buying or net selling) of capital and financial assets, and the sensitivity of equilibrium prices to changes in sector-wide state variables. The first two of the three sufficient statistics for distributive externalities can go in either direction. Depending on parameters, it is plausible to find economies in which differences in MRS and net trading positions take positive or negative values. In short, "anything goes," and distributive externalities cannot be signed in general.

The sign and magnitude of collateral externalities is also determined by the product of three sufficient statistics: the shadow value on the binding financial constraint, the sensitivity of the financial constraint to the asset price, and the sensitivity of the equilibrium asset price to changes in sector-wide state variables. The first two of the three sufficient statistics for collateral externalities are always positive. Under natural conditions, asset prices are increasing in net worth for each sector, pinning down the sign of the third sufficient statistic. This allows us to show that collateral externalities generally entail over-borrowing. Importantly, our characterization of both distributive and collateral externalities holds in a broad class of environments and is invariant to the precise nature of the underlying financial frictions.

We present several important implications of our general results in the form of corollaries. First, we show that there exists a relation between investment and financial distortions. Intuitively, because investing in capital and buying financial assets are both mechanisms for shifting resources across time, optimal policies must intervene in both margins in a consistent way.

We also show that the optimal corrective policy for an arbitrary security can be designed using an externality pricing kernel. This result provides a simple expression to guide financial regulators.

Next, we show that the existence of amplification effects is neither necessary nor sufficient for constrained inefficiency. They are not necessary because inefficiency also arises when there are pecuniary externalities that mitigate shocks rather than amplifying them. They are not sufficient because equilibrium is constrained efficient when there are only distributive externalities and insurance markets are complete, or when agents are in a corner solution. This result implies that policymakers have to be careful when arguing that fire sales and amplification effects justify policy intervention.

Finally, we identify three degrees of freedom in setting taxes to implement a given constrained optimal allocation. This allows a planner to restore constrained efficiency without intervening in each individual decision made by each agent. For example, we show that it is sufficient to intervene in the financial decisions of borrowers only, or that we can often combine taxes on borrowing and on investment into a single tax. Furthermore, when these degrees of freedom imply that the optimal tax on a decision margin can be set to zero, that decision can be interpreted as constrained efficient.

Subsequently, we study four applications of our general framework that illustrate the use of our sufficient statistics and how they can be traced back to the primitives of the economy. In doing so, we also provide specific examples of how some of our sufficient statistics may flip sign when the primitives of the model cross a defined threshold, corroborating the "anything goes" result of our general framework.

Our first application illustrates the possibility of constrained efficient fire sales. In an environment in which the financial constraint does not depend on prices, we show that fire sales and financial amplification effects of arbitrary magnitude are compatible with constrained efficiency. The reason is that complete ex-ante insurance markets in our example allow agents to equate their MRS so distributive effects do not lead to inefficiency. Our second and third applications consider environments in which there are distributive externalities that flip sign when certain primitives of the economy cross well-defined thresholds. In the second application, borrowers turn from net buyers into net sellers of capital when a productivity parameter crosses a certain threshold. In the third application, the difference in the MRS of borrowers and lenders switches sign as borrowers hit the upper versus the lower limit for trade in the constrained financial market when their endowment crosses two well-defined thresholds. When the sufficient statistics flip sign, the direction of inefficiency of financing and investment decisions switches sign as well. Our fourth application provides an example of a price-dependent collateral constraint in which collateral externalities cause over-borrowing and over-investment.

Before concluding, we use the general framework developed in the paper to place in context several results highlighted by previous literature. In particular, we classify papers according to whether they focus on distributive or collateral externalities or both.

Outline Section 2 describes the baseline model environment, characterizes the first best and solves for the decentralized equilibrium. We study the constrained efficiency properties of the equilibrium and present several corollaries in Section 3. In Section 4, we illustrate our findings in a number of specific applications. Section 5 relates our results to previous work, and Section 6 concludes. All proofs and derivations as well as several extensions are in the appendix.

2 Baseline Model

Our baseline model describes fire sales in an economy with two types of agents that we call borrowers and lenders. Borrowers are potentially more productive than lenders at using capital but are subject to financial constraints that may lead to fire sales. The model environment can be viewed as a three-date version of Kiyotaki and Moore (1997) with more general preferences, technology, and financial market structure.[2]

2.1 Environment

Time is discrete and there are three dates $t = 0, 1, 2$. There is a unit measure of borrowers and a unit measure of lenders, respectively denoted by $i \in I = \{b, \ell\}$. There are two types of goods, a homogeneous consumption good, which serves as numeraire, and a capital good. We denote by $\omega \in \Omega$ the state of nature realized at date 1, where Ω is the set of possible states.

Preferences/endowments Each agent i values consumption $c_t^i \geq 0$ according to a time separable utility function

$$U^i = \mathbb{E}_0 \left[\sum_{t=0}^{2} \beta^t u^i \left(c_t^i \right) \right] \tag{1}$$

where the flow utility function $u^i(c)$ is strictly increasing and weakly concave. We denote by $e_t^{i,\omega} \geq 0$ the endowment of consumption good that agent i receives at date t given a state ω.

Technology At date 0, agents can invest $h^i \left(k_1^i \right)$ units of consumption good to produce k_1^i units of date 1 capital goods, where the functions $h^i(k)$ are increasing and convex and satisfy $h^i(0) = 0$. The economy's total capital stock remains constant at $k_1^b + k_1^\ell$ after the initial investment. We denote by $k_2^{i,\omega}$ the amount of capital that agent i carries from date 1 to 2. Capital fully depreciates after date 2.

At dates 1 and 2, agent i employs capital to produce $F_t^{i,\omega}(k)$ units of the consumption good, where the production function is increasing and weakly concave and satisfies $F_t^{i,\omega}(0) = 0$. As is common in the literature on fire sales, we assume that the productivity of capital depends on who owns it (see e.g. Shleifer and Vishny, 1992). We will typically assume that borrowers have a superior use for capital goods than lenders in our applications.

[2] For expositional simplicity, our baseline model only features two agents and a specific production structure. We extend our main results to multiple agents with more general state-dependent utilities and a more general investment and production structure in the online appendix.

Market structure At date 0, agents trade one-period securities contingent on every state of nature $\omega \in \Omega$. We denote by $x_1^{i,\omega}$ the date 0 purchases of state ω contingent securities by agent i and by m_1^{ω} the date 0 state price density associated with such securities. If $x_1^{i,\omega} < 0$, agent i borrows against state ω. If $x_1^{i,\omega} > 0$, agent i saves towards state ω. The total amount spent by agent i at date 0 on state-contingent securities is $\mathbb{E}_0\left[m_1^{\omega} x_1^{i,\omega}\right]$. Because there is no further uncertainty at date 2, we denote by $x_2^{i,\omega}$ the date 1 holdings of uncontingent one-period bonds in state ω, which trade at a price m_2^{ω}. There is also a market to trade capital at a price q^{ω} at date 1 after production has taken place. There is no role for trading capital at date 2 because it fully depreciates.

The budget constraints capture that consumption, capital investment, and net purchases of capital and securities need to be covered by endowment income, security payoffs, and production income for each agent i in every state $\omega \in \Omega$

$$c_0^i + h^i\left(k_1^i\right) + \mathbb{E}_0\left[m_1^{\omega} x_1^{i,\omega}\right] = e_0^i \tag{2}$$

$$c_1^{i,\omega} + q^{\omega}\Delta k_2^{i,\omega} + m_2^{\omega} x_2^{i,\omega} = e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}\left(k_1^i\right),\ \forall\omega \tag{3}$$

$$c_2^{i,\omega} = e_2^{i,\omega} + x_2^{i,\omega} + F_2^{i,\omega}\left(k_2^{i,\omega}\right),\ \forall\omega \tag{4}$$

where $\Delta k_2^{i,\omega} := k_2^{i,\omega} - k_1^i$. All choice variables at dates 1 and 2 are contingent on the state of nature ω, which is realized at date 1.

Financial constraints The final ingredient of our model is a set of financial market imperfections that constrain borrowers' choices. We introduce these through two vector-valued functions $\Phi_1^b(\cdot)$ and $\Phi_2^{b,\omega}(\cdot)$.

At date 0, borrowers' security holdings $x_1^b = \left(x_1^{b,\omega}\right)_{\omega\in\Omega}$ are subject to a constraint of the form

$$\Phi_1^b\left(x_1^b, k_1^b\right) \geq 0 \tag{5}$$

which defines a convex set and satisfies $\Phi_{1x^{\omega}}^b := \frac{\partial\Phi_1^b}{\partial x_1^{b,\omega}} \geq 0$ and $\Phi_{1k}^b := \frac{\partial\Phi_1^b}{\partial k_1^b} \geq 0$. That is, less borrowing or more capital relax borrowers' financial constraint.

At date 1, borrowers' security holdings $x_2^{b,\omega}$ are subject to a possibly state-dependent constraint that is also a function of the asset price q^{ω}

$$\Phi_2^{b,\omega}\left(x_2^{b,\omega}, k_2^{b,\omega}; q^{\omega}\right) \geq 0,\ \forall\omega \tag{6}$$

with the same regularity properties as the date 0 constraint and $\Phi_{2q}^{b,\omega} := \frac{\partial\Phi_2^{b,\omega}}{\partial q^{\omega}} \geq 0$.[3] For instance, if borrowers have to collateralize their borrowing with a fraction $\phi^{\omega} \in [0,1]$ of

[3]For expositional simplicity, the financial constraint at date 0 does not depend on prices or other aggregate variables in our baseline model. We show in the online appendix that it is straightforward to extend our results to that case. We also show that it is straightforward to allow for constraints that depend on future aggregate state variables, which is appropriate when financial constraints depend directly on future asset prices.

their asset holdings, $\Phi_2^{b,\omega}(\cdot) := x_2^{b,\omega} + \phi^\omega q^\omega k_2^{b,\omega} \geq 0$. For symmetry of notation, we define $\Phi_1^\ell(\cdot) = \Phi_2^{\ell,\omega}(\cdot) := 0$ so the constraints are always trivially satisfied for lenders.

Interpretation of financial constraints This general specification allows us to consider a wide range of financial constraints.[4] Focusing on the date 0 constraints, one extreme, captured by the specification $\Phi_1^b\left(x_1^b, k_1^b\right) := 0$, is that agents face no constraints at date 0 and can trade in a complete market, since constraint (5) becomes redundant under this specification. This can be interpreted as very loose financial conditions and well-functioning risk markets. The opposite extreme, captured by the specification $\Phi_1^b\left(x_1^b, k_1^b\right) := \left(x_1^{b,\omega}\right)_{\omega\in\Omega}$ and the vector constraint $\Phi_1^b(\cdot) = 0$ with equality, implies that no financial trade is possible and borrowers have to satisfy $x_1^{b,\omega} = 0, \forall\omega$. This can be interpreted as a severe disruption of financial markets. Clearly, a planner who is subject to the same constraint cannot alter the financing decisions of agents who face this constraint.

The most interesting cases are in between, when borrowers face some market incompleteness but still have some meaningful financing and investment decisions. Our framework can flexibly accommodate intermediate degrees of financial market imperfections, including different types of market incompleteness. For example, if we specify $\Phi_1^b\left(x_1^b, k_1^b\right) := \left(x_1^{b,\omega} - x_1^{b,\omega_0}\right)_{\omega\in\Omega\backslash\omega_0}$, then the vector constraint $\Phi_1^b(\cdot) = 0$ describes that borrowers can only trade bonds at date 0 – all state-contingent payments have to be identical, $x_1^{b,\omega} = x_1^{b,\omega_0}$. Alternatively, for the specification $\Phi_1^b\left(x_1^b, k_1^b\right) := \left(x_1^{b,\omega} - \bar{x}\right)_{\omega\in\Omega}$, where $\bar{x} < 0$, the vector inequality constraint $\Phi_1^b(\cdot) \geq 0$ captures a form of limited commitment on date 1 repayments, such that borrowers cannot promise to repay more than $\bar{x}$.

Interpretation of environment Our baseline model captures a number of different situations in which financial constraints matter and fire sales may occur. We provide four natural interpretations. First, we can think of borrowers as entrepreneurs/firms who have a more productive use of capital goods than other agents in the economy. When financial constraints force them to sell, capital is diverted to a less efficient technology, leading to price declines. Second, borrowers can be interpreted as an amalgamate of financial intermediaries and firms that channel funds from savers/lenders into productive capital investment. If financial constraints force the intermediaries to reduce credit to the real sector, the firms are less able to externally finance their investments, leading to inefficient sales of capital. Third, we can also interpret borrowers as homeowners who hold mortgages. The transfer of houses from borrowers to lenders in case of foreclosure can accelerate house depreciation, causing declines in house prices.

[4] We have directly formulated financial constraints in the context of single-period claims. These types of constraints arise endogenously in some environments – see, for instance, the model of limited commitment without exclusion of Rampini and Viswanathan (2010) – however, multi-period constraints may arise in more general environments. The results of the paper can be adapted to that context. In particular, the sufficient statistics identified in this paper would remain valid in the more general case.

Finally, more broadly, when agents have heterogeneous preferences, we can interpret borrowers as financial specialists who place a higher value on risky assets than their lenders because they have a better capacity to bear risk, but who may be forced to unwind their positions at unusually low prices after a common negative shock.

2.2 First Best

A real allocation is a bundle of consumption vectors $\left(c_0^i, c_1^{i,\omega}, c_2^{i,\omega}\right)$ and capital holdings $\left(k_1^i, k_2^{i,\omega}\right)$ for all $\omega \in \Omega$ and $i \in I$. A real allocation is first-best if it maximizes the weighted sum of welfare $\sum_i \theta^i U^i$ for some welfare weights $\theta^b, \theta^\ell > 0$ subject to the resource constraints

$$\sum_i \left[c_0^i + h^i\left(k_1^i\right)\right] \leq \sum_i e_0^i \tag{7}$$

$$\sum_i c_t^{i,\omega} \leq \sum_i \left[e_t^i + F_t^{i,\omega}\left(k_t^{i,\omega}\right)\right], \text{ for } t = 1,2 \text{ and } \forall \omega \tag{8}$$

$$\sum_i k_2^{i,\omega} \leq \sum_i k_1^i, \ \forall \omega \tag{9}$$

It is easy to see that a real allocation is first-best if it satisfies the resource constraints, if the marginal rates of substitution (MRS) between the two sets of agents are equated across time and states,

$$\frac{u^{b\prime}\left(c_0^b\right)}{u^{\ell\prime}\left(c_0^\ell\right)} = \frac{u^{b\prime}\left(c_1^{b,\omega}\right)}{u^{\ell\prime}\left(c_1^{\ell,\omega}\right)} = \frac{u^{b\prime}\left(c_2^{b,\omega}\right)}{u^{\ell\prime}\left(c_2^{\ell,\omega}\right)}, \ \forall \omega$$

if the marginal cost of capital investment equals its discounted expected benefit, $u^{i\prime}\left(c_0^i\right) h^{i\prime}\left(k_1^i\right) = \mathbb{E}_0\left[\beta u^{i\prime}\left(c_1^{i,\omega}\right) F_1^{i,\omega\prime}\left(k_1^i\right) + \beta^2 u^{i\prime}\left(c_2^{i,\omega}\right) F_2^{i,\omega\prime}\left(k_2^{i,\omega}\right)\right], \forall i$, and if the marginal products of capital are equated at date 2, $F_2^{i,\omega\prime}\left(k_2^{i,\omega}\right) = F_2^{j,\omega\prime}\left(k_2^{j,\omega}\right), \forall i, j$.

2.3 Decentralized Equilibrium

A decentralized equilibrium consists of a real allocation $\left(c_0^i, c_1^{i,\omega}, c_2^{i,\omega}, k_1^i, k_2^{i,\omega}\right)$, a security allocation $\left(x_1^{i,\omega}, x_2^{i,\omega}\right)$, together with a set of prices $\left(m_1^\omega, m_2^\omega, q^\omega\right)$ such that both sets of agents solve their optimization problem and markets clear, i.e. equations (7), (8), and (9) hold, and $\sum_i x_t^{i,\omega} = 0$ holds at dates 1 and 2, $\forall \omega$. For the rest of the paper, we proceed under the presumption that there exists a unique equilibrium.[5] When financial constraints never bind, the real allocation of the decentralized equilibrium of our economy is first-best.

[5]At this level of generality, equilibrium existence and uniqueness are not guaranteed. Under regularity conditions, the generic existence results discussed, for instance, in Magill and Quinzii (2002) apply to our environment. We carefully establish the regularity properties of the model in each of our applications. We also provide examples of non-uniqueness in the appendix.

We solve for the decentralized equilibrium via backward induction, paying particular attention to date 1, which is when pecuniary externalities materialize.

Date 2 equilibrium Equilibrium at date 2 is simple. After production has taken place, agents settle their security positions and consume their holdings of consumption goods. Capital is worthless after date 2, since there is no further production in the economy.

Date 1 equilibrium The state of the economy at date 1 is fully described by two sets of state variables: the financial net worth

$$n^{i,\omega} := e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}\left(k_1^i\right) \tag{10}$$

and the capital holdings k_1^i of both groups of agents. The agents' net worth fully captures the impact of uncertainty on the economy. Note that $n^{i,\omega}$ may be negative if $x_1^{i,\omega}$ is sufficiently negative – in that case, the agents need to borrow and/or fire-sell at date 1 just to service existing debt.

It is useful to distinguish between individual state variables $\left(n^{b,\omega}, n^{\ell,\omega}, k_1^b, k_1^\ell\right)$ and sector-wide aggregate state variables $\left(N^{b,\omega}, N^{\ell,\omega}, K_1^b, K_1^\ell\right)$, which we denote by capitalized letters. In a symmetric equilibrium, it is always the case that $n^{i,\omega} = N^{i,\omega}$ and $k_1^i = K_1^i$, $\forall i, \omega$. However, the distinction matters because individual agents take sector-wide variables as given whereas they internalize that they can affect their own state variables through their date 0 actions. Sector-wide variables enter the welfare function of individual agents since they affect the prices of capital and financial securities. This plays a crucial role in our analysis of externalities below. In the following, we collect the sector-wide net worth and capital holdings of borrowers and lenders at date 1 in the two vectors $N^\omega = \left(N^{b,\omega}, N^{\ell,\omega}\right)$ and $K_1 = \left(K_1^b, K_1^\ell\right)$.

We describe the date 1 optimization problem of an individual agent i as a function of both sets of state variables

$$V^{i,\omega}\left(n^{i,\omega}, k_1^{i,\omega}; N^\omega, K_1\right) = \max_{c_1^{i,\omega}\geq 0, c_2^{i,\omega}\geq 0, k_2^{i,\omega}, x_2^{i,\omega}} u^i\left(c_1^{i,\omega}\right) + \beta u^i\left(c_2^{i,\omega}\right) \quad \text{s.t.} \quad (3), (4) \text{ and } (6) \tag{11}$$

where we denote by $\lambda_t^{i,\omega}$ the multipliers on the budget constraints (3) and (4), by $\kappa_2^{b,\omega}$ the multiplier on borrowers' financial constraint (6), and by $\eta_t^{i,\omega} \geq 0$ the multipliers on the non-negativity of consumption constraints.[6] We define $\kappa_2^{\ell,\omega} := 0$ to keep our notation symmetric. Since there is no uncertainty at date 2, financial contracts between dates 1 and 2 are uncontingent. The resulting Euler equation is

$$m_2^\omega \lambda_1^{i,\omega} = \beta\lambda_2^{i,\omega} + \kappa_2^{i,\omega}\Phi_{2x}^{i,\omega} \tag{12}$$

[6] The multiplier $\lambda_t^{i,\omega}$ corresponds to the marginal value of wealth for agent i in a given date/state and satisfies $\lambda_t^{i,\omega} = u^{i\prime}\left(c_t^{i,\omega}\right) + \eta_t^{i,\omega}$. If consumption is positive, $\lambda_t^{i,\omega}$ is identical to the marginal utility of consumption.

where $\Phi_{2x}^{i,\omega} := \frac{\partial \Phi_2^{i,\omega}}{\partial x_2^{i,\omega}}$. For borrowers, the multiplier on the borrowing constraint satisfies $\kappa_2^{b,\omega} \geq 0$, and they attach the shadow value $\kappa_{2x}^{b,\omega}\Phi_{2x}^{b,\omega}$ to the marginal unit of borrowing.

The optimal capital accumulation decision implies

$$q^\omega \lambda_1^{i,\omega} = \beta \lambda_2^{i,\omega} F_2^{i,\omega\prime}\left(k_2^{i,\omega}\right) + \kappa_2^{i,\omega}\Phi_{2k}^{i,\omega} \tag{13}$$

where $\Phi_{2k}^{i,\omega} := \frac{\partial \Phi_2^{i,\omega}}{\partial k_2^{i,\omega}}$. If the financial constraint on agent i is slack, then $\kappa_2^{i,\omega} = 0$ and the price of capital is simply its marginal value in the hands of agent i discounted by the market discount factor $m_2^\omega = \frac{\beta\lambda_2^{i,\omega}}{\lambda_1^{i,\omega}}$. This always holds for lenders. Borrowers, on the other hand, may be subject to a binding financial constraint. In that case, the term $\kappa_2^{i,\omega}\Phi_{2k}^{i,\omega}$ in their optimality condition reflects the marginal benefit of capital in relaxing the financial constraint. Furthermore, they discount the future payoff of capital at a higher rate than lenders $\frac{\beta\lambda_2^{b,\omega}}{\lambda_1^{b,\omega}} > m_2^\omega$. As a result, the binding constraint may force borrowers to sell capital even though they have a higher marginal valuation than lenders – this is what we call a fire sale.

In general equilibrium, optimality conditions (12) and (13) define the price of discount bonds $m_2^\omega\left(N^\omega, K_1\right)$ and capital $q^\omega\left(N^\omega, K_1\right)$ as functions of the aggregate state variables. Both prices are generally – but not always – increasing functions of the net worth $N^{i,\omega}$ of each sector, since higher net worth at date 1 implies that agents are more willing to carry wealth forward to date 2, and they value the instruments for doing this, including bonds and capital, more highly. Formally, we capture this in the following condition on the response of the asset price to sector i net worth.

Condition 1. (Asset price increasing in sectoral net worth) *The price of capital assets is increasing in the net worth of both sectors,*

$$\frac{\partial q^\omega}{\partial N^{i,\omega}} > 0 \quad \forall i \in \{b, \ell\}$$

Condition 1 is not necessary to derive the two main propositions of our paper. However, it is useful to determine the sign of pecuniary externalities. We impose assumptions on primitives that ensure that the condition is satisfied in each of our four applications in the main text, and we demonstrate in Appendix B.1 how to relate the condition to elasticities of utility and production functions in our first application.[7] We also consider the alternate case in two additional applications in the appendix to show that violations of the condition typically go hand in hand with backward-bending demand curves that lead to multiple and locally unstable equilibria.[8]

[7] The behavior of prices cannot be easily stated in terms of fundamentals in almost all general equilibrium models. This makes it useful to focus on sufficient statistics, as we do in our approach.

[8] Although a full analysis is outside of the scope of this paper, the index theorem results in Chapter 17 of Mas-Colell, Whinston and Green (1995) suggest that Condition 1 emerges naturally in models with well-behaved equilibria.

We analyze next how changes in the sector-wide date 1 state variables of the economy N^{ω} and K_1 affect the welfare of individual agents. Lemma 1 characterizes the properties of the date 1 equilibrium that are relevant for our efficiency analysis.

Lemma 1. (Uninternalized welfare effects of changes in sector-wide N^{ω} and K_1) *The effects of changes in the sector-wide state variables (N^{ω}, K_1) on agent i indirect utility at date 1 are given by*

$$V_{N^j}^{i,\omega} := \frac{dV^{i,\omega}(\cdot)}{dN^{j,\omega}} = \lambda_1^{i,\omega}\mathcal{D}_{N^j}^{i,\omega} + \kappa_2^{i,\omega}\mathcal{C}_{N^j}^{i,\omega} \tag{14}$$

$$V_{K^j}^{i,\omega} := \frac{dV^{i,\omega}(\cdot)}{dK_1^j} = \lambda_1^{i,\omega}\mathcal{D}_{K^j}^{i,\omega} + \kappa_2^{i,\omega}\mathcal{C}_{K^j}^{i,\omega} \tag{15}$$

where we refer to $\mathcal{D}_{N^j}^{i,\omega}$ and $\mathcal{D}_{K^j}^{i,\omega}$ as the **distributive effects** *of changes in $N^{j,\omega}$ and K_1^j for type i agents*

$$\mathcal{D}_{N^j}^{i,\omega} := -\frac{\partial q^{\omega}}{\partial N^{j,\omega}}\Delta K_2^{i,\omega} - \frac{\partial m_2^{\omega}}{\partial N^{j,\omega}}X_2^i \tag{16}$$

$$\mathcal{D}_{K^j}^{i,\omega} := F_1^{i,\omega\prime}\left(K_1^{i,\omega}\right)\mathcal{D}_{N^j}^{i,\omega} - \left[\frac{\partial q^{\omega}}{\partial K_1^j}\Delta K_2^{i,\omega} + \frac{\partial m_2^{\omega}}{\partial K_1^j}X_2^{i,\omega}\right] \tag{17}$$

and we refer to $\mathcal{C}_{N^j}^{i,\omega}$ and $\mathcal{C}_{K^j}^{i,\omega}$ as the **collateral effects** *of changes in $N^{j,\omega}$ and K_1^j for type i agents*

$$\mathcal{C}_{N^j}^{i,\omega} := \frac{\partial \Phi_2^{i,\omega}}{\partial q^{\omega}}\frac{\partial q^{\omega}}{\partial N^{j,\omega}} \tag{18}$$

$$\mathcal{C}_{K^j}^{i,\omega} := F_1^{i,\omega\prime}(\cdot)\,\mathcal{C}_{N^j}^{i,\omega} + \frac{\partial \Phi_2^{i,\omega}}{\partial q^{\omega}}\frac{\partial q^{\omega}}{\partial K_1^j} \tag{19}$$

As shown in equations (14) and (15), changes in the sector-wide net worth $N^{j,\omega}$ and capital K_1^j affect welfare through two distinct mechanisms that both occur because changes in $N^{j,\omega}$ and K_1^j affect the equilibrium prices $q^{\omega}(N^{\omega}, K_1)$ and $m_2^{\omega}(N^{\omega}, K_1)$: distributive effects and collateral effects. The effects of changes in $N^{j,\omega}$ and K_1^j on all other equilibrium variables in problem (11) drop out by the envelope theorem.

First, changes in $N^{j,\omega}$ and K_1^j affect the equilibrium prices $q^{\omega}(N^{\omega}, K_1)$ and $m_2^{\omega}(N^{\omega}, K_1)$ at which sector i agents trade capital and bonds. The *distributive effects* $\mathcal{D}_{N^j}^{i,\omega}$ and $\mathcal{D}_{K^j}^{i,\omega}$ capture the marginal wealth redistributions to sector i that result from price changes following a change in the sector-wide net worth $N^{j,\omega}$ or capital K_1^i. We use the terminology distributive effects, because they are zero-sum across agents on a state-by-state basis. Formally, exploiting market clearing

$$\sum_i \mathcal{D}_{N^j}^{i,\omega} = 0 \qquad \text{and} \qquad \sum_i \mathcal{D}_{K^j}^{i,\omega} = 0, \quad \forall\omega \tag{20}$$

Second, changes in the equilibrium price $q^{\omega}(N^{\omega}, K_1)$ directly affect the tightness of the financial constraint faced by borrowers. The *collateral effects* $\mathcal{C}_{N^j}^{i,\omega}$ and $\mathcal{C}_{K^j}^{i,\omega}$ capture the direct effect of changes in aggregate state variables on the tightness of $\Phi_2^{i,\omega}(\cdot)$. Unlike distributive effects, collateral effects are generally not zero-sum across agents. In a symmetric equilibrium, it must be that $n^{i,\omega} = N^{i,\omega}$ and $k_1^i = K_1^i$, $\forall i$. In that case, agent's i indirect utility is given by $V^{i,\omega}\left(N^{i,\omega}, K_1^i; N^\omega, K_1\right)$, and we can decompose the equilibrium effects of a change in sector i financial net worth $N^{i,\omega}$ on welfare into two parts

$$\frac{dV^{i,\omega}\left(N^{i,\omega}, K_1^i; N^\omega, K_1\right)}{dN^{i,\omega}} = V_n^{i,\omega}(\cdot) + V_{N^i}^{i,\omega}(\cdot)$$

The term $V_n^{i,\omega} := \frac{\partial V^{i,\omega}}{\partial n^{i,\omega}}$ represents the private marginal utility of wealth and is given by the envelope condition $V_n^{i,\omega}(\cdot) = \lambda_1^{i,\omega}$. This part is internalized by individual agents who choose how much wealth to carry into date 1. The term $V_{N^i}^{i,\omega}$ represents the effects of changes in sector-wide wealth that is *not internalized* by individual agents. A similar decomposition can be performed for the internalized and uninternalized effects of changes in sector-wide capital $k_1^i = K_1^i$. In our welfare analysis in Section 3, these uninternalized effects will represent pecuniary externalities.

Date 0 equilibrium We describe the date 0 optimization problem of agent i as

$$\max_{c_0^i \geq 0, k_1^i, x_1^{i,\omega}} u^i\left(c_0^i\right) + \beta \mathbb{E}_0\left[V^{i,\omega}\left(e_1^{i,\omega} + x_1^{i,\omega} + F_1^{i,\omega}\left(k_1^i\right), k_1^i; N^\omega, K_1\right)\right] \quad \text{s.t.} \quad (2), (5) \tag{21}$$

Using the envelope conditions $V_n^{i,\omega}(\cdot) = \lambda_1^{i,\omega}$ and $V_k^{i,\omega}(\cdot) = \lambda_1^{i,\omega} q^\omega$, we obtain a set of standard Euler equations and an optimal investment condition

$$m_1^\omega \lambda_0^i = \beta \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i, \quad \forall i, \omega \tag{22}$$

$$h^{i\prime}\left(k_1^i\right)\lambda_0^i = \mathbb{E}_0\left[\beta \lambda_1^{i,\omega}\left(F_1^{i,\omega\prime}\left(k_1^{i,\omega}\right) + q^\omega\right)\right] + \kappa_1^i \Phi_{1k}^i, \quad \forall i \tag{23}$$

where we define $\Phi_{1x^\omega}^i := \frac{\partial \Phi_1^i}{\partial x_1^{i,\omega}}$ and $\Phi_{1k}^i := \frac{\partial \Phi_1^i}{\partial k_1^i}$, we assign κ_1^b as the (vector) multiplier on the financial constraint of borrowers and, again, define $\kappa_1^\ell := 0$ for lenders to keep notation symmetric.

The Euler equations ensure that the intertemporal marginal rates of substitution of all agents are equated to the market prices m_1^ω and thus to each other in every state of nature, unless the financial constraint introduces a wedge. The optimal investment condition states that the marginal cost of capital investment equals its discounted marginal benefit, which consists of the marginal product $F_1^{i,\omega\prime}\left(k_1^{i,\omega}\right)$, the remaining value q^ω of capital, and the benefit of relaxing the constraint.

3 Efficiency Analysis

We set up a constrained social planner problem in the tradition of Stiglitz (1982) and Geanakoplos and Polemarchakis (1986) to determine if the decentralized equilibrium is constrained efficient.

The social planner chooses date 0 allocations subject to the same constraints as the private market, leaving all later decisions to private agents, and respecting that capital and security prices are market-determined.[9]

Formally, the constrained social planner maximizes the weighted sum of welfare of the two sets of agents for given Pareto weights $\left(\theta^b, \theta^\ell\right)$. The planner chooses date 0 allocations $\left(C_0^i, K_1^i, X_1^{i,\omega}\right)$, subject to the date 0 resource constraint. To emphasize that the planner chooses sector-wide variables, we denote her allocations by upper-case letters. Given our earlier definition of date 1 indirect utility functions $V^{i,\omega}(\cdot)$, the constrained planner's problem is

$$\max_{C_0^i, K_1^i, X_1^{i,\omega}} \sum_i \theta^i \left\{ u^i\left(C_0^i\right) + \beta \mathbb{E}_0\left[V^{i,\omega}\left(N^{i,\omega}, K_1^i; N^\omega, K_1\right)\right]\right\} \tag{24}$$

$$\begin{aligned} \text{s.t.} \qquad \sum_i \left[C_0^i + h^i\left(K_1^i\right) - e_0^i\right] &\leq 0 && (\nu_0) \\ \sum_i X_1^{i,\omega} &= 0,\ \forall \omega && (\nu_1^\omega) \\ \Phi_1^i\left(X_1^i, K_1^i\right) &\geq 0,\ \forall i && \left(\theta^i \kappa_1^i\right) \\ C_0^i &\geq 0,\ \forall i && \left(\theta^i \eta_0^i\right) \end{aligned}$$

where $N^{i,\omega} = e_1^i + X_1^{i,\omega} + F_1^{i,\omega}\left(K_1^i\right), \forall \omega, i$. We assign the shadow price ν_0 to the date 0 resource constraint, ν_1^ω to the intertemporal resource constraint for state ω, the vector of shadow prices $\theta^i \kappa_1^i$ to the financial constraint, and $\theta^i \eta_t^{i,\omega}$ to the multipliers on the non-negativity constraints of consumption.[10] We also denote the marginal value of wealth for agent i by $\lambda_t^{i,\omega} = u^{i\prime}\left(C_t^{i,\omega}\right) + \eta_t^{i,\omega}$ – it equals the marginal utility of consumption except when consumption is at a corner solution.

Proposition 1 characterizes constrained efficient allocations and shows how to implement them. Proposition 2 identifies the two distinct externalities that underlie inefficiency and establishes that each of them can be characterized as a function of a small set of variables that determine their sign and magnitude.

Proposition 1. a) (Constrained efficient allocations) *A date 0 allocation $\left(C_0^i, K_1^i, X_1^{i,\omega}\right)$ is constrained efficient if and only if there are positive welfare weights that satisfy $\frac{\theta^b}{\theta^\ell} = \frac{\lambda_0^\ell}{\lambda_0^b}$ and shadow prices ν_0, ν_1^ω, and κ_1^i such that the allocation respects the constraints in problem (24) and satisfies the*

[9]This setup is equivalent to the problem of a constrained Ramsey planner who chooses taxes on date 0 allocations plus transfers, as shown in Online Appendix C.1. We also discuss the possibility of finding Pareto improvements and allowing for transfers other than at date 0 in the Online Appendix C.2.

[10]We scale all agent-specific multiplier by θ^i to keep notation symmetric with the optimization problem of private agents.

following financing and investment conditions

$$\frac{\nu_1^\omega}{\nu_0}\lambda_0^i = \beta\lambda_1^{i,\omega} + \kappa_1^i\Phi_{1x^\omega}^i + \beta\sum_{j\in I}\frac{\theta^j}{\theta^i}V_{N^i}^{j,\omega},\ \forall i,\omega \tag{25}$$

$$h^{i\prime}\left(K_1^i\right)\lambda_0^i = \beta\mathbb{E}_0\left[\lambda_1^{i,\omega}\left(F_1^{i,\omega\prime}\left(K_1^i\right)+q^\omega\right)\right] + \kappa_1^i\Phi_{1k}^i + \beta\sum_{j\in I}\frac{\theta^j}{\theta^i}\mathbb{E}_0\left[F_1^{i,\omega\prime}\left(K_1^i\right)V_{N^i}^{j,\omega} + V_{K^i}^{j,\omega}\right],\ \forall i \tag{26}$$

where all variables at dates 1 *and* 2 *are determined by the optimization problem* (11) *and market clearing, and* $V_{N^i}^{j,\omega}$ *and* $V_{K^i}^{j,\omega}$ *are defined in Lemma* 1*.*

b) (Implementing constrained efficiency) *A planner can implement any constrained efficient allocation by setting taxes on state-contingent security purchases and capital investment that satisfy*

$$\tau_x^{i,\omega} = -\sum_{j\in I}MRS^{j,\omega}\mathcal{D}_{N^j}^{i,\omega} - \sum_{j\in I}\tilde{\kappa}_2^{i,\omega}\mathcal{C}_{N^j}^{i,\omega},\ \forall i,\omega \tag{27}$$

$$\tau_k^i = -\sum_{j\in I}\mathbb{E}_0\left[MRS^{j,\omega}\mathcal{D}_{K^j}^{i,\omega}\right] - \sum_{j\in I}\mathbb{E}_0\left[\tilde{\kappa}_2^{i,\omega}\mathcal{C}_{K^j}^{i,\omega}\right],\ \forall i \tag{28}$$

where $MRS^{j,\omega} := \frac{\beta\lambda_1^{j,\omega}}{\lambda_0^j}$ *and* $\tilde{\kappa}_2^{i,\omega} := \frac{\kappa_2^{i,\omega}}{\lambda_0^i}$*, and conducting lump-sum transfers* T^i *such that date 0 budget constraints* (2) *with taxes are met and the government budget constraint* $\sum_i T^i = \sum_i \mathbb{E}_0\left[\tau_x^{i,\omega}X_1^{i,\omega}\right] + \sum_i \tau_k^i K_1^i$ *is satisfied.*

Proposition 1.a) characterizes constrained efficient allocations through a set of Euler equations for financing and investment decisions, as in the decentralized case. The left hand side of equation (25) is the social marginal price of saving one unit of wealth. The right hand side is the associated social marginal benefit, consisting of the consumption value of an extra unit of net worth, the value of relaxing the financial constraint and the uninternalized welfare effects described in Lemma 1. Similarly, the left hand side of (26) reflects the social marginal cost of capital investment, and the social marginal benefits on the right hand side consist of the marginal product of capital, the continuation value of capital q^ω, the benefits of capital in relaxing the financial constraint, and the uninternalized welfare effects. A comparison of equations (22) and (23) with equations (25) and (26) highlights that the sole difference between the decentralized and the constrained efficient allocation is that the planner internalizes the general equilibrium effects captured by these uninternalized welfare effects, as described in Lemma 1.

Proposition 1.b) describes how to set the corrective tax instruments $\tau_x^{i,\omega}$ and τ_k^i to modify agents' date 0 decisions and implement constrained efficient allocations. Intuitively, judiciously chosen tax rates induce private agents to internalize the pecuniary externalities of their actions caused by both the distributive and collateral effects. A positive $\tau_x^{i,\omega}$ induces agent i to allocate less resources towards state ω – indicating that private agents underborrow in the decentralized equilibrium; a positive τ_k^i induces agent i to invest less in capital – indicating that private agents overinvest in the decentralized equilibrium – and vice versa for negative signs.

We can simplify each of the two additive terms in (27) and (28). For each of the first terms, corresponding to the distributive effects, we exploit market clearing, as in equation (20), and define $\Delta MRS^{ij,\omega} := MRS^{i,\omega} - MRS^{j,\omega}$ to express the distributive effects as a function of the difference between marginal rates of substitution between agents. For each of the second terms, corresponding to the collateral effects, we simply note that $\tilde{\kappa}_2^{\ell,\omega} = 0$ by construction. This allows us to equivalently express $\tau_x^{i,\omega}$and τ_k as follows:

$$\tau_x^{i,\omega} = -\Delta MRS^{ij,\omega}\mathcal{D}_{N^i}^{i,\omega} - \tilde{\kappa}_2^{b,\omega}\mathcal{C}_{N^i}^{b,\omega},\ \forall i,\omega \tag{29}$$

$$\tau_k^i = -\mathbb{E}_0\left[\Delta MRS^{ij,\omega}\mathcal{D}_{K^i}^{i,\omega}\right] - \mathbb{E}_0\left[\tilde{\kappa}_2^{b,\omega}\mathcal{C}_{K^i}^{b,\omega}\right],\ \forall i \tag{30}$$

Proposition 2 formally establishes the distinct nature of distributive and collateral externalities. For both types of externalities, the direction of the inefficiency is fully determined by a small set of sufficient statistics with a natural interpretation.

Proposition 2. (Distinct nature of externalities/sufficient statistics) *There are two distinct types of externalities: distributive externalities (D) and collateral externalities (C).*
The sign and magnitude of distributive externalities are determined by the product of three variables:

(D1) The difference in MRS of agents $\Delta MRS^{ij,\omega}$

(D2) The net trading positions (net buying or net selling) on capital $\Delta K_2^{i,\omega}$ *and financial assets* $X_2^{i,\omega}$

(D3) The sensitivity of equilibrium prices to changes in sector-wide state variables $\frac{\partial q^\omega}{\partial N^{j,\omega}}, \frac{\partial m_2^\omega}{\partial N^{j,\omega}}, \frac{\partial q^\omega}{\partial K_1^j}, \frac{\partial m_2^\omega}{\partial K_1^j}$

The sign and magnitude of collateral externalities are determined by the product of three variables:

(C1) The shadow value on the financial constraint $\tilde{\kappa}_2^{i,\omega}$

(C2) The sensitivity of the financial constraint to the asset price $\frac{\partial \Phi_2^{i,\omega}}{\partial q^\omega}$

(C3) The sensitivity of the equilibrium capital price to changes in sector-wide state variables $\frac{\partial q^\omega}{\partial N^{j,\omega}}, \frac{\partial q^\omega}{\partial K_1^j}$

Proposition 2 contains one of the main economic insights of this paper. A small number of sufficient statistics encapsulate the information needed to determine whether an economy is constrained efficient and how to correct any distortions. Distributive and collateral externalities are generically present in any competitive environment in which financial market imperfections nest into the form of equations (5) and (6).

Distributive externalities arise because agents do not internalize that their actions change equilibrium prices, affecting the amount received by other agents through capital or financial asset sales or purchases. When financial constraints inhibit optimal risk-sharing and prevent the equalization of MRS between agents across dates or states, independently of the reason why MRS are not equalized, a suitable change in the behavior of agents redistributes resources through price changes towards agents with higher MRS in a given date/state, improving efficiency.

Therefore, understanding the nature of distributive externalities requires to understand the difference in relative valuations of wealth (i.e. the MRS) of all agents across dates/states, their net trading positions, and how changes in sector-wide state variables affect equilibrium prices.

Collateral externalities arise because agents do not internalize that their actions change equilibrium prices, directly modifying the borrowing/saving capacity of other constrained agents. A suitable change in the behavior of agents modifies asset prices, relaxing financial constraints directly and changing the effective financial decisions of those agents for which the constraint binds.

Therefore, understanding the nature of collateral externalities requires to understand the welfare benefit of relaxing borrowers' financial constraint, the change in borrowing capacity due to a change in asset prices, and the sensitivity of equilibrium prices to changes in sector-wide state variables.

While distributive externalities work by changing the value of the flow of resources, collateral externalities work by directly affecting the financing capacity of constrained agents by changing the value of the stock of assets that serve as collateral, not only the flow of resources between agents. For this reason, only borrowers in our baseline model experience the effects of collateral externalities, while all agents experience the effects of distributive externalities.

As usual in normative problems, it is in general not feasible to characterize distortions or optimal corrective policies as a function of primitives.[11] Instead, Proposition 2 shows that, independently of the specific nature of the financial frictions, identifying the sign and magnitude of the externalities boils down to identifying a small number of sufficient statistics, which should guide the design of corrective policies. These variables will be invariant to the precise nature of the underlying distortions, e.g., uncontingent bonds, limited commitment, market segmentation, idiosyncratic risks, etc. In Section 4, we illustrate in specific applications how changes in primitives affect the sign and magnitude of the externalities through changes in these sufficient statistics. The sufficient statistics that we identify in Proposition 2 remain the key determinants of the sign of the externalities in more general environments with multiple agents and more general preferences and production technologies, as shown in the Online Appendix C.3.[12]

[11]Even the most elementary results in normative economics are expressed as a function of high level observables as opposed to primitives. For instance, Ramsey's characterization of optimal commodity taxes relies on demand elasticities, which are endogenous to the level of taxes.

[12]The online appendix also considers more general constraint sets $\Phi_t^i(\cdot)$ that depend directly on aggregate state variables, e.g., moral-hazard/incentive constraints or value-at-risk requirements. For further examples see Greenwald and Stiglitz (1986). We show that these are of the same nature as collateral externalities, although it may

An important application of our optimal tax formulas is to identify general circumstances under which equilibria with financially constrained agents and fire sales are *constrained efficient*. Distributive pecuniary externalities are zero whenever either (i) financially constrained agents face complete risk markets to insure against future fire sales so $\Delta MRS^{b\ell,\omega} = 0$ or (ii) the net trading position of capital and financial assets is zero or (iii) the prices of capital and financial assets are fixed, e.g. because of linear preferences and technologies. We will show an example of (i) below in Application 1, and examples of (ii) and (iii) below in Application 4. Collateral externalities are absent whenever (i) borrowers are unconstrained at date 1 so $\kappa_2^b = 0$ or (ii) their financial constraint only depends on individual-level variables so $\frac{\partial \Phi_2^{b,\omega}}{\partial q} = 0$ or (iii) the prices of capital assets are fixed. In Applications 1 to 3 below we assume (ii) to rule out collateral externalities and focus on distributive externalities. When neither type of externality is present, optimal taxes are zero and equilibrium is constrained efficient. These findings formalize a folk theorem in the field that has not been analytically spelled out.

In the following corollaries, we provide five general results that follow from our analysis. We further elaborate on those results in our applications in Section 4.

Sign of externalities In the existing literature on pecuniary externalities, it has proven remarkably difficult to provide general results on the direction of inefficiency – except in tightly-defined special cases. The following corollary rationalizes why.

Corollary 1. (Sign of externalities and "anything goes") *The collateral externalities of sector-wide net worth are non-negative under Condition 1. All distributive externalities as well as the collateral externalities of sector-wide capital holdings can naturally take on either sign, so "anything goes."*

The corollary states that in general, only the collateral externalities of financing decisions can be signed since the sufficient statistics *C1* and *C2* are by construction non-negative; the shadow value of borrowers' financial constraint is weakly positive and a higher asset price weakly relaxes the financial constraint. Furthermore, *C3* is positive for sector-wide net worth if and only if the natural Condition 1 is satisfied, implying that collateral externalities unambiguously lead to overborrowing in that case. We provide an illustration of this result in Application 4.

The sufficient statistics *D1* and *D2* can naturally take on either sign, as we illustrate in Applications 2 and 3; plausible configurations of primitives are consistent with positive or negative differences in MRS and with agents that can be net buyers or sellers. For example, if borrowers have a high relative valuation compared to lenders in a given state and they are net sellers of capital in that state, it will be optimal to subsidize their savings towards that state. Furthermore, the sufficient statistics *C3* and *D3* can take on either sign for the externalities of

be more appropriate to call them binding-constraint externalities instead of collateral externalities. In the appendix, we explain how to adjust the sufficient statistics for collateral effects to this more general case.

sector-wide capital holdings. As a result, "anything goes" for the sign of distributive externalities and the collateral externalities of sector-wide capital holdings.

Unpacking the optimal tax rates for distributive and collateral externalities into three sufficient statistics each is also helpful in spelling out explicit conditions under which they can be signed. This is useful if we are explicitly concerned with devising conditions under which the direction of inefficiency can be pinned down unambiguously, as a number of papers that we discuss in Section 5 have done.

For statistic *D1* ("difference in MRS"), a common mechanism to pin down the sign is to assume that uncontingent bonds are traded in date 0 financial markets so $\mathbb{E}\left[\Delta MRS^{b\ell,\omega}\right] = 0$; if borrowers are subject to a binary shock $\omega \in \{L, H\}$ but lenders aren't, then this implies $\Delta MRS^{b\ell,L} < 0 < \Delta MRS^{b\ell,H}$, illustrating that sign-reversals in $\Delta MRS^{b\ell}$ between different states of nature are common when insurance markets at date 0 are incomplete. The stated inequality makes it desirable to provide insurance to borrowers in state L. If fire sales occur in state L but capital does not change ownership in state H, then any action that reduces fire sales improves insurance and is socially desirable – as we illustrate in Application 2 below in the case in which borrowers fire-sell. Conversely, if capital is traded in multiple states of nature, the sign-reversal in $\Delta MRS^{b\ell}$ is one of the main reasons why it is difficult to sign distributive pecuniary externalities in general frameworks.

For statistic *D2* ("direction of trade"), a sufficient condition for $\Delta K_2^{b,\omega} \leq 0$, i.e. for capital to trade from borrowers to lenders only, is that lenders do not invest in capital at date 0 so $K_1^{\ell} = 0$ – we will use this assumption in Application 3 below to focus on sign reversals in *D1*. A sufficient condition for $\Delta K_2^{b,\omega} \geq 0$ is for lenders to have no use for capital at date 2 so $F_2^{\ell,\omega}(k) = 0, \forall \omega$. Imposing both assumptions, that lenders neither invest in nor have use for capital, ensures that $\Delta K_2^{b,\omega} = 0$ and "turns off" distributive externalities from capital trade – we will use this assumption in Application 4 to focus on collateral externalities. Moreover, sufficient conditions to ensure that distributive effects from bond trade can be signed are (i) that borrowers cannot save $\Phi_2^{b,\omega}(\cdot) := x_2^{b,\omega} \leq 0$, e.g. because of imperfect commitment by lenders, or (ii) that endowments are such that borrowers always borrow in equilibrium. A sufficient assumption to turn off distributive externalities from bond trade altogether is that date 1 financial markets are completely frozen $\Phi_2^{b,\omega}(\cdot) := x_2^{b,\omega} = 0$, as we assume in Applications 2 and 3.

Investment and financing distortion There exists a relation between the distortions in investment and financing decisions because investing in capital and financial assets are both mechanisms for shifting resources across time. However, capital investment changes the aggregate amount of resources available at dates 1 and 2 whereas financial decisions exclusively affect the allocation of resources between agents.

Increasing capital K_1^i in sector i has two general equilibrium effects. First, it increases output and sector i's net worth by $F_1^{i,\omega\prime}\left(K_1^i\right)$ at date 1. In that sense, increasing K_1^i is identical to saving by sector i while holding $N^{j,\omega}$ constant for the other sector. Secondly, additional capital increases the production opportunities of agents at date 2, which has general equilibrium effects on prices q^{ω} and m_2^{ω} and may in turn lead to distributive and collateral effects, as described in Lemma 1. The following corollary describes the relationship when the latter effects are absent.

Corollary 2. (Relationship between distortion in investment and financing decisions) *Whenever $\frac{\partial q^{\omega}}{\partial K_1^i} = \frac{\partial m_2^{\omega}}{\partial K_1^i} = 0$, the optimal corrective taxes $\tau_x^{i,\omega}$ and τ_k^i on financing and investment decisions satisfy*

$$\tau_k^i = \mathbb{E}_0 \left[\tau_x^{i,\omega} F_1^{i,\omega\prime} \left(K_1^i \right) \right] \tag{31}$$

Corollary 2 implies that an optimal policy must coordinate the distortions introduced in investment and saving decisions. When $\frac{\partial m_2^{\omega}}{\partial K_1^i} = \frac{\partial q^{\omega}}{\partial K_1^i} = 0$, the resulting relationship is simple and intuitive. The condition holds for example when production and utility functions are linear at date 2, as in Application 2 below. In that case, both distortions are tightly linked. The general case in which $\frac{\partial m_2^{\omega}}{\partial K_1^i}$ and $\frac{\partial q^{\omega}}{\partial K_1^i}$ can take any values is formally described in the appendix.

Externality pricing kernel The results from Proposition 1 extend to any financial asset, which can be thought of as a bundle of state-contingent securities. Formally, consider a financial security Z traded at date 0 with a state-contingent payoff profile $(Z^{\omega})_{\omega \in \Omega}$ at date 1. This asset can be viewed as a bundle of Arrow securities with weight Z^{ω} on the security contingent on state $\omega \in \Omega$. For example, a risk-free bond corresponds to the vector $Z^{\omega} = 1, \forall \omega$. To hold constant the set of trading opportunities, we require that total security holdings satisfy $\tilde{x}_1^{i,\omega} = x_1^{i,\omega} + \alpha_Z Z^{\omega}$, where $\Phi_1^i \left(\tilde{x}_1^{i,\omega}, k_1^i \right) \geq 0$ and α_Z denotes the holdings of security Z. Under this assumption, the security Z is redundant and no-arbitrage pricing allows us to establish Corollary 3.

Corollary 3. (Externality pricing kernel) *The optimal corrective tax on agent i's holdings of a financial security Z is given by*

$$\tau_Z^i = \mathbb{E}_0 \left[\tau_x^{i,\omega} Z^{\omega} \right], \tag{32}$$

where $\tau_x^{i,\omega}$ is given by equation (27).

Equation (32) reveals a close parallel between traditional security pricing and the pricing of pecuniary externalities. In fact, we can view the optimal state-contingent tax rates $\tau_x^{i,\omega}$ defined in Proposition 1 as an *externality pricing kernel*, which is used to determine the social cost of sector i holding a security with payoff profile Z^{ω}.

Corollary 3 provides a simple expression to guide financial regulators on the design of optimal corrective policies for any financial instrument. A comparison of Corollaries 2 and 3 shows that capital can be interpreted like a financial asset if it does not affect equilibrium at date 1 via general equilibrium effects that result from the expansion of date 2 production possibilities.

Amplification and welfare The role of amplification effects and its relation with efficiency of an economy are often intertwined in policy discussions. Our framework allows us to distinguish between amplification effects and pecuniary externalities.

In our context, it is natural to define financial amplification as when a decline in sector-wide wealth $N^{i,\omega}$ leads to general equilibrium effects that further reduce sector i wealth or that tighten price-dependent financial constraints for sector i. In particular, exploiting the definitions from Lemma 1, we can distinguish in our framework between two types of financial amplification, amplification via distributive effects, which occurs when $\mathcal{D}^{i,\omega}_{N^i} > 0$, and amplification via collateral effects, which occurs when $\mathcal{C}^{i,\omega}_{N^i} > 0$. However, the following corollary establishes that neither form of amplification is necessary or sufficient to generate constrained inefficiency.

Corollary 4. (Decoupling of amplification and inefficiency) *The existence of amplification effects ($\mathcal{D}^{i,\omega}_{N^i} > 0$ or $\mathcal{C}^{i,\omega}_{N^i} > 0$) is neither necessary nor sufficient for constrained inefficiency.*

Amplification effects are not necessary because constrained inefficiency can also arise when $\mathcal{D}^{i,\omega}_{N^i} < 0$, that is, when there are distributive effects that benefit a sector that experiences a decline in its sector-wide net worth, as we illustrate in one case of Application 2 below.

Amplification effects are not sufficient for inefficiency because there are several situations in which amplification effects are consistent with constrained efficiency. First, if there is amplification via distributive effects $\mathcal{D}^{i,\omega}_{N^i} > 0$ but decentralized agents face complete financial markets and equate their MRS, Proposition 1 implies that optimal taxes are zero and equilibrium is constrained efficient, as we stress in Application 1. Intuitively, decentralized agents insure optimally and a planner cannot improve on the outcome by inducing wealth redistributions. Second, if there is amplification via distributive effects $\mathcal{D}^{i,\omega}_{N^i} > 0$ or collateral effects $\mathcal{C}^{i,\omega}_{N^i} > 0$ but decentralized agents are in a corner solution, a planner may not be able to improve welfare, implying that equilibrium is constrained efficient. We describe such situations in further detail in Corollary 5.

Even when amplification leads to constrained inefficiency, Corollary 1 implies that the sign of the resulting distortion is indeterminate – amplification via distributive effects $\mathcal{D}^{i,\omega}_{N^i} > 0$ may be consistent with both over- and underborrowing, as we illustrate in Application 3 below.

Indeterminacy and simplified implementation results Proposition 1 provides the most transparent exposition of pecuniary externalities in that it attributes taxes to each decision margin and each sector according to the externalities it creates. This suggests that it is desirable to intervene in every single decision margin of private agents, which imposes a significant burden on regulators. However, the optimal tax formulas in the proposition are just one out of a continuum of alternative implementations. There are up to three dimensions of indeterminacy that allow us to normalize one or more of the policy instruments $\left\{\tau^{b,\omega}_x, \tau^{\ell,\omega}_x, \tau^b_k, \tau^\ell_k\right\}$ to zero in order to simplify the implementation, i.e. to implement the same constrained efficient allocations with fewer policy instruments.

This finding is also useful to relate our optimal tax formulas to the existing literature. For example, there are a number of papers in which constrained efficiency is achieved by taxing financing decisions only, even though both financing and investment decisions are distorted, as we will discuss in further detail in Section 5. In the following, we assume that all tax changes are performed in a wealth-neutral manner, i.e. any additional tax revenue raised from a given sector is returned to the same sector in the form of a lump-sum transfer.

Corollary 5. (Degrees of freedom in setting taxes/simplified implementation) *a) There are up to three degrees of freedom in setting taxes to implement a given constrained optimal allocation: (1) we can change the tax burden on borrowers vs. lenders in any state ω by varying $\tau_x^{i,\omega}$ and m_1^ω such that the sum $\left(\tau_x^{i,\omega} + m_1^\omega\right)$ remains unchanged for each agent $i \in I$; (2) if consumption is a corner solution ($\eta_0^i > 0$), we can change the tax burden on financing vs. investment decisions for any agent i by jointly varying $\tau_x^{i,\omega}$, τ_k^i and letting η_0^i adjust; (3) if the financial constraint is binding ($\kappa_1^b(z) > 0$ for the z'th element of the constraint function Φ_1^b), we can change the tax burden on financing vs. investment decisions for borrowers by jointly varying both τ_k^b (if $\Phi_{1k}^b(z) > 0$) and $\tau_x^{b,\omega}$ (for all ω for which $\Phi_{1x^\omega}^b(z) > 0$) and letting the shadow price $\kappa_1^b(z)$ adjust.*

b) These degrees of freedom allow us in case (1) to normalize $\tau_x^{i,\omega} = 0$ for one of the agents $\forall\omega \in \Omega$, in cases (2) and (3) to normalize either $\tau_k^i = 0$ or $\tau_x^{i,\omega} = 0$ for one $\omega \in \Omega$ (as long as the respective constraints are sufficiently binding).

The first degree of freedom, or indeterminacy, captures that agents only care about the after-tax price of financial securities when they trade – a parallel change in the tax rates on borrowers and lenders moves the pre-tax market prices m_1^ω but does not affect the after-tax prices $\left(\tau_x^{i,\omega} + m_1^\omega\right)$ faced by each sector. It also leaves the total wedge between borrowers and lenders $\left(\tau_x^{b,\omega} - \tau_x^{\ell,\omega}\right)$ unchanged. Assuming that the tax change is performed in a wealth-neutral manner, the resulting allocation is unaffected. This indeterminacy allows a financial regulator to impose taxes or regulation on the financing decisions of one sector (e.g. lenders) and leave the financing behavior of the other sector (e.g. borrowers) unregulated.

The second and third degrees of freedom, or indeterminacies of implementation, arise when either date 0 consumption is a corner solution (i.e. the non-negativity constraint on c_0^i is binding, $\eta_0^i > 0$) or when the date 0 financial constraint is binding (i.e. at least one element $\kappa_1^b(z) > 0$). In both cases, agents effectively face a single decision margin between borrowing and investing. Both $\tau_x^{i,\omega}$ and τ_k^i target that single decision margin and can substitute for each other. In particular, it is sufficient for a regulator to regulate only the financing of sector i and leave investment decisions unregulated. We provide an example after the proof in the appendix. Naturally, all three of the described degrees of freedom/strategies for simplifying implementation can be combined.

An important application of the corollary is that it describes situations in which private agents face corner solutions and a constrained efficient equilibrium can be implemented with zero taxes, even though the tax formulas of Proposition 1 imply non-zero tax rates. For example, in an economy in which both the financing and investment decisions of borrowers are fully determined by binding constraints and lenders do not engage in investment, the three indeterminacies together imply that we can set $\tau_x^{b,\omega} = \tau_x^{\ell,\omega} = \tau_k^b = 0$ to implement a constrained efficient allocation. Zero taxes signify that equilibrium is constrained efficient – this reflects that a planner cannot improve on the decentralized allocation if there are no free decision margins. Another example in which borrowing decisions are corner solutions and are therefore constrained efficient is given below in Application 3.

4 Applications

We present four specific applications that allow us to zero in on the efficiency results of Proposition 1 and illustrate how the sufficient statistics that underlie the sign of pecuniary externalities may easily flip sign, as described in Proposition 2 and Corollary 1.

Application 1 describes a setting in which there are fire sales but the economy is constrained efficient since risk markets are complete and financial constraints do not depend on prices. Applications 2 and 3 provide two distinct examples in which the sign of distributive externalities depends on the primitives of the model. In Application 2, we describe a setting in which borrowers switch from being net buyers to being net sellers of capital when their endowment crosses a threshold, which changes the sign of sufficient statistic *D2* and therefore the sign of the inefficiency. In Application 3, we describe a setting in which borrowers may be either constrained in their borrowing or in their saving, depending on their initial endowment. As their endowment crosses the relevant thresholds, the difference in the MRS of borrowers and lenders and, by implication, the sufficient statistic *D1* and the associated distributive externalities change sign. Application 4 provides an example of collateral externalities and overborrowing.

We illustrate our results graphically with a single figure for each application. The parameter values used to draw the figures are described in the appendix.

4.1 Efficient Fire Sales

Environment We build on the baseline model from Section 2, but use a specific formulation for investment and production technologies and financial constraints.

Regarding technologies, we assume that only borrowers can invest in capital at date 0. Formally, $h^{\ell}(k) = \infty$ for any positive level of investment. We also assume that borrower have

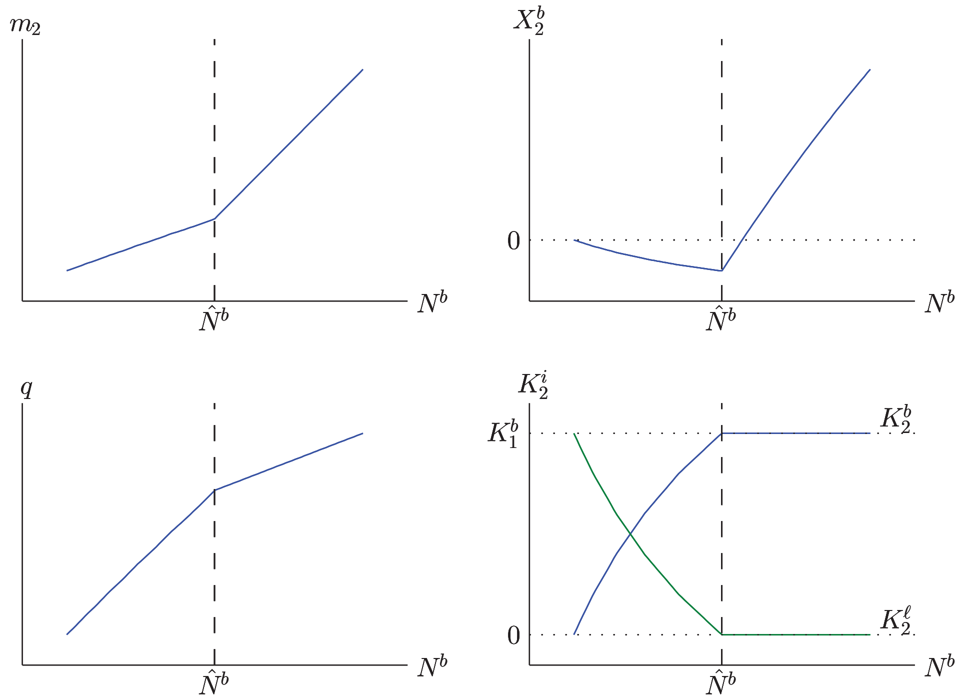

Figure 1: Date 1 Equilibrium

a superior use for capital. Formally, the production technology of borrowers is linear $F_t^{b,\omega}(k) = A_t^\omega k$, whereas that of lenders is equally productive for the first marginal unit $F_t^{\ell,\omega\prime}(0) = A_t^\omega$ but exhibits strictly decreasing returns $F_t^{\ell,\omega\prime\prime}(k) < 0$. Date 1 productivity A_1^ω is a random variable, with negative realizations representing reinvestment requirements, and date 2 productivity is given by a constant $A_2 > 0$.

Regarding financial constraints, we assume that borrowers face complete financial markets at date 0, but that they can only pledge at date 1 to repay at most a fraction ϕ of their date 2 production. Formally, borrowers' date 0 constraint is given by the degenerate function $\Phi_1^b\left(x_1^b, k_1^b\right) := 0$ while the constraint they face at date 1 is

$$\Phi_2^{b,\omega}\left(x_2^{b,\omega}, k_2^{b,\omega}\right) := x_2^{b,\omega} - \phi F_2^{b,\omega}\left(k_2^{b,\omega}\right)\ \forall\omega$$

Equilibrium and efficiency It is simplest to illustrate our results graphically. Figure 1 shows date 1 equilibrium prices m_2^ω and q^ω as well as saving $X_2^{b,\omega}$ and capital holdings $\left(K_2^{b,\omega}, K_2^{\ell,\omega}\right)$ as a function of borrowers' net worth $N^{b,\omega}$ for given K_1^b and $N^{\ell,\omega}$. We describe the general conditions under which the date 1 equilibrium of the economy is well-defined and unique as well as the specific parameters used in the figure in the appendix.

Figure 1 captures both optimal smoothing, when the financial constraint is slack, and fire sales, when the constraint is binding. For any pair $\left(K_1^b, N^{\ell,\omega}\right)$, we can define a threshold $\hat{N}^{b,\omega}$ such that, for $N^{b,\omega} \geq \hat{N}^{b,\omega}$, borrowers keep all capital and save a fraction of any additional net worth to smooth consumption between dates 1 and 2. Bond prices and capital prices increase (interest rates decrease) in parallel with borrowers' net worth, reflecting the greater abundance of wealth at date 1. If borrowers' net worth falls below the threshold, $N^{b,\omega} < \hat{N}^b$, the borrowing constraint binds, and borrowers sell some of their capital, which forces them to reduce their borrowing. The price functions m_2^ω and q^ω experience a kink at the threshold because the rate at which borrowers exchange date 1 and date 2 consumption goods with lenders becomes more disadvantageous when fire sales are involved.

At date 0, agents make investment and financing decisions optimally by solving problem (21). If these decisions lead to $N^{b,\omega} \geq \hat{N}^{b,\omega}$ for all states ω, then the financial constraint is always slack and the allocation is first-best. Otherwise, the financial constraint binds and fire sales occur in some states. Independently of whether fire sales occur at date 1 or not, we find the following result:

Application 1. (Efficient fire sales). *The decentralized equilibrium in the described economy is constrained efficient.*

The general lesson of this application is that when agents face complete financial markets between dates 0 and 1, the welfare effects of distributive externalities cancel in the decentralized equilibrium. Formally, choosing Pareto weights $\theta^i = \frac{1}{u^{i\prime}\left(c_0^i\right)}$, it is the case that

$$\sum_i \theta^i \lambda_1^{i,\omega} \mathcal{D}_{N^j}^{i,\omega} = 0 \; \forall \omega, j \tag{33}$$

and similar for the distributive effects of capital holdings.

Equation (33) is stronger than our earlier observation in equation (20) that distributive externalities are zero-sum, $\sum_i \mathcal{D}_{N^j}^{i,\omega} = 0$, at every date and state. It instead shows that distributive externalities evaluated at the planner's welfare weights net out to zero. Because complete financial markets allow agents to equalize their marginal valuations of wealth (MRS) across all states of nature there is no scope for the planner to increase efficiency by using distributive effects.

4.2 Distributive Externalities and Direction of Capital Trade

Environment In this application, we assume that both agents have linear utility $U^i = c_0^i + c_1^i + c_2^i$, with $c_t^i \geq 0$. Lenders have large endowments of the consumption good at each date while borrowers have no endowment. Both agents have access to an identical investment technology at date 0, given by $h(k) = \frac{\alpha k^2}{2}$. Borrowers' production function is linear $F_t^b(k) = A_t^\omega k$, while that of lenders is the same $F_1^\ell(k) = A_1^\omega k$ at date 1, but takes the value $F_2^\ell(k) = A_2 \log(1+k)$ at date 2. A binary shock $\omega \in \Omega = \{L, H\}$ that affects solely productivity A_1^ω is realized at date 1. In the first-best, borrowers and lenders invest such that $h'\left(k^i\right) = \alpha k^i = \mathbb{E}\left[A_1^\omega\right] + A_2$ or $k_1^i = \frac{\mathbb{E}\left[A_1^\omega\right] + A_2}{\alpha}$ for $i = b, \ell$. Because borrowers have the more efficient production technology at date 2, they hold all the capital, so $k_2^{b,\omega} = 2k_1^i$ and $k_2^{\ell,\omega} = 0$.

Regarding financial constraints, we assume that only uncontingent bonds are available for trade at date 0 and that no borrowing or lending is possible at date 1, capturing an extreme disruption of financial markets. Formally, borrowers' date 0 constraint is given by $\Phi_1^b\left(x_1^b, k_1^b\right) := \left(x_1^{b,L} - x_1^{b,H}\right) = 0$, while their date 1 constraint is given by $\Phi_2^b\left(x_2^b, k_2^b\right) := x_2^b = 0$.

Equilibrium and efficiency The date 1 demand for capital assets by lenders is given by their optimality condition

$$q^\omega = F_2^{\ell\prime}\left(k_2^{\ell,\omega}\right) = \frac{A_2}{1 + k_2^{\ell,\omega}} \tag{34}$$

Given borrowers' net worth $n^{b,\omega} = A_1^\omega k_1^b + x_1^b$, their date 1 budget constraint, financial constraint, and non-negativity constraint on consumption can be combined into

$$n^{b,\omega} + q^\omega\left(k_1^b - k_2^{b,\omega}\right) \geq 0 \tag{35}$$

Therefore, borrowers' date 1 value function is

$$V^{b,\omega}\left(n^{b,\omega}, k_1^b; N^\omega, K_1\right) = \max_{k_2^{b,\omega}} A_2 k_2^{b,\omega} + \lambda_1^{b,\omega}\left[n^{b,\omega} + q^\omega\left(k_1^b - k_2^{b,\omega}\right)\right] \tag{36}$$

We define the threshold $\hat{N}^b := A_2 K_1^\ell$ and observe that the date 1 financial constraint is slack if $N^{b,\omega} \geq \hat{N}^b$ and binding otherwise. If the constraint is slack, borrowers buy up all capital in the economy $K_2^{b,\omega} = K_1^b + K_1^\ell$ at a price $q^\omega = V_k^{b,\omega} = A_2$, which is independent of sectoral net worth so all distributive effects are zero, $\mathcal{D}_{N^j}^{i,\omega} = \mathcal{D}_{K^j}^{i,\omega} = 0$. The marginal value of date 1 borrower wealth in that case is $\lambda_1^{b,\omega} = V_n^{b,\omega} = 1$.

If $N^{b,\omega} < \hat{N}^b$, borrowers' financial constraint binds, causing them to reduce their capital holdings below the efficient level, $K_2^{b,\omega} < K_1^b + K_1^\ell$. Combining lenders' demand (34) with borrowers' constraint (35) yields the equilibrium capital holdings and price of capital

$$K_2^{b,\omega} = K_1^b + \frac{N^{b,\omega}\left(1 + K_1^\ell\right)}{N^{b,\omega} + A_2} \qquad \text{and} \qquad q^\omega = \frac{N^{b,\omega} + A_2}{1 + K_1^\ell}$$

where a well-defined equilibrium with strictly positive capital holdings exists when $N^{b,\omega} \geq N^{b,\min} = -\frac{A_2 K_1^b}{1 + K_1^b + K_1^\ell}$. The equilibrium price of capital depends exclusively on two aggregate state variables $q^\omega\left(N^{b,\omega}, K_1^\ell\right)$ and satisfies $\frac{\partial q^\omega}{\partial N^{b,\omega}} > 0$ and $\frac{\partial q^\omega}{\partial K_1^\ell} < 0$. The resulting distributive effects, as defined in Lemma 1, are

$$\mathcal{D}_{N^b}^{b,\omega} = -\frac{\partial q^\omega}{\partial N^{b,\omega}} \Delta K_2^{b,\omega} = -\frac{N^{b,\omega}}{N^{b,\omega} + A_2}$$

$$\mathcal{D}_{K^b}^{b,\omega} = A_1^\omega \mathcal{D}_{N^b}^{b,\omega} = -\frac{A_1^\omega N^{b,\omega}}{N^{b,\omega} + A_2}$$

$$\mathcal{D}_{K^\ell}^{b,\omega} = -\frac{\partial q^\omega}{\partial K_1^\ell} \Delta K_2^{b,\omega} = \frac{N^{b,\omega}}{1 + K_1^\ell}$$

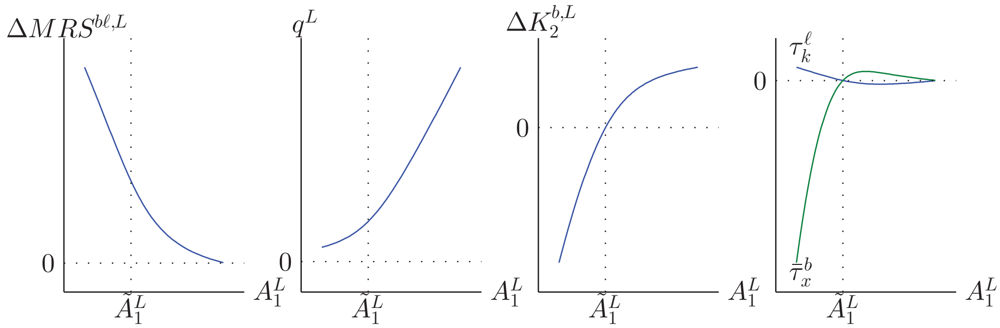

Figure 2: Components of Optimal Taxes $\bar{\tau}_x^b$, τ_k^ℓ in Application 2

By contrast, lenders' net worth does not have distributive effects, $\mathcal{D}_{N^\ell}^{i,\omega} = 0$. Depending on the value of $N^{b,\omega}$, the signs of the distributive externality terms can take all possible values, as we explain next.

When the financial constraint binds, we distinguish three regions for the date 1 equilibrium. First, if $N^{b,\omega} \in \left[N^{b,\min}, 0\right)$, borrowers fire-sell assets, so $K_2^{b,\omega} < K_1^b$. In this region, higher borrowers' net worth and lower lenders' capital raise the price at which constrained borrowers are forced to sell their capital, which distributes resources towards borrowers. Therefore, the distributive effects of additional borrowers' net worth or capital are positive $\mathcal{D}_{N^b}^{b,\omega}, \mathcal{D}_{K^b}^{b,\omega} > 0$, while those of additional lenders' capital are negative $\mathcal{D}_{K^\ell}^{b,\omega} < 0$. Second, if $N^{b,\omega} = 0$, borrowers neither buy nor sell additional capital so $K_2^{b,\omega} = K_1^b$, even though the marginal products satisfy $F_2^{\ell\prime}\left(K_2^{\ell,\omega}\right) < F_2^{b\prime}\left(K_2^{b,\omega}\right)$. In this knife-edge case there is no trade in capital goods and all distributive effects are zero, $\mathcal{D}_{N^b}^{b,\omega} = \mathcal{D}_{K^b}^{b,\omega} = \mathcal{D}_{K^\ell}^{b,\omega} = 0$. Third, if $N^{b,\omega} \in \left(0, \hat{N}^b\right)$, borrowers purchase additional capital but less than the optimal amount so $K_1^b < K_2^{b,\omega} < K_1^b + K_1^\ell$. In this region, the distributive effects of extra borrowers' net worth or capital are negative $\mathcal{D}_{N^b}^{b,\omega}, \mathcal{D}_{K^b}^{b,\omega} < 0$ but those of extra lenders' capital are positive $\mathcal{D}_{K^\ell}^{b,\omega} > 0$.

We provide a full characterization of the date 0 equilibrium in the appendix. The interesting case to focus on – for which we provide sufficient conditions in the appendix – is when the financial constraint is slack in the high state and binding in the low state. In that case, we summarize our findings on how distributive effects determine efficiency as follows.

Application 2. (Changing sign of capital trade). *There is a threshold value $\tilde{A}_1^L$ such that a) if $A_1^L < \tilde{A}_1^L$, the economy exhibits overborrowing and underinvestment by borrowers as well as overinvestment by lenders, b) if $A_1^L = \tilde{A}_1^L$, the economy is constrained efficient, and c) if $A_1^L > \tilde{A}_1^L$, the economy exhibits underborrowing and overinvestment by borrowers as well as underinvestment by lenders.*

Intuitively, case a) represents a traditional scenario in which constrained borrowers fire-sell assets: a marginal reduction in borrowing, or additional date 1 capital income, pushes up the price that borrowers fetch for their fire-sales of capital. Since borrowers are constrained in the low state and were unable to arrange contingent insurance towards that state, the positive distributive effects, captured by $\mathcal{D}^{b,\omega}_{N^b}, \mathcal{D}^{b,\omega}_{K^b} > 0$, improve insurance. A marginal reduction in investment by lenders also raises the fire-sale price and has similar effects, captured by $\mathcal{D}^{b,\omega}_{K^\ell} < 0$. Case b) corresponds to the knife-edge in which borrowers are constrained but neither buy nor sell assets; as a result, marginal changes to either borrowing or investment do not affect welfare. In case c), borrowers are also constrained but have positive date 1 net worth, which allows them to purchase some assets from lenders. Higher borrowing (lower investment by borrowers) leads to lower borrowers' net worth and pushes down the price of capital, making it cheaper for constrained borrowers to buy assets, captured by $\mathcal{D}^{b,\omega}_{N^b}, \mathcal{D}^{b,\omega}_{K^b} < 0$. Similarly, more investment by lenders reduces the price of capital, implying $\mathcal{D}^{b,\omega}_{K^\ell} > 0$.

Figure 2 depicts the key variables that drive the sign and magnitude of the distributive externalities as we vary A^L_1. The first panel shows the difference in MRS between agents in the low state of nature, which is decreasing in A^L_1 and always weakly positive in state L since borrowers are weakly constrained. The second panel depicts the price of capital in the low state q^L, which is increasing in the shock realization A^L_1 as higher income implies smaller fire sales. The third panel shows borrowers' net trading position of capital $\Delta K^{b,L}_2$ in the low state, which changes sign at the productivity threshold $\tilde{A}^L_1$. The fourth panel illustrates the resulting tax rates: for $A^L_1 < \tilde{A}^L_1$, it is optimal to tax borrowing (i.e. subsidize saving so $\bar{\tau}^b_x < 0$) and tax investment by lenders (so $\tau^\ell_k > 0$) and vice versa for $A^L_1 > \tilde{A}^L_1$. We denote by $\bar{\tau}^b_x$ the optimal tax on uncontingent bond holdings since the date 0 financial constraint implies that only uncontingent bonds can be traded.

The general lesson of this application is that constrained borrowers may be either buyers or sellers of capital and financial assets. They may switch from one to the other in response to small changes in fundamentals. As a result, sufficient statistic *D2* can take on either sign. It is straightforward to enrich this application allowing for multiple states of nature in which borrowers are constrained, some of which satisfy $A^\omega_1 < \tilde{A}^L_1$ and others $A^\omega_1 > \tilde{A}^L_1$. In that case, there will be overborrowing towards some states and underborrowing towards others.

4.3 Distributive Externalities and Sign of ΔMRS

Environment In this application, we assume that borrowers have concave utility $U^b = \sum_{t=0}^{2} \log\left(c^b_t\right)$, and that lenders have linear utility $U^\ell = c^\ell_0 + c^\ell_1 + c^\ell_2$, with $c^\ell_t \geq 0$. This is a perfect foresight economy with no uncertainty. Lenders have large endowments of the consumption good at each date. Borrowers have non-negative endowments $e^b_t \geq 0$ that we vary as our main experiment; we set $e^b_2 = 0$ to obtain closed-form solutions. Only borrowers invest at date 0. Formally, borrowers' investment technology is given by $h^b(k) = \frac{\alpha k^2}{2}$, while lenders' technology corresponds to $h^\ell(k) = \infty$, for $k > 0$. Both borrowers and lenders are unproductive at date 1, but produce according to $F^b_2(k) = Ak$ and $F^\ell_2(k) = A\log(k)$ at date 2, where $A > \alpha$. In the first-best, borrowers' date 0 investment corresponds to $k_1^b = \frac{A}{\alpha}$. At date 1, borrowers hold $k_2^\ell = 1$ and $k_2^b = k_1^b - 1$. Borrowers' consumption is equalized across all dates.

Regarding financial constraints, we assume that borrowers are unable to save at date 0 and face a fixed limit ϕ on the amount they can borrow at date 0. As in Application 2, no borrowing or lending is possible at date 1. Formally, borrowers' date 0 financial constraint is given by the vector inequality

$$\Phi_1^b\left(x_1^b, k_1^b\right) := \begin{pmatrix} x_1^b + \phi \\ -x_1^b \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

while borrowers' date 1 financial constraint is given by $\Phi_2^b\left(x_2^b, k_2^b\right) := x_2^b = 0$.

Equilibrium and efficiency The date 1 demand for capital assets by lenders is given by their optimality condition

$$q = F_2^{\ell\prime}\left(K_2^\ell\right) = \frac{A}{K_2^\ell} \tag{37}$$

Given borrowers' date 1 net worth $n^b = e_1^b + x_1^b$, their date 1 budget constraint and financial constraint can be combined to $c_1^b = n^b + q\left(k_1^b - k_2^b\right)$. Therefore, borrowers' date 1 value function is

$$V^b\left(n^b, k_1^b; N, K_1\right) = \max_{k_2^b} u^b\left(n^b + q\left(k_1^b - k_2^b\right)\right) + u^b\left(Ak_2^b\right), \tag{38}$$

Their optimality condition for capital holdings is given by

$$q = \frac{n^b}{2k_2^b - k_1^b} \tag{39}$$

Combining lenders' demand (37) for capital with borrowers' supply (39) yields the equilibrium capital holdings and price of capital

$$K_2^b = \frac{A + N^b}{2A + N^b} K_1^b \qquad \text{and} \qquad q = \frac{2A + N^b}{K_1^b}$$

where a well-defined equilibrium with non-negative capital holdings exist when $N^b \geq -A$. Consequently, the date 1 equilibrium level of consumption for borrowers corresponds to $C_1^b = N^b + q\left(K_1^b - K_2^b\right) = N^b + A$. Borrowers receive a constant amount A for their sales of capital, independently of the number of units of sold.

The equilibrium asset price depends exclusively on two aggregate state variables $q\left(N^b, K_1^b\right)$ and satisfies $\frac{\partial q}{\partial N^b} < 0$ and $\frac{\partial q}{\partial K_1^b} > 0$. The distributive effects in this economy, as defined in Lemma 1, are given by

$$\mathcal{D}_{N^b}^b = -\frac{\partial q}{\partial N^b} \Delta K_2^b > 0$$
$$\mathcal{D}_{K^b}^b = -\frac{\partial q}{\partial K_1^b} \Delta K_2^b < 0$$

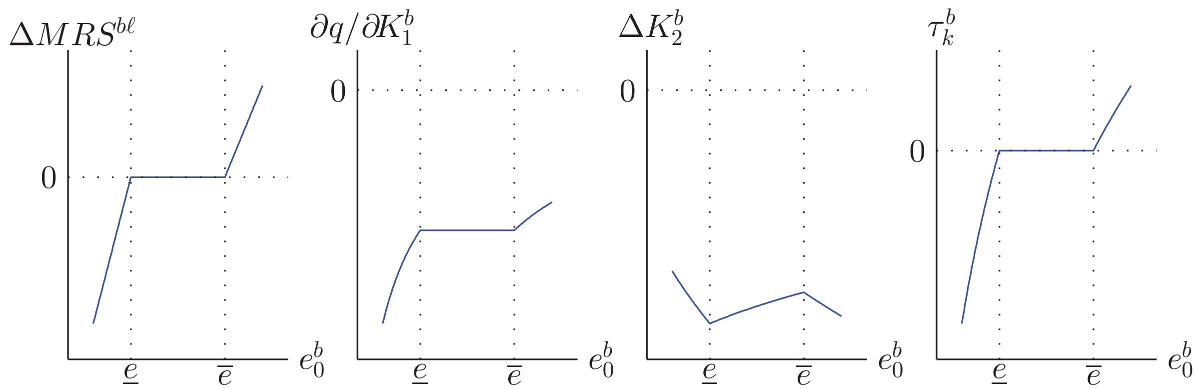

Figure 3: Components of Optimal Tax τ_k^b in Application 3

where we use the fact that borrowers are always net sellers of capital, that is, $\Delta K_2^b = -\frac{A}{2A+N^b} < 0$ is negative. As in our previous application, lenders' net worth does not have distributive effects, so $\mathcal{D}^i_{N^\ell} = 0$. Since lenders do not hold capital, characterizing $\mathcal{D}^i_{K^\ell}$ is irrelevant.

We provide a full characterization of the date 0 equilibrium in the appendix. We show that the optimal unconstrained financial decision by borrowers' is given by

$$X_1^* = \frac{e_0^b - e_1^b - 2A}{2.5}$$

Therefore, depending on the difference between date 0 and date 1 borrowers' endowments, $e_0^b - e_1^b$, the date 0 equilibrium can take three different forms. First, when $X_1^* \in [-\phi, 0]$, the financial constraint is slack, so $X_1^b = X_1^*$. Second, when $X_1^* < -\phi$, then borrowers hit their borrowing limit, so $X_1^b = -\phi$. Third, when $X_1^* > 0$, then borrowers hit their saving limit, so $X_1^b = 0$.

Because borrowers are constrained in their choice of x_1^b whenever the financial constraint is binding, Corollary 5 implies that the same financing decision x_1^b can be implemented by setting $\tau^b = 0$. Borrowing is therefore always constrained efficient in this economy, and it is sufficient to focus on the optimal corrective policy for investment τ_k^b. Out of the three sufficient statistics from Proposition 2, our application restricts the signs of *D2* and *D3*, since borrowers are always net sellers and asset prices increase with the level of K_1^b. We summarize our findings on how the form of the date 0 equilibrium affects *D1* and efficiency of capital investment as follows.

Application 3. (Changing sign of ΔMRS) *There are two thresholds $\underline{e}$ and $\overline{e}$ for the value of the difference in borrowers' endowments $e_0^b - e_1^b$ such that a) if $e_0^b - e_1^b < \underline{e}$, then $\Delta MRS^{b\ell} > 0$, and the economy exhibits under-investment, b) if $\underline{e} \leq e_0^b - e_1^b \leq \overline{e}$, the economy is constrained efficient, and c) if $e_0^b - e_1^b > \overline{e}$, then $\Delta MRS^{b\ell} < 0$, and the economy exhibits over-investment. Borrowing decisions are always constrained efficient.*

In case a), when $e_0^b - e_1^b < \underline{e}$, borrowers hit their date 0 borrowing limit, and $\Delta MRS^{b\ell} < 0$ – they value wealth at date 1 compared to date 0 relatively less than lenders. This makes it desirable to allocate more wealth to lenders at date 1 and more to borrowers at date 0. The borrowing constraint prevents the financial market from performing this operation. However, the planner increases capital investment by borrowers, which reduces the price at which borrowers sell capital at date 1 and effectively redistributes resources to lenders at date 1. Moreover, the planner provides a lump-sum transfer from lenders to borrowers at date 0. Taken together, these two interventions constitute a second-best way for the planner to emulate the effects of borrowing at date 0. Conversely, in case c), borrowers would like to save more wealth than the saving limit allows for and $\Delta MRS^{b\ell} > 0$. A planner can circumvent this constraint through reduced capital investment by borrowers, which increases the date 1 price of capital and redistributes resources to borrowers, combined with a lump-sum transfer from borrowers to lenders at date 0.

Figure 3 depicts the key variables that drive the sign and magnitude of the distributive externalities as we vary the endowment parameter e_0^b for $e_1^b = 0$. The first panel illustrates that $\Delta MRS^{b\ell}$ is weakly increasing over the entire domain of e_0^b – it is negative for $e_0^b < \underline{e}$ and positive for $e_0^b > \bar{e}$. The second and third panels depict the response of the asset price to capital $\frac{\partial q}{\partial K_1^b}$ and borrowers' net trading position of capital ΔK_2^b, both of which are always negative. The fourth panel combines the three sufficient statistics from the first three panels and reports the resulting tax rate τ_K^b, which is negative for low e_0^b (implying a subsidy) and positive for high e_0^b.

The general lesson of this application is that the relative intertemporal valuation of resources by agents – captured by $\Delta MRS^{b\ell}$ – can take on either sign, and borrowers may switch from having a higher valuation of resources to having a lower valuation of resources in response to small changes in fundamentals. In the described application, distributive effects arise in a single state of nature since the example is set in perfect foresight. A planner can only improve efficiency if she also has access to date 0 lump-sum transfers so that the combination of distributive effect plus transfer substitute for incomplete date 0 financial markets – otherwise, the distributive effects would constitute mere movements along a constrained Pareto frontier. For an elaboration of this point, see Davila (2014). In more general stochastic settings, incomplete risk markets may give rise to differences in marginal rates of substitution for different agents across multiple states of nature, and the planner can employ distributive effects to improve efficiency by improving risk-sharing between agents, even when lump-sum transfers are ruled out.

4.4 Collateral Externalities

Environment Our last application illustrates the workings of collateral externalities. We assume that borrowers have quasilinear utility $U^b = \log c_0^b + \log c_1^b + c_2^b$, and that lenders have linear utility $U^\ell = c_0^\ell + c_1^\ell + c_2^\ell$, with $c_t^i \geq 0$. This is a perfect foresight economy with no uncertainty. Lenders have large endowments of the consumption good at each date while borrowers have endowments $e_0^b, e_1^b \in [0,1]$ and $e_2^b = 0$. Only borrowers invest at date 0.

Formally, borrowers' investment technology is given by $h^b(k) = \frac{\alpha k^2}{2}$, while lenders' technology corresponds to $h^\ell(k) = \infty$, for $k > 0$. Borrowers have a linear production technology while lenders have no use for capital. Formally, $F_t^b(k) = Ak$ and $F_t^\ell(k) = 0$ for $t = 1, 2$ where we assume $A \leq \frac{\alpha}{2}$. Our assumptions on utility and the technology of lenders imply that all distributive effects are zero $\mathcal{D}^i_{N^j} = \mathcal{D}^i_{K^b_1} = 0$, focusing our analysis exclusively on collateral externalities. In the first-best, as shown in the appendix, $C_0^b = C_1^b = 1$, $K_t^{b*} = \frac{2A}{\alpha}$ and $q = A$.

Regarding financial constraints, we assume that borrowers are unconstrained at date 0. We also assume that borrowers can only borrow up to a fraction ϕ of the value of their asset holdings at date 1, where $\phi \in \left(0, \frac{1}{A}\right)$ to ensure equilibrium is well-defined. Formally, $\Phi_1^b := 0$ and

$$\Phi_2^b\left(x_2^b, k_2^b; q\right) := x_2^b + \phi q k_2^b \geq 0 \tag{40}$$

Equilibrium and efficiency Since capital always remains in the hands of borrowers, the price of capital is pinned down by borrowers' optimality condition for capital holdings

$$q = \frac{A}{u^{b\prime}\left(c_1^b\right) + \phi \kappa_2^b} = \frac{A c_1^b}{1 - \phi + \phi c_1^b} \tag{41}$$

In an unconstrained equilibrium, borrowers consume $C_1^b = 1$ and save $X_2^b = N^b - 1$ at date 1, resulting in an asset price of $q = F_2^{b\prime}(\cdot) = A$. This allocation is feasible as long as $X_2^b \geq -\phi q K_2^b$ or, equivalently, $N^b \geq 1 - \phi A K_1^b$.

Otherwise, if $N^b \in \left(0, 1 - \phi A K_1^b\right)$, then borrowing is constrained to $X_2^b = -\phi q K_1^b$. The date 1 budget constraint then implies that

$$C_1^b = N^b + \phi q K_1^b = N^b + \phi K_1^b \frac{A C_1^b}{1 - \phi + \phi C_1^b} \tag{42}$$

This equation defines a unique level of consumption C_1^b since the right-hand side is increasing in C_1^b at a slope of less than one given our assumption $\phi A < 1$ and given $K_1^b \leq K_1^{b*}$ in equilibrium. Consumption is increasing in both N^b and K_1^b. Substituting this consumption level into (41), the price of capital is a function $q\left(N^b, K_1^b\right)$ that is also increasing in both N^b and K_1^b. Equilibrium is independent of N^ℓ, and $K_t^\ell = 0$ at all times.

Since the collateral constraint (40) depends on the price of capital, changes in the sector-wide state variables $\left(N^b, K_1^b\right)$ have collateral effects, as defined in Lemma 1

$$\begin{aligned} \mathcal{C}^b_{N^b} &= \phi K_1^b \frac{\partial q}{\partial N^b} > 0 \\ \mathcal{C}^b_{K^b} &= \phi K_1^b \left(A \frac{\partial q}{\partial N^b} + \frac{\partial q}{\partial K_1^b}\right) > 0 \end{aligned}$$

We provide a full characterization of the date 0 equilibrium in the appendix. Given the date 1 allocation, equilibrium is determined by the private Euler equation $C_0^b = C_1^b$ and the following

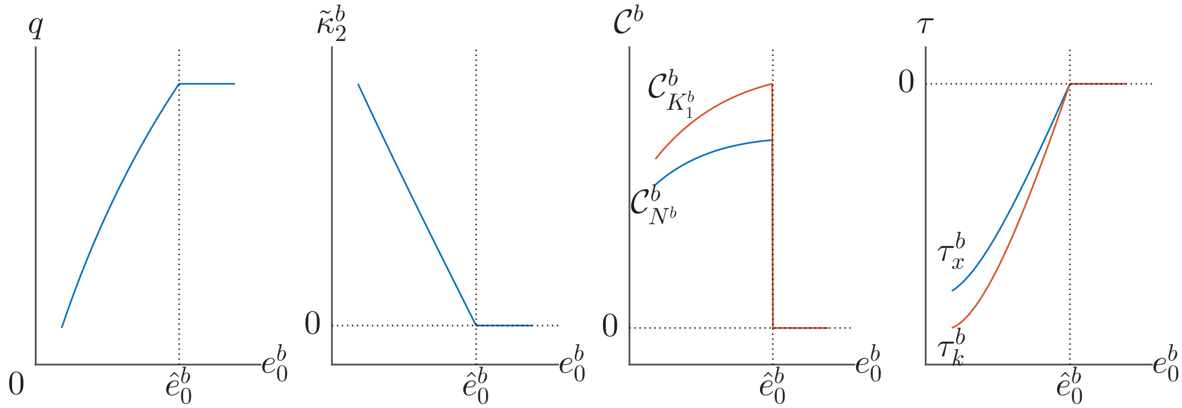

Figure 4: Components of Optimal Taxes τ_x^b, τ_k^b in Application 4

optimality condition for capital investment, which equates the marginal cost of investment to its private marginal benefit

$$h^{b\prime}\left(k_1^b\right) = 2k_1^b = A + q \tag{43}$$

The following statement is a simple application of Corollary 1 to the described economy.

Application 4. (Collateral externalities). *If the parameters of the economy satisfy $e_0^b + e_1^b + \frac{2\phi A^2}{\alpha} < 2$, the financial constraint binds, and there is overborrowing and underinvestment by borrowers. In the converse case, the financial constraint is slack and the equilibrium allocation is first-best efficient.*

The inequality captures whether borrowers' endowments plus the collateralizable part of their first-best level of date 2 production $\left(\phi A K_1^{b*} = \frac{2\phi A^2}{\alpha}\right)$ is sufficient to cover their first-best consumption $C_t^{b*} = 1$ at $t = 0, 1$. When the financial constraint binds, the overborrowing result simply reflects our general findings for collateral externalities in Corollary 1 – intuitively, a marginal unit of net worth in the hands of borrowers at date 1 increases the price of capital, which generates positive collateral effects and allows constrained borrowers to borrow more at date 1. Similarly, additional capital investment generates positive collateral effects – both by generating more date 1 net worth and by providing more collateral for the borrower sector.

Figure 4 depicts the key variables that drive the magnitude of the collateral externalities as we vary the endowment parameter e_0^b for $e_1^b = 0$. (The parameter values used are described in the appendix.) The first panel shows the price of capital q, which is increasing in the endowment when the constraint binds but constant otherwise. The second panel depicts the shadow price on the date 1 collateral constraint, which declines in e_0^b and reaches zero when the constraint becomes slack at $\hat{e}_0^b$. The third panel shows the collateral effects $\mathcal{C}_{N^b}^b$ and $\mathcal{C}_{K^b}^b$ as defined in Lemma 1 which are always positive when the constraint binds and turn zero when it becomes slack. The fourth panel reports the tax wedges on bonds and capital investment, which represent the product of the variables shown in panels 2 and 3. The two are declining in absolute value and go to zero as e_0^b reaches $\hat{e}_0^b$ and the constraint ceases to bind.

This application illustrates the result of Corollary 1 that collateral externalities lead to overborrowing. Since asset prices are increasing in sectoral net worth under the natural Condition 1 and since higher asset prices relax collateral constraints, it is desirable for a planner to induce private agents to save more when faced with collateral externalities, i.e. there is overborrowing in the decentralized equilibrium. Furthermore, since capital generates additional date 1 net worth in this application, there is also underinvestment.

5 Related Literature

Our paper is part of a strand of literature that analyzes pecuniary externalities in settings with fire sales and financial amplification. Hart (1975) and Stiglitz (1982) were the first to identify pecuniary externalities that give rise to inefficiency when financial markets are incomplete in the sense that the set of available assets is exogenously limited. Geanakoplos and Polemarchakis (1986) generalized their results and showed that competitive equilibrium is generically constrained inefficient in such a setting. This inefficiency is the basis of what we call distributive externalities: changes in allocations influence market prices in a way that improves risk-sharing or intertemporal smoothing. Greenwald and Stiglitz (1986) showed that pecuniary externalities also arise when private agents are subject to other constraints that depend on market prices such as selection or incentive constraints. This inefficiency is closely related to what we call collateral externalities: changes in allocations influence markets prices in a way that relaxes binding price-dependent constraints.[13]

The analysis of financial amplification and fire sale effects as positive phenomena dates back to at least Fisher (1933) and includes seminal contributions by Bernanke and Gertler (1990), Shleifer and Vishny (1992), and Kiyotaki and Moore (1997). See Krishnamurthy (2010), Shleifer and Vishny (2011), and Brunnermeier and Oehmke (2013) for recent surveys. The main observation in these works is that changes in the net worth of borrowers may be amplified by price changes that further reduce their net worth, corresponding to distributive amplification effects (Shleifer and Vishny, 1992), or that tighten binding constraints on borrowers, corresponding to collateral amplification effects (cf. Corollary 4). The remainder of this section relates the existing literature on financial amplification and pecuniary externalities to our framework with the objective of highlighting the precise mechanisms that lead to inefficiency through the lens of our sufficient statistics results.

[13] Prescott and Townsend (1984) show how such pecuniary externalities can be overcome when agents can directly contract consumption levels and no anonymous re-trading is allowed. In a similar vein, Kilenthong and Townsend (2014) propose to create segregated security exchanges with entry fees/subsidies for the exclusive right and obligation to trade in a particular exchange, representing a Coasian solution to restore efficiency in pecuniary externality problems.

Gromb and Vayanos (2002) analyze financially constrained agents who arbitrage between segmented markets in an environment with incomplete risk markets (because investors other than arbitrageurs can only trade uncontingent bonds) and price-dependent collateral constraints (which limit the borrowing of arbitrageurs). This gives rise to both distributive and collateral externalities in our terminology. As a result, borrowing and risk-taking by arbitrageurs can be either excessive or, when distributive externalities dominate, insufficient.

Caballero and Krishnamurthy (2003) analyze risk-taking by emerging market agents who insure against aggregate shocks but face uninsurable idiosyncratic shocks. Retrading among domestic agents after the idiosyncratic shock is realized generates distributive externalities that lead to excessive aggregate risk-taking and overinvestment. Lorenzoni (2008) considers fire sales in an economy in which borrowers are financially constrained and limited commitment by lenders constrains insurance provision, similar to Application 3. This generates distributive externalities that give rise to overinvestment and excessive borrowing against the good state of nature. In both papers, efficiency can be restored by solely taxing borrowing, even though both borrowing and investment decisions are distorted. This is an application of our Corollary 5.

Distributive externalities also arise in the literature that studies liquidity provision and the coexistence of financial intermediaries and markets since the possibility of spot retrading in financial markets, together with market incompleteness, reduces risk sharing opportunities. See for example Jacklin (1987), Bhattacharya and Gale (1987), Allen and Gale (2004) and Farhi, Golosov and Tsyvinski (2009). Kehoe and Levine (1993) endogenize financial constraints from limited commitment and exclusion from intertemporal markets. Rampini and Viswanathan (2010) endogenously derive state-dependent collateral constraints through limited commitment without exclusion. Other recent papers that consider distributive externalities include Davila et al. (2012), Hart and Zingales (2015) and He and Kondor (2016).

Jeanne and Korinek (2010*a*,*b*) and Bianchi and Mendoza (2012) consider borrowers in a dynamic setting who are subject to a price-dependent collateral constraint that introduces collateral externalities which lead to excessive borrowing, as implied by our Corollary 1. They assume that lenders cannot hold capital and have linear utility so no distributive effects arise, similar to Application 4. Other recent papers with collateral externalities include Gersbach and Rochet (2012), Stein (2012), Benigno et al. (2013) and Kilenthong and Townsend (2014).

There is also a complementary strand of literature that focuses on aggregate demand externalities in the presence of nominal price rigidities. See e.g. Farhi and Werning (2016), Korinek and Simsek (2016) and Schmitt-Grohé and Uribe (2016). These externalities are qualitatively different from the pecuniary externalities that we study. However, Korinek and Simsek (2016) illustrate that the two types of externalities interact and may mutually reinforce each other. Farhi and Werning (2016) provide an integrated welfare analysis of both aggregate demand and pecuniary externalities.

6 Conclusion

This paper develops a general framework to characterize the pecuniary externalities that arise in economies with financially constrained agents and fire sales. We identify two distinct externalities, distributive and collateral externalities, and show that each of the two types can be quantified as a function of three intuitive sufficient statistics. Distributive externalities occur when a planner can employ changes in prices to allocate wealth to agents who are underinsured because of incomplete insurance markets. Collateral externalities occur when a planner can employ changes in prices to relax binding collateral constraints that depend on market prices. Incomplete insurance markets and financial constraints that depend on prices are pervasive in both theory and practice, suggesting that our findings have broad applicability.

Although this paper focuses mostly on fire sales, our general framework and our dichotomy between distributive and collateral externalities can be applied to any economy with welfare-relevant pecuniary externalities to provide simple and intuitive formulas for optimal policy intervention. Examples include externalities arising from wage changes, terms of trade fluctuations, or exchange rate movements. The general principle is that a planner wants to tax actions that redistribute wealth away from imperfectly insured agents or that tighten binding financial constraints in proportion to how much the action improves insurance or smoothing.

However, even when pecuniary externalities are present, our paper shows that determining their sign is not straightforward. Two of the three sufficient statistics that determine the sign of distributive pecuniary externalities can flip sign in response to changes in fundamental parameters, as we carefully illustrate in our applications, making it impossible to sign pecuniary externalities in general. Furthermore, even though there is a close relationship between fire sales/financial amplification and the distributive and collateral effects that underlie pecuniary externalities, we show that amplification is neither necessary nor sufficient to obtain inefficient pecuniary externalities.

Our results also provide direct guidance to financial regulators who work on designing so-called "macroprudential" financial regulations with the goal of reducing fire sales and financial amplification to enhance financial stability. Our paper shows that fire sales are only constrained inefficient if they occur between agents who are imperfectly insured or if financial constraints depend on the prices of fire-sale assets. Regulators should thus focus their attention on improving insurance of financially constrained agents (e.g. by promoting contingent forms of financing) and on stabilizing the value of assets used as collateral (e.g. by adjusting margins in response to asset price movements). This also suggests that it is dangerous if policymakers impose regulatory constraints that explicitly depend on market prices. We hope that our findings help to discipline the ongoing debate on the design of our financial architecture and macroprudential regulation.

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Appendix

A Proofs and Derivations from Sections 2 and 3

Proof of Lemma 1 Equation (14) follows from taking the partial derivatives of the value function (11), exploiting investors' privately optimal decisions, and applying the definitions of $\mathcal{D}^{i,\omega}_{Nj}$, $\mathcal{D}^{i,\omega}_{Kj}$, $\mathcal{C}^{i,\omega}_{Nj}$, and $\mathcal{C}^{i,\omega}_{Kj}$ in equations (16) to (19). Equation (15) uses the definition of net worth from equation (10), which implies that the total derivative of the value function with respect to K^i_1 has two components. The first one captures the marginal increase in net worth that leads to the same redistribution $\mathcal{D}^i_{Nj}$ effect as any other change in sector-wide net worth. The second one results from the direct effect of K^i_1 on market prices.

Proof of Proposition 1 The Lagrangian corresponding to problem (24) can be written as

$$\mathcal{L} = \sum_i \theta^i \left\{ u^i\left(C^i_0\right) + \eta^i_0 C^i_0 + \beta \mathbb{E}_0 \left[V^{i,\omega}\left(N^{i,\omega}, K^i_1; N^\omega, K_1\right)\right] + \kappa^i_1 \Phi^i_1\left(X^i_1, K^i_1\right)\right\}$$
$$- \nu_0 \sum_i \left[C^i_0 + h^i\left(K^i_1\right) - e^i_0\right] - \sum_\omega \nu^\omega_1 \sum_i X^{i,\omega}_1$$

where $N^{i,\omega} = e^{i,\omega}_1 + F^{i,\omega}_1\left(K^i_1\right) + X^{i,\omega}_1$ and $N^\omega = \left(N^{b,\omega}, N^{\ell,\omega}\right)$. The set of necessary conditions for the optimality of the constrained planner's problem are

$$\frac{d\mathcal{L}}{dC^i_0} = \theta^i \left[u^{i\prime}\left(C^i_0\right) + \eta^i_0\right] - \nu_0 = 0,\ \forall i$$
$$\frac{d\mathcal{L}}{dX^{i,\omega}_1} = -\nu^\omega_1 + \theta^i \beta V^{i,\omega}_n + \theta^i \kappa^i_1 \Phi^i_{1x^\omega} + \beta \sum_j \theta^j V^{j,\omega}_{N^i} = 0,\ \forall i,\omega$$
$$\frac{d\mathcal{L}}{dK^i_1} = -\nu_0 h^{i\prime}\left(K^i_1\right) + \theta^i \beta \mathbb{E}_0\left[V^{i,\omega}_n F^{i,\omega\prime}_1\left(K^i_1\right) + V^{i,\omega}_k\right] + \theta^i \kappa^i_1 \Phi^i_{1k} + \beta \sum_j \theta^j \mathbb{E}_0 \left[V^{j,\omega}_{N^i} F^{i,\omega\prime}_1\left(K^i_1\right) + V^{j,\omega}_{K^i}\right] = 0,\ \forall i$$

a) Using the definition of $\lambda^{i,\omega}_t$, the first optimality condition implies $\nu_0 = \theta^i \lambda^i_0$, $\forall i$. This implies that $\frac{\theta^b}{\theta^\ell} = \frac{\lambda^\ell_0}{\lambda^b_0}$ as stated in the proposition. Equation (25) follows from dividing the second optimality condition by θ^i and using $\theta^i = \frac{\nu_0}{\lambda^i_0}$ from the first optimality condition as well as the envelope condition $V^{i,\omega}_n = \lambda^{i,\omega}_1$. Equation (26) follows from substituting ν_0 from the first optimality condition into the third optimality condition and using the envelope condition $V^{i,\omega}_k = \mathbb{E}_0\left[\lambda^{i,\omega}_1 q^\omega\right]$.

b) Substituting the tax rates from the proposition into the optimality conditions of private agents with taxes[14]

$$\left(m^\omega_1 + \tau^{i,\omega}_x\right)\lambda^i_0 = \beta \lambda^{i,\omega}_1 + \kappa^i_1 \Phi^i_{1x^\omega}, \quad \forall i,\omega$$
$$\left[h^{i\prime}\left(k^i_1\right) + \tau^i_k\right]\lambda^i_0 = \mathbb{E}_0\left[\beta\lambda^{i,\omega}_1\left(F^{i,\omega\prime}_1\left(k^{i,\omega}_1\right) + q^\omega\right)\right] + \kappa^i_1 \Phi^i_{1k}, \quad \forall i$$

[14] In the particular case of collateral constraints, there exists a relation between the shadow value of a binding collateral constraint and the MRS across dates/states of a given agent, but this is not a robust feature of models with binding price-dependent constraints. Importantly, because collateral effects are not zero-sum on the aggregate, collateral externalities cannot in general be expressed as a *difference* of MRS between agents.

replicates the planner's optimality conditions (25) and (26). The lump sum transfers ensure that the budget constraints of private agents are met for the desired allocation. In conjunction with the government budget constraint, this guarantees that the date 0 resource constraint holds. The resulting allocation is thus constrained efficient.

Proof of Proposition 2 It follows from equations (29) and (30) by substituting the definition of distributive and collateral effects from equations (16) to (19).

Proof of Corollary 1 For collateral externalities, sufficient statistic *C1*, corresponding to the shadow value of the financial constraint $\tilde{\kappa}_2^{b,\omega}$, is by definition non-negative; *C2*, corresponding to the price derivative of the constraint $\frac{\partial \Phi_2^{b,\omega}}{\partial q^\omega}$, is by construction non-negative; *C3* for the effect of financial net worth on the price of capital $\frac{\partial q^\omega}{N^{b,\omega}}$ is non-negative under Condition 1. Therefore the product of the three is non-negative under the condition. Sufficient statistic *C3* for the effect of sector-wide capital holdings on the price of capital $\frac{\partial q^\omega}{\partial K_1^b}$ cannot be signed in general – in Application 3 we provide an example of $\frac{\partial q}{\partial K_1^b} < 0$ whereas in Application 4 we provide an example of $\frac{\partial q}{\partial K_1^b} > 0$. The collateral externalities of capital can thus take on either sign.

For distributive externalities, sufficient statistic *D1*, corresponding to $\Delta MRS^{ij,\omega}$, can take positive or negative values, as we illustrate in Application 3; *D2*, corresponding to $\Delta K_2^{i,\omega}$ and X_2^i, can take positive or negative values, as we illustrate in Application 2. Even though *D3* pins down $\frac{\partial q^\omega}{N^{b,\omega}}$, the product of the three sufficient statistics can take on either sign; thus "anything goes" for distributive externalities.

Proof of Corollary 2 The optimal corrective taxes $\tau_x^{i,\omega}$ and τ_k^i on financing and investment decisions in the general case satisfy

$$\tau_k^i = \mathbb{E}_0 \left[F_1^{i,\omega\prime}\left(\cdot\right) \tau_x^{i,\omega} \right] + \Xi_i, \ \forall i \tag{A.1}$$

where we define Ξ_i as the direct effect of the level of capital on collateral and distributive externalities for a given level of net worth

$$\Xi_i := \mathbb{E}_0 \left[\Delta MRS^{ij,\omega} \left(\frac{\partial q^\omega}{\partial K_1^i} \Delta K_2^{i,\omega} + \frac{\partial m_2^\omega}{\partial K_1^i} X_2^{i,\omega} \right) - \tilde{\kappa}_2^{i,\omega} \frac{\partial \Phi_2^{b,\omega}}{\partial q^\omega} \frac{\partial q^\omega}{\partial K^b} \right]$$

Equation (A.1) follows by combining equations (18) and (19) with (29) and (30). Equation (31) in the text is a special case of equation (A.1) when $\Xi_i = 0$.

Proof of Corollary 3 It follows from no-arbitrage considerations.

Proof of Corollary 4 It follows from Propositions 1 and 2.

Proof of Corollary 5 Explicitly substituting λ_0^i, we can write the planner's two main optimality conditions as

$$\left[m_1^\omega + \tau_x^{i,\omega} \right] \left[u^{i\prime}\left(c_0^i\right) + \eta_0^i \right] = \beta \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i \quad \forall i, \omega \tag{A.2}$$

$$\left[h^{i\prime}\left(k_1^i\right) + \tau_k^i \right] \left[u^{i\prime}\left(c_0^i\right) + \eta_0^i \right] = \mathbb{E}_0 \left[\beta u^{i\prime}\left(c_1^{b,\omega}\right) \left(F_1^{i,\omega\prime}\left(k_1^{i,\omega}\right) + q^\omega \right) \right] + \kappa_1^i \Phi_{1k}^i \ \forall i \tag{A.3}$$

For a given real allocations in the economy, these two optimality conditions as well as all the constraints on the planner's problem continue to be satisfied if

1. we vary m_1^ω and $\tau_x^{i,\omega}$ in a given state ω such that the sum $\left[m_1^\omega + \tau_x^{i,\omega}\right]$ remains unchanged for all $i \in I$,

2. if $\eta_0^i > 0$ and we jointly vary $\tau_x^{i,\omega}$, τ_k^i and η_0^i such that both $\left[m_1^\omega + \tau_x^{i,\omega}\right]\left[u^{i\prime}\left(c_0^i\right) + \eta_0^i\right]$, $\forall\omega$ and $\left[h^{i\prime}\left(k_1^i\right) + \tau_k^i\right]\left[u^{i\prime}\left(c_0^i\right) + \eta_0^i\right]$ remain unchanged for a given agent i, or

3. if $\kappa_1^b(z) > 0$ for the z'th element of the vector κ_1^b and we jointly vary $\kappa_1^i(z)$ and both τ_k^b (if $\Phi_{1k}^b(z) > 0$) and $\tau_x^{b,\omega}$ (for all ω for which $\Phi_{1x^\omega}^b(z) > 0$) such that both $\kappa_1^b(z)\,\Phi_{1x^\omega}^b(z) - \left(m_1^\omega + \tau_x^{b,\omega}\right)\lambda_0^b$, $\forall\omega$ and $\kappa_1^b(z)\,\Phi_{1k}^b(z) - \left[h^{b\prime}\left(k_1^b\right) + \tau_k^b\right]\lambda_0^b$ remain unchanged.

Each of the three described variations of implementation consist of changes in tax rates, market prices and shadow prices such that a given optimal real allocation continues to satisfy the planner's optimality conditions, proving the indeterminacy part of the corollary. It follows a fortiori that the planner can employ the three described degrees of freedom to

1. set $\tau_x^{i,\omega} = 0$ for one of the types of agents $i \in \{b, \ell\}$ in each $\omega \in \Omega$,

2. if $\eta_0^i > 0$, set either $\tau_k^i = 0$ (as long as $\eta_0^i + \frac{\tau_k^i}{h^{i\prime}\left(k_1^i\right)}\left[u^{i\prime}\left(c_0^i\right) + \eta_0^i\right] \geq 0$ at the original implementation) or $\tau_x^{i,\omega} = 0$ for one $\omega \in \Omega$ (as long as $\eta_0^i + \frac{\tau_x^{i,\omega}}{m_1^\omega}\left[u^{i\prime}\left(c_0^i\right) + \eta_0^i\right] \geq 0$ at the original implementation),

3. if $\kappa_1^b(z) > 0$, set either $\tau_k^b = 0$ if $\Phi_{1k}^b(z) > 0$ (as long as $\tau_k^b \leq \frac{\kappa_1^i(z)}{\lambda_0^b}\Phi_{1k}^b(z)$ at the original allocation) or set $\tau_x^{b,\omega} = 0$ in any state ω for which $\Phi_{1x^\omega}^b(z) > 0$ (as long as $\tau_x^{b,\omega} \leq \frac{\kappa_1^i(z)}{\lambda_0^b}\Phi_{1x^\omega}^b(z)$ at the original allocation),

while adjusting the remaining policy instruments and prices to satisfy the planner's optimality conditions (A.2) and (A.3). The conditions in parentheses ensure that the respective binding constraints in points 2. and 3. continue to be binding so shadow prices do not become negative, i.e. that $c_0^i = 0$ continues to be satisfied in point 2. and that the z'th element of the constraint continues to bind, i.e. $\Phi_1^b(z) = 0$ in point 3. Moreover, in point 3., only those decision variables that are affected by the binding constraint are included in the indeterminacy, i.e. only the tax on financing decisions in those states of nature ω for which $\Phi_{1x^\omega}^b(z) > 0$ is indeterminate, and the tax on investment is only indeterminate if $\Phi_{1k}^b(z) > 0$.

A simple example of the third indeterminacy is when the borrowing constraint $\Phi_1^b\left(x_1^{b,\omega}, k_1^b\right) := \left(x_1^{b,\omega} + \phi\right) \geq 0$ is strictly binding in a given state ω. In that case, a marginal change in the tax rate $\tau_x^{b,\omega}$ changes the shadow price κ_1^b but does not have any real effects. If the constraint is tight enough, the tax can be set to $\tau_x^{b,\omega} = 0$ in that state without any effect on the real allocation, and the financing decision for state ω can be considered constrained efficient. In this example, the constraint does not depend on capital, $\Phi_{1k}^b = 0$, so the optimal tax rate on capital is unchanged when we vary $\tau_x^{b,\omega}$. We utilize this example in Application 3.

B Proofs and Derivations of Applications in Section 4

B.1 Analytic Details on Application 1

Characterizing uniqueness at date 1 Assume that borrowers and lenders enter date 1 with state variables $(n, k_1; N, K_1)$ where $n = N$ and $k_1 = K_1$. Let us denote by z the amount of resources that borrowers receive from lenders at date 1 – both via borrowing and fire sales – and by $\rho(z)$ the resulting payoff received by lenders at date 2 – both from the repayment of borrowing and from production using fire-sold assets

$$z = m_2 x_2^\ell + q k_2^\ell \tag{A.4}$$
$$\rho(z) = x_2^\ell + F^\ell\left(k_2^\ell\right) \tag{A.5}$$

Given this notation, we can describe $\rho(z)$ as the "supply of funds" of lenders. Let us also denote by $\gamma(z)$ the total resources given up by borrowers at date 2 – both as a repayment and because of production foregone – and by $\delta(z)$ the deadweight loss of fire sales that results from the lower productivity of lenders

$$\begin{aligned}\gamma(z) &= x_2^\ell + A_2 k_2^\ell \\ \delta(z) &= \gamma(z) - \rho(z) = A_2 k_2^\ell - F^\ell\left(k_2^\ell\right)\end{aligned}$$

The market prices m_2 and q are pinned down by the optimality conditions of lenders

$$m_2 = \frac{u'\left(e_2^\ell + \rho(z)\right)}{u'\left(n^\ell - z\right)} \tag{A.6}$$
$$q = m_2 F^{\ell\prime}\left(k_2^\ell\right) \tag{A.7}$$

Our goal is to formally describe the conditions under which the "supply of funds" curve of lenders is well-behaved so as to lead to a unique equilibrium. Given our assumptions on production technology, there are two distinct regions for the date 1 equilibrium.

Unconstrained equilibrium When the financial constraint is slack, then equations (A.4) to (A.7) together with $k_2^\ell = 0$ (or, equivalently, $z = m_2 x_2^\ell$) define a system of 5 equations in 6 variables $\left(z, \rho, m_2, q, x_2^\ell, k_2^\ell\right)$. We reduce the system to a single implicit equation,

$$z u'\left(n^\ell - z\right) = \rho u'\left(e_2^\ell + \rho\right)$$

which defines a supply of funds curve $\rho = \rho(z)$ by lenders that satisfies

$$\frac{\partial \rho}{\partial z} = \frac{u'\left(c_1^\ell\right) - z u''\left(c_1^\ell\right)}{u'\left(c_2^\ell\right) + \rho u''\left(c_2^\ell\right)} > 0$$

and is non-degenerate as long as

$$\eta_{c_2} \frac{\rho}{c_2^\ell} < 1 \tag{A.8}$$

where $\eta_{c_2} := -c_2^\ell \frac{u''\left(c_2^\ell\right)}{u'\left(c_2^\ell\right)}$. If condition (A.8) is satisfied, there exists a unique equilibrium.

Constrained equilibrium When the constraint is binding, then equations (A.4) to (A.7) together with $x_2^\ell = m_2 \phi A_2 \left(k_1^b - k_2^\ell\right)$ defines a system of 5 equations in 6 variables $\left(z, \rho, m_2, q, x_2^\ell, k_2^\ell\right)$. Combining them, we find

$$zu'\left(n^\ell - z\right) = u'\left(e_2^\ell + \phi A_2\left(k_1^b - k_2^\ell\right) + F^\ell\left(k_2^\ell\right)\right)\left[\phi A_2\left(k_1^b - k_2^\ell\right) + k_2^\ell F^{\ell\prime}\left(k_2^\ell\right)\right]$$

This equation implicitly defines a "demand for fire sales" curve $k_2^\ell = k(z)$ that satisfies

$$\frac{\partial k}{\partial z} = \frac{u'\left(c_1^\ell\right) - zu''\left(c_1^\ell\right)}{u'\left(c_2^\ell\right)\left[F^{\ell\prime}\left(k_2^\ell\right) - \phi A_2 + k_2^\ell F^{\ell\prime\prime}\left(k_2^\ell\right)\right] + u''\left(c_2^\ell\right)\left[F^{\ell\prime}\left(k_2^\ell\right) - \phi A_2\right]\rho} > 0$$

and is non-degenerate as long as the denominator is positive, which requires two conditions,

$$\eta_{qk_2} + \frac{\phi A_2}{F^{\ell\prime}\left(k_2^\ell\right)} < 1 \tag{A.9}$$

$$\eta_{c_2}\frac{\rho}{c_2^\ell}\left[1 - \frac{\eta_{qk_2}}{1 - \frac{\phi A_2}{F^{\ell\prime}\left(k_2^\ell\right)} - \eta_{qk_2}}\right] < 1 \tag{A.10}$$

where $\eta_{qk_2} := -\frac{k_2^\ell F^{\ell\prime\prime}\left(k_2^\ell\right)}{F^{\ell\prime}\left(k_2^\ell\right)}$ so that the first square bracket is positive, and where the term in square brackets derives from $\frac{F^{\ell\prime}\left(k_2^\ell\right) - \phi A_2}{F^{\ell\prime}\left(k_2^\ell\right) - \phi A_2 + k_2^\ell F^{\ell\prime\prime}\left(k_2^\ell\right)} = 1 - \frac{\frac{k_2^\ell F^{\ell\prime\prime}\left(k_2^\ell\right)}{F^{\ell\prime}\left(k_2^\ell\right)}}{1 - \frac{\phi A_2}{F^{\ell\prime}\left(k_2^\ell\right)} - \eta_{qk_2}}$.

Under the three conditions (A.8), (A.9), and (A.10), the function $k(z)$ is well-behaved and captures how much capital borrowers have to give up to raise z units of funds at date 1 when the constraint is binding. The function is defined up to an upper limit $z^{\max}$ that is given by $k\left(z^{\max}\right) = k_1^b$.

It is now straightforward to express the supply of funds curve $\rho(z)$ of lenders as well as the deadweight loss curve $\delta(z)$ and the resources given up by borrowers $\gamma(z)$ by substituting $x_2^\ell = \phi A_2\left(k_1^b - k_2^\ell\right)$ and $k_2^\ell = k(z)$ into the three expressions to obtain

$$\begin{aligned}
\rho(z) &= x_2^\ell + F^\ell\left(k_2^\ell\right) = \phi A_2\left(k_1^b - k(z)\right) + F^\ell\left(k(z)\right) \\
\delta(z) &= A_2 k_2^\ell - F^\ell\left(k_2^\ell\right) = A_2 k(z) - F^\ell\left(k(z)\right) \\
\gamma(z) &= x_2^\ell + A_2 k_2^\ell = \phi A_2 k_1^b + (1-\phi) A_2 k(z)
\end{aligned}$$

Given that $k(z)$ is non-degenerate and strictly increasing, all three functions are well-defined and strictly increasing. Note that condition (A.9) implies that $F^{\ell\prime}\left(k_2^\ell\right) > \phi A_2$.

In combination, under the stated assumptions the two regions, constrained and unconstrained, define a supply of funds curve $\rho(z)$ of lenders that is strictly increasing and continuous over the interval $z \in \left[0, z^{\max}\right]$, which is sufficient to guarantee existence and equilibrium uniqueness at date 1. In conjunction with equations (A.6) and (A.7), this also ensures that Condition 1 is satisfied.

Date 0 Given the lack of financial frictions, it is trivial to close the model at date 0 after defining conditions for uniqueness of the date 1 equilibrium. Hence, for brevity, we omit the details of the date 0 characterization.

B.2 Analytic Details on Application 2

Date 1 lenders' problem The date 1 value function of lenders is

$$V^{\ell,\omega}\left(n^{\ell,\omega},k_1^{\ell};N^{\omega},K_1\right)=\max_{k_2^{\ell,\omega}} n^{\ell,\omega}+q^{\omega}\left(k_1^{\ell}-k_2^{\ell,\omega}\right)+F_2^{\ell}\left(k_2^{\ell,\omega}\right)$$

Their optimality condition is given by equation (34). The partial derivatives of $V^{\ell,\omega}$ internalized by private agents are given by

$$\begin{aligned}V_n^{\ell,\omega}&=\lambda_1^{\ell}=1\\ V_k^{\ell,\omega}&=q^{\omega}\end{aligned}$$

Date 1 borrowers' problem and equilibrium The value function of borrowers is given by equation (36). If $N^{b,\omega}\geq\hat{N}$, then equilibrium is unconstrained so $q^{\omega}=A_2$ and the value function of borrowers can be simplified to

$$V^{b,\omega}\left(n^{b,\omega},k_1^{b};N^{\omega},K_1\right)\;=\;n^{b,\omega}+A_2k_1^{b}$$

where the partial derivatives of $V^{b,\omega}$internalized by private agents are given by $V_n^{b,\omega}=\lambda_1^{b,\omega}=1$, $V_k^{b,\omega}=A_2$, and $V_{N^j}^{b,\omega}=V_{K_1^j}^{b,\omega}=0$.

If $N^{b,\omega}\leq\hat{N}$, then equilibrium is constrained, and the value function of borrowers is

$$V^{b,\omega}\left(n^{b,\omega},k_1^{b};N^{b,\omega},K_1^{\ell}\right)\;=\;A_2\left(k_1^{b}+\frac{n^{b,\omega}}{q\left(N^{b,\omega},K_1^{\ell}\right)}\right)$$

The partial derivatives of this value function that are internalized by private agents are

$$\begin{aligned}V_n^{b,\omega}&=\frac{A_2}{q\left(N^{b,\omega},K_1^{\ell}\right)}=\frac{1+K_1^{\ell}}{1+\frac{N^{b,\omega}}{A_2}}\\ V_k^{b,\omega}&=A_2\end{aligned}$$

The uninternalized distributive effects are

$$\begin{aligned}\mathcal{D}_{N^b}^{b,\omega}&=-\frac{\partial q^{\omega}}{\partial N^{b,\omega}}\left(K_2^{b,\omega}-K_1^{b}\right)=-\frac{1}{1+K_1^{\ell}}\frac{N^{b,\omega}\left(1+K_1^{\ell}\right)}{N^{b,\omega}+A_2}=-\frac{N^{b,\omega}}{N^{b,\omega}+A_2}\\ \mathcal{D}_{K^{\ell}}^{b,\omega}&=-\frac{\partial q^{\omega}}{\partial K_1^{\ell}}\left(K_2^{b,\omega}-K_1^{b}\right)=\frac{N^{b,\omega}+A_2}{\left(1+K_1^{\ell}\right)^2}\frac{N^{b,\omega}\left(1+K_1^{\ell}\right)}{N^{b,\omega}+A_2}=\frac{N^{b,\omega}}{1+K_1^{\ell}}\end{aligned}$$

It can easily be verified that $V_{N^b}^{b,\omega}=\lambda_1^{b}\mathcal{D}_{N^b}^{b,\omega}$ and $V_{K^{\ell}}^{b,\omega}=\lambda_1^{b}\mathcal{D}_{K^{\ell}}^{b,\omega}$. As described in the text, whenever $N^{b,\omega}\geq N^{b,\min}$, there exists a unique equilibrium at date 1.

Date 0 equilibrium For both sets of agents, optimal date 0 capital investment is determined by

$$\max_{k_1^i}\mathbb{E}\left[V^{i,\omega}\left(A_1^{\omega}k_1^i-h\left(k_1^i\right),k_1^i;N^b,K_1^{\ell}\right)\right]$$

with optimality condition

$$\mathbb{E}\left[\lambda_1^{i,\omega}\left[A_1^\omega - h'\left(k_1^i\right)\right] + V_k^{i,\omega}\right] = 0$$

Under the assumption that the financial constraint is slack in the high state and binding in the low state. For lenders, the optimality condition is then

$$h'\left(K_1^\ell\right) = \alpha K_1^\ell = \mathbb{E}\left[A_1^\omega + q^\omega\right] = \mathbb{E}\left[A_1^\omega\right] + (1-\pi)A_2 + \pi q^L\left(N^{b,L}, K_1^\ell\right) \tag{A.11}$$

The left-hand side of this expression is increasing in K_1^ℓ and the right-hand side is decreasing in K_1^ℓ (since $\frac{\partial q}{\partial K_1^\ell} < 0$), pinning down a unique solution for K_1^ℓ. For borrowers, the date 0 optimality condition is

$$(1-\pi)\left[A_1^H - \alpha K_1^b + A_2\right] = \pi\left\{\frac{A_2}{q^L\left(N^{b,L}, K_1^\ell\right)}\left[\alpha K_1^b - A_1^L\right] - A_2\right\} \tag{A.12}$$

The left-hand side captures the marginal gains from additional capital investment in the high state of nature, which, in equilibrium, must be positive and must offset the marginal losses from additional investment in the low state of nature, captured by the right-hand side. The optimum thus needs to satisfy $\alpha K_1^b \in \left[A_1^L, A_1^H + A_2\right]$. Since the left-hand side is decreasing in K_1^b and the right-hand side is increasing in K_1^b, the optimality condition pins down a unique solution within this interval.

The condition under which the constraint is indeed slack in a given state is $N^{b,\omega} = A_1^\omega K_1^b - \frac{\alpha\left(K_1^b\right)^2}{2} \geq A_2 K_1^\ell = \hat{N}^b$ or

$$A_1^\omega \geq \frac{A_2 K_1^\ell}{K_1^b} + \frac{\alpha K_1^b}{2}$$

Intuitively, the return on capital in the high state needs to cover both the additional capital purchases from lenders (per unit of borrower capital) and the average cost of investment. We assume that this inequality is satisfied in the high state but violated in the low state of nature.[15]

Proof of Application 2 Given all other parameters, we define the threshold $\hat{A}_1^L$ such that $N^{b,L} = 0$ or, equivalently,

$$\hat{A}_1^L = \frac{\alpha K_1^b}{2}$$

This condition together with the optimality conditions (A.11) and (A.12) pins down a unique level of $\hat{A}_1^L$. By construction, $N^{b,L} = 0$ for $A_1^L = \hat{A}_1^L$, proving case 2 of the proposition. Standard stability conditions imply that $\frac{dN^{b,L}}{dA_1^L} > 0$. As a result, $A_1^L < \hat{A}_1^L$ implies $N^{b,L} < \hat{N}^b$ and $A_1^L > \hat{A}_1^L$ leads to $N^{b,L} > \hat{N}^b$, proving the other two cases. In the limit case $\pi \to 0$, the threshold is easy to characterize, $\hat{A}_1^L = \frac{\alpha K_1^b}{2} = \frac{A_1^H + A_2}{2}$.

[15] In the limit case $\pi \to 0$, optimal investment implies $\alpha K_1^i = A_1^H + A_2$ for both agents and the condition simplifies to $A_1^\omega - \frac{A_1^H + A_2}{2} \gtreqless A_2$ or

$$A_1^H \geq A_2 \quad \text{and} \quad A_1^L < \frac{A_1^H + 3A_2}{2}$$

B.3 Analytic Details on Application 3

Date 1 lenders' problem The date 1 value function of lenders is

$$V^\ell\left(n^\ell, k_1^\ell; N, K_1\right) = \max_{k_2^\ell} n^\ell + q\left(k_1^\ell - k_2^\ell\right) + F_2^\ell\left(k_2^\ell\right)$$

Their optimality condition is given by equation (37). The partial derivatives of V^ℓ internalized by private agents are given by

$$V_n^\ell = \lambda_1^\ell = 1 \quad \text{and} \quad V_k^\ell = q$$

Date 1 borrowers' problem and equilibrium The definition of date 1 borrowers' net worth $n^b = e_1^b + x_1^b$ together with their date 1 budget constraint and financial constraint implies $c_1^b = n^b + q\left(k_1^b - k_2^b\right)$. The value function of borrowers is thus given by equation (38) and the partial derivatives of this value function that are internalized by private agents are

$$V_n^b = u'\left(c_1^b\right) \quad \text{and} \quad V_k^b = qu'\left(c_1^b\right)$$

The optimal capital holdings of borrowers are given by

$$q = \frac{u'\left(c_2^b\right)}{u'\left(c_1^b\right)}A = \frac{n^b + q\left(k_1^b - k_2^b\right)}{Ak_2^b}A = \frac{n^b}{2k_2^b - k_1^b}$$

which corresponds to equation (39). Combining lenders' demand (37) and borrowers' supply (39) for capital implies

$$K_2^b = \frac{A + N^b}{2A + N^b}K_1^b \qquad \text{and} \qquad q = \frac{2A + N^b}{K_1^b}$$

Date 0 equilibrium At date 0, borrowers solve

$$\max_{x_1^b, k_1^b} u\left(e_0^b - h^b\left(k_1^b\right) - x_1^b\right) + V^b\left(e_1^b + x_1^b, k_1^b; K, N\right) + \underline{\lambda}\left(x_1^b + \phi\right) - \overline{\lambda}x_1^b$$

Their optimality conditions are given by

$$u'\left(c_0^b\right) = u'\left(c_1^b\right) + \underline{\lambda} - \overline{\lambda}$$

$$h^{b\prime}\left(k_1^b\right)u'\left(c_0^b\right) = qu'\left(c_1^b\right)$$

Substituting for the date 0 and 1 budget constraints, the second condition can be re-written as $h^{b\prime}\left(K_1^b\right)u'\left(e_0^b - h\left(K_1^b\right) - X_1^b\right) = qu'\left(e_1^b + X_1^b + A\right)$ and, using the equilibrium q from above, solved for

$$K_1^b = \sqrt{\frac{\left(2A + e_1^b + X_1^b\right)\left(e_0^b - X_1^b\right)}{\alpha\left(1.5e_1^b + 1.5X_1^b + 2A\right)}}$$

If the date 0 financial constraints are slack, we find $\underline{\lambda} = \overline{\lambda} = 0$. The Euler equation then implies $c_0^b = c_1^b$, and the expression for capital investment simplifies to $K_1^{b*} = \sqrt{\frac{2A + e_1^b + X_1^b}{\alpha}}$. We can solve for the optimal unconstrained level of saving

$$X_1^{b*} = \frac{e_0^b - e_1^b - 2A}{2.5}$$

The two constraints on borrowing and saving are indeed slack if $X_1^{b*} \in [-\phi, 0]$. Otherwise, if the constraint on borrowing (saving) is binding, then $X_1^b = -\phi$ (or $X_1^b = 0$). The threshold values $\underline{e}$ and $\bar{e}$ for the difference in endowments $\left(e_0^b - e_1^b\right)$ at which the two constraints become binding are defined by $X_1^{b*} = -\phi$ and $X_1^{b*} = 0$, respectively.

Welfare analysis The sensitivities of the equilibrium price of capital $q = \frac{2A+N^b}{K_1^b}$ to $\left(K_1^b, N^b\right)$ are given by

$$\frac{\partial q}{\partial K_1^b} = -\frac{2A+N^b}{\left(K_1^b\right)^2} < 0$$
$$\frac{\partial q}{\partial N^b} = \frac{1}{K_1^b} > 0$$

and the respective distributive effects are $\mathcal{D}_{N^b}^{b,\omega} = -\frac{\partial q^\omega}{\partial N^{b,\omega}} \Delta K_2^{b,\omega} = -\frac{\Delta K_2^{b,\omega}}{K_1^b} > 0$ since $\Delta K_2^{b,\omega} < 0$ always holds. Note that $MRS^\ell = 1$ and $MRS^b = \frac{u'\left(c_1^b\right)}{u'\left(c_0^b\right)} = 1 - \frac{\underline{\lambda}-\overline{\lambda}}{u'\left(c_0^b\right)}$. Consequently,

$$\Delta MRS^{b\ell} = \frac{\overline{\lambda} - \underline{\lambda}}{u'\left(c_0^b\right)}$$

Therefore, if borrowers are borrowing-constrained at date 0, then $\underline{\lambda} > 0$ and $\overline{\lambda} = 0$, which implies that $\Delta MRS^{b\ell} < 0$ and $\tau_k^b < 0$, so there is under-investment in that case. Instead, if borrowers are saving-constrained at date 0, then $\underline{\lambda} = 0$ and $\overline{\lambda} > 0$, which implies that $\Delta MRS^{b\ell} > 0$ and $\tau_k^b > 0$, so there is over-investment in that case.

B.4 Analytic Details on Application 4

Because lenders are risk neutral and have no use for capital, they simply pin down the equilibrium value of $m_2 = 1$. We thus focus exclusively on the borrowers' problem.

Date 1 borrowers' problem and equilibrium The date 1 value function of is given by

$$V^b\left(n^b, k_1^b; N, K_1\right) = \max_{x_2^b, k_2^b} u\left(n^b - q\Delta k_2^b - x_2^b\right) + x_2^b + Ak_2^b + \kappa_2^b\left(x_2^b + \phi q k_2^b\right)$$

With optimality conditions

$$q\left(u'\left(c_1^b\right) - \phi\kappa_2^b\right) = A$$
$$u'\left(c_1^b\right) = 1 + \kappa_2^b$$

Which combined yield equation (41) in the text and defines a function $q\left(C_1^b\right)$ with $q'\left(C_1^b\right) > 0$.

In an unconstrained equilibrium, borrowers consume $C_1^b = 1$ and save $X_2^b = N^b - 1$ at date 1, resulting in a price of capital $q = F_2^{b\prime}(\cdot) = A$. This allocation is first-best and is feasible as long as $X_2^b \geq -\phi q K_2^b$ or, equivalently, $N^b \geq 1 - \phi A K_1^b$.

Otherwise, if $N^b \in \left(0, 1 - \phi A K_1^b\right)$, then borrowing is constrained to $X_2^b = -\phi q K_1^b$. The date 1 budget constraint then implies $C_1^b = N^b + \phi q K_1^b$ which leads to equation (42) in the text and implicitly defines a function $C_1^b\left(N^b, K_1^b\right)$ that satisfies $\frac{\partial C_1^b}{\partial N^b}, \frac{\partial C_1^b}{\partial N^b} > 0$. In combination with equation (41), this pins down the price of capital $q\left(N^b, K_1^b\right)$ as a function that is strictly increasing in both arguments.

The date 0 optimality conditions of individual agents are given by the standard Euler equation $u'\left(C_0^b\right) = u'\left(C_1^b\right)$ or equivalently $C_0^b = C_1^b$ and the optimality condition for capital investment (43). Combining the condition $N^b = e_1^b + F\left(K_1^b\right) + X_1^b \geq 1 - \phi A K_1^b$ for a slack financial constraint at date 1 with the date 0 budget constraint, we observe that the unconstrained (and first-best) allocation is feasible if $e_0^b + e_1^b \geq 2 - \phi A K_1^{b*} = 2 - \frac{2\phi A^2}{\alpha}$.

B.5 Distributive Externalities and Multiple Equilibria

Applications 5 and 6 illustrate that violations of Condition 1 are typically associated with backward-bending demand curves that lead to multiple and locally unstable equilibria.

Environment We modify Application 3 to introduce multiple equilibria. We now assume that borrowers have CRRA utility $U^b = \sum_{t=0}^2 u\left(c_t^b\right)$, where $u\left(\cdot\right) = \frac{c^{1-\theta}}{1-\theta}$, and that lenders have linear utility $U^\ell = c_0^\ell + c_1^\ell + c_2^\ell$, with $c_t^\ell \geq 0$. This is a perfect foresight economy with no uncertainty. Lenders have large endowments of the consumption good at each date while borrowers have non-negative endowments $e_0^b \geq 0$, $e_1^b = e_2^b = 0$. Only borrowers invest at date 0. Formally, borrowers' investment technology is given by $h^b\left(k\right) = \frac{\alpha k^2}{2}$, while lenders' technology corresponds to $h^\ell\left(k\right) = \infty$, for $k > 0$. Both borrowers and lenders are unproductive at date 1, but they produce according to $F_2^b\left(k\right) = Ak$ and $F_2^\ell\left(k\right) = A\frac{(k+\delta)^\eta}{\eta}$ at date 2, where $\eta < 1$, $A > \alpha$ and $\delta \gtreqless 0$, for $k_2^\ell > \delta$. Conceptually, this formulation introduces more curvature into the model. Agents face the same financial constraints as in Application 3.

In the first-best, borrowers' date 0 investment corresponds to $k_1^b = \frac{A}{\alpha}$. At date 1, borrowers hold $k_2^\ell = 1 - \delta$ and $k_2^b = k_1^b - 1 + \delta$. Borrowers' consumption is equalized across all dates.

Date 1 equilibrium and multiplicity The date 1 demand for capital assets by lenders is given by their optimality condition

$$q = F_2^{\ell\prime}\left(K_2^\ell\right) = A\left(K_2^\ell + \delta\right)^{\eta-1} = A\left(K_1^b - K_2^b + \delta\right)^{\eta-1} \tag{A.13}$$

Given borrowers' net worth, which corresponds to $n^b = x_1^b$, their date 1 budget constraint, and financial constraint imply $c_1^b = n^b + q\left(k_1^b - k_2^b\right)$. Therefore, borrowers' date 1 value function corresponds to

$$V^b\left(n^b, k_1^b; N, K_1\right) = \max_{k_2^b} u\left(n^b + q\left(k_1^b - k_2^b\right)\right) + u\left(Ak_2^b\right), \tag{A.14}$$

Their optimal capital holdings for borrowers are given by

$$qu'\left(n^b + q\left(k_1^b - k_2^b\right)\right) = Au'\left(Ak_2^b\right) \tag{A.15}$$

Setting $A = 1$ without loss of generality, because of the homogeneity of the problem, we show that the sensitivity of borrowers' demand for capital to prices is determined by

$$\frac{\partial k_2^b}{\partial q} = -\frac{u'\left(c_1^b\right)\left(\theta\frac{q\left(k_1^b - k_2^b\right)}{c_1^b} - 1\right)}{q^2 u''\left(c_1^b\right)\left(k_1^b - k_2^b\right) + u''\left(c_2^b\right)}$$

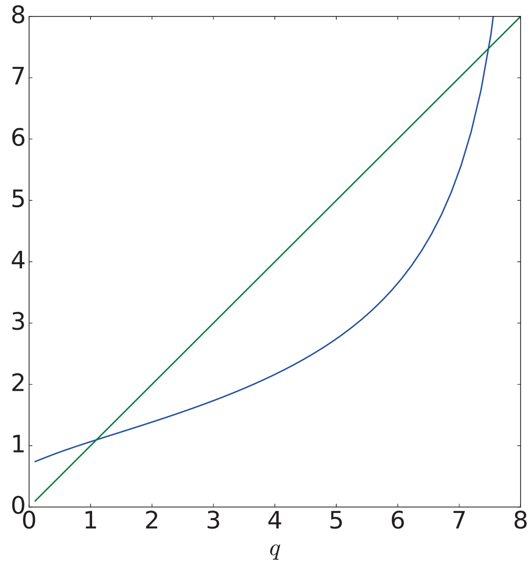

Figure A.1: Multiple solutions for (A.17)

When income effects are sufficiently strong, $\frac{\partial k_2^b}{\partial q}$ can take on positive values, implying that lower prices reduce the demand for capital of borrowers. Formally, this occurs when the curvature of the utility function is sufficiently large

$$\theta - 1 > \frac{n^b}{q\left(k_1^b - k_2^b\right)}$$

We can solve equation (A.15) for K_2^b to explicitly find an expression for the demand for capital given N^b and K_1^b

$$K_2^b = \frac{N^b + qK_1^b}{q^{\frac{1}{\theta}} + q} \tag{A.16}$$

The equilibrium of the economy is then fully characterized by the solution to equations (A.13) and (A.16). Combining both equations we can directly characterize $q\left(N^b, K_1^b\right)$ as the solution to the following equation

$$q = \left(K_1^b - \frac{N^b + qK_1^b}{q^{\frac{1}{\theta}} + q} + \delta\right)^{\eta - 1} \tag{A.17}$$

For given N^b and K_1^b, equation (A.17) may have multiple solutions, as illustrated in Figure A.1, which depicts the left-hand side and right-hand side of the equation for the parameter values reported at the end of Appendix B. It also follows that that $\frac{\partial q}{\partial N^b} > 0$ in the equilibrium with low q, which is the one that survives for any value of θ whenever an equilibrium exists.[16] The equilibrium with high price features $\frac{\partial q}{\partial N^b} < 0$, which violates Condition 1 in the text.

Our assumptions guarantee that borrowers are net sellers of capital, so $\Delta K_2^b < 0$. The distributive effects of borrowers' net worth, defined in Lemma 1, are given by

$$\mathcal{D}_{N^b}^b = -\frac{\partial q}{\partial N^b} \Delta K_2^b$$

[16]In this specific example, because we have assumed that $\delta < 0$, there is also a possibility of nonexistence of equilibrium when N^b is sufficiently high.

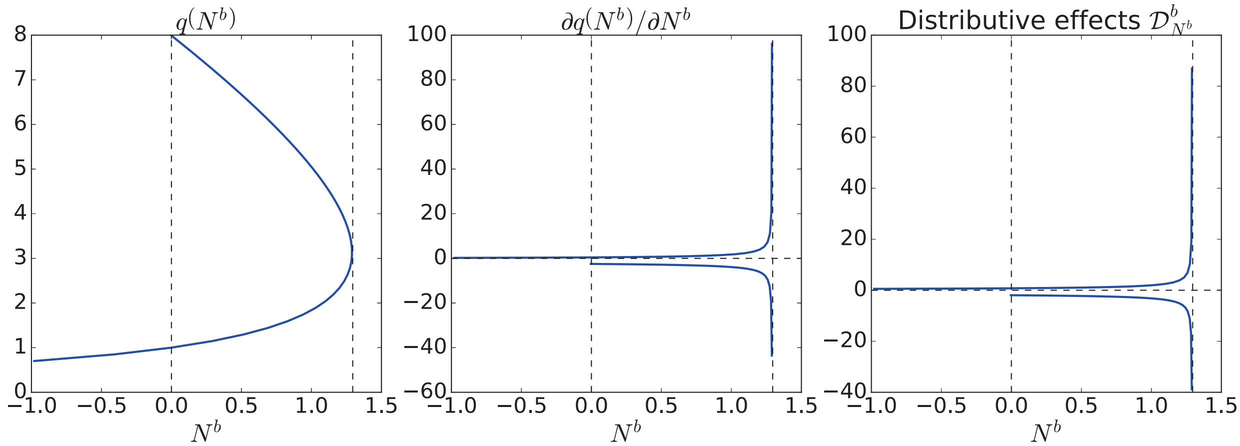

Figure A.2: Equilibrium correspondences

and will therefore have the same sign as $\frac{\partial q}{\partial N^b}$. Consequently, for a given sign of $\Delta MRS^{b\ell}$, whose determination is extensively discussed in Application 3, if one equilibrium features overborrowing, the other one will feature underborrowing.

Application 5. (Changing sign of $\frac{\partial q}{\partial N^b}$, distributive externalities). *For sufficiently large values of θ, there exist N^b and K_1^b such that the economy features multiple equilibria, each of them with different signs of $\frac{\partial q}{\partial N^b}$. Therefore, for a given sign of $\Delta MRS^{b\ell}$, if one equilibrium features overborrowing, the other one will feature underborrowing, and vice versa.*

For brevity, we do not repeat the date 0 characterization of the equilibrium, which follows the same steps as in Application 3, after accounting for the expectation of equilibrium selection.

Propositions 1 and 2 are valid to characterize the planner's constrained optimum regardless of whether there is a unique equilibrium or multiple equilibria. However, the solution to the constrained planning problem will be generically unique for a given equilibrium selection rule, or if the planner has sufficiently rich policy instruments to implement a specific equilibrium. We leave a more detailed analysis of the implementation of optimal corrective policies with multiple equilibria to future research.

B.6 Collateral Externalities and Multiple Equilibria

Environment We modify Application 4 to introduce multiple equilibria. We show that one of them violates Condition 1 in that the price derivative $\frac{\partial q}{\partial N^b}$ is positive. We continue to assume a perfect foresight economy in which lenders have large endowments, cannot invest at date 0 so $h^\ell(k) = \infty$ for $k > 0$, have no use for capital $F_t^\ell(k) := 0$, $\forall t$ and linear utility $U^\ell = c_0^\ell + c_1^\ell + c_2^\ell$ with $c_t^i \geq 0$. Given this utility and technology, all distributive effects are zero $\mathcal{D}^i_{N^j} = \mathcal{D}^i_{K_1^b} = 0$.

For borrowers, we modify the date 0 and 1 period utility functions to be CRRA instead of log so $U^b = u\left(c_0^b\right) + u\left(c_1^b\right) + c_2^b$ with $u(c) = \frac{c^{1-\theta}}{1-\theta}$. We will focus on the case $\theta > 1$, since this increases the curvature of the utility function and naturally prepares the ground for multiplicity in our setting. We assume borrowers have endowments $e_0^b, e_1^b \in (0,1)$ and $e_2^b = 0$; they invest at date 0 according to the cost function $h^b(k) = \frac{\alpha k^2}{2}$ and have linear production function $F_t^b(k) = Ak$ where we assume $A \leq \sqrt{\frac{\alpha}{2}}$. The collateral constraints are the same as in Application 4, $\Phi_1^b := 0$

and $\Phi_2^b\left(x_2^b, k_2^b; q\right) := x_2^b + \phi q k_2^b \geq 0$ with $\phi \in (0,1)$. The first-best exhibits $C_0^b = C_1^b = 1$, $K_t^{b*} = \frac{2A}{\alpha}$ and $q = A$.

Date 1 equilibrium and multiplicity The date 1 optimization problem of borrowers is

$$V^b\left(n^b, k_1^b; N, K_1\right) = \max_{c_1^b, c_2^b, x_2^b, k_2^b} u\left(c_1^b\right) + c_2^b \qquad \text{s.t. (3),(4),(40)}$$

Since capital always remains in the hands of borrowers, the price of capital as a function of borrower consumption C_1^b is pinned down by their optimality condition for capital holdings

$$q\left(C_1^b\right) = \frac{A}{u'\left(C_1^b\right) + \phi\kappa_2^b} = \frac{A}{(1-\phi)\left(C_1^b\right)^{-\theta} + \phi}$$

In an unconstrained equilibrium, borrowers consume $C_1^b = 1$ and save $X_2^b = N^b - 1$ at date 1, resulting in a price of capital of $q = F_2^{b\prime}(\cdot) = A$. This allocation is feasible as long as $X_2^b \geq -\phi q K_2^b$ or, equivalently, $N^b \geq 1 - \phi A K_1^b$.

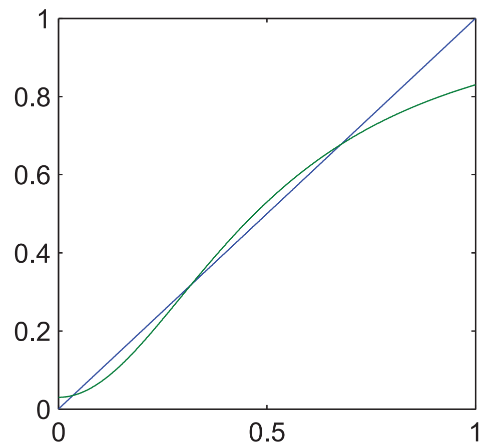

Figure A.3: Multiple solutions for (A.18)

In a constrained equilibrium $X_2^b = -\phi q K_1^b$ and $C_1^b < 1$. The date 1 budget constraint then implies that

$$C_1^b = N^b + \phi q\left(C_1^b\right) K_1^b = N^b + \frac{\phi A K_1^b}{(1-\phi)\left(C_1^b\right)^{-\theta} + \phi} \tag{A.18}$$

For given N^b and K_1^b, this is an implicit equation in C_1^b that may have multiple solutions, as illustrated in an example in Figure A.3, in which the right-hand side of (A.18) is the curved line. Formally, as we vary C_1^b over the interval $[0,1]$, the collateral term on the right-hand goes from $\phi q(0) K_1^b = 0$ (for $\theta > 1$) to $\phi q(1) K_1^b < 1$. Multiplicity arises for some values of N^b if there exists a range of C_1^b for which the slope of the right-hand side of (A.18) exceeds unity, as in the figure. The slope is highest at the inflection point of the price function, i.e. at the value $\tilde{C}$ that satisfies $q''\left(\tilde{C}\right) = 0$ and is given by $\tilde{C} = \left[\frac{(\theta-1)(1-\phi)}{(\theta+1)\phi}\right]^{\frac{1}{\theta}}$, which we assume w.l.o.g. to be in the unit interval. At that point, the slope of the right-hand side of (A.18) is

$$\phi q'\left(\tilde{C}\right) K_1^b = A K_1^b \left(\frac{\phi}{1-\phi}\right)^{\frac{1}{\theta}} \frac{(\theta+1)^{\frac{\theta+1}{\theta}} (\theta-1)^{\frac{\theta-1}{\theta}}}{4\theta}$$

The first multiplicative term on the right-hand side of this expression is bounded to $AK_1^b < 1$ by our earlier assumptions; the second and third terms are increasing functions of ϕ and θ. If they are chosen sufficiently large, the slope exceeds unity at $\tilde{C}$, and there is a neighborhood of borrowers' net worth around $\tilde{N}^b = \tilde{C} - \phi q\left(\tilde{C}\right) K_1^b$ for which equation (A.18) has multiple equilibria.

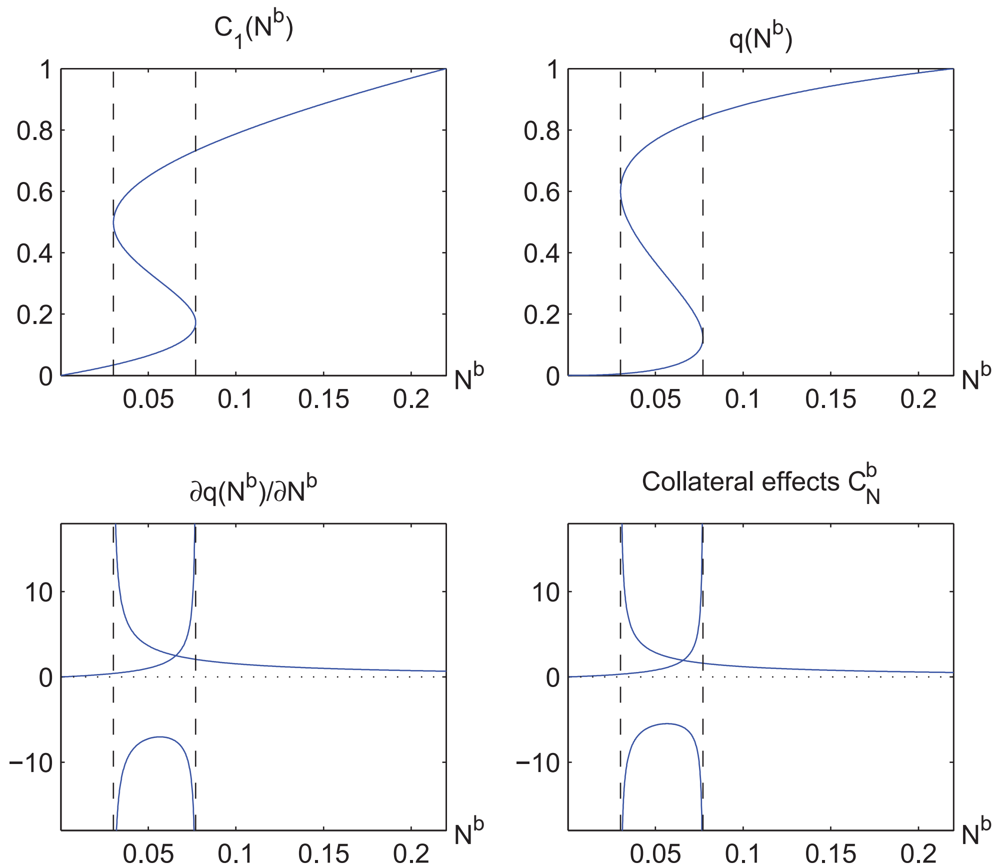

Figure A.4: Equilibrium correspondences

In this area of multiplicity, a given set of state variables (N, K_1) is consistent with multiple solutions $C_1^b\left(N^b, K_1^b\right)$ and $q\left(N^b, K_1^b\right)$, as illustrated in Figure A.4. The two bottom panels of the figure also show the price derivative $\frac{\partial q}{\partial N^b}$ with respect to borrowers' net worth and the collateral effects $\mathcal{C}_N^b$. In the region between the two dashed vertical lines, three equilibria and three possible values for the price derivative and collateral effects exist. As can be seen from the top right panel, the price q is an increasing function of banker net worth N^b for two of the three equilibria, but a decreasing function of banker net worth for the middle equilibrium. In the first two equilibria, the standard results on excessive borrowing hold; in the third equilibrium, net worth has negative collateral effects and the decentralized equilibrium exhibits insufficient borrowing.

Application 6. (Changing sign of $\frac{\partial q}{\partial N^b}$, collateral externalities). *If the parameters ϕ and θ are chosen to be sufficiently large to satisfy*

$$AK_1^b \left(\frac{\phi}{1-\phi}\right)^{\frac{1}{\theta}} \frac{(\theta+1)^{\frac{\theta+1}{\theta}} (\theta-1)^{\frac{\theta-1}{\theta}}}{4\theta} > 1$$

then there is a neighborhood of borrowers' net worth around $\tilde{N}^b$ such that the economy exhibits three equilibria. Two of the three are stable and feature overborrowing and underinvestment; the third equilibrium is unstable and features underborrowing and overinvestment.

In the described setting of multiplicity, the equilibrium with the highest C_1^b is Pareto-superior to the other two equilibria – the utility of lenders is always constant $U^\ell = \sum_{t=0}^2 e_t^\ell$ since there are no distributive effects. However, this is not a general feature – when lenders have concave utility or production technologies, distributive effects arise, and lenders may be better off in those equilibria in which borrowers are worse off. A planner can rule out multiple equilibria if she has the capacity to coordinate private agents on one specific equilibrium, for example by committing to a contingent tax/subsidy scheme that makes it suboptimal for individuals to choose inferior equilibria.

For brevity, we do not repeat the date 0 characterization of the equilibrium, which follows the same steps as in Application 4, after accounting for the expectation of equilibrium selection.

Parameters Used for Figures in Applications

Figure 1 To illustrate Application 1, we assume date 2 production technologies $F_2^{b,\omega}(k) = k$ and $F_2^{\ell,\omega}(k) = \log(1+k)$ and period utility functions $u^i(c) = \log(c)$ with no discounting for both agents. Furthermore we set $\phi = 0.2$ and date 2 endowments $e_2^b = 3$ and $e_2^\ell = 10$. The figure depicts date 1 equilibrium for $N^\ell = 2.5$, $K_1^b = 0.5$, $K_1^\ell = 0$ and $N^b \in \left[N^{b,\min}, 1.30\right]$ where the the minimum admissible borrower wealth (at which borrowers fire-sell all their capital holdings and obtain non-negative consumption) is $N^{b,\min} = 0.33$. The wealth threshold on which the financial constraint on borrowers is marginally binding is $\hat{N}^b = 0.8$.

Figure 2 To illustrate Application 2, we set the parameters $\alpha = 1$, $A_1^H = 3$ and $A_2 = 1$ and the probability of the low state $\pi = 5\%$. We vary $A_1^L \in [1.2, 3]$ and compute the resulting equilibria, which we trace out in the four panels of the figure. The net trading position of capital $\Delta K_2^{b,L}$ switches sign at $\tilde{A}_1^L = 1.8$.

Figure 3 To illustrate Application 3, we set the parameters $\alpha = \phi = \frac{1}{2}$, $A = 1$ and $e_1^b = 0$. We vary $e_0^b \in [.25, 2.5]$ and compute the resulting equilibria, tracing out the three sufficient statistics and the resulting tax rate in the four panels of the figure. The borrowing constraint is binding when $e_0^b < \underline{e}_0^b = 0.75$ and the constraint on saving is binding when $e_0^b > \bar{e}_0^b = 2$.

Figure 4 To illustrate Application 4, we set the parameters $\alpha = 2$, $A = 1$, $\phi = \frac{1}{3}$ and $e_1^b = 0$. We vary $e_0^b \in [0, 2.5]$ and compute the resulting equilibria, which we trace out in the four panels of the figure. We observe that the collateral constraint is binding when $e_0^b < \hat{e}_0^b = \frac{5}{3}$.

Figures A.1 and A.2 To illustrate Application 5, we set the parameters $\eta = 0.4$, $\theta = 2.5$, and $\delta = -0.75$. Figure A.1 plots the left- and right-hand-side of equation (A.17) for $N^b = 0.2$ and $K_1^b = 3.5$. For Figure A.2, we vary $N^b \in [-1, 1.5]$ and compute all resulting equilibria, which we trace out in the two panels of the figure.

Figures A.3 and A.4 To illustrate Application 6, we set the parameters $\alpha = 2$, $A = 1$, $\theta = 2$, $\phi = .8$ and $e_1^b = 0$. Figure A.3 plots the left- and right-hand-side of equation (A.18) for $N^b = 0.03$. For Figure A.4, we vary $N^b \in [0, 0.22]$ and compute all resulting equilibria, which we trace out in the four panels of the figure.

C Online Appendix

C.1 Equivalence Ramsey and Constrained Social Planner

The optimization problem of a Ramsey planner who has the tax instruments $\left(\tau_x^{i,\omega}\right)$ and τ_k^i on date 0 security purchases and capital investment and agent specific transfers T^i is equivalent to the problem of a constrained social planner who directly chooses the economy's date 0 allocations $\left(C_0^i, K_1^i, X_1^{i,\omega}\right)$, or equivalently $\left(C_0^i, K_1^b, N^{i,\omega}\right)$ for $i \in I$ and $\omega \in \Omega$.

The budget constraint that incorporates taxes and transfers faced by agent i at date 0 is formally given by

$$c_0^i + h^i\left(k_1^i\right) + \tau_k^i k_1^i + \mathbb{E}_0\left[\left(m_1^\omega + \tau_x^{i,\omega}\right) x_1^{i,\omega}\right] = e_0^i + T_0^i$$

Formally, the optimality conditions of agents in a decentralized equilibrium with tax instruments $\left(\tau_x^{i,\omega}\right)$ and $\left(\tau_k^i\right)$ are

$$\left(m_1^\omega + \tau_x^{i,\omega}\right)\lambda_0^i = \beta\lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i \quad \forall i, \omega$$
$$\left(h^{i\prime}\left(k_1^i\right) + \tau_k^i\right)\lambda_0^i = \beta\mathbb{E}_0\left[\lambda_1^{i,\omega}\left(F_1^{i,\omega\prime}\left(k_1^{i,\omega}\right) + q^\omega\right)\right] + \kappa_1^i \Phi_{1k}^i, \quad \forall i$$

Any date 0 allocation $\left(C_0^i, K_1^b, X_1^{i,\omega}\right)$ chosen by a social planner can be replicated by a Ramsey planner who sets the tax instruments

$$\tau_x^{i,\omega} = \frac{\beta\lambda_1^{i,\omega}}{\lambda_0^i} + \frac{\kappa_1^i}{\lambda_0^i}\Phi_{1x}^{i,\omega} - m_1^\omega$$
$$\tau_k^i = \mathbb{E}_0\left[\frac{\beta\lambda_1^{i,\omega}}{\lambda_0^i}\left(F_1^{i,\omega\prime}\left(k_1^{i,\omega}\right) + q^\omega\right)\right] + \frac{\kappa_1^i}{\lambda_0^i}\Phi_{1k}^i - h^{i\prime}\left(k_1^i\right)$$

and who imposes transfers so that the date 0 budget constraints of individual agents are satisfied, where the state price densities equal the increase in the social planner's shadow prices on the resource constraint, $m_1^\omega = \frac{\nu_1^\omega}{\nu_0}$.[17]

Conversely, any set of tax instruments and transfers will result in a date 0 allocation $\left(c_0^i, k_1^i, x_1^{i,\omega}\right)$ for $i \in I$ and $\omega \in \Omega$ that satisfies the economy's resource constraints. A constrained social planner can therefore replicate the allocation by setting her date 0 allocation $\left(C_0^i, K_1^i, X_1^{i,\omega}\right)$ equal to that chosen by decentralized agents under the planner's optimal tax instruments.

C.2 Welfare Weights and Pareto Improvements

Proposition 1 characterizes constrained efficient allocations for given Pareto weights $\left(\theta^b, \theta^\ell\right)$ to describe the entire Pareto frontier of the economy, but it does not explicitly describe how to achieve a Pareto improvement.

[17] See application 4.2 and Davila (2014) for more detailed discussions regarding the role of transfers.

Imposing the optimal tax rates (27) and (28) and rebating all tax revenue to the set of agents from whom it is obtained does not guarantee a Pareto improvement.[18] For an example, consider an economy in which the set of taxes $\left\{\tau_x^{i,\omega}, \tau_k^i\right\}$ implements a Pareto efficient allocation. Now assume that we move from an inefficient set of taxes where one tax rate $\hat{\tau}_x^{i,\omega_0}$ is marginally below its efficient level τ_x^{i,ω_0} to the described efficient set of taxes. Since we were close to efficiency, the overall welfare improvement is second-order, but the change in tax rates leads to a first-order change in equilibrium prices and a first-order redistribution between sectors. One sector will experience a first-order welfare gain, the other a first-order loss, implying that the policy change does not generate a Pareto improvement, even though it moves the economy from an inefficient allocation to a constrained Pareto efficient allocation.

To achieve a Pareto improvement, a planner has to maximize the welfare of one sector subject to the constraint of not making the other sector worse off. For example, denote the welfare level of the lender sector in the decentralized equilibrium $U^{\ell,DE}$. Then the following result, stated as a corollary of Propositions 1 and 2, holds.

Corollary. (Pareto improvement) *A planner can achieve a Pareto improvement by solving*

$$\max U^b \qquad s.t. \qquad U^\ell \geq U^{\ell,DE}$$

and subject to the constraints of problem (24)*.*

If we assign the Lagrange multiplier θ^ℓ to the constraint $U^\ell \geq U^{\ell,DE}$ and normalize $\theta^b = 1$, it is easy to see that this maximization problem is identical to problem (24) up to a constant. Since the decentralized allocation is feasible for the planner but the allocations she chooses generally differ from the decentralized allocation when the date 0 financial constraint is binding, the planner must achieve a Pareto improvement with $U^b > U^{b,DE}$.

In short, adding the constraint $U^\ell \geq U^{\ell,DE}$ to the planner's optimization problem provides us with an endogenous way of picking Pareto weights $\left(\theta^b, \theta^\ell\right)$ to guarantee a Pareto improvement. The resulting allocation can then be implemented as described in Proposition 1.[19]

C.3 Generalizations of Baseline Model

This appendix generalizes our baseline model along several dimensions and shows that the optimal corrective tax formulas (27) and (28) continue to apply. First, we allow for a more general set of agents with general discount factors and state-contingent utility that can capture subjective probabilities. Second, we allow for a more general investment and production structure. Finally, we also allow for a more general specification of financial constraints that now apply to all agents.

[18] Rebating all tax revenue to the set of agents from whom it is obtained replicates the same allocation that would be obtained if the planner used quantity restrictions on date 0 allocations – therefore our statement on Pareto efficiency not guaranteeing a Pareto improvement extends to quantity regulations.

[19] It is not generally possible to obtain a simple expression for the transfers required to obtain a Pareto improvement. However, the planner can approximate the wealth transfer that occurred in response to an intervention using the marginal distributive effects $\mathcal{D}_{N^i}^{j,\omega}$ times the change in sector-wide net worth $\Delta N^{i,\omega}$, and use the $MRS^{j,\omega}$ to discount this to date 0. If a macroprudential intervention increases the net worth of sector i by $\Delta N^{i,\omega}$, this suggests that a date 0 transfer of $-\mathbb{E}_0[MRS^{j,\omega}\mathcal{D}_{N^i}^{j,\omega}\Delta N^{i,\omega}]$ would leave type j agents approximately indifferent. A similar expression can be obtained for changes in sector-wide capital K_1^i.

Preferences/endowments Assume that there is a set of agents is given by $I = \{1, ... |I|\}$. Agents have heterogeneous preferences, given by

$$U^i = \mathbb{E}_0 \left[\sum_{t=0}^{2} \left(\beta^i\right)^t u_t^i \left(c_t^i; \omega\right) \right] \tag{A.19}$$

which allows both for agent-specific discount factors β^i and arbitrary time separable utility functions that may or may not satisfy an Inada condition $\lim_{c \to 0} u_t^{i\prime} (c; \omega) = \infty$. Since $u_t^i (\cdot; \omega)$ depends on the state of nature, the setup is also able to capture heterogeneous beliefs, i.e. agents may value consumption in some states of nature more highly because they believe them to be more likely. We continue to denote agent i endowment at date t given state ω by $e_t^{i,\omega}$. As in the baseline model, consumption must be non-negative, so $c_t^i \geq 0$.

Technology Assume that each agent i is initially endowed with k_0^i units of capital and chooses to create ι_0^i capital goods at date 0 at a cost $h_0^i \left(\iota_0^i, k_0^i\right)$ and invest/disinvest into capital ι_1^i at cost $h_1^i \left(\iota_1^i, k_1^i\right)$ at date 1, where the cost functions satisfy $h_t^i \left(0, k_t^i\right) = 0$ and are increasing and convex in ι_t^i and decreasing in k_t^i. This formulation nests fixed capital endowments if we assume that $h_t^i \left(\iota_t^i, k_t^i\right) = \infty$ for $\iota_t^i \neq 0$ for $t = 0, 1$. It also nests standard quadratic adjustment costs if we assume that $h_1^i \left(\iota_1^i, k_1^i\right) = \iota_1^i \left(1 + \frac{\varphi}{2} \frac{\iota_1^i}{k_1^i}\right)$, as well as a range of other models of endogenous investment with and without adjustment costs. We continue to assume that capital fully depreciates at the end of date 2.

Regarding production, we assume that type i agents have access to a production technology given by the function $F_t^{i,\omega} (k)$ that is increasing and weakly concave and depends on the agent type i, the date t, and the state of nature ω. As we discussed in the main text, it is common in the literature on fire sales to assume that the productivity of capital depends on who owns it and differs between different agents. This rules out that capital is owned by one agent but rented out and used in another agent's production function. A typical justification for this assumption is agency frictions, i.e. that efficient use of capital requires ownership to ensure proper incentives (see e.g. Shleifer and Vishny, 1992).

The budget constraints of type i agents, including tax instruments and transfers, which generalize equations (2) to (4) in the text, are given by

$$\begin{aligned} c_0^i + h_0^i \left(\iota_0^i, k_0^i\right) + \left(q_0 + \tau_k^i\right) \left(k_1^i - k_0^i - \iota_0^i\right) & \\ + \mathbb{E}_0 \left[\left(m_1^\omega + \tau_x^{i,\omega}\right) x_1^{i,\omega} \right] &= e_0^i + F_0^i \left(k_0^i\right) + T^i \end{aligned} \tag{A.20}$$

$$c_1^{i,\omega} + h_1^i \left(\iota_1^{i,\omega}, k_1^i\right) + q_1^\omega \left(k_2^{i,\omega} - k_1^i - \iota_1^{i,\omega}\right) + m_2^\omega x_2^{i,\omega} = e_1^{i,\omega} + x_1^\omega + F_1^{i,\omega} \left(k_1^i\right), \ \forall \omega \tag{A.21}$$

$$c_2^{i,\omega} = e_2^{i,\omega} + x_2^\omega + F_2^{i,\omega} \left(k_2^{i,\omega}\right), \ \forall \omega \tag{A.22}$$

Financial constraints We generalize financial constraints along three dimensions. First, we assume that $\Phi_2^{i,\omega} (\cdot)$ depends directly on the aggregate state variables N^ω and K_1, not only through the date 1 asset price. This more general formulation emphasizes the role played by the dependence of financial constraints on aggregate state variables and allows for example the bond price m_2^ω to enter the constraint. It can similarly capture more general moral hazard and incentive constraints. Secondly, we assume that $\Phi_1^{i,\omega}$ also depends on the entire vector of future aggregate state variables N^ω and K_1. This formulation allows us to capture date 0 financial constraints that depend on future date 1 prices, which can be written as a function of these aggregate state variables. Third, we assume that all agents, not only borrowers, are potentially subject to financial constraints. Formally, agent i faces vector-valued financial constraints

$$\Phi_1^i\left(x_1^i, k_1^i; N^\omega, K_1\right) \geq 0 \tag{A.23}$$

$$\Phi_2^{i,\omega}\left(x_2^i, k_2^i; N^\omega, K_1\right) \geq 0,\ \forall\omega \tag{A.24}$$

Equilibrium at dates 1 and 2 Agent i maximizes the generalized utility function (A.19) subject to the set of budget constraints (A.20), (A.21), (A.22) and the set of financial constraints (A.23) and (A.24).

As in the baseline model, the vectors $N^\omega = \left(N^{1,\omega}, ...N^{|I|,\omega}\right)$ and $K_1 = \left(K_1^1, ...K_1^{|I|}\right)$, representing aggregate net worth and capital holdings of type i agents, become aggregate state variables. We continue to index indirect utility by the state ω, to capture the direct effect of the state on investment opportunities.

The date 1 continuation utility $V^{i,\omega}\left(n^i, k_1^i; N^\omega, K_1\right)$ of type i agents is

$$V^{i,\omega}\left(n^{i,\omega}, k_1^{i,\omega}; N^\omega, K_1\right) = \max_{c_1^{i,\omega}\geq 0, c_2^{i,\omega}\geq 0, x_2^{i,\omega}, \iota_1^{i,\omega}, k_2^{i,\omega}} u_1^i\left(c_1^{i,\omega};\omega\right) + \beta^i u_2^i\left(c_2^{i,\omega};\omega\right) \text{ s.t. (A.21), (A.22) and (A.24)}$$

Date 1 market prices are now functions $q_1^\omega\left(N^\omega, K_1\right)$ and $m_2^\omega\left(N^\omega, K_1\right)$ of the net worth and capital holding vectors of all sectors. Agent i date 1 optimality conditions for borrowing/saving and capital holdings can be expressed as

$$\lambda_1^{i,\omega} m_2^\omega = \beta^i \lambda_2^{i,\omega} + \kappa_2^{i,\omega}\Phi_{2x}^{i,\omega}$$

$$\lambda_1^{i,\omega} q_1^\omega = \beta^i \lambda_2^{i,\omega} F_2^{i,\omega\prime}\left(k_2^{i,\omega}\right) + \kappa_2^{i,\omega}\Phi_{2k}^{i,\omega}$$

The optimality condition for investment is given by $q_1^\omega = \frac{\partial h^i}{\partial \iota_1^{i,\omega}}$. Equations (14) and (15) remain valid in this more general environment, with the exception that the collateral effects now correspond to

$$\mathcal{C}_{N^j}^{i,\omega} := \frac{\partial \Phi_2^{i,\omega}}{\partial N^{j,\omega}}$$

$$\mathcal{C}_{K^j}^{i,\omega} := F_1^{i,\omega\prime}\left(\cdot\right)\mathcal{C}_{N^j}^{i,\omega} + \frac{\partial \Phi_2^{i,\omega}}{\partial K_1^j}$$

Date 0 decentralized equilibrium At date 0, private agents of type i solve the maximization problem

$$\max_{c_0^i\geq 0, \iota_0^i, k_1^i, x_1^{i,\omega}} u_0^i\left(c_0^i\right) + \beta^i \mathbb{E}_0\left[V^{i,\omega}\left(x_1^{i,\omega} + e_1^{i,\omega} + F_1^{i,\omega}\left(k_1^i\right), k_1^i; N^\omega, K_1\right)\right] \quad \text{s.t.} \quad \text{(A.20), (A.23)}$$

At date 0, agent i optimality conditions can be expressed as

$$\left(m_1^\omega + \tau_x^{i,\omega}\right) u_0^{i\prime}\left(c_0^i\right) = \beta^i u_1^{i\prime}\left(c_1^{i,\omega};\omega\right) + \kappa_1^i \Phi_{1x}^{i,\omega},\ \forall\omega \tag{A.25}$$

$$\left(q_0 + \tau_k^i\right) u_0^{i\prime}\left(c_0^i\right) = \mathbb{E}_0\left[\beta^i u_1^{i\prime}\left(c_1^{i,\omega};\omega\right)\left(F_1^{i,\omega\prime}\left(k_1^i\right) + q_1^\omega - \frac{\partial h_1^i}{\partial k_1^i}\right)\right] + \kappa_1^i \Phi_{1k}^i \tag{A.26}$$

The optimality condition for investment is given by $q_0 + \tau_k^i = \frac{\partial h^i}{\partial i_0^i}$.

Date 0 constrained planner allocation The Lagrangian corresponding to the problem solved by a constrained planner who leaves date 1 allocations to the market but determines date 0 allocations is given by

$$\mathcal{L} = \sum_i \theta^i \left\{ u^i \left(C_0^i \right) + \eta_0^i C_0^i + \beta^i \mathbb{E}_0 \left[V^{i,\omega} \left(N^{i,\omega}, K_1^i; N^\omega, K_1 \right) \right] + \kappa_1^i \Phi_1^i \left(X_1^i, K_1^i; N^\omega, K_1 \right) \right\}$$
$$- \nu_0 \sum_i \left[C_0^i + h_0^i \left(\iota_0^i, K_0^i \right) - e_0^i - F_0^i \left(K_0^i \right) \right] - \mu_0 \sum_i \left[K_1^i - K_0^i - \iota_0^i \right] - \sum_\omega \nu_1^\omega \sum_i X_1^{i,\omega}$$

We assign the new shadow price μ_0 to the new constraint on capital accumulation. The optimality conditions of the planner problem are given by

$$\frac{d\mathcal{L}}{dC_0^i} = \theta^i \left[u_0^{i\prime} \left(C_0^i \right) + \eta_0^i \right] - \nu_0 = 0, \ \forall i$$
$$\frac{d\mathcal{L}}{dX_1^{i,\omega}} = -\nu_1^\omega + \theta^i \beta^i V_n^{i,\omega} + \theta^i \kappa_1^i \Phi_{1x}^{i,\omega} + \sum_j \theta^j \beta^j V_{N^i}^{j,\omega} + \sum_j \theta^j \kappa_1^j \frac{\partial \Phi_1^j}{\partial N^{i,\omega}} = 0, \ \forall i, \omega$$
$$\frac{d\mathcal{L}}{dK_1^i} = -\mu_0 + \theta^i \beta^i \mathbb{E}_0 \left[V_k^{i,\omega} \right] + \theta^i \kappa_1^i \Phi_{1k}^i + \beta \sum_j \theta^j \mathbb{E}_0 \left[V_{N^i}^{j,\omega} F_1^{i,\omega\prime} \left(K_1^i \right) + V_{K^i}^{j,\omega} \right]$$
$$+ \sum_j \theta^j \kappa_1^j \left(\sum_\omega \frac{\partial \Phi_1^j}{\partial N^{i,\omega}} F_1^{i,\omega\prime} \left(\cdot \right) + \frac{\partial \Phi_1^j}{\partial K_1^i} \right) = 0, \ \forall i$$

Where also $\frac{d\mathcal{L}}{d\iota_0^i} = -\nu_0 \frac{\partial h_0^i}{\partial \iota_0^i} + \mu_0$, and we denote $V_n^{i,\omega} = u_1^{i\prime} \left(C_1^{i,\omega} \right)$ and $V_k^{i,\omega} = F_1^{i,\omega\prime} \left(k_1^i \right) + q^\omega - \frac{\partial h_1^i}{\partial k_1^i}$. Rearranging the optimality conditions and the date 1 envelope conditions, we find the analogous equations to (25) and (26) characterizing constrained efficiency in this more general version

$$\frac{\nu_1^\omega}{\nu_0} \lambda_0^i = \beta^i \lambda_1^{i,\omega} + \kappa_1^i \Phi_{1x^\omega}^i + \sum_{j \in I} \frac{\theta^j}{\theta^i} \beta^j V_{N^i}^{j,\omega}, \ \forall i, \omega \tag{A.27}$$

$$\frac{\partial h_0^i}{\partial i_0^i} \lambda_0^i = \mathbb{E}_0 \left[\beta^i \lambda_1^{i,\omega} \left(F_1^{i,\omega\prime} \left(k_1^i \right) + q^\omega - \frac{\partial h_1^i}{\partial k_1^i} \right) \right] \tag{A.28}$$
$$+ \kappa_1^i \Phi_{1k}^i + \sum_j \frac{\theta^j}{\theta^i} \beta^j \mathbb{E}_0 \left[V_{N^i}^{j,\omega} F_1^{i,\omega\prime} \left(K_1^i \right) + V_{K^i}^{j,\omega} \right]$$
$$+ \sum_j \frac{\theta^j}{\theta^i} \kappa_1^j \left(\sum_\omega \frac{\partial \Phi_1^j}{\partial N^{i,\omega}} F_1^{i,\omega\prime} \left(\cdot \right) + \frac{\partial \Phi_1^j}{\partial K_1^i} \right)$$

The only difference with the baseline model is a new collateral externality term, capturing the direct effect of changes in aggregate state variables on the date 1 financial constraints – this is a straightforward generalization of the results in the text. As in the baseline model, the planner can set optimal corrective taxes to ensure that the private optimality conditions (A.25) and (A.26) replicate the planner's optimality conditions (A.27) and (A.28). Propositions 1 and 2 and all their implications, including the characterization of optimal corrective taxes in (27) and (28) and the corollaries, remain valid in this more general environment after accounting for the new collateral externality term. In the case with I agents, $I - 1$ differences in MRS are needed to express the optimal tax wedges as in equations (29) and (30).