Criminal Network Formation and Optimal Detection Policy: The Role of Cascade of Detection*

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Abstract

This paper investigates the effect of cascade of detection, that is, how detection of a criminal triggers detection of his network neighbors, on criminal network formation. We develop a model in which criminals choose both links and actions. We show that the degree of cascade of detection plays an important role in shaping equilibrium criminal networks. Surprisingly, greater cascade of detection could reduce *ex ante* social welfare. In particular, we prove that full cascade of detection yields a weakly denser criminal network than that under partial cascade of detection. We further characterize the optimal allocation of the detection resource and demonstrate that it should be highly asymmetric among ex ante identical agents.

Keywords: Criminal network, cascade of detection, network formation, local complementarity, detection policy

JEL Classification: A14, C70, D85, K42

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1 Introduction

Criminal decision-making is often interdependent. Social interaction is both theoretically and empirically identified as an important channel through which neighborhood criminal behavior affects individual criminal behavior. While the structure of social networks plays a key role in facilitating crimes, it can also be utilized by law enforcement agencies to trace linked criminals. In a criminal network, detection of an agent could potentially trigger further detection of his network neighbors. We call this triggering effect cascade of detection. In this paper, we study how cascade of detection affects ex ante social welfare in the presence of endogenous network formation among criminals. Interestingly, we find that a *higher degree of cascade of detection may backfire*. The relationship between the degree of cascade and ex ante social welfare is non-monotonic. Although enhancing cascade of detection is ex post efficient, it could be ex ante sub-optimal precisely because criminal network formation adjusts upon the degree of cascade. Under a higher degree of cascade, as additional cost of connecting to an indirect network neighbor becomes lower, criminals become less selective in choosing their linking partners, thus rendering a denser equilibrium network. Our work highlights that the degree of cascade of detection has very nuanced implication on social welfare, thereby shedding light on the nexus between law enforcement and criminal networks.

In our model, the government first announces a detection policy. It consists of two components, the degree of cascade and the allocation of the detection budget. The former is the key innovating feature of our model. After observing the detection policy, agents play a two-stage game. In the first stage, they propose links to each other and it requires bilateral consent to form a link. Creating a new link does not incur any explicit cost, but a well-connected agent tends to be more likely to be detected. In the second stage, agents play a game with local complementarities in the fashion of Ballester et al. (2006). The payoff of an agent increases with his centrality in the network. Therefore, each agent is faced with the trade-off between increase of his centrality in the network and being more likely to be detected. Under a given detection policy, we consider two equilibrium notions, pairwise stable Nash equilibrium and its refinement, strongly stable Nash equilibrium. We say an equilibrium is pairwise stable if it is stable against bilateral coordination of link formation (Jackson and Wolinsky, 1996; Hiller, 2014) and an equilibrium is strongly stable if it is stable against multilateral coordination of link formation (Jackson and van den Nouweland, 2005).

As a starting point, we consider three scenarios: (1) no cascade of detection – detection of an agent does not trigger any further detection; (2) partial cascade of detection – detection of an agent only triggers detection of his direct network neighbors; (3) full cascade of detection – detection of an agent triggers detection of every agent who is directly or indirectly connected with him. We show that the equilibrium network in any pairwise stable Nash equilibrium, including the strongly stable Nash equilibrium if any, under partial cascade of detection is weakly sparser than the equilibrium network in the unique strongly stable Nash equilibrium under full cascade of detection. This result holds for any allocation of the detection budget.

Using the unique strongly stable Nash equilibrium under full cascade of detection as a benchmark, we also fully characterize the optimal allocation of detection budget. We show that the optimal budget allocation is highly asymmetric among ex ante identical agents. Intuitively, when the government is unable to refrain agents from linking to each other, the best strategy is to minimize the number of linked agents. To achieve this, the government needs to create certain gradient in terms of scrutiny among agents such that a subset of agents will be excluded from link formation.

Our model is built upon the framework of Baccara and Bar-Isaac (2008). Using terrorist networks as a motivating example, they investigate the optimal information structure in a criminal organization and its implication on the optimal detection policy. Our work complements theirs from two aspects. First, we focus on individual incentive to form networks, while the notion of the optimal information structure in Baccara and Bar-Isaac (2008) is from a group perspective. Different from their centralized view of the organized crime, we take a decentralized approach to tackle criminal networks. Second, this paper examines in detail the cascade of detection. With very few exceptions[1], most of the existing work in the literature on detection policy of criminal networks assumes either no cascade of detection or full cascade of detection. Our work offers insights on how the degree of cascade of detection could affect *ex ante* social welfare in a surprising direction.

Our paper adds to the literature that investigates crime and punishment in a network framework. Economic modeling of crime and punishment can be dated back to Becker (1968), but it is only until very recently that social networks, which have long been perceived as a crucial ingredient in crime decisions, are explicitly formulated in this context[2]. Calvó-Armengol and Zenou (2004) is among the first few papers that introduce the network geometry into a model of criminal activity. As a seminal paper, Ballester et al. (2006) pro-

[1] A notable exception is the follow-up work by Baccara and Bar-Isaac (2009), but again they focus on the efficient network from a group perspective, which is more applicable to highly organized criminal networks like terrorist networks.

[2] Garoupa (1997) provides an excellent review of the literature prior to the introduction of the network framework.

vide a tractable model in which agents play a network game with local complementarities. Their paper spurs a series of subsequent work that explores from the standpoint of a social planner which player in the criminal network should be removed so as to achieve the greatest reduction in aggregate criminal activity, namely, the "key player" policy[3]. Following Beckerian incentive approach, Ballester et al. (2010) use a model of delinquent networks to derive the key player policy and extend it to target the key group as well as the key link. Liu et al. (2012) structurally estimate a key-player model and find that the key player policy achieves sizable reduction of criminal activity. In a more recent work, Chen et al. (2015) further extend Ballester et al. (2006) to allow agents to have multiple types of interdependent activities and demonstrate that isolating the criminal activity from other activities could render the key-player policy mis-targeted.

Our paper also ties into the growing literature that integrates network formation with a network game with local complementarities. The structure of our model is closely related to Hiller (2014) in which two-sided link formation is introduced before agents play a network game. In a parallel study, Baetz (2015) incorporates one-sided link formation into a model with strategic complementarities. In both papers, certain types of social hierarchy endogenously emerge as equilibrium outcome. The idea of combining network formation with network games is also studied in a dynamic setting by König et al. (2014) and Lagerås and Seim (2015). By imposing myopic assumptions on individual decision and introducing stochastic arrival of linking opportunity, these two papers point out the prominent role played by the so-called nested split graphs as equilibrium network structures.

From a substantive point of view, our work shares with Garoupa (2007) the insights that stricter law enforcement could have unintended consequences, albeit through very different channels. Garoupa (2007) argues that more severe punishment tends to change the internal organization of criminal networks and consequently reduces effectiveness of the policy. Using a very different framework, our work explicitly accounts for the network structure among criminals and its formation. With a network grounding, our model captures how aggregate criminal activity reacts to the cascade of detection and explains why ex ante social welfare could be dampened under stricter law enforcement.

The rest of the paper is organized as follows. We present the baseline model and define equilibrium notions in section 2. In section 3, we study criminal network formation under different degrees of cascade of detection. In light of emergence of a unique strongly stable equilibrium under full cascade of detection, we characterize the optimal detection policy in

[3]Zenou (2014) provides an extensive survey of the recent literature on key players in networks.

section 4. Extensions are discussed in section 5. We conclude in section 6.

2 Model

There are finite agents and a government which acts as an external authority. Denote the set of agents by $N = \{1, 2, ..., n\}$. The timing structure of the game closely follows Baccara and Bar-Isaac (2008). As is illustrated by Figure 1, the government first announces its detection policy which consists of the allocation of the detection budget and the degree of cascade. The government decides the allocation of the detection budget, while the degree of cascade is an exogenous feature of the law enforcement institutions that is not freely chosen by the government. The core of our theoretical exercise is to understand efficacy of law enforcement under different degrees of cascade. After observing the detection policy, agents play a two-stage game. At the first stage, agents propose and form links with each other. At the second stage, agents decide the effort level of criminal activity, which is modeled as a game with local complementarities on the criminal network à la Ballester et al. (2006).

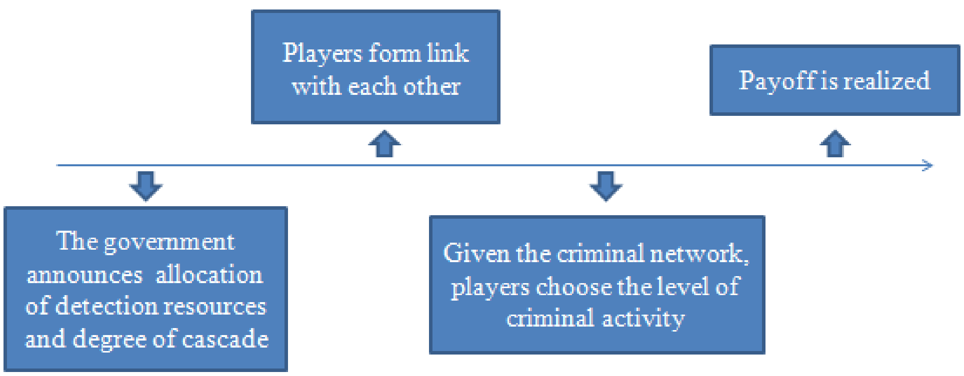

Figure 1: Timing Structure

2.1 Government Policy

The government detection policy has two dimensions: the detection resource allocation and the degree of cascade. First, the government allocates a fixed amount of the detection budget B over n agents. Denote the probability of agent i being directly detected by $\beta_i \in [0, 1]$. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n)$. Following Baccara and Bar-Isaac (2008), we assume that the detection technology is linear, and therefore $\sum_{i=1}^{n} \beta_i \leq B$. Without loss of generality, n agents are ranked such that $\beta_1 \leq \beta_2 \leq ... \leq \beta_n$. Second, the degree of cascade of criminal detection concerns how detection of an agent triggers further detection of his network neighbors. It is the key departure from the existing literature, but notice that the cascade of detection is *not* a choice variable of the government. Once a given agent i is detected, we consider three scenarios: (1) no cascade of detection, i.e., detection of agent i will not affect anyone directly or indirectly connected to him; (2) full cascade of detection, i.e., those who are directly or indirectly connected to agent i will also be detected; (3) partial cascade of detection, i.e., only those who are directly connected to agent i will be detected. Admittedly stark as these three scenarios are, we will demonstrate that the main intuition can be captured without losing tractability of the model. We also discuss other degrees of cascade in Section 5.1. Before we formally specify the degree of cascade, we first introduce network notation and terminology.

2.2 Network Formation

Denote the set of all n-by-n *symmetric* Boolean matrices with zeros on the diagonal by $\overline{G}$. A criminal network can be fully characterized by an adjacency matrix $\overline{g} \in \overline{G}$ with $\overline{g}_{ij} = 0$ if two agents i and j are not linked and $\overline{g}_{ij} = 1$ if they are linked. Following notational convention, $\overline{g}_{ii} = 0$ for any $i \in N$. We define the *distance* d_{ij} between agent i and j as the length of the shortest path connecting i and j. By definition, $d_{ii} = 0$ for any $i \in N$ and $d_{ij} < n$ for any pair i and j connected by a path. If there is no path connecting agent i and j, the distance is defined as ∞. A network $\overline{g}$ is said to be *complete* if $\overline{g}_{ij} = 1$ for any $i, j \in N$ such that $i \neq j$. A network $\overline{g}$ is said to be *empty* if $\overline{g}_{ij} = 0$ for any $i, j \in N$. A *component* consists of a subset of agents $C \subset N$ and links among them such that any pair of agents in C are connected by a path and there is no path connecting an agent in C with an agent outside C. We say a criminal network $\overline{g}$ is *sparser* than another criminal network $\overline{h}$ if the set of links in $\overline{g}$ is a subset of the set of links in $\overline{h}$, or to put it in another way, $\overline{h}_{ij} = 0$ implies $\overline{g}_{ij} = 0$ for any $i, j \in N$.

Given a criminal network $\overline{g}$, the probability of agent i *not* being detected is given by

$$p_i(\overline{g}; \boldsymbol{\beta}, d) = \Pi_{j \in N, d_{ij} \leq d}(1 - \beta_j),$$

where d is the degree of cascade and $d = 0, 1, n$ corresponds to the three cases aforementioned. If $d = 0$, $p_i = 1 - \beta_i$. This is the case of no cascade of detection. The probability of an agent not being detected is the same as the probability of an agent not being directly detected. If $d = 1$, $p_i = (1-\beta_i) \cdot \Pi_{d_{ij}=1}(1-\beta_j)$, which means agent i will be detected if and only if either himself or his direct network neighbors get directly caught. We call this partial cascade of detection. If $d = n$, anyone who is directly or indirectly connected to agent i shares the same probability of being detected. This corresponds to full cascade of detection. In the network shown by Figure 2, it can be easily seen that $p_1(\overline{g}; \boldsymbol{\beta}, 0) = 1 - \beta_1$, $p_1(\overline{g}; \boldsymbol{\beta}, 1) = \Pi_{i=1}^{4}(1-\beta_i)$, and $p_1(\overline{g}; \boldsymbol{\beta}, 6) = \Pi_{i=1}^{6}(1-\beta_i)$.

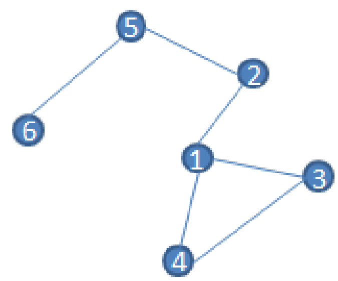

Figure 2: Degree of Cascade: An Example

After the government announces the detection policy, the detection resource allocation $\boldsymbol{\beta}$ and the degree of cascade d become common knowledge among all agents. They then make their link proposals contingent on $\boldsymbol{\beta}$ and d. Denote the set of all n-by-n (symmetric or asymmetric) Boolean matrices with zeros on the diagonal by G. Link proposals by n agents can be fully characterized by an adjacency matrix $g \in G$ with $g_{ij} = 1$ if agent i proposes a link to agent j and $g_{ij} = 0$ otherwise. Following notational convention, $g_{ii} = 0$ for any $i \in N$. Link formation is bilateral. A link between agent i and j is formed if and only if both of them agree to form that link. Therefore, a criminal network $\overline{g} \in \overline{G}$ is given by $\overline{g}(g) \equiv \min(g, g')$[4] for any adjacency matrix of link proposals $g \in G$. Denote by G_i the set of all n-by-1 Boolean vector with i-th element to be zero. G_i is the set of all possible link proposals by agent i.

2.3 A Game with Local Complementarities

Once a criminal network $\overline{g} \in \overline{G}$ is formed, agents play a game with local complementarities on the network. Each agent chooses an effort level. Denote agent i's effort level by $x_i \in \mathbb{R}_+$.

[4]Throughout the paper, the transpose of a matrix M is denoted by M'.

Let $\boldsymbol{x} = (x_1, x_2, ..., x_n)$. Denote by $\overline{\pi}_i$ the payoff to agent i in the stage game. Following Ballester et al. (2006), the payoff function is of the form

$$\overline{\pi}_i(\boldsymbol{x}, \overline{g}; \lambda) = x_i - \frac{1}{2}x_i^2 + \lambda \sum_{j=1}^{n} \overline{g}_{ij} x_i x_j$$

where $\lambda \in (0, 1/(n-1))$ measures the degree of complementarities, capturing the interdependence of criminal activity documented in the empirical literature. With a positive λ, effort levels exerted by network neighbors reinforce each other.

We assume that an agent gets zero payoff if he is caught by the government[5]. Therefore, agent i's net payoff π_i is given by

$$\pi_i(\boldsymbol{x}, \overline{g}; \boldsymbol{\beta}, \lambda, d) = p_i(\overline{g}; \boldsymbol{\beta}, d) \cdot \overline{\pi}_i(\boldsymbol{x}, \overline{g}; \lambda).$$

2.4 Definition of Equilibrium

The timing structure of the model can be formally written as follows.

1. The government announces the allocation of the detection budget $\boldsymbol{\beta}$ and the degree of cascade d.

2. After observing $\boldsymbol{\beta}$ and d, agents propose link with each other. A criminal network $\overline{g}$ is formed via bilateral agreement.

3. Given the network $\overline{g}$, each agent chooses his effort level x_i.

4. Payoff is realized.

Denote the set of all mappings from $\overline{G}$ to $\mathbb{R}_+$ by X. A given agent i's strategy is a pair of a vector of link proposals $g_i \in G_i$ and an effort mapping $x_i(\cdot) \in X$. Let $\boldsymbol{x}(\cdot) = (x_1(\cdot), x_2(\cdot), ..., x_n(\cdot))$. Given a strategy profile $(\boldsymbol{x}(\cdot), g)$, agent i's payoff can be rewritten as a function of the strategy profile

$$\Pi_i(\boldsymbol{x}(\cdot), g; \boldsymbol{\beta}, \lambda, d) \equiv \pi_i(\boldsymbol{x}(\overline{g}(g)), \overline{g}(g); \boldsymbol{\beta}, \lambda, d).$$

As $\boldsymbol{\beta}$, λ, and d are treated as parameters in the n-player two-stage game, we will omit them in the payoff function if it does not cause any confusion.

[5] We will discuss later how our results change if each agent has an outside option with positive payoff.

We first define two standard notions of equilibrium.

Definition 1. *A Nash equilibrium is a strategy profile* $(\boldsymbol{x}^*(\cdot), g^*)$ *such that*

$$\Pi_i(\boldsymbol{x}^*(\cdot), g^*) \geq \Pi_i(x_i(\cdot), x^*_{-i}(\cdot), g_i, g^*_{-i}), \ \ \forall i \in N, \ x_i(\cdot) \in X, \ g_i \in G_i.$$

Definition 2. *A subgame-perfect Nash equilibrium is a strategy profile* $(\boldsymbol{x}^*(\cdot), g^*)$ *such that a Nash equilibrium is played for every subgame.*

Before defining a stronger notion of equilibrium that is more suitable for network formation, we introduce an additional matrix operator. For any $g \in G$, $g \oplus (i,j)$ sets (i,j)-element to be one with all the other elements in g unchanged[6]. Our next equilibrium definition follows Hiller (2014).

Definition 3. *A pairwise stable Nash equilibrium (PSNE) is a strategy profile* $(\boldsymbol{x}^*(\cdot), g^*)$ *such that*

1. $(\boldsymbol{x}^*(\cdot), g^*)$ *is a subgame-perfect Nash equilibrium.*

2. *There is no profitable bilateral deviation at the stage of link formation. For any* (i,j)*-pair such that* $\overline{g}(g^*)_{ij} = 0$ $(i \neq j)$*,*

$$\Pi_i(\boldsymbol{x}^*(\cdot), g^* \oplus (i,j) \oplus (j,i)) > \Pi_i(\boldsymbol{x}^*(\cdot), g^*)$$

implies

$$\Pi_j(\boldsymbol{x}^*(\cdot), g^* \oplus (i,j) \oplus (j,i)) < \Pi_j(\boldsymbol{x}^*(\cdot), g^*).$$

Throughout this paper, we restrict our attention to coordination of link formation, so we only allow agents to coordinate in the first stage of the game. This assumption effectively restricts the set of deviation strategies. In the presence of strategic complementarities, if we allow two agents to coordinate with each other in terms of effort levels, both of them will achieve higher payoffs by choosing higher-than-equilibrium effort levels.

3 Criminal Network Formation

In this section, we characterize and refine equilibria under a given detection policy $(\boldsymbol{\beta}, d)$. We solve the game by backward induction. The equilibrium characterization of the stage

[6] If $g_{ij} = 1$, $g \oplus (i,j) = g$.

game with local complementarities can be found in Ballester et al. (2006). Proposition 1 guarantees a unique equilibrium in this stage game.

Proposition 1. *Given a criminal network $\overline{g} \in \overline{G}$, if $\lambda \in (0, 1/(n-1))$, there exists a unique interior Nash equilibrium for the stage game with local complementarities. In particular, the equilibrium effort level is of the form*

$$\boldsymbol{x}(\overline{g}) = (\mathbf{I} - \lambda \overline{g})^{-1} \cdot \mathbf{1},$$

where $\mathbf{I}$ is an n-dimensional identity matrix and $\mathbf{1}$ is an n-by-1 vector with all elements equal to one. Moreover, agent i's net payoff is given by $p_i(\overline{g})x_i^2(\overline{g})/2$.

All the proofs are delegated to the appendix. Because of the uniqueness of the equilibrium in the second-stage game and its analytical tractability, we can mainly focus our analysis on strategic network formation among agents. The central trade-off faced by each agent is between connectivity and riskiness. Due to local complementarities, connecting with more agents implies higher payoff in the stage game, while it is also riskier to be well connected. In the following three subsections, we will characterize and compare equilibrium under different degrees of cascade ($d = 0, 1, n$).

3.1 No Cascade of Detection ($d = 0$)

It becomes entirely costless for each agent to form links, if there is no cascade of detection at all. Because of strategic complementarity, it is always beneficial to have more links. The following proposition states this simple result.

Proposition 2. *If there is no cascade of detection ($d = 0$), there exists a generically unique pairwise stable Nash equilibrium in which agents form a complete network*[7]*.*

When detection is purely individual-based, the equilibrium criminal network is complete except some polar cases, and as a result, the equilibrium network structure is independent from the detection budget allocation $\boldsymbol{\beta}$.

3.2 Full Cascade of Detection ($d = n$)

In this subsection, we turn to the other extreme by considering full cascade of detection ($d = n$). Detection of agent i triggers detection of any agent that is in the same component

[7] A sufficient condition for the uniqueness of equilibrium is that $\beta_i < 1$ for any $i \in N$.

as agent i. This implies that once agent i chooses to form a link with agent j, there is no additional cost for him to add more links with agents who are in the same component as agent j. Therefore, two agents who are indirectly connected always have incentive to form a direct link with each other. The following lemma formalizes this intuition.

Lemma 1. *Under full cascade of detection ($d = n$), if $\beta_i < 1$ for any $i \in N$, each component of the criminal network is complete in a pairwise stable Nash equilibrium.*

An interesting observation can be drawn from this simple lemma. Though our setting is purely network based, our equilibrium solution resembles a problem of coalitional formation. Because the equilibrium network is always component-wise complete, it can be equivalently expressed as a partition of agents.

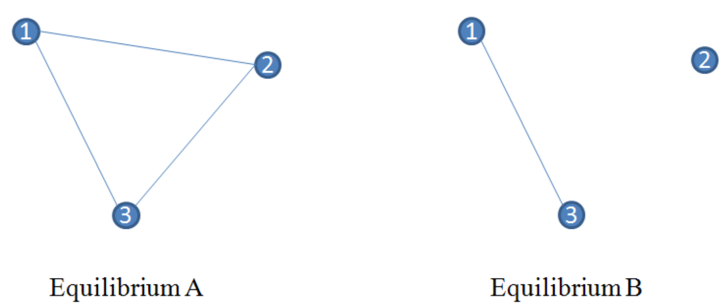

Figure 3: Multiple Pairwise Stable Nash Equilibria ($\beta_1 = \beta_2 = \beta_3 = 0.18,\ \lambda = 0.1$)

However, there typically exists multiple PSNE. For example, figure 3 illustrates two PSNE networks in a three-agent setting with $\beta_1 = \beta_2 = \beta_3 = 0.18$ and $\lambda = 0.1$. It can be calculated that equilibrium A Pareto dominates equilibrium B (from criminals' point of view), but equilibrium B is still pairwise stable because agent 1 has no incentive to form a link to agent 2 if agent 3 stays unconnected with agent 2 and vice versa. Under equilibrium A, all three agents will be able to fully reap the benefit of a complete network by adding two links between agent 2 and the other two agents. Hence, if multilateral coordination of link formation is allowed, only equilibrium A is stable.

To further sharpen our equilibrium characterization, we need to introduce a stronger notion of equilibrium. Following Jackson and van den Nouweland (2005), we say a network $\overline{h} \in \overline{G}$ is *obtainable* from $\overline{g} \in \overline{G}$ via deviations by a nonempty subset $S \subset N$ if the following two conditions are satisfied.

1. $\overline{g}_{ij} = 0$ and $\overline{h}_{ij} = 1$ implies $i, j \in S$;

2. $\overline{g}_{ij} = 1$ and $\overline{h}_{ij} = 0$ implies $\{i, j\} \cap S \neq \emptyset$.

In words, for each link addition, both partners are required to be within this subgroup S, while for each link deletion, at least one partner has to be in S. For example, in the upper panel of Figure 4, network B is obtainable from network A via deviations by $S = \{2, 5, 6\}$, while in the lower panel, network D is not obtainable from network C via deviations by $S = \{2, 5, 6\}$, because a new link is added between agent 1 and 6, but agent 1 is not from the deviation group S.

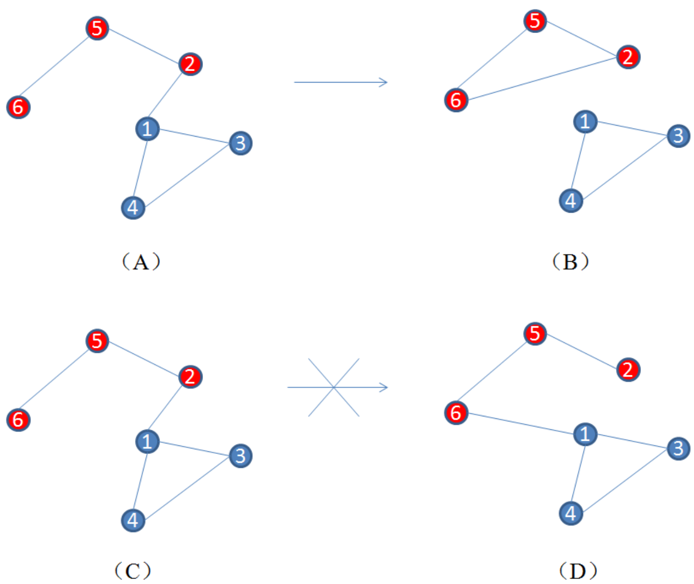

Figure 4: Obtainability

We now define the strongly stable Nash equilibrium à la Jackson and van den Nouweland (2005).

Definition 4. *A strongly stable Nash equilibrium (SSNE) is a strategy profile* $(\boldsymbol{x}^*(\cdot), g^*)$ *such that*

1. $(\boldsymbol{x}^*(\cdot), g^*)$ *is a subgame-perfect Nash equilibrium.*

2. *For any nonempty* $S \subset N$, $\overline{h} \in \overline{G}$ *that is obtainable from* $\overline{g}(g^*)$ *via deviations by* S, *and* $i \in S$ *such that* $\pi_i(\boldsymbol{x}^*(\overline{h}), \overline{h}) > \pi_i(\boldsymbol{x}^*(\overline{g}(g^*)), \overline{g}(g^*))$, *there exists* $j \in S$ *such that* $\pi_j(\boldsymbol{x}^*(\overline{h}), \overline{h}) < \pi_j(\boldsymbol{x}^*(\overline{g}(g^*)), \overline{g}(g^*))$.

By definition, a strongly stable Nash equilibrium is always a pairwise stable Nash equilibrium, because it allows not only bilateral coordination but also multilateral coordination of link formation. In a pairwise stable Nash equilibrium, an agent is not allowed to add several links simultaneously even though doing so is mutually beneficial for all agents involved in link addition. A strongly stable Nash equilibrium rules out this implausible limitation by allowing multilateral coordination. The following result indicates that the criminal network in a strongly stable Nash equilibrium is formed assortatively: each agent tends to be connected with agents with similar probability of being directly detected.

Lemma 2. *In any strongly stable Nash equilibrium, the equilibrium partition of agents*[8] *"preserves" the order of detection probability*

$$\left\{\{1,2,...,n_1\},\{n_1+1,n_1+2,...,n_1+n_2\},...,\{\sum_{i=1}^{k-1}n_i+1,\sum_{i=1}^{k-1}n_i+2,...,\sum_{i=1}^{k}n_i\}\right\}$$

where $n\equiv\sum_{i=1}^{k}n_i$ *and agents are labeled such that* $\beta_1\leq\beta_2\leq...\leq\beta_n$.

The central idea in proving this lemma is to show that if agent i is connected with agent j, they must be connected with any agent with detection probability in between β_i and β_j. Intuitively, the least risky agent always wants to pick the second least risky agent if he is ever willing to connect, and this incentive of connection is also aligned with that of the second least risky agent. The lemma is a direct generalization of this intuition. The next proposition establishes the existence of a unique strongly stable Nash equilibrium and fully characterizes the equilibrium network structure, or equivalently, the equilibrium partition. This characterization is particularly useful, because it operationalizes the key target of an optimal detection policy, that is, the size of the non-singleton component.

Proposition 3. *There exists a generically unique strongly stable Nash equilibrium with the equilibrium partition* $\{\{1,2,...,n_0\},\{n_0+1\},\{n_0+2\},...,\{n\}\}$ *and*

$$n_0=\max\left\{\arg\max_{k\in N}\pi^k\right\}, \tag{1}$$

[8]Recall that the equilibrium network under full cascade of detection is always component-wise complete, so the equilibrium network can be equivalently expressed as the equilibrium partition of agents.

where π^k is the individual payoff of a complete component formed by the first k agents,

$$\pi^k = \frac{1}{2}\left(\frac{1}{1-(k-1)\lambda}\right)^2 \Pi_{i=1}^{k}(1-\beta_i). \tag{2}$$

More precisely, this equilibrium is unique if at least one of the following conditions holds:

$$(a)\ n_0 = 1;\ (b)\ \beta_1 < \beta_{n_0};\ (c)\ \frac{1-\beta_1}{2} < \frac{1}{2}\left(\frac{1}{1-(n_0-1)\lambda}\right)^2 \Pi_{i=1}^{n_0}(1-\beta_i).$$

If none of these three conditions holds, there exists another equilibrium in which the equilibrium network is empty.

The proof of this proposition consists of three steps. We first show that in any SSNE agents are divided into two groups: the first group form a complete component, while each agent in the second group is isolated. This network structure is also known as the dominant group architecture[9] (Goyal and Joshi, 2003). We then demonstrate that the equilibrium partition constructed above yields an SSNE. As a last step, we provide necessary and sufficient conditions to guarantee the uniqueness of equilibrium. Since there exists a unique SSNE except some rather restrictive cases, we are able to further analyze the optimal detection policy without concerning issues of equilibrium selection.

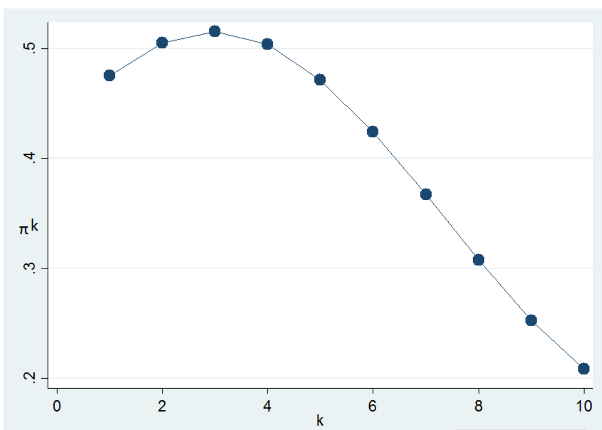

Figure 5: Individual Payoff of the Largest Component ($n = 10; \beta_k = k/20; \lambda = 0.08$)

This proposition also suggests how to find a strongly stable Nash equilibrium via simple

[9]Formally, the dominant group architecture is characterized by a complete non-singleton component and a set of isolated nodes.

calculation. Given $\boldsymbol{\beta}$ and λ, the individual payoff of a complete component consisting of the first k agents, π^k, can be readily calculated, and n_0 is simply given by the largest maximizer of π^k. As a numerical example, consider a 10-agent game with $\beta_k = k/20$ and $\lambda = 0.08$. This example is illustrated by Figure 5. It can be seen that $k = 3$ is the unique maximizer of π^k, and therefore the equilibrium partition is obtained as $\{\{1,2,3\},\{4\},\{5\},...,\{10\}\}$.

3.3 Partial Cascade of Detection ($d = 1$)

Under partial cascade of detection ($d = 1$), each agent is faced with a more nuanced trade-off: unlike full cascade of detection, adding a new link always brings about additional risk of being detected, so each agent becomes more selective in link formation. This selection motive tends to reduce the number of links each agent is willing to form. On the other hand, compared with full cascade of detection, agents become less vulnerable to link formation, because each agent is only exposed to the risk of his direct neighbors. Interestingly, it turns out that the selection motive dominates. In particular, any PSNE, including SSNE if any[10], yields a criminal network weakly *sparser* than that of the unique SSNE under full cascade of detection. The following proposition states one of the central results in this paper.

Proposition 4. *Those players who are isolated in the strongly stable Nash equilibrium under full cascade of detection remain isolated in any pairwise-stable Nash equilibrium under partial cascade of detection.*

The key step in the proof is to show that in any PSNE no one wants to be directly connected with those agents who are isolated in the unique SSNE under full cascade of detection. Recall that the individual effort level increases with the number of links an agent has. Since any PSNE network under partial cascade of detection consists of a subset of links of the SSNE criminal network under full cascade of detection, the aggregate criminal activity in any PSNE under partial cascade of detection is also weakly lower than that in the SSNE under full cascade of detection. Although we do not have a complete equilibrium characterization of a PSNE under partial cascade of detection, this proposition provides an “upper bound” in terms of the equilibrium network structure as well as the aggregate criminal activity. Moreover, the proposition also suggests that criminal network formation exhibits strong discontinuity with respect to the degree of cascade: when the detection policy

[10] Under partial cascade of detection, SSNE may not exist. For example, when $n = 3$, $\beta_1 = \beta_2 = \beta_3 = 0.195$, and $\lambda = 0.1$, there exists a unique PSNE which yields an empty network and there is no SSNE.

switches from no cascade to partial cascade, it tends to achieve a social outcome that is as desirable as, if not better than, that of the unique SSNE under full cascade of detection.

4 Optimal Detection Policy

The optimal allocation of detection resources hinges on the degree of cascade of detection. When there is no cascade of detection, detection resource allocation plays little role in shaping the criminal network formation; in the presence of partial or full cascade of detection, optimal allocation of detection resource further depends on which equilibrium n agents choose to play. To highlight the core trade-off the government is faced with, we consider a very specific form of the decision problem as follows

$$\min_{\boldsymbol{\beta} \in \mathbb{R}_+^n} n_0(\boldsymbol{\beta}), \quad \text{s.t.} \quad \sum_{i=1}^{n} \beta_i \leq B,$$

where $n_0(\boldsymbol{\beta})$ is defined in Equation 1 and it is interpreted as the size of the largest component of the criminal network in the unique SSNE under full cascade of detection given the detection resource allocation $\boldsymbol{\beta}$. We take the unique SSNE under full cascade of detection as a benchmark for two reasons. First, uniqueness of SSNE guarantees the minimization problem is well-defined. Second, since this decision problem is equivalent to minimizing the total effort level in the SSNE under full cascade of detection[11], in light of Proposition 4, this formulation of the government decision problem can be alternatively interpreted as a minimax problem: it is to minimize the upper bound of the aggregate criminal activity in any PSNE under partial cascade of detection.

Apparently, if the total detection resource B is sufficiently large, the government is always able to keep the SSNE criminal network empty. We therefore consider a variant of the original problem: if the government wants every agent to be isolated in the SSNE, what is the minimum detection resource required? Specifically, we have

$$\min_{\boldsymbol{\beta} \in \mathbb{R}_+^n} \sum_{i=1}^{n} \beta_i, \quad \text{s.t.} \quad \pi^k(\boldsymbol{\beta}) \leq \pi^1(\boldsymbol{\beta}), \quad k \in N,$$

[11] Formally, it is equivalent to assume that the government has the following decision problem under full cascade of detection

$$\min_{\boldsymbol{\beta} \in \mathbb{R}_+^n} \sum_{i=1}^{n} x_i^* \quad \text{s.t.} \quad \sum_{i=1}^{n} \beta_i \leq B,$$

where x_i^* is the effort level chosen by player i in the SSNE and it depends on the resource allocation $\boldsymbol{\beta}$.

where $\pi^k(\boldsymbol{\beta})$ is the individual payoff of a complete size-k component defined in Equation 2. According to Proposition 3, the criminal network in the SSNE is empty if and only if $\pi^k(\boldsymbol{\beta}) < \pi^1(\boldsymbol{\beta})$ for $k = 2, 3, ..., n$. Therefore, this minimization problem yields a lower bound of detection resource to ensure an empty criminal network. The following proposition establishes this lower bound and specifies one optimal allocation of the detection budget provided that the detection budget exceeds the lower bound.

Proposition 5. *Under full cascade of detection, the government can keep each agent isolated in the strongly stable Nash equilibrium if and only if*

$$B > B_1 \equiv n - 1 - \sum_{k=2}^{n} \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda} \right)^2,$$

and the optimal allocation of the detection budget is given by $\beta_1 = 0$ *and*

$$\beta_k = 1 - \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda} \right)^2 + \frac{B - B_1}{n-1}, \quad k = 2, 3, ..., n.$$

Although the proof of this proposition is complicated by the implicit ordering of $\boldsymbol{\beta}$, the underlying logic is very intuitive. The government first allocates its detection budget such that $\pi^1(\boldsymbol{\beta}) = \pi^k(\boldsymbol{\beta})$ for any $k \in N$ and let the first agent be free of risk, $\beta_1 = 0$. In that $\pi^k(\boldsymbol{\beta})$ is convex with respect to k, β_k also has to be convex with respect to k for $k \geq 2$. This allocation gives rise to a knife-edge situation in which a complete size-n criminal network is formed, but this equilibrium is not robust even to a small perturbation of $\boldsymbol{\beta}$. Imposing ε-increment of scrutiny on each agent except agent 1, the government will achieve the first-best by making $\pi^1(\boldsymbol{\beta}) > \pi^k(\boldsymbol{\beta})$ for any $k \geq 2$. In the optimal allocation of the detection budget we choose, the government simply divide the extra budget, $B - B_1$, equally among $n - 1$ agents.

Without enough detection resource ($B \leq B_1$), the government has to tolerate certain degree of networking among agents. If the government makes the compromise and only attempts to keep the size of the largest component in the SSNE network to be $S \in \{2, 3, ..., n-1\}$, what is the minimum requirement of the detection budget? Similarly, this question can be formulated as

$$\min_{\boldsymbol{\beta} \in \mathbb{R}_+^n} \sum_{i=1}^{n} \beta_i, \quad \text{s.t.} \quad \pi^i(\boldsymbol{\beta}) \leq \pi^S(\boldsymbol{\beta}), \quad \forall i \in N.$$

Solving this minimization problem and applying the same argument as above yields the following generalization of Proposition 5.

Corollary 1. *Under full cascade of detection, the government can keep the size of the largest component of the criminal network in the strongly stable Nash equilibrium to be* $S \in \{2, 3, ..., n-1\}$ *if and only if*

$$B > B_S \equiv n - S - \sum_{k=S+1}^{n} \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda} \right)^2 ,$$

and the optimal allocation of the detection budget is given by $\beta_k = 0$ *for* $k \leq S$ *and*

$$\beta_k = 1 - \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda} \right)^2 + \frac{B - B_S}{n - S}, \quad k = S+1, S+2, ..., n.$$

Despite agents being ex ante identical, our results say that they are subject to heterogeneous levels of scrutiny under the optimal detection policy. This finding complements earlier theoretical results by Baccara and Bar-Isaac (2008) in which they derive the optimal detection resource allocation in consideration of a criminal network of information sharing from a group perspective. In contrast, we obtain our optimal allocation rules in a model featuring local complementarities and individual incentives of network formation. Although our model is admittedly stylized, we believe our results have broad policy implications. Our model highlights that local complementarities have very important implication on the optimal resource allocation. In the presence of local complementarities, agents tend to exert more effort when connecting with more other agents. This "scale effect" gives rise to the asymmetric allocation of the detection budget, as a sequence of increasing scrutiny levels is introduced to deter agents from forming increasingly larger networks. Moreover, we can show that $\partial B_S / \partial \lambda > 0$ for any $S \in \{1, 2, ..., n-1\}$. The comparative statics says the minimum requirement of the detection resource monotonically strictly increases with the degree of local complementarities. Intuitively, if effort levels among neighbors tend to significantly reinforce each other and thereby agents have strong incentive to connect, it becomes more difficult for the government to restrict criminal network formation. Therefore, to inform the detection policy, future empirical work needs to quantify the degree of local complementarities in different types of criminal networks.

5 Extension and Discussion

5.1 Degree of Cascade

In the benchmark model, we have focused on three specific cases of the degree of cascade: $d = 0, 1, n$. A natural question is whether our results, Proposition 4 in particular, are robust under other degrees of cascade. The answer is of affirmative.

Proposition 6. *Those players who are isolated in the strongly stable Nash equilibrium under full cascade of detection ($d = n$) remain isolated in any strongly stable Nash equilibrium under positive degree of cascade ($d \in \{1, 2, ..., n\}$).*

The proposition says that as long as there is positive degree of cascade, any SSNE yields a criminal network weakly sparser than the one under full cascade of detection. It should be noticed that this proposition is not a strict generalization of Proposition 4 in that it focuses exclusively on SSNE[12]. However, if we restrict our attention to SSNE, this proposition reassures that the equilibrium aggregate criminal activity under full cascade of detection serves well as an upper bound of the aggregate criminal activity under any positive degree of cascade.

5.2 Outside Option

In our baseline setting, agents have no outside option. A more plausible assumption is that each agent is allowed to opt out if the payoff of the criminal activity is sufficiently low. Denote the payoff of the outside option by π_0. We assume that after the criminal network $\overline{g} \in \overline{G}$ is formed and before each agent exerts any effort, they can choose to opt out. We first consider full cascade of detection. Recall that π^k is defined as the individual payoff of a complete component formed by the first k agents. Proposition 3 suggests that $\max_{k \in N} \pi^k$ is the individual payoff of the largest component in the unique SSNE. In the presence of the outside option, the equilibrium network structure solely depends on the comparison between π_0 and $\max_{k \in N} \pi^k$. If the outside option is sufficiently attractive such that $\pi_0 > \max_{k \in N} \pi^k$, there exists a unique SSNE that gives rise to an empty criminal network[13]. Every agent opts out in the equilibrium because even the highest attainable payoff in a criminal network is strictly less than the payoff of the outside option. On the other hand, if $\pi_0 < \max_{k \in N} \pi^k$,

[12] In fact, for $d \geq 2$, we can always find examples such that PSNE gives rise to a criminal network with more links than that of the unique SSNE under full cascade of detection.

[13] In this case, the unique SSNE also coincides with the *unique* PSNE.

Proposition 3 continues to hold with a slight modification: the size of the largest component is still determined by Equation 1, while isolated agents may choose to opt out given their own individual payoffs. If $\pi_0 = \max_{k\in N} \pi^k$, the criminal network in SSNE can be either empty or dominant group architecture. Therefore, a small change of π_0 around $\max_{k\in N} \pi^k$ may considerably change the equilibrium network. Under partial cascade of detection, it can still be shown that anyone who is isolated in the unique SSNE under full cascade of detection remains isolated in any PSNE. The proof carries over because introducing the outside option does not alter the core trade-off each agent is faced with. However, the outside option does add another layer to the optimal detection policy, as the government can obtain the socially optimal outcome by inducing each agent to opt out if the maximum payoff under the criminal network is dominated by the outside option.

Proposition 7. *Let $B_n = 0$. If the detection budget $B \in [B_{\ell+1}, B_\ell)$*[14] *for $\ell \in \{1, 2, ..., n-1\}$, the government can incentivize all agents to opt out if and only if $\frac{1-(B-B_{\ell+1})}{2(1-\ell\lambda)^2} < \pi_0$ with the allocation of the detection budget given by $\beta_k = 0$ for $k \leq \ell$, $\beta_{\ell+1} = B - B_{\ell+1}$, and $\beta_k = 1 - \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2$ for $k > \ell + 1$.*

This proposition[15] suggests that the government may achieve the first best even though its detection budget is insufficient to keep each agent isolated in the equilibrium criminal network. Compared with 1, the allocation of the detection budget looks quite similar. They are actually two sides of the same coin. In the absence of the outside option, the government focuses on the size of the equilibrium criminal network, formally, the maximizer of $\max_{k\in N} \pi^k$; in the presence of the outside option, the government also cares the individual payoff under the equilibrium network, that is, the maximum of $\max_{k\in N} \pi^k$. That is why two optimal allocations share the same ingredient in spite of their different objectives.

5.3 Exogenous Linking Cost

We now introduce exogenous linking cost into the benchmark model. Denote the cost of forming a link by c. The new payoff function can be written as

$$\tilde{\Pi}_i(\boldsymbol{x}(\cdot), g; \boldsymbol{\beta}, \lambda, d) \equiv \pi_i(\boldsymbol{x}(\overline{g}(g)), \overline{g}(g); \boldsymbol{\beta}, \lambda, d) - \eta_i(\overline{g}(g))c,$$

[14]Recall that $B_S \equiv n - S - \sum_{k=S+1}^{n} \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2$ for $S \in \{1, 2, ..., n-1\}$.

[15]If $B > B_1$, the optimal allocation rule needs to be solved recursively. It is entirely due to the fact that the implicit constraint $\beta_1 \leq \beta_2 \leq ... \leq \beta_n$ becomes binding. Discussion is omitted because it does not give additional insight.

where $\eta_i(\overline{g}(g))$ is the number of links agent i has in the criminal network $\overline{g}(g)$. Under no cascade of detection, the results in Hiller (2014) naturally carry over: when exogenous linking cost is sufficiently low, a complete criminal network is formed in any PSNE; when exogenous linking cost is sufficiently high, the unique PSNE network is empty; when exogenous linking cost is in intermediate level, hierarchy may arise in a PSNE[16] with more central agents exerting higher effort levels. Under full cascade of detection, the results in Hiller (2014) apply component-wisely to the criminal network in any PSNE. However, the model becomes much less tractable under partial cascade of detection, because the cost of adding a new link enters the payoff function both multiplicatively (increasing probability of being detected) and additively (exogenous linking cost). This modification substantially complicates the trade-off faced by each agent.

5.4 Timing Structure

In our model, partial cascade of detection is ex ante more desirable, or at least as desirable as, full cascade of detection, while full cascade of detection is ex post optimal. On the other hand, the asymmetric allocation of the detection budget also leaves the government room of manipulation. These two channels of dynamic inconsistency echo earlier discussion in Baccara and Bar-Isaac (2008). Without commitment technology, the government has incentive to re-optimize their detection policy after criminal network formation. This re-optimization motive could have significant impact on equilibrium network structures. However, similar to Baccara and Bar-Isaac (2008), we argue that the system of law enforcement is relatively rigid. The timing structure can be justified if individuals have expectation that the detection policy is not amenable to change in the short run and crackdowns are not frequently implemented.

6 Conclusion

In this paper, we study the optimal detection policy in the presence of criminal network from the ex ante point of view. Using the criminal network in the unique SSNE under full cascade of detection as a benchmark, we have two main findings. The cascade of detection is identified as an important channel through which the detection policy could shape the

[16]Formally, any equilibrium network is a *nested split graph*. For discussions about nested split graphs and its applications in network economics, see König et al. (2014) and references therein.

criminal network. We show that stronger cascade of detection could backfire. This reminds policy makers of the importance of endogenous network formation among criminals. We also derive the optimal allocation of the detection budget. In the presence of strategic complementarities, the optimal budget allocation tends to be asymmetric across ex ante identical agents.

We believe three directions of the future work are very promising. The cost structure of criminal activity in our model is very simple. In our benchmark setting, there is no explicit bilateral linking cost. It would be very interesting to extend our framework by incorporating more flexible bilateral cost specification like Belhaj et al. (2015). Second, the police system and criminal networks are evolving over time. Studying the optimal detection policy in a dynamic model is technically challenging but of particular interest (Jackson and Zenou, 2014). Finally, it is important to have systematic understanding of how the detection policy affects the criminal network in actual practice. We envisage that empirical investigation in this line will be fruitful.

A Appendix

A.1 Proof of Proposition 1

Proof can be found in Ballester et al. (2006).

A.2 Proof of Proposition 2

Because detection is purely individual-based, connecting with other agents does not increase probability of being detected. According to Ballester et al. (2006), $(\mathbf{I} - \lambda \overline{g})^{-1} = \sum_{k=0}^{\infty} \lambda^k \overline{g}^k$ if $\lambda \in (0, 1/(n-1))$. Therefore, the equilibrium effort level given by Proposition 1 increases with the number of network walks. Since stage-game payoff is an increasing function of each agent's own effort level, each agent prefers to form as many links as possible so as to increase network walks. In particular, agent i's payoff strictly increases with the number of links he forms if and only if $\beta_i < 1$. Therefore, if $\beta_i < 1$ for any $i \in N$, a unique complete criminal network emerges.

A.3 Proof of Lemma 1

The proof of this lemma closely follows the proof of Proposition 2. Recall that probability of being detected remains unchanged when an agent forms a link with someone who is in the same component. Because the equilibrium payoff of the stage game increases with the number of network walks, each agent has incentive to be directly connected with everyone who is in the same component. Therefore, the criminal network under any pairwise stable Nash equilibrium has to be component-wise complete.

A.3.1 Proof of Lemma 2

To prove this lemma, we first prove the following claim.

Claim 1. *In any strongly stable Nash equilibrium, if there exist three agents with $\beta_i < \beta_k < \beta_j$ and agent i is connected with agent j ($g_{ij} = g_{ji} = 1$), then agent k must be connected with agent i and j ($g_{ik} = g_{ki} = g_{jk} = g_{kj} = 1$).*

We prove this claim by contraposition. Suppose there exist agent i, j, and k with $\beta_i < \beta_k < \beta_j$ such that i and j are connected while k is not connected with either of them. Denote the number of agents who are in the same component as agent i and j by m. Because the equilibrium network must be component-wise complete, each agent who is in the same component shares a common payoff. Denote agent i and j's payoff by π. Now consider that agent k joins i's component by connecting with everyone in agent i's component and dropping all his existing links. Under this deviation, agent i's payoff, which is equal to agent k's payoff, is given by

$$\pi_{\oplus k} = \left(1 + \frac{\lambda}{1 - m\lambda}\right)^2 (1 - \beta_k)\pi.$$

It is noticed that $\pi_{\oplus k} > \pi$, because otherwise each agent except j in agent i's component will be better off by excluding agent j, which can be seen more clearly by

$$\pi = \left(1 + \frac{\lambda}{1 - (m-1)\lambda}\right)^2 (1 - \beta_j)\pi_{\ominus j},$$

where $\pi_{\ominus j}$ is the individual payoff in i's component when agent j is excluded. Since $\beta_k < \beta_j$ implies that $\left(1 + \frac{\lambda}{1-m\lambda}\right)^2 (1 - \beta_k) > \left(1 + \frac{\lambda}{1-(m-1)\lambda}\right)^2 (1 - \beta_j)$, $\pi \geq \pi_{\ominus j}$ implies $\pi_{\oplus k} > \pi$. The remaining question is whether agent k is willing to join i's component. Suppose he is not willing to change his current linking choice. Therefore, his current payoff $\pi' \geq \pi_{\oplus k} > \pi$.

This implies that agent k cannot be isolated with no connections, because otherwise agent i will be better off by dropping all his links. If agent k is connected with someone, the same argument above applies here. That being said, everyone in agent k's component must be willing to connect with agent i because $\beta_i < \beta_k$. Given $\pi' > \pi$, agent i will be strictly better off by joining agent k's component. It contradicts strong stability.

Next, we want to argue that agents with the same probability of being directly detected must be connected with each other if they choose not be isolated. Again, we prove by contraposition. Suppose there exist two agents i and j with $\beta_i = \beta_j$ and they are not connected with each other ($g_{ij} = g_{ji} = 0$). Without loss of generality, we assume agent i is not isolated and he is also connected with agent k. Denote agent i' payoff by π_i and agent j's payoff by π_j. The similar reasoning applies. Everyone in agent i's component will be willing to add agent j into their component. Therefore, it must be the case that agent j is not willing to join i's component. This implies $\pi_j > \pi_i$, which further implies agent j is not isolated. Since $\pi_j > \pi_i$, agent i and everyone in j's component will be better off by forming a larger component together. Contradiction.

In sum, each component of the equilibrium network in a strongly stable Nash equilibrium can only be one of the following two cases: (1) singleton; (2) Given any two agents i and j from the same component ($\beta_i \leq \beta_j$), any agent with detection probability within $[\beta_i, \beta_j]$ must also be in that component. This completes our proof of Lemma 2.

A.3.2 Proof of Proposition 3

We first show that in any strongly stable Nash equilibrium, the equilibrium partition can be written as

$$\{\{1, 2, ..., n_0\}, \{n_0 + 1\}, \{n_0 + 2\}, ..., \{n\}\}.$$

That being said, the equilibrium consists of at most one non-singleton component.

We prove by contraposition. Suppose there are two components each consisting of more than one agents in a strongly stable Nash equilibrium. Denote the first component by $\{i_1, i_2, ..., i_{\ell+1}\}$ and the second component by $\{j_1, j_2, ..., j_{m+1}\}$. In that any equilibrium network is component-wise complete, the individual payoff of the first component is given by

$$\pi = \frac{1}{2}\left(\frac{1}{1-\ell\lambda}\right)^2 \Pi_{h=1}^{\ell+1}(1-\beta_{i_h}),$$

while the individual payoff of the second component is given by

$$\pi' = \frac{1}{2}\left(\frac{1}{1-m\lambda}\right)^2 \Pi_{h=1}^{m+1}(1-\beta_{j_h}).$$

Without loss of generality, we assume that $\pi \geq \pi'$. Suppose these two components are regrouped into a larger component $\{i_1, i_2, ..., i_{\ell+1}, j_1, j_2, ..., j_m\}$ and a singleton $\{j_{m+1}\}$. Now consider the individual payoff of the larger component, which is given by

$$\begin{aligned}
\pi'' &= \frac{1}{2}\left(\frac{1}{1-(\ell+m)\lambda}\right)^2 \Pi_{h=1}^{\ell+1}(1-\beta_{i_h}) \cdot \Pi_{h=1}^{m}(1-\beta_{j_h}) \\
&> \frac{1}{2}\left(\frac{1}{1-\ell\lambda}\right)^2 \Pi_{h=1}^{\ell+1}(1-\beta_{i_h}) \cdot \left(\frac{1}{1-m\lambda}\right)^2 \Pi_{h=1}^{m}(1-\beta_{j_h}) \\
&= \pi \cdot \left(\frac{1}{1-m\lambda}\right)^2 \Pi_{h=1}^{m}(1-\beta_{j_h})
\end{aligned}$$

The equilibrium condition implies that no one in the second component $\{j_1, j_2, ..., j_{m+1}\}$ can be better off by being isolated, *i.e.*, $\pi' \geq (1-\beta_{j_h})/2$ for any $h = 1, 2, ..., m+1$. This further implies that $\left(\frac{1}{1-m\lambda}\right)^2 \Pi_{h=1}^{m}(1-\beta_{j_h}) \geq 1$. According to the inequality above, we have $\pi'' > \pi \geq \pi'$. Therefore, everyone in the new, larger component will be strictly better off, contracting to strong stability.

Next we argue that agent 1 must be included in the largest component if that component is not a singleton. Suppose this is not true. Given the conclusion above, agent 1 must be isolated. Denote the greatest component by $\{n_1+1, n_1+2, ..., n_1+n_0\}$ with $n_1 > 1$ and $n_0 > 1$. Similarly, it can be shown that it is mutually beneficial to form a complete component $\{1, n_1+1, n_1+2, ..., n_1+n_0-1\}$ by including agent 1 and excluding agent (n_1+n_0).

The next step is to show that there indeed exists a strongly stable Nash equilibrium with an equilibrium partition $\{\{1, 2, ..., n_0\}, \{n_0+1\}, \{n_0+2\}, ..., \{n\}\}$. For a given partition $\{\{1, 2, ..., k\}, \{k+1\}, \{k+2\}, ..., \{n\}\}$, the individual payoff of the size-k component is given by

$$\pi^k = \frac{1}{2}\left(\frac{1}{1-(k-1)\lambda}\right)^2 \Pi_{h=1}^{k}(1-\beta_h), \quad k \in N. \tag{A.3}$$

Let n_0 be the maximal size of the largest component such that the individual payoff of the largest component is maximized

$$n_0 = \max\left\{\arg\max_{k\in N} \frac{1}{2}\left(\frac{1}{1-(k-1)\lambda}\right)^2 \Pi_{h=1}^{k}(1-\beta_h)\right\} \equiv \max\{\arg\max_{k\in N} \pi^k\}.$$

We need to show this partition $\{\{1,2,...,n_0\},\{n_0+1\},\{n_0+2\},...,\{n\}\}$ with n_0 defined above gives rise to a strongly stable Nash equilibrium. Arbitrarily pick an alternative criminal network that is obtainable from the network implied by this partition via deviations by $S \subset N$. There are three possible cases. (1) $S \subset \{1,2,...,n_0\}$. By definition of n_0, no agent in S can be strictly better off under this alternative network. (2) $S \subset \{n_0+1, n_0+2, ..., n\}$. Suppose agent $i \in S$ becomes strictly better off. Denote the set of agents in his component by $\{v_1, v_2, ..., v_{k_0+1}\}$ with $v_{k_0+1} = i$. According to the proof of Proposition 2, agent i's payoff under the alternative network is weakly less than

$$\frac{1}{2}\left(\frac{1}{1-k_0\lambda}\right)^2 \Pi_{h=1}^{k_0+1}(1-\beta_{v_h}),$$

which further implies

$$\left(\frac{1}{1-k_0\lambda}\right)^2 \Pi_{h=1}^{k_0}(1-\beta_{v_h}) > 1.$$

If we allow $\{1,2,...,n_0,v_1,v_2,...,v_{k_0}\}$ to form a complete component, the individual payoff of this component is given by

$$\frac{1}{2}\left(\frac{1}{1-(n_0+k_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h)\Pi_{h=1}^{k_0}(1-\beta_{v_h}).$$

This yields a contradiction to the definition of n_0 because

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{1-(n_0+k_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h)\Pi_{h=1}^{k_0}(1-\beta_{v_h}) \\
&> \frac{1}{2}\left(\frac{1}{1-(n_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h)\left(\frac{1}{1-k_0\lambda}\right)^2 \Pi_{h=1}^{k_0}(1-\beta_{v_h}) \\
&> \frac{1}{2}\left(\frac{1}{1-(n_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h).
\end{aligned}$$

(3) $S \cap \{1,2,...,n_0\} \neq \emptyset$ and $S \cap \{n_0+1, n_0+2, ..., n\} \neq \emptyset$. If there is no new link created between an agent in $\{1,2,...,n_0\}$ and an agent in $\{n_0+1, n_0+2, ..., n\}$, our argument in Case (1) and (2) applies. If there exists a new link connecting an agent i in $\{1,2,...,n_0\}$ and

an agent j in $\{n_0+1, n_0+2, ..., n\}$, we argue that agent i must be strictly worse off under the alternative network. Denote the set of agents in agent i's component by $\{w_1, w_2, ..., w_{k_1}\}$ with $w_{k_1} = i$. Agent i's payoff is weakly less than

$$\frac{1}{2}\left(\frac{1}{1-(k_1-1)\lambda}\right)^2 \Pi_{h=1}^{k_1}(1-\beta_{w_h}).$$

If $k_1 > n_0$, by definition of n_0, we have

$$\frac{1}{2}\left(\frac{1}{1-(k_1-1)\lambda}\right)^2 \Pi_{h=1}^{k_1}(1-\beta_{w_h}) < \frac{1}{2}\left(\frac{1}{1-(n_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h).$$

If $k_1 \leq n_0$, we have

$$\begin{aligned} \frac{1}{2}\left(\frac{1}{1-(k_1-1)\lambda}\right)^2 \Pi_{h=1}^{k_1}(1-\beta_{w_h}) &< \frac{1}{2}\left(\frac{1}{1-(k_1-1)\lambda}\right)^2 \Pi_{h=1}^{k_1}(1-\beta_h) \\ &\leq \frac{1}{2}\left(\frac{1}{1-(n_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h), \end{aligned}$$

where the first inequality holds because $\beta_j > \beta_{n_0}$[17] for $n_0 \geq 2$.

The last step is to establish conditions that guarantee the uniqueness of a strongly stable Nash equilibrium. Suppose there exist two equilibria with the following equilibrium partitions: $\{\{1, 2, ..., n_0^1\}, \{n_0^1+1\}, \{n_0^1+2\}, ..., \{n\}\}$ and $\{\{1, 2, ..., n_0^2\}, \{n_0^2+1\}, \{n_0^2+2\}, ..., \{n\}\}$ $(n_0^1 > n_0^2 > 1)$. If the individual payoff of the component $\{1, 2, ..., n_0^1\}$ is strictly less than that of the component $\{1, 2, ..., n_0^2\}$, the first equilibrium is not stable because a subset of agents from $\{1, 2, ..., n_0^1\}$ can increase their payoff by forming a complete component by themselves. If the individual payoff of the component $\{1, 2, ..., n_0^1\}$ is weakly greater than that of the component $\{1, 2, ..., n_0^2\}$, the second equilibrium is not stable because it is mutually beneficial for agents $\{1, 2, ..., n_0^2\}$ and isolated agents $\{n_0^2+1, n_0^2+2, ..., n_0^1\}$ to form a larger complete component. Therefore, we have shown that there exist at most two strongly stable Nash equilibria: one has the equilibrium partition $\{\{1, 2, ..., n_0\}, \{n_0+1\}, \{n_0+2\}, ..., \{n\}\}$ and the other has an empty network. When $n_0 = 1$, these two equilibria coincides. When $n_0 > 1$, if $\beta_1 < \beta_{n_0}$ or $\frac{1-\beta_1}{2} < \frac{1}{2}\left(\frac{1}{1-(n_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h)$, agents $\{\{1, 2, ..., n_0\}$ always have incentive to deviate from the empty network by forming a complete component. Multiplicity of strongly stable Nash equilibria arises only if $n_0 > 1$, $\beta_1 = \beta_{n_0}$, and $\frac{1-\beta_1}{2} = \frac{1}{2}\left(\frac{1}{1-(n_0-1)\lambda}\right)^2 \Pi_{h=1}^{n_0}(1-\beta_h)$.

[17] Recall $j \in \{n_0+1, n_0+2, ..., n\}$, so $\beta_j = \beta_{n_0}$ yields a contradiction to the definition of n_0.

This completes the proof of Proposition 3.

A.3.3 Proof of Proposition 4

To establish this proposition, we proceed by first proving two claims.

Claim 2. *For any $\overline{g} \in \overline{G}$, let $L(\overline{g}) \equiv \max_{i\in N} \sum_{j=1}^{n} \overline{g}_{ij}$. We have*

$$x_i(\overline{g}) \leq \frac{1}{1-L(\overline{g})\lambda}, \quad \forall i \in N,\ \overline{g} \in \overline{G}.$$

In words, if the most connected agent in a criminal network has L links, we claim that the highest individual effort level is weakly less than that in a complete network of size L. If each agent in a network $\overline{g} \in \overline{G}$ has exactly L links, it can be easily verified that $x_i = 1/(1-L\lambda)$. Now we consider the general case in which each agent is connected to L agents at maximum. We have

$$\begin{aligned}
\boldsymbol{x}(\overline{g}) - \frac{1}{1-L(\overline{g})\lambda}\mathbf{1} &= \left[(I-\lambda\overline{g})^{-1} - (I-\lambda L(\overline{g})I)^{-1}\right]\mathbf{1} \\
&= \lambda(I-\lambda\overline{g})^{-1}(\overline{g}-L(\overline{g})I)(I-\lambda L(\overline{g})I)^{-1}\mathbf{1} \\
&= \frac{\lambda}{1-\lambda L(\overline{g})}(I-\lambda\overline{g})^{-1}(\overline{g}\mathbf{1}-L(\overline{g})\mathbf{1})
\end{aligned}$$

By definition, $L(\overline{g}) \geq \sum_{j=1}^{n} \overline{g}_{ij}$ for any $i \in N$, so each element in $(\overline{g}\mathbf{1}-L(\overline{g})\mathbf{1})$ is non-positive. Because $(I-\lambda\overline{g})$ is an M-matrix[18] for $\lambda < 1/(n-1)$, $(I-\lambda\overline{g})^{-1}$ is a non-negative matrix (Plemmons, 1977). Therefore, $(I-\lambda\overline{g})^{-1}(\overline{g}\mathbf{1}-L(\overline{g})\mathbf{1})$ is non-positive, which implies the inequality in Claim 2

Consider an arbitrary network $\overline{g} \in \overline{G}$. Suppose agent i drops ℓ of his existing links with agents $j_1, j_2, ..., j_\ell$ and denote by $\overline{h} \in \overline{G}$ the new criminal network obtained. we can prove the following inequality always holds.

Claim 3.

$$\frac{x_i(\overline{h})}{x_i(\overline{g})} \geq \frac{1-L(\overline{g})\lambda}{1-(L(\overline{g})-\ell)\lambda}.$$

Denote by e_{ij} the Boolean matrix only taking value of one for elements (i,j) and (j,i).

[18] Definition of an M-matrix can be found in Plemmons (1977): "An $n\times n$ matrix A that can be expressed in the form $A = sI - B$, where $B = (b_{ij})$ with $b_{ij} \geq 0$, $i \leq i, j \leq n$, and $s \geq \rho(B)$, the maximum of the moduli of the eigenvalues of B, is called an M-matrix."

To simplify notation, let $\boldsymbol{x}(\overline{g}) \equiv \mathbf{x}$ and $\boldsymbol{x}(\overline{h}) = \mathbf{y}$. We have

$$\begin{aligned}
\mathbf{x} - \mathbf{y} &= \left((I - \lambda\overline{g})^{-1} - \left(I - \lambda\overline{h}\right)^{-1}\right) \cdot \mathbf{1} \\
&= \left((I - \lambda\overline{g})^{-1} - \left(I - \lambda\left(\overline{g} - \sum_{m=1}^{\ell} e_{ij_m}\right)\right)^{-1}\right) \cdot \mathbf{1} \\
&= \left(I - \lambda\left(\overline{g} - \sum_{m=1}^{\ell} e_{ij_m}\right)\right)^{-1} \cdot \left(\lambda \sum_{m=1}^{\ell} e_{ij_m}\right) \cdot (I - \lambda\overline{g})^{-1} \cdot \mathbf{1} \\
&= \lambda\left(I - \lambda\overline{h}\right)^{-1} \cdot \sum_{m=1}^{\ell} e_{ij_m} \cdot \mathbf{x}.
\end{aligned}$$

Let $(I - \lambda\overline{g})^{-1} \equiv \{x_{ij}\}_{n\times n}$ and $(I - \lambda\overline{h})^{-1} \equiv \{y_{ij}\}_{n\times n}$. The equation above implies

$$x_k - y_k = \lambda \sum_{m=1}^{\ell} (y_{ki}x_{j_m} + y_{kj_m}x_i), \quad \forall k \in N,$$

where $x_k = \sum_{i=1}^{n} x_{ki}$ and $y_k = \sum_{i=1}^{n} y_{ki}$. Because $(I - \lambda\overline{h})$ is an M-matrix for $\lambda < 1/(n-1)$, $(I - \lambda\overline{h})^{-1}$ is a non-negative matrix (Plemmons, 1977), i.e., $y_{ij} \geq 0$ for any i, j. Combined with the inequality in Claim 2, we have

$$\begin{aligned}
x_k - y_k &\leq \frac{\lambda}{1 - L(\overline{g})\lambda} \sum_{m=1}^{\ell} (y_{ki} + y_{kj_m}) \\
&\leq \frac{\ell\lambda}{1 - L(\overline{g})\lambda} y_k, \quad \forall k \in N.
\end{aligned}$$

Rearranging this inequality and picking $k = i$, we obtain the inequality in Claim 3. Intuitively, this inequality provides a lower bound of the effort level as well as the individual payoff at the stage game when an agent decides to drop a subset of existing links.

Now we proceed to prove the proposition. Under full cascade of detection, Proposition 3 shows there is a unique strongly stable Nash equilibrium. Denote the set of agents who are in the complete, largest component in that equilibrium by $N_C = \{1, 2, ..., n_0\}$ and the set of isolated agents by $N_I = \{n_0 + 1, n_0 + 2, ..., n\}$. If $n_0 = n$, the proposition trivially holds, so we focus on the case that $n_0 < n$.

Suppose there exists a pairwise stable Nash equilibrium in which an agent $j \in N_I$ is no longer isolated under partial cascade of detection. Denote the equilibrium criminal network by $\overline{g}$ and define $L(\overline{g})$ as before. Given $j \in N_I$, Proposition 3 implies that

$$\left(\frac{1-(n_0-1)\lambda}{1-n_0\lambda}\right)^2 (1-\beta_j) < 1.$$

If $L(\overline{g}) < n_0$, Claim 3 suggests that the agent who is connected with agent j has incentive to drop that link, because by dropping that link, his payoff increases at least by the factor of

$$\left(\frac{1-L(\overline{g})\lambda}{1-(L(\overline{g})-1)\lambda}\right)^2 \frac{1}{1-\beta_j} > \left(\frac{1-L(\overline{g})\lambda}{1-(L(\overline{g})-1)\lambda}\right)^2 \left(\frac{1-(n_0-1)\lambda}{1-n_0\lambda}\right)^2 > 1.$$

If $L(\overline{g}) \geq n_0$, the most connected agent must be connected with at least $(L(\overline{g})-n_0+1)$ agents in N_I. Similarly, we can show that the most connected agent has incentive to drop his links with $(L(\overline{g})-n_0+1)$ agents with the highest β_i. Contradiction to the definition of a pairwise stable Nash equilibrium.

A.3.4 Proof of Proposition 5

We first solve the minimization problem

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^n_+} \sum_{i=1}^{n} \beta_i, \quad \text{s.t.} \quad \pi^k(\boldsymbol{\beta}) \leq \pi^1(\boldsymbol{\beta}), \quad k = 2,3,...,n.$$

The key step is to check if these weak inequalities have to be binding to attain the optimum. First, the last weak inequality $\pi^n(\boldsymbol{\beta}) \leq \pi^1(\boldsymbol{\beta})$ must be binding. We prove by contraposition. Suppose $\pi^n(\boldsymbol{\beta}) < \pi^1(\boldsymbol{\beta})$, which implies $\beta_n > 0$. If $\beta_n > \beta_{n-1}$, the government can always save the detection resource by reducing β_n by a small amount and maintain the order of $\boldsymbol{\beta}$. If $\beta_n = \beta_{n-1}$, we define $i_0 \equiv \min\{i \in N : \beta_i = \beta_n\}$. If $i_0 = 1$, the government can reduce β_1 to zero and still have every constraint satisfied. If $i_0 > 1$, we need to further consider two possible cases. If $\pi^i(\boldsymbol{\beta}) < \pi^n(\boldsymbol{\beta})$ for any $i \in \{i_0, i_0+1, ..., n\}$, the government can reduce β_{i_0}, β_{i_0+1},..., and β_n uniformly by a small amount and still maintain the order of $\boldsymbol{\beta}$. If there exists $j \in \{i_0, i_0+1, ..., n\}$ such that $\pi^j(\boldsymbol{\beta}) = \pi^n(\boldsymbol{\beta})$, we then have $\pi^{j-1}(\boldsymbol{\beta}) \leq \pi^n(\boldsymbol{\beta}) = \pi^j(\boldsymbol{\beta})$ which implies $1-\beta_n > \left(\frac{1-(j-1)\lambda}{1-(j-2)\lambda}\right)^2$. Therefore,

$$\begin{aligned}(1-\beta_n)^{n-j} &> \left(\frac{1-(j-1)\lambda}{1-(j-2)\lambda}\right)^{2(n-j)} \\ &> \left(\frac{1-(n-1)\lambda}{1-(n-2)\lambda}\right)^2 \left(\frac{1-(n-2)\lambda}{1-(n-3)\lambda}\right)^2 \cdots \left(\frac{1-j\lambda}{1-(j-1)\lambda}\right)^2 = \left(\frac{1-(n-1)\lambda}{1-(j-1)\lambda}\right)^2,\end{aligned}$$

which yields $\pi^n(\boldsymbol{\beta}) > \pi^j(\boldsymbol{\beta})$, a contradiction to our previous assumptions.

We prove other inequalities are binding by induction. Suppose there exists $\pi^{\ell+1}(\boldsymbol{\beta}) = \pi^1(\boldsymbol{\beta})$ and $\pi^{\ell}(\boldsymbol{\beta}) < \pi^1(\boldsymbol{\beta})$ for $\ell \in 2, ..., n-1$. Consider an alternative detection resource allocation $\boldsymbol{\beta'}$ as follows

$$\begin{aligned} &\beta'_\ell = \beta_\ell - \varepsilon,\ \beta'_{\ell+1} = \beta_{\ell+1} + \varepsilon, \\ &\beta'_k = \beta_k, \quad k \in N,\ k \neq \ell, k \neq \ell + 1. \end{aligned}$$

First of all, $\beta_\ell > 0$, because otherwise $\pi^\ell(\boldsymbol{\beta}) > \pi^1(\boldsymbol{\beta})$. For sufficiently small ε, β'_ℓ is therefore well-defined. We consider two scenarios.

1. Perturbation of $\boldsymbol{\beta}$ does not change ordering. Since two resource allocation differs only in β_ℓ and $\beta_{\ell+1}$, the first $\ell - 1$ inequalities will not be affected, i.e., $\pi^k(\boldsymbol{\beta'}) = \pi^k(\boldsymbol{\beta}) \leq \pi^1(\boldsymbol{\beta}) = \pi^1(\boldsymbol{\beta'})$ for $k < \ell$. If $\varepsilon > 0$ is sufficiently small, $\pi^\ell(\boldsymbol{\beta'}) \leq \pi^1(\boldsymbol{\beta'})$ in that $\pi^\ell(\boldsymbol{\beta}) < \pi^1(\boldsymbol{\beta})$. Moreover, for $k > \ell$, we have

$$\frac{\pi^k(\boldsymbol{\beta'})}{\pi^k(\boldsymbol{\beta})} = \frac{(1 - \beta_\ell + \varepsilon)(1 - \beta_{\ell+1} - \varepsilon)}{(1 - \beta_\ell)(1 - \beta_{\ell+1})} = 1 - \frac{\varepsilon(\beta_{\ell+1} - \beta_\ell) + \varepsilon^2}{(1 - \beta_\ell)(1 - \beta_{\ell+1})} < 1,$$

which implies $\pi^k(\boldsymbol{\beta'}) < \pi^1(\boldsymbol{\beta'})$ for $k > \ell$. In particular, $\pi^n(\boldsymbol{\beta'}) < \pi^1(\boldsymbol{\beta'})$, so the government can further save their budget by applying the same argument as above.

2. Perturbation of $\boldsymbol{\beta}$ changes ordering. First, consider $\beta_\ell = \beta_{\ell-1}$. If $\beta_\ell = \beta_1$, it is not optimal because β_1 can be reduced to be zero. If $\beta_\ell > \beta_1$, again we can show $\pi^\ell(\boldsymbol{\beta}) < \pi^1(\boldsymbol{\beta})$ implies $\pi^i(\boldsymbol{\beta}) < \pi^1(\boldsymbol{\beta})$ for any i such that $\beta_i = \beta_\ell$. Therefore, we can pick a sufficiently small ε such that $\pi^i(\boldsymbol{\beta'}) \in (\pi^i(\boldsymbol{\beta}), \pi^1(\boldsymbol{\beta}))$ for any i such that $\beta_i = \beta_\ell$. Then the argument in scenario 1 follows. Second, consider $\beta_{\ell+1} = \beta_{\ell+2}$. An ε-increase of $\beta_{\ell+1}$ always leaves the first ℓ agents unchanged and the rest weakly worse off (including someone strictly worse off), so the change of ordering will not affect our results in scenario 1 as well.

In sum, we have shown that $\pi^{\ell+1}(\boldsymbol{\beta}) = \pi^1(\boldsymbol{\beta})$ implies $\pi^\ell(\boldsymbol{\beta}) = \pi^1(\boldsymbol{\beta})$ for $\ell \in 2, ..., n-1$. Combined with $\pi^n(\boldsymbol{\beta}) = \pi^1(\boldsymbol{\beta})$, we have $\pi^k(\boldsymbol{\beta}) = \pi^1(\boldsymbol{\beta})$ for any $k \in N$. Using the definition of $\pi^k(\boldsymbol{\beta})$, we have

$$\frac{1}{2}\left(\frac{1}{1 - (k-1)\lambda}\right)^2 \Pi_{h=1}^k (1 - \beta_h) = \frac{1 - \beta_1}{2}, \quad k \in N,$$

which implies

$$\beta_k = 1 - \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2, \quad k = 2, 3, ..., n.$$

It can be easily verified that β_k strictly increases with k and $\beta_1 \le \beta_2 = 1 - (1-\lambda)^2$. Picking $\beta_1 = 0$, we obtain the solution to this minimization problem. If $B > B_1 \equiv n - 1 - \sum_{k=2}^{n} \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2$, the government can make above $n-1$ inequalities strict by having $\beta_1 = 0$ and

$$\beta_k = 1 - \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2 + \frac{B - B_1}{n-1}, \quad k = 2, 3, ..., n.$$

According to Proposition 3, this allocation of detection resource yields an empty criminal network in the strongly stable Nash equilibrium under full cascade of detection.

A.3.5 Proof of Corollary 1

Consider the minimization problem

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^n_+} \sum_{i=1}^{n} \beta_i, \quad \text{s.t.} \quad \pi^i(\boldsymbol{\beta}) \le \pi^S(\boldsymbol{\beta}), \quad \forall i \in N.$$

Similar to the proof of Proposition 5, it can be shown that $\pi^k = \pi^S$ for $k > S$. If $B > B_S$, the government can restrict the largest component of SSNE criminal network to be of size S. The government achieves this by allocating the detection budget as follows: $\beta_k = 0$ for $k \le S$ and $\beta_k = 1 - \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2 + \frac{B-B_S}{n-S}$ for $k > S$. Under this allocation, S is the unique maximizer to $\max_{k \in N} \pi^k(\boldsymbol{\beta})$.

A.3.6 Proof of Proposition 6

Like the proof of Proposition 4, we first establish an inequality result regarding link deletion. Pick an arbitrary network $\overline{g} \in \overline{G}$. Denote the set of agents in agent ℓ's component by $\{\ell_0, \ell_1, ..., \ell_L\}$ with $\ell_0 = \ell$. Denote by $\overline{h}$ the network in which agent ℓ severs all his links in $\overline{g}$.

Claim 4. *The inequality*

$$\frac{x_{\ell_k}(\overline{h})}{x_{\ell_k}(\overline{g})} \ge \frac{1 - L\lambda}{1-(L-1)\lambda}$$

holds for any $k = 1, 2, ..., L$.

Denote by $\overline{h}_L$ the subgraph of $\overline{h}$ induced by $\{\ell_1, \ell_2, ..., \ell_L\}$. Since agents in $\{\ell_1, \ell_2, ..., \ell_L\}$ do not share any link with the rest of agents under $\overline{h}$, their payoff $(x_{\ell_1}(\overline{h}), x_{\ell_2}(\overline{h}), ..., x_{\ell_L}(\overline{h}))' = (I - \lambda\overline{h}_L)^{-1}\mathbf{1}$. We also define

$$(y_{\ell_1}, y_{\ell_2}, ..., y_{\ell_L}, y_{\ell_0})' \equiv \left(I - \lambda \begin{pmatrix} \overline{h}_L & \mathbf{1}_{L\times 1} \\ \mathbf{1}_{1\times L} & 0 \end{pmatrix} \right)^{-1} \mathbf{1}.$$

Using block matrix inversion, we have

$$\begin{aligned}(y_{\ell_1}, y_{\ell_2}, ..., y_{\ell_L}, y_{\ell_0})' &= \begin{pmatrix} I - \lambda\overline{h}_L & -\lambda\mathbf{1}_{L\times 1} \\ -\lambda\mathbf{1}_{1\times L} & 1 \end{pmatrix}^{-1} \mathbf{1} \\ &= \begin{pmatrix} (I - \lambda\overline{h}_L - \lambda^2\mathbf{1}_{L\times 1}\mathbf{1}_{1\times L})^{-1} & \frac{(I-\lambda\overline{h}_L)^{-1}\lambda\mathbf{1}_{L\times 1}}{1-\lambda^2\mathbf{1}_{1\times L}(I-\lambda\overline{h}_L)^{-1}\mathbf{1}_{L\times 1}} \\ \lambda\mathbf{1}_{1\times L}(I - \lambda\overline{h}_L - \lambda^2\mathbf{1}_{L\times 1}\mathbf{1}_{1\times L})^{-1} & (1 - \lambda^2\mathbf{1}_{1\times L}(I - \lambda\overline{h}_L)^{-1}\mathbf{1}_{L\times 1})^{-1} \end{pmatrix} \mathbf{1}\end{aligned}$$

Let $\{x_{ij}\}_{L\times L} \equiv (I - \lambda\overline{h}_L)^{-1}$ and $\{z_{ij}\}_{L\times L} \equiv (I - \lambda\overline{h}_L - \lambda^2\mathbf{1}_{L\times 1}\mathbf{1}_{1\times L})^{-1}$. The equation

$$(I-\lambda\overline{h}_L)^{-1}-(I-\lambda\overline{h}_L-\lambda^2\mathbf{1}_{L\times 1}\mathbf{1}_{1\times L})^{-1} = (I-\lambda\overline{h}_L-\lambda^2\mathbf{1}_{L\times 1}\mathbf{1}_{1\times L})^{-1}(-\lambda^2\mathbf{1}_{L\times 1}\mathbf{1}_{1\times L})(I-\lambda\overline{h}_L)^{-1}$$

implies

$$x_{ij} - z_{ij} = -\lambda^2 z_i x_{\ell_j}(\overline{h}),$$

with $z_i = \sum_{m=1}^{L} z_{im}$ and $x_{\ell_j}(\overline{h}) = \sum_{m=1}^{L} x_{im}$. Adding up equations with respect to j, we have

$$z_i = \frac{x_{\ell_i}(\overline{h})}{1 - \lambda^2 \sum_{j=1}^{L} x_{\ell_j}(\overline{h})}.$$

According to the matrix equation above,

$$\begin{aligned} y_{\ell_k} &= z_k + \frac{\lambda x_{\ell_k}(\overline{h})}{1 - \lambda^2\mathbf{1}_{1\times L}(I - \lambda\overline{h}_L)^{-1}\mathbf{1}_{L\times 1}} \\ &= \frac{(1+\lambda)x_{\ell_k}(\overline{h})}{1 - \lambda^2 \sum_{j=1}^{L} x_{\ell_j}(\overline{h})}, \end{aligned}$$

for $k = 1, 2, ..., L$. Notice that y_{ℓ_k} is the effort level exerted by agent ℓ_k if agent ℓ is directly connected to everyone in his component under $\overline{g}$, so $y_{\ell_k} \geq x_{\ell_k}(\overline{g})$, and as a result,

$$\frac{x_{\ell_k}(\overline{h})}{x_{\ell_k}(\overline{g})} \geq \frac{x_{\ell_k}(\overline{h})}{y_{\ell_k}} = \frac{1 - \lambda^2 \sum_{j=1}^{L} x_{\ell_j}(\overline{h})}{1+\lambda} \geq \frac{1 - \lambda^2 L/(1-(L-1)\lambda)}{1+\lambda} = \frac{1 - L\lambda}{1-(L-1)\lambda}.$$

Denote the set of agents who are in the complete, largest component in the strongly stable Nash equilibrium under full cascade of detection by $N_C = \{1, 2, ..., n_0\}$ and the set of isolated agents by $N_I = \{n_0 + 1, n_0 + 2, ..., n\}$. If $n_0 = n$, the proposition trivially holds, so we focus on the case that $n_0 < n$. Consider an arbitrarily given positive degree of cascade $d \in \{1, 2, ..., n\}$. Suppose there exists a strongly stable Nash equilibrium in which agent $j \in N_I$ is no longer isolated. Denote the equilibrium criminal network by $\overline{g}$ and the number of agents who are directly or indirectly connected to agent j by L. Given $j \in N_I$, Proposition 3 implies that

$$\left(\frac{1-(n_0-1)\lambda}{1-n_0\lambda}\right)^2 (1-\beta_j) < 1.$$

If $L < n_0$, Claim 4 suggests that agents who are directly connected with agent j have incentive to simultaneously drop that link, because by dropping that link, each agent's payoff increases at least by the factor of

$$\left(\frac{1-L\lambda}{1-(L-1)\lambda}\right)^2 \frac{1}{1-\beta_j} > \left(\frac{1-L\lambda}{1-(L-1)\lambda}\right)^2 \left(\frac{1-(n_0-1)\lambda}{1-n_0\lambda}\right)^2 > 1.$$

If $L \geq n_0$, similarly, we can show that agents who are directly connected to $(L - n_0 + 1)$ agents with the highest β_i have the incentive to simultaneously drop their links with these $(L - n_0 + 1)$ agents. Contradiction to the definition of a strongly stable Nash equilibrium.

A.3.7 Proof of Proposition 7

Consider the following minimax problem

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^n_+} \max_{k\in N} \pi^k(\boldsymbol{\beta}) \qquad \text{s.t.} \qquad \sum_{i\in N} \beta_i \leq B.$$

In words, the government tries to minimize the maximum individual payoff that can be derived from an SSNE criminal network under full cascade of detection. If the solution to this problem yields a payoff lower than the outside option, the government will be able to induce all agents to opt out. Recall that $B_S \equiv n - S - \sum_{k=S+1}^{n} \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2$ for $S \in \{1, 2, ..., n-1\}$. We consider two possible cases.

Case I: $B < B_{n-1}$.

In this case, suggested by Corollary 1, the SSNE criminal network is always complete regardless of the allocation of the detection budget. Therefore, $\max_{k\in N} \pi^k(\boldsymbol{\beta}) = \pi^n(\boldsymbol{\beta})$. The minimax problem is simplified to $\min_{\boldsymbol{\beta}\in\mathbb{R}^n_+} \pi^n(\boldsymbol{\beta})$ subject to $\sum_{i\in N} \beta_i \leq B$. The optimal allocation is simply obtained as $\beta_i = 0$ for any $i \in \{1, 2, ..., n-1\}$ and $\beta_n = B$. The minimum is given by $\frac{1-B}{2(1-(n-1)\lambda)^2}$.

Case II: $B \in [B_{\ell+1}, B_\ell)$ for $\ell \in \{1, 2, ..., n-2\}$.

We first prove the following claim

Claim 5. *Let $i_0 = \min\{i \in N : \beta_i > 0\}$. If $\boldsymbol{\beta}$ is the solution to the minimax problem, $\pi^{i_0}(\boldsymbol{\beta}) = \max_{k \in N} \pi^k(\boldsymbol{\beta})$.*

Suppose the claim is not true. Let $j_0 = \min\{j \in N : \pi^j(\boldsymbol{\beta}) = \max_{k \in N} \pi^k(\boldsymbol{\beta})\}$. We first consider that $j_0 > i_0$. By definition, $\pi^{j_0}(\boldsymbol{\beta}) = \max_{k \in N} \pi^k(\boldsymbol{\beta}) > \pi^{i_0}(\boldsymbol{\beta})$. Consider an alternative detection resource allocation $\boldsymbol{\beta'}$: $\beta'_{i_0} = \beta_{i_0} - \varepsilon$, $\beta'_{j_0} = \beta_{j_0} + \varepsilon$, and $\beta'_k = \beta_k$, for $k \in N/\{i_0, j_0\}$. If $j_0 = n$, a sufficiently small ε guarantees the order of $\boldsymbol{\beta}$ will not be changed under $\boldsymbol{\beta'}$. If $j_0 < n$, $\pi^{j_0}(\boldsymbol{\beta}) > \pi^{j_0-1}(\boldsymbol{\beta})$ and $\pi^{j_0}(\boldsymbol{\beta}) \geq \pi^{j_0+1}(\boldsymbol{\beta})$ imply that $\beta_{j_0} < \beta_{j_0+1}$. Again, this guarantees the scrutiny ordering will be unchanged if ε is sufficiently small. We can show that $\pi^k(\boldsymbol{\beta'}) < \pi^k(\boldsymbol{\beta})$ for $k \geq j_0$, $\pi^k(\boldsymbol{\beta'}) = \pi^k(\boldsymbol{\beta})$ for $k < i_0$, and given a sufficiently small ε, $\pi^k(\boldsymbol{\beta'}) \in (\pi^k(\boldsymbol{\beta}), \pi^{j_0}(\boldsymbol{\beta}))$ for $i_0 \leq k < j_0$. This yields a contradiction to optimality of $\boldsymbol{\beta}$.

We still need to consider the case that $j_0 < i_0$ if $i_0 > 1$. As $\beta_k = 0$ for $k < i_0$, $j_0 = i_0 - 1$. Using the same argument as above, we first eliminate other maximizers (if any) of $\max_{k \in N} \pi^k(\boldsymbol{\beta})\}$ which are greater than i_0 without changing the maximum. Consider an alternative allocation $\boldsymbol{\beta''}$: $\beta''_{i_0} = \beta_{i_0} - \varepsilon$, $\beta''_{i_0-1} = \beta_{i_0-1} + \varepsilon$, and $\beta''_k = \beta_k$, for $k \in N/\{i_0, i_0 - 1\}$. We can show that $\pi^k(\boldsymbol{\beta''}) = \pi^k(\boldsymbol{\beta})$ for $k < i_0 - 1$, $\pi^{i_0-1}(\boldsymbol{\beta''}) < \pi^{i_0-1}(\boldsymbol{\beta})$, and given a sufficiently small ε, $\pi^k(\boldsymbol{\beta''}) \in (\pi^k(\boldsymbol{\beta}), \pi^{i_0-1}(\boldsymbol{\beta}))$ for $k \geq i_0$. Contradiction.

Claim 5 greatly simplifies our analysis. The minimax problem can be solved in two steps. First, for a given $i_0 \in N$ and a set of compatible allocation rules, we solve the optimal allocation. In each sub-problem, the government only needs to find the maximum β_{i_0} that is consistent with Claim 5 and follows the increasing order. Second, we pick i_0 that attains the minimum among all sub-problems. $B < B_\ell$, so the detection budget is insufficient to restrict the size of the largest component of the SSNE network to be ℓ. In other words, there is not enough detection budget to guarantee that $\pi^\ell(\boldsymbol{\beta}) \geq \pi^k(\boldsymbol{\beta})$ for any $k > \ell$. It suggests i_0 must be greater than ℓ. If $i_0 = \ell + 1$, we can show that the optimal allocation is given by $\beta_k = 0$ for $k \leq \ell$, $\beta_{\ell+1} = B - B_{\ell+1}$, and $\beta_k = 1 - \left(\frac{1-(k-1)\lambda}{1-(k-2)\lambda}\right)^2$ for $k > \ell + 1$, under which $\min_{\boldsymbol{\beta}: i_0 = \ell+1} \max_{k \in N} \pi^k(\boldsymbol{\beta}) = \frac{1-(B-B_{\ell+1})}{2(1-\ell\lambda)^2}$. If $i_0 > \ell + 1$, we know from Claim 5 that $\pi^{i_0}(\boldsymbol{\beta}) = \max_{k \in N} \pi^k(\boldsymbol{\beta}) \geq \pi^{i_0-1}(\boldsymbol{\beta})$. By definition, $\pi^{i_0-1}(\boldsymbol{\beta}) = \frac{1}{2(1-(i_0-2)\lambda)^2} > \frac{1-(B-B_{\ell+1})}{2(1-\ell\lambda)^2}$ for $i_0 > \ell + 1$. Therefore, $i_0 = \ell + 1$ yields the optimal allocation with the minimum $\frac{1-(B-B_{\ell+1})}{2(1-\ell\lambda)^2}$.

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