Dynamic Decisions under Subjective Beliefs: A Structural Analysis*

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Abstract

Rational expectations of agents on state transitions are crucial but restrictive in Dynamic Discrete Choice (DDC) models. This paper analyzes DDC models where agents' beliefs about state transition allowed to be different from the objective state transition. We show that the single agent's subjective beliefs in DDC models can be identified and estimated from multiple periods of observed conditional choice probabilities. Besides the widely-used assumptions, our results require that the agent's subjective beliefs corresponding to each choice to be different and that the conditional choice probabilities contain enough variations across time in the finite horizon case, or vary enough with respect to other state variables in which subjective beliefs equals objective ones in the infinite horizon case. Furthermore, our identification of subjective beliefs is nonparametric and global as they are expressed as a closed-form function of the observed conditional choice probabilities. Given the identified subjective beliefs, the model primitives may be estimated using the existing conditional choice probability approach.

Keywords: Identification, estimation, dynamic discrete choice models, subjective beliefs.
JEL: C14

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1 Introduction

For many years great effort has been devoted to the study of identifying dynamic discrete choice (DDC) models. A ubiquitous assumption for identification in this literature is rational expectations of agents, i.e., they have perfect expectations on law of motion for state variables. This strong assumption is inconsistent with some recent empirical evidence on comparison between agents' subjective expectations and the objective probabilities of state transition.[1] Failure of the rational expectations assumption may induce biased estimation of model primitives, e.g., agents' preference, thus prediction of counterfactual experiments would also be biased. A popular solution in empirical studies is to employ data on agents' subjective expectations instead of objective probabilities of transition in estimation. Unfortunately, in many empirical contexts the data on agents' subjective expectations are not available.

In this paper, we analyze the nonparametric identification of DDC models where the assumption of rational expectations is relaxed and agents' subjective expectations are unobserved. We show that the single agent's subjective beliefs in DDC models can be identified and estimated from observed conditional choice probabilities in both finite-horizon and infinite-horizon cases. Based on the insight of the underidentification results, e.g., Rust (1994) and Magnac and Thesmar (2002), we address identification of DDC models by assuming the distribution of agents' unobserved preference shocks and the discount factor are known. Our methodology then identifies agents' subjective probabilities on state transition as a closed-form solution to a set of nonlinear moment conditions that are induced from Bellman equations using the insight in Hotz and Miller (1993). Identifying subjective beliefs in the case of finite-horizon relies on the variation of agents' conditional choice probabilities (CCP) in multiple time periods while the subjective beliefs are time-invariant. In infinite-horizon DDC models, stationarity provides no variation of CCP. Our identification strategy is to introduce additional an additional state variable whose transition probabilities are known to the agent. We then investigate the moment conditions induced from Bellman equations by varying the realizations of this state variable.

A great advantage of our methodology is that agents' subjective probabilities are nonparametrically identified as a closed-form function of observed CCP. This implies a multi-step procedure to estimate DDC models. We first follow the identification result to obtain a nonparametric and global estimator of subjective beliefs from observed CCP. Given the estimated subjective probabilities of state transition, the model primitives can be estimated using the existing CCP approach (e.g., see Hotz and Miller (1993)).

Relaxing rational expectations in DDC models, or more generally in decision models is of both both theoretical and empirical importance (see Aguirregabiria and Mira (2010) for further discussions). Manski (2004) summarizes and advocates using data of subjective

[1] In a DDC setting, Wang (2014) finds some differences between the objective and the subjective probabilities of two-year survival probabilities. Cruces et al. (2013) provide evidence of agents' biased perception of income distribution.

expectations in empirical decision models. The literature along this line are growing recently. For example, Van der Klaauw and Wolpin (2008) study Social Security and savings using a DDC model where agents' subjective expectations on their own retirement age and longevity and future changes in Social Security policy are from surveys. Nevertheless, to the best of our knowledge, this paper is the first to investigate the identifiability of DDC models with subjective beliefs. In a different context, Aguirregabiria and Magesan (2015) consider identification and estimation of dynamic games by assuming players' beliefs about other players' actions are not at equilibrium while rational expectations on state transition are still assumed to hold.[2]

This paper contributes to a growing literature on (nonparametric) identification of dynamic discrete choice models. Rust (1994) provide some non-identification results for the case of infinite-horizon. Magnac and Thesmar (2002) further determine the exact degree of underidentification and explore the identifying power of some exclusion restrictions. Fang and Wang (2015) also employ exclusion restrictions to identify a DDC model with hyperbolic discounting. Hu and Shum (2012) consider identification of DDC models with unobserved state variables. Abbring (2010) presents excellent review of on identification of DDC models. Our paper is fundamentally different from these papers in that they assume rational expectations to achieve identification. For the first time, we provide rigorous identification results for DDC model with agents having subjective beliefs. Not surprisingly, our results of identification and estimation can be applied to a wide array of empirical studies where agents' subjective expectations are crucial for their decisions but unobserved.

2 DDC models with subjective beliefs

We consider a single agent DDC model with subjective expectations. In period t, an agent makes the choice a_t from a finite set of actions $\mathcal{A} = \{1, \cdots, K\}, K \geq 2$ to maximize her expected lifetime utilities, based on her expectations of future state transitions.

State variables that the agent considers consist of both observable and unobservable components, x_t and ϵ_t, respectively. The observed state variable x_t takes values in $\mathcal{X} \equiv \{1, \cdots, J\}, J \geq 2$ and the unobserved state variable $\epsilon_t(a_t)$ may depend on choice a_t with $\epsilon_t = (\epsilon_t(1), \cdots, \epsilon_t(K))$ and they are random preference shocks to actions. At the beginning of each period, the state variables (x_t, ϵ_t) are revealed to the agent who then chooses an action $a_t \in \mathcal{A}$. The instantaneous period utility function is $u(x_t, a_t, \epsilon_t)$. Then the state variables of the next period $(x_{t+1}, \epsilon_{t+1})$ are drawn conditional on (x_t, ϵ_t) as well as the

[2]Relying on exclusion restrictions on payoff functions and the existence of a subset of state variables on which subjective beliefs are equal to the objective ones, Aguirregabiria and Magesan (2015) identifies payoff functions first, then recover subjective beliefs. Our paper is different from theirs in several aspects. First, our paper relaxes rational expectations of agents by assuming agents' subjective beliefs on state transition may be different from the actual transition probabilities. Second, we explore variation of CCPs to identify subjective beliefs of agents without information of payoff functions.

agent's decision a_t. The *objective* state transitions are denoted as $f(x_{t+1}, \epsilon_{t+1}|x_t, \epsilon_t, a_t)$. For simplicity, we impose the following widely used assumption regarding state transition for DDC models.

Assumption 1 *(i) The state variables evolve $\{x_t, \epsilon_t\}$ following a first-order Markov process; (ii) $f(x_{t+1}, \epsilon_{t+1}|x_t, \epsilon_t, a_t) = f(x_{t+1}|x_t, a_t)f(\epsilon_{t+1})$.*

In each period, the agent maximizes her expected utility as follows:

$$\max_{a_t} \sum_{\tau=t,t+1,\dots} \beta^{\tau-t} E\big[u(x_\tau, a_\tau, \epsilon_\tau)|x_t, a_t, \epsilon_t\big]$$

where $\beta \in [0, 1]$ is the discount factor. The expectation is taken using the agent's subjective beliefs.

Following Rust (1987), we make the following assumptions concerning the unobservable component in the preferences.

Assumption 2 *(i) $u(x_t, a_t, \epsilon_t) = u(x_t, a_t) + \epsilon_t(a_t)$ for any $a_t \in \mathcal{A}$; (ii) $\epsilon_t(a)$ for all t and all $a \in \mathcal{A}$ are i.i.d. draws from mean zero type-I extreme value distribution*

The additive separability of agents' utility imposed in Assumption 2 (i) is widely used in the literature. Assuming a known distribution of ϵ_t is due to the non-identificatification results in Rust (1994) and Magnac and Thesmar (2002). The mean zero type-I extreme value distribution is assumed for ease of exposition. Our identification holds for any known distribution of ϵ_t.

Since the discount factor is not the focus of this paper, we assume β is known. We refer to Magnac and Thesmar (2002) and Abbring and Daljord (2016) for the identification of discount factor β.

Assumption 3 *The discount factor β is known.*

Let $s(x_{t+1}|x_t, a_t)$ denote the agent's subjective beliefs about future state transitions conditional on her action a_t. In a standard DDC model, agents are assumed to have correct beliefs about the state transition (rational expectations), i.e., their subjective beliefs $s(x_{t+1}|x_t, a_t)$ are the same as the objective state transition $f(x_{t+1}|x_t, a_t)$. We deviate from such a setting and allow the subjective beliefs to be different from the objective beliefs. The subjective beliefs are a complete set of conditional probabilities that satisfy Assumption 1 and the following two properties.

Assumption 4 *(i) $\sum_{x_{t+1}\in\mathcal{K}} s(x_{t+1}|x_t, a_t) = 1$. (ii) $s(x_{t+1}|x_t, a_t) \geq 0$.*

Notice that in the DDC mode, the state transitions are still governed by the objective probabilities $f(x_{t+1}|x_t, a_t)$. Nevertheless, the observed choices $\{a_t\}_{t=1,2,\cdots}$ in general would have different distributions from the case where agents have rational expectations.

3 Closed-form identification of subjective beliefs

This section shows that the subjective beliefs are identified with a closed-form expression and focuses on how the subjective beliefs may be uniquely determined by the conditional choice probabilities. We consider a dynamic discrete choice model of finite horizon where an agent has subjective beliefs about the state transition. We further impose the following restriction on the utility function and the subjective beliefs.

Assumption 5 *The subjective belief $s(x'|x,a)$ and the utility function $u(x_t,a_t)$ are time-invariant.*

This assumption of time-invariant $s(x'|x,a)$ is consistent with some theoretical explanation, e.g., in Brunnermeier and Parker (2005) subjective beliefs are rationalized as a solution of agents' maximization problem. It is also widely imposed in the recent empirical literature where agents' subjective beliefs are one-time self-reported, e.g., see Wang (2014).

In this dynamic setting, the optimal choice a_t in period t is

$$a_t = \arg\max_{a\in\mathcal{A}}\{v_t(x_t,a)+\epsilon_t(a)\}$$

where $v_t(x,a)$ is the choice-specific value function and the additively separability of $v_t(x,a)$ and $\epsilon_t(a)$ is due to the assumption of additive separability of instantaneous period utility function. Under assumptions 1-2 above, the *ex ante* value function at t can be expressed as

$$\begin{aligned} V_t(x_t) &= -\log p_t(a_t=K|x_t)+v_t(x_t,a_t=K) \\ &\equiv -\log p_{t,K}(x_t)+v_{t,K}(x_t), \end{aligned}$$

where the choice K can be substituted by any other choice in $\mathcal{A}$.

Given that the state variable x_t has support $\{1,2,...,J\}$, we define the vector of $J-1$ independent subjective probabilities as follows:

$$\boldsymbol{S}_{a_t}(x_t) = [s(x_{t+1}=1|x_t,a_t),...,s(x_{t+1}=J-1|x_t,a_t)]. \tag{1}$$

Similarly, we define

$$\begin{aligned} -\log \boldsymbol{p}_{t,K} &= [-\log p_{t,K}(x_t=1),...,-\log p_{t,K}(x_t=J-1)]'-(-\log p_{t,K}(J)) \\ \boldsymbol{v}_{t,K} &= [v_{t,K}(x_t=1),...,v_{t,K}(x_t=J-1)]'-v_{t,K}(J) \end{aligned} \tag{2}$$

The choice-specific value function may then be expressed as follows:

$$\begin{aligned} & v_t(x_t,a_t) = u(x_t,a_t)+\beta\int V_t(x_{t+1})s(x_{t+1}|x_t,a_t)dx_{t+1} \\ = \;& u(x_t,a_t)+\beta\boldsymbol{S}_{a_t}(x_t)(-\log\boldsymbol{p}_{t+1,K}+\boldsymbol{v}_{t+1,K})+\beta\big[-\log p_{t+1,K}(J)+v_{t+1,K}(J)\big]. \end{aligned}$$

We take the difference of the choice-specific value function above between $a_t = i$ and $a_t = K$, and apply the results in Hotz and Miller (1993),

$$\log\left(\frac{p_{t,i}(x_t)}{p_{t,K}(x_t)}\right) = \beta[\boldsymbol{S}_i(x_t) - \boldsymbol{S}_K(x_t)][\boldsymbol{m}_{t+1} + \boldsymbol{v}_{t+1}] + [u(x_t, i) - u(x_t, K)], \quad (3)$$

where $t = 1, 2, \cdots, T-1$. (3) allows us to further get rid of the utility function in the relationship between choice-specific value function and CCPs,

$$\begin{aligned} \Delta\xi_{t,i,K}(x_t) &\equiv \log\left(\frac{p_{t,i}(x_t)}{p_{t,K}(x_t)}\right) - \log\left(\frac{p_{t-1,i}(x_t)}{p_{t-1,K}(x_t)}\right) \\ &= \beta[\boldsymbol{S}_i(x_t) - \boldsymbol{S}_K(x_t)][-\Delta\log\boldsymbol{p}_{t+1,K} + \Delta\boldsymbol{v}_{t+1,K}], t = 1, \cdots, T-1, \quad (4) \end{aligned}$$

where $\Delta\boldsymbol{p}_{t+1,K} = \log\boldsymbol{p}_{t+1,K} - \log\boldsymbol{p}_{t,K}$ and $\Delta\boldsymbol{v}_{t+1,K} = \boldsymbol{v}_{t+1,K} - \boldsymbol{v}_{t,K}$. This equation holds for each of $x \in \{1, 2, ..., J\}$. Next we put the equation above for all the values of x_t in the matrix form with the following definitions

$$\boldsymbol{S}_a = \begin{pmatrix} \boldsymbol{S}_a(x_t = 1) \\ \boldsymbol{S}_a(x_t = 2) \\ \vdots \\ \boldsymbol{S}_a(x_t = J) \end{pmatrix} \quad (5)$$

and

$$\boldsymbol{\Delta\xi}_{t,i,K} = [\Delta\xi_{t,i,K}(1), ..., \Delta\xi_{t,i,K}(J)]'.$$

Notice that matrix $\boldsymbol{S}_a$ is of dimension $J \times (J-1)$. We then have a matrix version of (4)

$$\Delta\boldsymbol{\xi}_{t,i,K} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K][-\Delta\log\boldsymbol{p}_{t+1,K} + \Delta\boldsymbol{v}_{t+1,K}]. \quad (6)$$

We now focus on the value function corresponding to choice $a_t = K$ and $\Delta\boldsymbol{v}_{t+1,K}$. In the matrix form, we have

$$\Delta\boldsymbol{v}_{t,K} = \beta\widetilde{\boldsymbol{S}}_K(-\Delta\log\boldsymbol{p}_{t+1,K} + \Delta\boldsymbol{v}_{t+1,K}), \quad (7)$$

where $\widetilde{\boldsymbol{S}}_K$ is a $(J-1)\times(J-1)$ matrix

$$\tilde{\boldsymbol{S}}_K = \begin{pmatrix} 1 & 0 & ... & 0 & -1 \\ 0 & 1 & ... & 0 & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -1 \end{pmatrix} \boldsymbol{S}_K.$$

In summary, the choice probabilities are associated with subjective beliefs and value functions through equation (6), and the choice-specific value function evolves as in equation (7). By eliminating the value functions in these two equations, we are able to find the direct relationship between the observed choice probabilities and the subjective beliefs.

To proceed, we first impose a rank condition to the primitive matrices $\boldsymbol{S}_i$ and $\boldsymbol{S}_K$,

Assumption 6 *The $J \times (J-1)$ matrix $\boldsymbol{S}_i - \boldsymbol{S}_K$ has a full column rank of $J-1$ for all $i \in \mathcal{A}, i \neq K$.*

Assumption 6 guarantees that the generalized inverse of $\boldsymbol{S}_i - \boldsymbol{S}_K$, denoted as $[\boldsymbol{S}_i - \boldsymbol{S}_K]^+$ has a closed-form. Eliminating $\Delta \boldsymbol{v}_{t+1,K}$ in equation (6) leads to

$$\beta \widetilde{\boldsymbol{S}}_K [\boldsymbol{S}_i - \boldsymbol{S}_K]^+ \Delta \boldsymbol{\xi}_{t,i,K} - [\boldsymbol{S}_i - \boldsymbol{S}_K]^+ \Delta \boldsymbol{\xi}_{t-1,i,K} \;=\; \beta \Delta \log \boldsymbol{p}_{t,K}, t = 3, \cdots, T-1. \quad (8)$$

This equation implies that the choice probabilities are directly associated with the subjective beliefs through a nonlinear system, which enables us to solve the subjective beliefs with a closed-form. The nonlinear system above contains $J-1$ equations for a given t, and there are $(J-1)\times(J-1)$ and $(J-1)\times J$ unknown parameters in $\widetilde{\boldsymbol{S}}_K$ and $[\boldsymbol{S}_i - \boldsymbol{S}_K]^+$, respectively.

Suppose we observe data for $T = 2J+2$ consecutive periods, denote by $t_1, \cdots, t_{2J}$. We assume that the conditional choice probabilities satisfy

Assumption 7 *Matrix $\Delta \boldsymbol{\xi}_{i,K}$ is invertible, where*

$$\Delta \boldsymbol{\xi}_{i,K} \;=\; \begin{bmatrix} \Delta \boldsymbol{\xi}_{t_1,i,K} & \Delta \boldsymbol{\xi}_{t_2,i,K} & \ldots & \Delta \boldsymbol{\xi}_{t_{2J},i,K} \\ \Delta \boldsymbol{\xi}_{t_1-1,i,K} & \Delta \boldsymbol{\xi}_{t_2-1,i,K} & \ldots & \Delta \boldsymbol{\xi}_{t_{2J-1},i,K} \end{bmatrix}.$$

This assumption is imposed on the observed probabilities and therefore directly testable. However, this assumption also rules out the infinite horizon case, where the choice probabilities are time-invariant. Under this assumption, $[\boldsymbol{S}_i - \boldsymbol{S}_K]$ and $\widetilde{\boldsymbol{S}}_K$ are solved with a closed-form expression as follows:

$$\left[\widetilde{\boldsymbol{S}}_K [\boldsymbol{S}_i - \boldsymbol{S}_K]^+, \quad -\beta^{-1} [\boldsymbol{S}_i - \boldsymbol{S}_K]^+ \right] \;=\; \Delta \log \boldsymbol{p}_K \Delta \boldsymbol{\xi}_{i,K}^{-1}, \quad (9)$$

where $\Delta \log \boldsymbol{p}_K = \left[\Delta \log \boldsymbol{p}_{t_1,K}, \Delta \log \boldsymbol{p}_{t_2,K}, ..., \Delta \log \boldsymbol{p}_{t_{2J},K} \right]$. We may then solve for $\widetilde{\boldsymbol{S}}_K$ and $[\boldsymbol{S}_i - \boldsymbol{S}_K]$ from the nonlinear system above. Once $\widetilde{\boldsymbol{S}}_K$ is identified, we have obtained $\boldsymbol{S}_K(x) - \boldsymbol{S}_K(J)$ for all $x \in \{1, 2, \cdots, J\}, x \neq J$. In order to fully recover $\boldsymbol{S}_K(x)$, we need to pin down $\boldsymbol{S}_K(J)$, then all the subjective probabilities are identified.

Assumption 8 *There exist a state $x = J$ and action $a = K$ under which the agent's subjective beliefs are known, i.e., $s(x_{t+1}|x_t = J, a_t = K)$ or $\boldsymbol{S}_K(J)$ are known.*

The restriction of known subjective beliefs imposed in assumption 8 is only required to hold for a certain state and action. For example, the agent might have correct beliefs in some extreme states, i.e., $s(x_{t+1}|J,K) = f(x_{t+1}|J,K)$.

Finally, all the subjective beliefs $s(x_{t+1}|x_t, a_t)$ are identified with a closed-form. We summarize our identification results as follows:

Theorem 1 *Suppose that Assumptions 1–8 hold. Then the subjective beliefs* $s(x_{t+1}|x_t, a_t)$ *for* $x_t, x_{t+1} \in \{1, 2, ..., J\}$ *and* $a_t \in \{1, 2, ..., K\}$ *are identified as a closed-form function of conditional choice probabilities* $p_t(a_t|x_t)$, $p_{t-1}(a_{t-1}|x_{t-1})$, *and* $p_{t-2}(a_{t-2}|x_{t-2})$ *for* $t = t_1, t_2, ..., t_{2J}$.

Proof : See the Appendix.

Our identification results require at least $2J + 2$ consecutive periods of observations or $2J$ spells of 3 consecutive periods. Notice that we do not need to specify what the last period T is, nor need the usual normalization of the utility function, e.g. $u(x, a = K) = 0$. In empirical applications, we may focus on subjective beliefs of part of the state variables and let other state variables follow the objective transition under the restriction

$$s(x_{t+1}, w_{t+1}|x_t, w_t, a_t) = s(x_{t+1}|x_t, a_t)f(w_{t+1}|w_t, a_t).$$

Such a restriction may help relieve the curse of dimensionality.

Alternatively, we show that the last $J + 1$ periods of observations may be sufficient to identify the subjective beliefs if the conditional distribution $s(x_{t+1}|x_t, a_t = K)$, and therefore, $\widetilde{\boldsymbol{S}}_K$, are known. We assume

Assumption 9 *(i)* $s(x_{t+1}|x_t, a_t = K) = f(x_{t+1}|x_t, a_t = K)$; *(ii)* $u(x, a = K) = 0$.

Assumption 9(i) is stronger than Assumption 8 because it normalizes the whole conditional distribution $s(x_{t+1}|x_t, a_t = K)$ for all the values of x_t. Nevertheless, the advantage of such a normalization is that it reduces the number of observed periods required for identification. Assumption 9(ii) is widely-used in this literature to identify DDC models, e.g., Fang and Wang (2015) and Bajari et al. (2015).

Under Assumption 9, we may show

$$\Delta \boldsymbol{v}_{T,K} \quad = \quad -\beta \widetilde{\boldsymbol{S}}_K(-\log \boldsymbol{p}_{T,K}), \tag{10}$$

and all the value functions $\Delta \boldsymbol{v}_{t,K}$ can be solved for through equation (7) recursively.

We then define

$$\Delta \boldsymbol{\xi}_{i,K}^{T-J+1} = \begin{bmatrix} \Delta \boldsymbol{\xi}_{T-J+1,i,K} & \Delta \boldsymbol{\xi}_{T-J+2,i,K} & ... & \Delta \boldsymbol{\xi}_{T-1,i,K} \end{bmatrix} \tag{11}$$

$$\Delta \boldsymbol{v}^{T-J+1} = \begin{bmatrix} \Delta \boldsymbol{v}_{T-J+1} & \Delta \boldsymbol{v}_{T-J+2} & ... & \Delta \boldsymbol{v}_{T-1} \end{bmatrix} \tag{12}$$

where $\Delta \boldsymbol{v}_t \equiv (-\Delta \log \boldsymbol{p}_{t,K} + \Delta \boldsymbol{v}_{t,K})$. Equation (6) with $t = T - J + 1, T - J + 2, ..., T$ may be written as

$$\Delta \boldsymbol{\xi}_{i,K}^{T-J+1} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K]\Delta \boldsymbol{v}^{T-J+1} \tag{13}$$

We may then solve for $\boldsymbol{S}_i$ under the following assumption:

Assumption 10 *The* $(J-1)\times(J-1)$ *matrix* $\Delta \boldsymbol{v}^{T-J+1}$ *is invertible.*

As shown above, Assumption 10 is imposed on the observed choice probabilities, and therefore, is directly testable from the data. Given that we have identified the subjective beliefs, the utility function $u(x,a)$ is also identified. We summarize the result as follows:

Theorem 2 *Suppose that Assumptions 1-6, 9, and 10 hold. Then the subjective belief* $s(x_{t+1}|x_t,a_t)$ *for* $x_t, x_{t+1} \in \{1,2,...,J\}$ *and* $a_t \in \{1,2,...,K\}$*, together with the utility function* $u(x,w,a)$*, is identified as a closed-form function of conditional choice probabilities* $p_t(a_t|x_t)$ *for* $t = T-J, T-J+1, ..., T$*.*

Proof : See the Appendix.

In addition, combination of Theorems 2 and 2 implies that we may relax Assumptions 6 and 9(i) if the last $2J+2$ periods of observations are available. We may relax Assumption 6 as follows: There exists an $i \in \mathcal{A}$ such that the $J\times(J-1)$ matrix $\boldsymbol{S}_i - \boldsymbol{S}_K$ has a full column rank of $J-1$. Following the same proof of Theorem 1, one can show that this new assumption is sufficient to identify $\widetilde{\boldsymbol{S}}_K$ using the last $2J+2$ periods of observations. Thus, there is no need to normalize the conditional distribution in this case as in Assumption 9(i) in Theorem 2.

4 The infinite horizon case

The previous identification strategy makes use of variations in conditional choice probability across time. In the infinite horizon case, unfortunately, such variations across time are not available. Therefore, different assumptions are needed for the identification of the subjective beliefs. We consider the case where there is an additional state variable $w_t \in \{w_1,...,w_L\}$, on which the subjective beliefs are equal to the objective ones, i.e., rational expectations hold for w_t.

We assume that the observed state variables includes $\{x_t, w_t\}$ and that both the subjective beliefs and the objective probabilities follow a first order Markov process. We update the relevant assumptions as follows:

Assumption 1' *(i) The state variables* $\{x_t, w_t, \epsilon_t\}$ *follows a first-order Markov process; (ii)* $f(x_{t+1}, w_{t+1}, \epsilon_{t+1}|x_t, w_t, \epsilon_t, a_t) = f(x_{t+1}, w_{t+1}|x_t, w_t, a_t)f(\epsilon_{t+1})$*; (iii)* $u(x_t, w_t, a_t, \epsilon_t) = u(x_t, w_t, a_t) + \epsilon_t(a_t)$ *for any* $a_t \in \mathcal{A}$*; (iv)* $\epsilon_t(a)$ *for all* t *and all* $a \in \mathcal{A}$ *are i.i.d. draws from mean zero type-I extreme value distribution.*

Since the choice probabilities become stationary in the infinite horizon case, the previous identification strategy for the finite horizon case is no longer applicable. To be specific, Assumption 7 does not hold in this case. We have to focus on a class of models where the subjective beliefs are equal to the objective state transitions for part of the state variables, i.e., w_t. We assume

Assumption 2' *The subjective beliefs satisfy*

$$s(x_{t+1}, w_{t+1}|x_t, w_t, a_t) = s(x_{t+1}|x_t, a_t)s(w_{t+1}|w_t, a_t) = s(x_{t+1}|x_t, a_t)f(w_{t+1}|w_t, a_t) \quad (14)$$

For simplicity, we keep Assumption 9, which normalizes the utility function $u(x, w, a = K) = 0$ and $s(x_{t+1}|x_t, a_t = K) = f(x_{t+1}|x_t, a_t = K)$.

The choice-specific value function then becomes

$$\begin{aligned} & v(x, w, a) \\ = \; & u(x, w, a) + \beta \iint \big[-\log p_K(x', w') + v_K(x', w') \big] s(x'|x, a)s(w'|w, a)dx'dw'. \end{aligned} \quad (15)$$

For $x \in \{x_1, ..., x_J\}$ and $w \in \{w_1, ..., w_L\}$, we define vector

$$\boldsymbol{u}_a = [u(x_1, w_1, a), ..., u(x_1, w_L, a), u(x_2, w_1, a), ..., u(x_2, w_L, a), ..., u(x_J, w_1, a), ..., u(x_J, w_L, a)]^T.$$

We define vectors $\boldsymbol{v}_a$, and $\log \boldsymbol{p}_K$ analogously. Let $\boldsymbol{S}_a^x = [s(x_{t+1} = x_j|x_t = x_i, a_t)]_{i,j}$ and $\boldsymbol{F}_a^w = [s(w_{t+1} = w_j|w_t = w_i, a_t)]_{i,j}$. In matrix form,

$$\boldsymbol{v}_a = \boldsymbol{u}_a + \beta[\boldsymbol{S}_a^x \otimes \boldsymbol{F}_a^w]\big[-\log \boldsymbol{p}_K + \boldsymbol{v}_K \big]. \quad (16)$$

Similar to the case of finite horizon, we need a rank condition to identify the value functions.

Assumption 3' $[I - \beta(\boldsymbol{F}_K^x \otimes \boldsymbol{F}_K^w)]$ *is invertible.*

Under this assumption, the value function $\boldsymbol{v}_K$ corresponding to action K is identified with a closed-form

$$\boldsymbol{v}_K = [I - \beta(\boldsymbol{F}_K^x \otimes \boldsymbol{F}_K^w)]^{-1}[\beta(\boldsymbol{F}_K^x \otimes \boldsymbol{F}_K^w)(-\log \boldsymbol{p}_K)] \quad (17)$$

where $\boldsymbol{F}_K^x = [f(x_{t+1} = x_j|x_t = x_i, a_t = K)]_{i,j}$.

Assumption 2' implies that the state transition is separable with respect to x_t and w_t so that we can consider the value function corresponding to action a as follows:

$$v(x, w, a) \;=\; u(x, w, a) + \beta \int \big[-\log \tilde{p}_K^a(x', w) + \tilde{v}_K^a(x', w) \big] s(x'|x, a)dx' \quad (18)$$

where

$$\begin{aligned} \log \tilde{p}_K^a(x', w) &= \int \log p_K(x', w')f(w'|w, a)dw' \\ \tilde{v}_K^a(x', w) &= \int v_K(x', w')f(w'|w, a)dw'. \end{aligned} \quad (19)$$

Therefore, we have

$$\begin{aligned}\xi_{i,K}(x,w) &= \log\Big(\frac{p_i(x,w)}{p_K(x,w)}\Big) \\ &= \beta \boldsymbol{S}_i^x(x)[-\log \tilde{\boldsymbol{p}}_K^i(w) + \tilde{\boldsymbol{v}}_K^i(w)] - \beta \boldsymbol{S}_K^x(x)[-\log \tilde{\boldsymbol{p}}_K^K(w) + \tilde{\boldsymbol{v}}_K^K(w)] \\ &\quad + u(x,w,i), \end{aligned} \tag{20}$$

where

$$\begin{aligned}\boldsymbol{S}_{a_t}^x(x_t) &= [s(x_{t+1}=1|x_t,a_t),...,s(x_{t+1}=J|x_t,a_t)] \\ \log \tilde{\boldsymbol{p}}_K^i(w) &= [\log \tilde{p}_K^i(1,w),...,\log \tilde{p}_K^i(J,w)]^T \\ \tilde{\boldsymbol{v}}_K^i(w) &= [\tilde{v}_K^i(1,w),...,\tilde{v}_K^i(J,w)]^T.\end{aligned}$$

For certain class of utility function, we are able to eliminate the utility function and reveal a direct relationship between the subjective beliefs and the observed choice probabilities. We consider a class of utility functions, which are linear in w, as follows:

Assumption 4' *$u(x,w,a) = u^1(x,a) + u^2(x,a)w$.*

Suppose the support of w contains $\{0,1,2,...,J+1\}$. The second order difference of $u(x,w,a)$ with respect to w is zero with

$$\Delta_w^2 f(w) \equiv [f(w) - f(w-1)] - [f(w+1) - f(w)].$$

Taking the second order difference with respect to w leads to

$$\begin{aligned}\Delta_w^2 \xi_{i,K}(x,w) &= \beta \boldsymbol{S}_i^x(x)[-\Delta_w^2 \log \tilde{\boldsymbol{p}}_K^i(w) + \Delta_w^2 \tilde{\boldsymbol{v}}_K^i(w)] \\ &\quad -\beta \boldsymbol{S}_K^x(x)[-\Delta_w^2 \log \tilde{\boldsymbol{p}}_K^K(w) + \Delta_w^2 \tilde{\boldsymbol{v}}_K^K(w)] \\ &\equiv \beta \boldsymbol{S}_i^x(x)\tilde{\boldsymbol{v}}^i(w) - \beta \boldsymbol{S}_K^x(x)\tilde{\boldsymbol{v}}^K(w)\end{aligned} \tag{21}$$

With $w \in \{1,2,...,J\}$, we may obtain enough restrictions to solve for $\boldsymbol{S}_i^x(x)$ under an invertibility condition imposed on the observables. We assume

Assumption 5' *The $J \times J$ matrix $\tilde{\boldsymbol{V}}^i$ is invertible, where $\tilde{\boldsymbol{V}}^i = [\tilde{\boldsymbol{v}}^i(1), \tilde{\boldsymbol{v}}^i(2), ..., \tilde{\boldsymbol{v}}^i(J)]$.*

This assumption is directly testable from the data because matrix $\tilde{\boldsymbol{V}}^i$ only contains directly estimable entries. Under assumption 5', we may solve for $\boldsymbol{S}_i^x(x)$, i.e., $s(x_{t+1}|x_t,a_t)$ with a closed-form as follows:

$$\boldsymbol{S}_i^x(x) = \beta^{-1}\Big(\Delta_w^2 \boldsymbol{\xi}_{i,K}(x) + \beta \boldsymbol{S}_K^x(x)\tilde{\boldsymbol{V}}^K\Big)\Big(\tilde{\boldsymbol{V}}^i\Big)^{-1}$$

where $\Delta_w^2 \boldsymbol{\xi}_{i,K}(x) = [\Delta_w^2 \xi_{i,K}(x,1), \Delta_w^2 \xi_{i,K}(x,2), ..., \Delta_w^2 \xi_{i,K}(x,J)]$. Given that we have identified the subjective beliefs, the utility function $u(x,w,a)$ is also identified from equation (20). We summarize the results as follows:

Theorem 3 *Suppose that Assumptions 3, 4, 5, 9, 1', 2', 3', 4', and 5' hold. Then, the subjective belief $s(x_{t+1}|x_t,a_t)$ for $x_t, x_{t+1} \in \{1,2,...,J\}$ and $a_t \in \{1,2,...,K\}$, together with the utility function $u(x,w,a)$, is identified as a closed-form function of conditional choice probabilities $p_t(a_t|x_t,w_t)$ and objective state transition $f(w_{t+1}|w_t,a_t)$.*

Proof : See the Appendix.

5 Heterogeneous beliefs

Agents may display heterogenous beliefs about transition of the same state variable. We show in this section that a DDC model with agents holding heterogenous subjective beliefs can also be identified using the results in previous sections.

Suppose agents can be classified into $L \geq 2$ types based on their heterogenous beliefs and let $\tau \in \{1, 2, ..., L\}$ denote the unobserved type (heterogeneity). The subjective beliefs can then be described as $s(x_{t+1}|x_t, a_t, \tau)$. In the meanwhile, the conditional choice probability also depends on the heterogeneity τ as $p_t(a_t|x_t, \tau)$. We employ an identification methodology for measurement error models to show that the observed joint distribution of state variables and agents' actions uniquely determines the conditional choice probability $p_t(a_t|x_t, \tau)$ for all $\tau \in \{1, 2, \cdots, L\}$. Given the conditional choice probability $p_t(a_t|x_t, \tau)$, we can apply the results in Theorems 1,2, or 3 to identify the heterogeneous beliefs $s(x_{t+1}|x_t, a_t, \tau)$.

We start our identification with the following assumption.

Assumption 11 $\{a_t, x_t, \tau\}$ *follows a first-order Markov process.*

The first-order Markov property of action and state variable are widely assumed in the literature of DDC models.

The observed joint distribution is then associated with the unobserved ones as follows:

$$\begin{aligned} & f(a_{t+l}, ..., a_{t+1}, x_{t+1}, a_t, x_t, a_{t-1}, ..., a_{t-l}) \\ = & \sum_{\tau} f(a_{t+l}, ..., a_{t+1}|x_{t+1}, \tau) f(x_{t+1}, a_t|x_t, \tau) f(\tau, x_t, a_{t-1}, ..., a_{t-l}). \end{aligned} \tag{22}$$

Let l be an integer such that $J \leq K^l$, where K and J are numbers of possible realizations of a_t and τ, respectively.[3] Suppose $h(\cdot)$ is a known function that maps the support of $(a_{t+l}, ..., a_{t+1})$, $\mathcal{A}^l$ to that of τ, i.e., $\{1, 2, ..., L\}$. This mapping We define

$$\begin{aligned} a_{t+} &= h(a_{t+l}, ..., a_{t+1}), \\ a_{t-} &= h(a_{t-1}, ..., a_{t-l}). \end{aligned}$$

For a fixed pair (x_t, x_{t+1}), we may consider a_{t+}, a_t, and a_{t-} as three measurements of the unobserved heterogeneity τ and use the results in Hu (2008) to identify the objective $f(x_{t+1}, a_t|x_t, \tau)$, which leads to conditional choice probability $p_t(a_t|x_t, \tau)$.[4] It is worth noting that we maintain that the support of a_{t+} and a_{t-} is the same as that of τ just for simplicity of our identification argument. Our results can be generalized straightforwardly to that case where the support of a_{t+} and a_{t-} is larger than that of τ.

For a given pair x_t and x_{t+1} in $\mathcal{X}$, we define a matrix

$$M_{a_{t+}, x_{t+1}, x_t, a_{t-}} = \left[f(a_{t+} = i, x_{t+1}, x_t, a_{t-} = j) \right]_{i,j}.$$

[3]This restriction is satisfied in most of the empirical application where $K = 2$ and J is not very large.

[4] Similar results can also be found in Kasahara and Shimotsu (2009).

Assumption 12 *For all $(x_{t+1}, x_t) \in \mathcal{X} \times \mathcal{X}$, matrix $M_{a_{t+}, x_{t+1}, x_t, a_{t-}}$ has a full rank of L.*

This identification strategy in Hu (2008) requires an eigenvalue-eigenvector decomposition of an observed matrix. The uniqueness of such a decomposition requires that the eigenvalues are distinctive as follows:

Assumption 13 *For all $(x_{t+1}, x_t) \in \mathcal{X} \times \mathcal{X}$, there exists a known function $\omega(\cdot)$ such that*

$$E\big[\omega(a_t)|x_{t+1}, x_t, \tau_1\big] \neq E\big[\omega(a_t)|x_{t+1}, x_t, \tau_2\big]$$

for any $\tau_1 \neq \tau_2 \in \{1, 2, ..., L\}$. Without loss of generality, we assume $E\big[\omega(a_t)|x_{t+1}, x_t, \tau\big]$ is increasing in τ.

To illustrate Assumption 13, we consider a binary choice, i.e, $a_t \in \{0, 1\}$ and $\omega(\cdot)$ is an identity function. Then $E\big[\omega(a_t)|x_{t+1}, x_t, \tau\big] = \Pr(a_t = 1|x_{t+1}, x_t, \tau)$. Suppose $\tau = 1, 2, \cdots, L$ are ordered such that agents whose type is $\tau = L$ have the most "accurate" subjective beliefs in the sense that their beliefs are the closest to the objective state transition while $\tau = 1$ have the lest accurate ones. Assumption 13 states that given that state variable x_t and x_{t+1} the probability of choosing action $a_t = 1$ is higher if an agent's subjective beliefs are closer to the objective ones.

We summarize the identification result in the following theorem.

Theorem 4 *Suppose that Assumptions 11, 12, and 13 hold. Then, the joint distribution $f(a_{t+l}, ..., a_{t+1}, a_t, x_{t+1}, x_t, a_{t-1}, ..., a_{t-l})$ uniquely determines the conditional choice probability $p_t(a_t|x_t, \tau)$.*

Proof : See the Appendix.

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6 Appendix

Proof of Theorem 1

In this dynamic setting, the optimal choice a_t in period t is

$$a_t = \arg\max_{a\in\mathcal{A}}\{v_t(x_t, a) + \epsilon_t(a)\} \tag{23}$$

where $v_t(x_t, a)$ is the choice-specific value function. The ex ante value function at t can be expressed as

$$\begin{aligned} V_t(x_t) &= \int \sum_{a\in\mathcal{A}} \mathbb{1}\{a = a_t\}\big[v_t(x_t, a) + \epsilon_t(a)\big] g(\epsilon_t) d\epsilon_t \\ &= \log\left\{\sum_{a\in\mathcal{A}} \exp\big[v_t(x_t, a)\big]\right\}, \end{aligned} \tag{24}$$

where the second equality is obtained under the assumption that ϵ_t is distributed according to a mean zero type-I extreme value distribution. The conditional choice probability is for $i \in \mathcal{A}$

$$p_t(a_t = i|x_t) = \frac{\exp\big[v_t(x_t, i)\big]}{\sum_{a\in\mathcal{A}} \exp\big[v_t(x_t, a)\big]}. \tag{25}$$

We may further simplify $V_t(x_t)$ with $i = K$ as follows:

$$\begin{aligned} V_t(x_t) &= -\log p_t(a_t = K|x_t) + v_t(x_t, a_t = K) \\ &\equiv -\log p_{t,K}(x_t) + v_{t,K}(x_t) \end{aligned} \tag{26}$$

Given that the state variable x_t has support $\mathcal{X} = \{1, 2, ..., J\}$, we define a row vector of $J - 1$ independent subjective beliefs as follows:

$$\boldsymbol{S}_a(x_t) = [s(x_{t+1} = 1|x_t, a), s(x_{t+1} = 2|x_t, a), ..., s(x_{t+1} = J - 1|x_t, a)] \tag{27}$$

Notice that $\boldsymbol{S}_a(x_t)$ contains the same information as $s(x_{t+1}|x_t, a)$. We consider the choice-specific value function

$$
\begin{aligned}
& v_t(x_t, a_t) \\
= \; & u(x_t, a_t) + \beta \int V_{t+1}(x_{t+1}) s(x_{t+1}|x_t, a_t) dx_{t+1} \\
= \; & u(x_t, a_t) + \beta \sum_{x_{t+1}=1}^{J} \big[-\log p_{t+1,K}(x_{t+1}) + v_{t+1,K}(x_{t+1}) \big] s(x_{t+1}|x_t, a_t) \\
= \; & u(x_t, a_t) + \beta \sum_{x_{t+1}=1}^{J-1} \big[-\log p_{t+1,K}(x_{t+1}) + v_{t+1,K}(x_{t+1}) \big] s(x_{t+1}|x_t, a_t) \\
& + \beta \big[-\log p_{t+1,K}(J) + v_{t+1,K}(J) \big] \big[1 - \sum_{x_{t+1}=1}^{J-1} s(x_{t+1}|x_t, a_t) \big] \\
= \; & u(x_t, a_t) + \beta \sum_{x_{t+1}=1}^{J-1} \big[-\log p_{t+1,K}(x_{t+1}) + \log p_{t+1,K}(J) \\
& + v_{t+1,K}(x_{t+1}) - v_{t+1,K}(J) \big] s(x_{t+1}|x_t, a_t) + \beta \big[-\log p_{t+1,K}(J) + v_{t+1,K}(J) \big] \quad (28)
\end{aligned}
$$

For convenience, we define

$$
-\log \boldsymbol{p}_{t+1,K} = \begin{pmatrix} -\log p_{t+1,K}(x_{t+1} = 1) + \log p_{t+1,K}(J) \\ -\log p_{t+1,K}(x_{t+1} = 2) + \log p_{t+1,K}(J) \\ \dots \\ -\log p_{t+1,K}(x_{t+1} = J-1) + \log p_{t+1,K}(J) \end{pmatrix} \quad (29)
$$

$$
\boldsymbol{v}_{t+1,K} = \begin{pmatrix} v_{t+1,K}(x_{t+1} = 1) - v_{t+1,K}(J) \\ v_{t+1,K}(x_{t+1} = 2) - v_{t+1,K}(J) \\ \dots \\ v_{t+1,K}(x_{t+1} = J-1) - v_{t+1,K}(J) \end{pmatrix} \quad (30)
$$

The choice-specific value function may then be expressed as follows:

$$
\begin{aligned}
& v_t(x_t, a_t) \\
= \; & u(x_t, a_t) + \beta \boldsymbol{S}_{a_t}(x_t)(-\log \boldsymbol{p}_{t+1,K} + \boldsymbol{v}_{t+1,K}) + \beta \big[-\log p_{t+1,K}(J) + v_{t+1,K}(J) \big] \quad (31)
\end{aligned}
$$

The observed choice probabilities are associated with the choice-specific value function

$$
\begin{aligned}
\xi_{t,i,K}(x) &= \log \left(\frac{p_{t,i}(x)}{p_{t,K}(x)} \right) \\
&= v_t(x, a_t = i) - v_t(x, a_t = K) \\
&= \beta [\boldsymbol{S}_i(x) - \boldsymbol{S}_K(x)][-\log \boldsymbol{p}_{t+1,K} + \boldsymbol{v}_{t+1,K}] + [u(x, i) - u(x, K)]. \quad (32)
\end{aligned}
$$

Since the one-period utility function is time-invariant, we may further consider

$$\begin{aligned}\Delta\xi_{t,i,K}(x) &= \log\left(\frac{p_{t,i}(x)}{p_{t,K}(x)}\right) - \log\left(\frac{p_{t-1,i}(x)}{p_{t-1,K}(x)}\right) \\ &= \beta[\boldsymbol{S}_i(x) - \boldsymbol{S}_K(x)][-\log \boldsymbol{p}_{t+1,K} + \log \boldsymbol{p}_{t,K} + \boldsymbol{v}_{t+1,K} - \boldsymbol{v}_{t,K}] \\ &= \beta[\boldsymbol{S}_i(x) - \boldsymbol{S}_K(x)][-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}], \end{aligned} \tag{33}$$

where

$$\begin{aligned}\Delta \log \boldsymbol{p}_{t+1,K} &= \log \boldsymbol{p}_{t+1,K} - \log \boldsymbol{p}_{t,K} \\ \Delta \boldsymbol{v}_{t+1,K} &= \boldsymbol{v}_{t+1,K} - \boldsymbol{v}_{t,K} \end{aligned} \tag{34}$$

This equation hold for each of $x \in \{1, 2, ..., J\}$. Next we put all these equations with different values of x in the matrix form with the following definitions

$$\boldsymbol{S}_a = \begin{pmatrix} \boldsymbol{S}_a(x_t = 1) \\ \boldsymbol{S}_a(x_t = 2) \\ ... \\ \boldsymbol{S}_a(x_t = J) \end{pmatrix} \tag{35}$$

and

$$\boldsymbol{\Delta\xi}_{t,i,K} = [\Delta\xi_{t,i,K}(1), ..., \Delta\xi_{t,i,K}(J)]^T \tag{36}$$

Notice that matrix $\boldsymbol{S}_a$ is a $J \times (J-1)$ matrix. We then have

$$\Delta\boldsymbol{\xi}_{t,i,K} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K][-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}]. \tag{37}$$

We now focus on the value function corresponding to choice $a_t = K$ and $\Delta\boldsymbol{v}_{t+1,K}$, where

$$\begin{aligned} & \Delta\boldsymbol{v}_{t,K} \\ = \; & \boldsymbol{v}_{t,K} - \boldsymbol{v}_{t-1,K} \\ = \; & \begin{pmatrix} v_{t,K}(x_t = 1) - v_{t,K}(J) \\ v_{t,K}(x_t = 2) - v_{t,K}(J) \\ ... \\ v_{t,K}(x_t = J-1) - v_{t,K}(J) \end{pmatrix} - \begin{pmatrix} v_{t-1,K}(x_{t-1} = 1) - v_{t-1,K}(J) \\ v_{t-1,K}(x_{t-1} = 2) - v_{t-1,K}(J) \\ ... \\ v_{t-1,K}(x_{t-1} = J-1) - v_{t-1,K}(J) \end{pmatrix} \end{aligned} \tag{38}$$

Equation (31) implies

$$\begin{aligned} & [v_t(x, K) - v_t(J, K)] \\ = \; & [u(x, K) - u(J, K)] + \beta\big[\boldsymbol{S}_K(x) - \boldsymbol{S}_K(J)\big](-\log \boldsymbol{p}_{t+1,K} + \boldsymbol{v}_{t+1,K}) \end{aligned} \tag{39}$$

Each element in the $(J-1)$-by-1 vector $\Delta\boldsymbol{v}_{t,K}$ is for $x \in \{1, 2, ..., J-1\}$

$$\begin{aligned} & [v_t(x, K) - v_t(J, K)] - [v_{t-1}(x, K) - v_{t-1}(J, K)] \\ = \; & \beta\big[\boldsymbol{S}_K(x) - \boldsymbol{S}_K(J)\big](-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta\boldsymbol{v}_{t+1,K}) \end{aligned} \tag{40}$$

In the matrix form, we have

$$\begin{aligned}\Delta \boldsymbol{v}_{t,K} &= \boldsymbol{v}_{t,K} - \boldsymbol{v}_{t-1,K} \\ &= \beta \widetilde{\boldsymbol{S}}_K(-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}) \end{aligned} \tag{41}$$

where $\widetilde{\boldsymbol{S}}_K$ is a $(J-1) \times (J-1)$ matrix

$$\widetilde{\boldsymbol{S}}_K = \begin{pmatrix} \boldsymbol{S}_K(1) - \boldsymbol{S}_K(J) \\ \boldsymbol{S}_K(2) - \boldsymbol{S}_K(J) \\ \vdots \\ \boldsymbol{S}_K(J-1) - \boldsymbol{S}_K(J) \end{pmatrix}. \tag{42}$$

In fact,

$$\widetilde{\boldsymbol{S}}_K = \begin{pmatrix} 1 & 0 & ... & 0 & -1 \\ 0 & 1 & ... & 0 & -1 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -1 \end{pmatrix} \boldsymbol{S}_K. \tag{43}$$

In summary, the choice probabilities are associated with subjective beliefs and value functions through

$$\Delta \boldsymbol{\xi}_{t,i,K} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K][-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}] \tag{44}$$

And the choice-specific value function evolves as follows

$$\Delta \boldsymbol{v}_{t,K} = \beta \widetilde{\boldsymbol{S}}_K(-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}) \tag{45}$$

By eliminating the value functions in these two equations, we are able to find the direct relationship between the observed choice probabilities and the subjective beliefs.

Define $J \times (J-1)$ matrix

$$\boldsymbol{M} = [\boldsymbol{S}_i - \boldsymbol{S}_K] \tag{46}$$

The full rank condition in Assumption 6 guarantees that the generalized inverse $\boldsymbol{M}^+$ of $\boldsymbol{M}$ is

$$\boldsymbol{M}^+ = (\boldsymbol{M}^T \boldsymbol{M})^{-1} \boldsymbol{M}^T \tag{47}$$

with $\boldsymbol{M}^+ \boldsymbol{M} = I$. Therefore,

$$\begin{aligned} \boldsymbol{M}^+ \Delta \boldsymbol{\xi}_{t,i,K} &= \beta(-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}) \\ \boldsymbol{M}^+ \Delta \boldsymbol{\xi}_{t-1,i,K} &= \beta(-\Delta \log \boldsymbol{p}_{t,K} + \Delta \boldsymbol{v}_{t,K}) \\ &= \beta[-\Delta \log \boldsymbol{p}_{t,K} + \beta \widetilde{\boldsymbol{S}}_K(-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K})]. \end{aligned} \tag{48}$$

Eliminating $(-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K})$ leads to

$$\widetilde{\boldsymbol{S}}_K \boldsymbol{M}^+ \Delta \boldsymbol{\xi}_{t,i,K} - \beta^{-1} \boldsymbol{M}^+ \Delta \boldsymbol{\xi}_{t-1,i,K} \quad = \quad \Delta \log \boldsymbol{p}_{t,K}. \tag{49}$$

That is

$$[\widetilde{\boldsymbol{S}}_K \boldsymbol{M}^+ \quad -\beta^{-1} \boldsymbol{M}^+] \left(\begin{array}{c} \Delta \boldsymbol{\xi}_{t,i,K} \\ \Delta \boldsymbol{\xi}_{t-1,i,K} \end{array} \right) \quad = \quad \Delta \log \boldsymbol{p}_{t,K}. \tag{50}$$

This equation implies that the choice probabilities may be directly associated with the subjective beliefs. Furthermore, we may solve for the subjective beliefs with a closed-form. In these J equations, $\widetilde{\boldsymbol{S}}_K$ contains $(J-1) \times (J-1)$ unknowns and $\boldsymbol{M}^+$ has $(J-1) \times J$.

Suppose we observe data for $2J$ equations. Define a $(2J) \times (2J)$ matrix

$$\Delta \boldsymbol{\xi}_{i,K} \equiv \left[\begin{array}{cccc} \Delta \boldsymbol{\xi}_{t_1,i,K} & \Delta \boldsymbol{\xi}_{t_2,i,K} & \dots & \Delta \boldsymbol{\xi}_{t_{2J},i,K} \\ \Delta \boldsymbol{\xi}_{t_1-1,i,K} & \Delta \boldsymbol{\xi}_{t_2-1,i,K} & \dots & \Delta \boldsymbol{\xi}_{t_{2J-1},i,K} \end{array} \right] \tag{51}$$

and a $(J-1) \times (2J)$ matrix

$$\Delta \log \boldsymbol{p}_K \equiv \Big[\Delta \log \boldsymbol{p}_{t_1,K}, \Delta \log \boldsymbol{p}_{t_2,K}, \dots, \Delta \log \boldsymbol{p}_{t_{2J},K} \Big] \tag{52}$$

We have

$$\Big[\widetilde{\boldsymbol{S}}_K \boldsymbol{M}^+, \quad -\beta^{-1} \boldsymbol{M}^+ \Big] \Delta \boldsymbol{\xi}_{i,K} \quad = \quad \Delta \log \boldsymbol{p}_K \tag{53}$$

Under Assumption 7, $[\boldsymbol{S}_i - \boldsymbol{S}_K]$ and $\widetilde{\boldsymbol{S}}_K$ are solved with a closed-form expression with a known β as follows:[5]

$$\Big[\widetilde{\boldsymbol{S}}_K \boldsymbol{M}^+, \quad -\beta^{-1} \boldsymbol{M}^+ \Big] \quad = \quad \Delta \log \boldsymbol{p}_K [\Delta \boldsymbol{\xi}_{i,K}]^{-1}, \tag{54}$$

$$\begin{aligned} [\boldsymbol{S}_i - \boldsymbol{S}_K] \quad &= \quad \boldsymbol{M} \\ &= \quad (\boldsymbol{M}^+)^T [\boldsymbol{M}^+ (\boldsymbol{M}^+)^T]^{-1}, \end{aligned} \tag{55}$$

and

$$\widetilde{\boldsymbol{S}}_K \quad = \quad (\widetilde{\boldsymbol{S}}_K \boldsymbol{M}^+) \boldsymbol{M} \tag{56}$$

Given the definition of $\widetilde{\boldsymbol{S}}_K$, we have identified $\boldsymbol{S}_K(x) - \boldsymbol{S}_K(J)$ for $x \in \{1, 2, ..., J-1\}$. Assumption 8 normalizes $\boldsymbol{S}_K(J)$ to a known distribution, and therefore, we fully recover $\boldsymbol{S}_K(x)$ for $x \in \{1, 2, ..., J\}$, i.e., $\boldsymbol{S}_K$. Therefore, all the subjective probabilities $\boldsymbol{S}_i$ are identified from $[\boldsymbol{S}_i - \boldsymbol{S}_K]$ with a closed-form. QED.

[5] This equation also implies that $\boldsymbol{S}_i$ and β can be identified if $\boldsymbol{S}_K$ is known.

Proof of Theorem 2

We observe the last $J+1$ periods, i.e., $t = T-J, T-J+1, ..., T$. Assumption 9 implies that $\boldsymbol{S}_K$ is known and that $u(x,K)=0$. The choice-specific value function may then be expressed as follows:

$$\begin{aligned} v_T(x,K) &= u(x,K) = 0. \\ v_{T-1}(x,K) &= u(x,K) + \beta \boldsymbol{S}_K(x)(-\log \boldsymbol{p}_{T,K} + \boldsymbol{v}_{T,K}) + \beta\big[-\log p_{T,K}(J) + v_{T,K}(J)\big] \\ &= \beta \boldsymbol{S}_K(x)(-\log \boldsymbol{p}_{T,K}) + \beta\big[-\log p_{T,K}(J)\big] \end{aligned} \tag{57}$$

Each element in the $(J-1)$-by-1 vector $\Delta \boldsymbol{v}_{T,K}$ is

$$\begin{aligned} & [v_T(x,K) - v_T(J,K)] - [v_{T-1}(x,K) - v_{T-1}(J,K)] \\ =\ & 0 - [v_{T-1}(x,K) - v_{T-1}(J,K)] \\ =\ & -\beta[\boldsymbol{S}_K(x) - \boldsymbol{S}_K(J)](-\log \boldsymbol{p}_{T,K}) \end{aligned} \tag{58}$$

Therefore, the vector $\Delta \boldsymbol{v}_{T,K}$ is

$$\Delta \boldsymbol{v}_{T,K} \ = \ -\beta \tilde{\boldsymbol{S}}_K(-\log \boldsymbol{p}_{T,K}) \tag{59}$$

Since $\boldsymbol{S}_K$, and therefore, $\widetilde{\boldsymbol{S}}_K$, are known, we may identify $\Delta \boldsymbol{v}_{t,K}$ for all $t = T-J+1, T-J+2, ..., T$ recursively from

$$\Delta \boldsymbol{v}_{t,K} \ = \ \beta \widetilde{\boldsymbol{S}}_K(-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}). \tag{60}$$

As shown in the proof of Theorem 1, the choice probabilities are associated with subjective beliefs and value functions through

$$\Delta \boldsymbol{\xi}_{t,i,K} \ = \ \beta[\boldsymbol{S}_i - \boldsymbol{S}_K][-\Delta \log \boldsymbol{p}_{t+1,K} + \Delta \boldsymbol{v}_{t+1,K}] \tag{61}$$

Define

$$\Delta \boldsymbol{\xi}^{T-J+1}_{i,K} = \begin{bmatrix} \Delta \boldsymbol{\xi}_{T-J+1,i,K} & \Delta \boldsymbol{\xi}_{T-J+2,i,K} & \ldots & \Delta \boldsymbol{\xi}_{T-1,i,K} \end{bmatrix} \tag{62}$$

$$\Delta \boldsymbol{v}^{T-J+1} = \begin{bmatrix} \Delta \boldsymbol{v}_{T-J+1} & \Delta \boldsymbol{v}_{T-J+2} & \ldots & \Delta \boldsymbol{v}_{T-1} \end{bmatrix} \tag{63}$$

where

$$\Delta \boldsymbol{v}_t \equiv (-\Delta \log \boldsymbol{p}_{t,K} + \Delta \boldsymbol{v}_{t,K}).$$

Equation (61) for $t = T-J+1, T-J+2, ..., T$ may be written as

$$\Delta \boldsymbol{\xi}^{T-J+1}_{i,K} = \beta[\boldsymbol{S}_i - \boldsymbol{S}_K]\Delta \boldsymbol{v}^{T-J+1}. \tag{64}$$

Therefore, we may solve for $\boldsymbol{S}_i$ as

$$\boldsymbol{S}_i = \boldsymbol{S}_K + \beta^{-1}\Delta \boldsymbol{\xi}^{T-J+1}_{i,K}\big[\Delta \boldsymbol{v}^{T-J+1}\big]^{-1}$$

under assumption 10. Given the subjective beliefs and $u(x,K)=0$, we can solve for the utility function from

$$\log\left(\frac{p_{t,i}(x)}{p_{t,K}(x)}\right) = \beta[\boldsymbol{S}_i(x)-\boldsymbol{S}_K(x)][-\log\boldsymbol{p}_{t+1,K}+\boldsymbol{v}_{t+1,K}]+[u(x,i)-u(x,K)]. \quad (65)$$

QED.

Proof of Theorem 3

In the infinite horizon case, the choice-specific value function then becomes

$$\begin{aligned} & v(x,w,a) \\ = \;& u(x,w,a)+\beta\int\int V(x',w')s(x',w'|x,w,a)dx'dw' \\ = \;& u(x,w,a)+\beta\int\int\big[-\log p_K(x',w')+v_K(x',w')\big]s(x',w'|x,w,a)dx'dw' \quad (66)\end{aligned}$$

In particular, for choice $a=K$ with $u(x,w,K)=0$,

$$v(x,w,K) = \beta\int\int\big[-\log p_K(x',w')+v_K(x',w')\big]s(x',w'|x,w,K)dx'dw'. \quad (67)$$

For $x\in\{x_1,...,x_J\}$ and $w\in\{w_1,...,w_L\}$, we define vector

$\boldsymbol{u}_a=[u(x_1,w_1,a),...,u(x_1,w_L,a),u(x_2,w_1,a),...,u(x_2,w_L,a),...,u(x_J,w_1,a),...,u(x_J,w_L,a)]^T$.

Similarly, we define vectors $\boldsymbol{v}_a$, and $\log\boldsymbol{p}_K$. Let $\boldsymbol{S}_a^x=[s(x_{t+1}=x_j|x_t=x_i,a_t)]_{i,j}$ and $\boldsymbol{F}_a^w=[s(w_{t+1}=w_j|w_t=w_i,a_t)]_{i,j}$. In matrix form, the equation above becomes

$$\boldsymbol{v}_a = \boldsymbol{u}_a+\beta[\boldsymbol{S}_a^x\otimes\boldsymbol{F}_a^w]\big[-\log\boldsymbol{p}_K+\boldsymbol{v}_K\big] \quad (68)$$

We impose Assumption 9 to normalize the subjective belief corresponding to $a=K$ to be the objective ones, i.e., $s(x_{t+1}|x_t,K)=f(x_{t+1}|x_t,K)$. The rank condition in Assumption 3' implies that the value function $\boldsymbol{v}_K$ corresponding to action K is identified with a closed-form

$$\boldsymbol{v}_K = [I-\beta(\boldsymbol{F}_K^x\otimes\boldsymbol{F}_K^w)]^{-1}[\beta(\boldsymbol{F}_K^x\otimes\boldsymbol{F}_K^w)(-\log\boldsymbol{p}_K)] \quad (69)$$

where $\boldsymbol{F}_K^x=[f(x_{t+1}=x_j|x_t=x_i,a_t=K)]_{i,j}$.

Assumption 2' implies that the state transition is separable with respect to x_t and w_t so that we can consider the value function corresponding to action a as follows:

$$\begin{aligned} & v(x,w,a) \\ = \;& u(x,w,a)+\beta\int\int\big[-\log p_K(x',w')+v_K(x',w')\big]s(x',w'|x,w,a)dx'dw' \\ = \;& u(x,w,a)+\beta\int\int\big[-\log p_K(x',w')+v_K(x',w')\big]s(x'|x,a)f(w'|w,a)dx'dw' \\ = \;& u(x,w,a)+\beta\int\big[-\log\tilde{p}_K^a(x',w)+\tilde{v}_K^a(x',w)\big]s(x'|x,a)dx' \quad (70)\end{aligned}$$

where

$$\begin{aligned}
\log \tilde{p}_K^a(x', w) &= \int \log p_K(x', w') f(w'|w, a) dw' \\
\tilde{v}_K^a(x', w) &= \int v_K(x', w') f(w'|w, a) dw'. \qquad (71)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\xi_{i,K}(x, w) &= \log\Big(\frac{p_i(x, w)}{p_K(x, w)}\Big) \\
&= v(x, w, i) - v(x, w, K) \\
&= \beta \boldsymbol{S}_i^x(x)[-\log \tilde{\boldsymbol{p}}_K^i(w) + \tilde{\boldsymbol{v}}_K^i(w)] - \beta \boldsymbol{S}_K^x(x)[-\log \tilde{\boldsymbol{p}}_K^K(w) + \tilde{\boldsymbol{v}}_K^K(w)] \\
&\quad + u(x, w, i), \qquad (72)
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{S}_{a_t}^x(x_t) &= [s(x_{t+1} = 1|x_t, a_t), ..., s(x_{t+1} = J|x_t, a_t)] \\
\log \tilde{\boldsymbol{p}}_K^i(w) &= [\log \tilde{p}_K^i(1, w), ..., \log \tilde{p}_K^i(J, w)]^T \\
\tilde{\boldsymbol{v}}_K^i(w) &= [\tilde{v}_K^i(1, w), ..., \tilde{v}_K^i(J, w)]^T.
\end{aligned}$$

Assumption 4' imposes a linear structure on the utility function such that $u(x, w, a) = u^1(x, a) + u^2(x, a)w$. Suppose the support of w contains $\{0, 1, 2, ..., J + 1\}$. The second order difference of $u(x, w, a)$ with respect to w is zero with

$$\Delta_w^2 f(w) \equiv [f(w) - f(w - 1)] - [f(w + 1) - f(w)].$$

Taking the second order difference with respect to w leads to

$$\begin{aligned}
\Delta_w^2 \xi_{i,K}(x, w) &= \beta \boldsymbol{S}_i^x(x)[-\Delta_w^2 \log \tilde{\boldsymbol{p}}_K^i(w) + \Delta_w^2 \tilde{\boldsymbol{v}}_K^i(w)] \\
&\quad - \beta \boldsymbol{S}_K^x(x)[-\Delta_w^2 \log \tilde{\boldsymbol{p}}_K^K(w) + \Delta_w^2 \tilde{\boldsymbol{v}}_K^K(w)] \qquad (73)
\end{aligned}$$

With $w \in \{1, 2, ..., J\}$, we may obtain enough restrictions to solve for $\boldsymbol{S}_i^x(x)$ under an invertibility condition imposed on the observables. Define

$$\begin{aligned}
\tilde{\boldsymbol{V}}^i &= [\tilde{\boldsymbol{v}}^i(1), \tilde{\boldsymbol{v}}^i(2), ..., \tilde{\boldsymbol{v}}^i(J)], \\
\tilde{\boldsymbol{v}}^i(w) &= [-\Delta_w^2 \log \tilde{\boldsymbol{p}}_K^i(w) + \Delta_w^2 \tilde{\boldsymbol{v}}_K^i(w)], \\
\Delta_w^2 \boldsymbol{\xi}_{i,K}(x) &= [\Delta_w^2 \xi_{i,K}(x, 1), \Delta_w^2 \xi_{i,K}(x, 2), ..., \Delta_w^2 \xi_{i,K}(x, J)],
\end{aligned}$$

Equation (73) for $w \in \{1, 2, ..., J\}$ can be written as

$$\Delta_w^2 \boldsymbol{\xi}_{i,K}(x) = \beta \boldsymbol{S}_i^x(x) \tilde{\boldsymbol{V}}^i - \beta \boldsymbol{S}_K^x(x) \tilde{\boldsymbol{V}}^K$$

Under Assumption 5', we may solve for $\boldsymbol{S}_i^x(x)$, i.e., $s(x_{t+1}|x_t, a_t)$ with a closed-form as follows:

$$\boldsymbol{S}_i^x(x) = \beta^{-1}\Big(\Delta_w^2 \boldsymbol{\xi}_{i,K}(x) + \beta \boldsymbol{S}_K^x(x) \tilde{\boldsymbol{V}}^K\Big)\Big(\tilde{\boldsymbol{V}}^i\Big)^{-1}.$$

Given the subjective beliefs, we can solve for the utility function $u(x, w, a)$ from equation (72). QED.

Proof of Theorem 4 The first-order Markov process $\{a_t, x_t, \tau\}$ satisfies

$$f(a_{t+}, x_{t+1}, a_t, x_t, a_{t-}) \;=\; \sum_{\tau} f(a_{t+}|x_{t+1}, \tau) f(x_{t+1}, a_t|x_t, \tau) f(\tau, x_t, a_{t-}), \tag{74}$$

with $a_{t+} = h(a_{t+l}, ..., a_{t+1})$ and $a_{t-} = h(a_{t-1}, ..., a_{t-l})$. Integrating with respect to $\omega(a_t)$ leads to

$$\begin{aligned} & \int \omega(a_t) f(a_{t+}, x_{t+1}, a_t, x_t, a_{t-}) da_t \\ =\; & \sum_{\tau} f(a_{t+}|x_{t+1}, \tau) \Big[\int \omega(a_t) f(x_{t+1}, a_t|x_t, \tau) da_t \Big] f(\tau, x_t, a_{t-}). \end{aligned} \tag{75}$$

Note that we have reduced the support of $a_{t+l}, ..., a_{t+1}$ to that of τ by the mapping $h(\cdot)$. We define the following matrices

$$\begin{aligned} M_{a_{t+}, x_{t+1}, \omega, x_t, a_{t-}} &= \Big[\int \omega(a_t) f(a_{t+} = i, x_{t+1}, a_t, x_t, a_{t-} = j) da_t \Big]_{i,j} \\ M_{a_{t+}, x_{t+1}, \tau} &= \Big[f(a_{t+} = i, x_{t+1}, \tau = j) \Big]_{i,j} \\ M_{\tau, x_t, a_{t-}} &= \Big[f(\tau = i, x_t, a_{t-} = j) \Big]_{i,j} \\ D_{x_{t+1}, \omega|x_t, \tau} &= \mathrm{Diag}\Big\{ \int \omega(a_t) f(x_{t+1}, a_t|x_t, \tau = 1) da_t, ..., \int \omega(a_t) f(x_{t+1}, a_t|x_t, \tau = L) da_t \Big\} \\ D_{x_{t+1}|x_t, \tau} &= \mathrm{Diag}\Big\{ f(x_{t+1}|x_t, \tau = 1), ..., f(x_{t+1}|x_t, \tau = L) \Big\} \\ D_{\omega|x_{t+1}, x_t, \tau} &= \mathrm{Diag}\Big\{ \int \omega(a_t) f(a_t|x_{t+1}, x_t, \tau = 1) da_t, ..., \int \omega(a_t) f(a_t|x_{t+1}, x_t, \tau = L) da_t \Big\}. \end{aligned} \tag{76}$$

Equation (75) is equivalent to

$$M_{a_{t+}, x_{t+1}, \omega, x_t, a_{t-}} \;=\; M_{a_{t+}, x_{t+1}, \tau} D_{x_{t+1}, \omega|x_t, \tau} M_{\tau, x_t, a_{t-}}. \tag{77}$$

Similarly, we have

$$M_{a_{t+}, x_{t+1}, x_t, a_{t-}} \;=\; M_{a_{t+}, x_{t+1}, \tau} D_{x_{t+1}|x_t, \tau} M_{\tau, x_t, a_{t-}}, \tag{78}$$

where the matrices are defined analogously to those in (76) based on the following equality

$$\int f(a_{t+}, x_{t+1}, a_t, x_t, a_{t-}) da_t = \sum_{\tau} f(a_{t+}|x_{t+1}, \tau) \Big[\int f(x_{t+1}, a_t|x_t, \tau) da_t \Big] f(\tau, x_t, a_{t-}).$$

Assumption 12 implies that matrices $M_{a_{t+},x_{t+1},\tau}$, $D_{x_{x+1}|x_t,\tau}$, and $M_{\tau,x_t,a_{t-}}$ are all invertible. We may then consider

$$\begin{aligned} M_{a_{t+},x_{t+1},\omega,x_t,a_{t-}} M^{-1}_{a_{t+},x_{t+1},x_t,a_{t-}} &= M_{a_{t+},x_{t+1},\tau} D_{x_{x+1},\omega|x_t,\tau} D^{-1}_{x_{t+1}|x_t,\tau} M^{-1}_{a_{t+},x_{t+1},\tau} \\ &= M_{a_{t+},x_{t+1},\tau} D_{\omega|x_{t+1},x_t,\tau} M^{-1}_{a_{t+},x_{t+1},\tau} \end{aligned} \quad (79)$$

This equation above shows an eigenvalue-eigenvector decomposition of an observed matrix on the left-hand side. Assumptions 13 guarantee that this decomposition is unique. Therefore, the eigenvector matrix $M_{a_{t+},x_{t+1},\tau}$, i.e., $f(a_{t+}|x_{t+1},\tau)$ is identified. The distribution $f(x_{t+1},a_t|x_t,\tau)$, and therefore $f(a_t|x_t,\tau) = p_t(a_t|x_t,\tau)$, can then identified from equation (74) due to the invertibility of matrix $M_{a_{t+},x_{t+1},\tau}$. QED.