Size Premium, Distress Risk and Distress Anomaly

Daniel Kim

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Abstract

As documented in Fama and French (1992), small firms' expected equity returns are usually larger than big firms.' Notably, Fama and French (1995) attributed this return pattern, dubbed as *size premium*, to a notion that small firms are assigned a higher risk premium because they face greater risk of distress. However, "distress anomaly" papers including Campbell, Hilscher, and Szilagyi (2008) empirically have shown that firms with greater distress risk tend to generate lower expected returns thus imply that small firms are assigned a *lower* risk premium (empirically confirmed by Fama and French (2015)). In this paper, I attempt to reconcile these two seemingly contradicting set of results. I assume that firm's true distress risk exposure is determined by Z-score (Altman (1968)) and firm size. To the extent that distress risk is sensitive to firm size, the model captures size premium. Moreover, I assume that the firm size's sensitivity to distress risk changes over Z-score. This leads to a hump-shaped return profile over Z-score and, more interestingly, implies hump-shaped size premium over Z-score. Moreover, consistent with the model's implication, I empirically find that unconditional size premium increases by more than three times after I exclude low-Z firms. Lastly, I extend the model to show that size premium depends on government debt. I empirically find that government's debt and size premium are negatively correlated.

Keywords: Expected Returns, Financial Distress, Distress Risk, Distress Anomaly, Size Premium

1 Introduction

What explains size premium? One prominent explanation is that investors demand a positive premium for holding stocks of small firms because small firms face greater risk of distress relative to big firms. Consistent with this view, Fama and French (1995) show that small firms have persistently low earnings, higher financial leverage, more earnings uncertainty and are more likely to reduce dividends compared to big firms. In support of Fama and French (1995)'s claim, Ferguson and Shockley (2003) empirically show that expected returns are positively correlated with distress risk. In contrast, Dichev (1998) and Campbell, Hilscher, and Szilagyi (2008) empirically show that expected returns are negatively correlated with distress risk thus argue that size premium cannot be explained by distress risk. The latter results have posed a challenge to a rational equilibrium explanation and have been dubbed "distress anomaly."

In this paper, I attempt to reconcile these two seemingly contradicting set of results. Similar to Fama and French (1995), I associate high distress risk exposure with high default probability. Then, I assume that firm's true distress risk exposure (which is denoted as β_i^D) is determined by Z-score (Altman (1968)) and firm size. Depending on the relative position of Z-score to a cutoff point, β_i^D is positively or negatively correlated with its Z-score. This leads to a hump-shaped return profile over Z-score. To the extent that distress risk is sensitive to firm size, the model captures size premium. Moreover, I assume that the firm size's sensitivity to distress risk changes over Z-score. Based on a reduced form model that captures the above intuition, I show that it implies hump-shaped size premium over Z-score and I document a empirical evidence and note a striking resemblance between the model's implication and the empirical counterpart. In the rest of the introduction, I explain the implementation in detail.

As mentioned above, I use two firm characteristics to represent β_i^D: firm size and Z-score. Firm size (calculated as market price) reflects investors' expectations about a firm's future performance. As a result, they contain forward-looking information, which is well suited for calculating the likelihood that a firm may default in the future. Moreover, size of firms that are close to bankruptcy is typically discounted by traders. Thus, firm size should be related to firm's β_i^D, i.e. big size should be associated with low β_i^D whereas small size should be associated with high β_i^D. Next, I argue that Z-score should be an explanatory variable for β_i^D[1]. When firms are financially sound, larger Z-score implies lower β_i^D. This is not surprising because Z-score has been used as a popular measure to predict β_i^D ever since Altman published his paper in 1968. On the contrary, when firms are financially distressed, larger Z-score implies higher β_i^D and this is consistent with "distress anomaly" finding such as Griffin and Lemmon (2002) and Campbell, Hilscher, and Szilagyi (2008) which rely on market mispricing. Namely, due to information asymmetry, investors are incapable of fully assessing the prospects of firms with high distress risk and hence fail to demand a sufficient premium to compensate for distress risk. Therefore, highly distressed firms tend to be overpriced thus their expected returns tend to be lower than less-distressed firms.

Using the above ingredients, I write a model to explain relation between expected returns and Z-score. As expected, the model implies that the relation between expected returns and Z-score is not monotonic. Financially sound firms' equities returns are negatively correlated with Z-score whereas financially distressed firms' equities returns are positively correlated with Z-score. This naturally generates a hump-shaped return profile over Z-score. This result is similar in spirit to Garlappi, Shu, and Yan (2008) and Garlappi and Yan (2011) which imply that expected returns are hump-shaped in default probability in the presence of shareholder recovery.

I extend the model above to show how size premium changes over Z-score. Main ingredient for this extension is that firm size's sensitivity to β_i^D changes over Z-score. When firms are financially sound, firm size's sensitivity to β_i^D is negative and becomes less negative over Z-score. In other words, it is less beneficial to be large when firms already have good Z-score. This is reasonable if marginal benefit of the firm size decreases as Z-score increases. When firms are financially distressed, firm size's sensitivity to distress risk is negative and becomes more negative over Z-score. In other words, it is more beneficial to be large when firms already have good Z-score. This can be explained by extending market mispricing explanation. Because it is easier to acquire information about big firms than small firms, big firms are subject to less mispricing than small firms are. Thus, big firms' expected equities returns are less sensitive to Z-score than small firms' are. Taken

[1]As implied by various distress measures that were introduced after Z-score, Z-score does not fully capture firm size. This "omitted variable problem" actually helps me to uncover interesting relation between size premium and Z-score

together, these two ingredients lead us to obtain big firms' smaller sensitivity of Z-score to expected returns compared to small firms'.

This generates non-monotonic relation between size premium and Z-score and the intuition is as follows. Consider two identical firms, i.e. A and B have the same Z-score and firm size. Suppose firm A experiences a positive shock to its stock price and firm B contemporaneously experiences a negative shock. For illustrative purpose, if we assume that changes in firm size do not alter Z-score, these differential shocks lead firm A to become larger in size than firm B and both firms to still have the same Z-score. Then, firm A faces flatter return profile over Z-score compared to B (see **Figure** 2). If we construct a portfolio that is long B and short A, then the expected return on this portfolio, $E[R_B] - E[R_A]$, will depend crucially on the relationship between expected returns and Z-score. If this relationship monotonically decreases, as in typical distress risk story ($Z > Z^*$ in **Figure** 2), then the spread $E[R_B] - E[R_A]$ always decreases over Z-score. On the contrary, if this relationship monotonically increases, as in distress anomaly ($Z < Z^*$ in **Figure** 2), then the spread $E[R_B] - E[R_A]$ always increases over Z-score. Combining these two intuitions naturally implies a hump-shaped size premium in Z-score.

Lastly, I extend the model to show that size premium varies over government debt. As argued before, big firms face smaller distress risk than small firms. Thus, in the spirit of Friedman (1978) and Greenwood, Hanson, and Stein (2010), government bond and big firms' bond are similar in nature (in terms of distress risk) and the bond market is segmented in a way that increasing supply of the former increases the cost of bond for the latter. As government issues bond, big firms cannot enjoy the benefit of being big as much and are forced to reduce their investment/capital, which is empirically confirmed by Graham, Leary, and Roberts (2014). To make matter worse for big firms, government debt issuance tends to coincide with economic downturn. This naturally leads to higher distress cost for big firms, implying higher expected equity returns. However, small firms' bonds are not as substitutable for government bonds thus do not face the same effect. In sum, larger level of government debt decreases big firms' expected equity returns yet leaving small firms' relatively unchanged. In other words, larger level of government debt leads to smaller size premium as big firms' increased-return eats away size premium that would have existed with lower government debt.

Compared to the existing literature, my paper is novel in several dimensions. First, instead of attempting to introduce another distress measure, I make use of the fact that Z-score suffers from omitted variable problem (e.g. size) in measuring distress. Second, instead of assuming monotonic relation between distress measure and expected equity returns, I assume and show that the relation is hump-shaped. In fact, previous papers' results do document hump-shaped relation although they intentionally choose to ignore it. For instance, Vassalou and Xing (2004)'s Table IV and Campbell, Hilscher, and Szilagyi (2008)'s Table VI imply that SMB (Small-minus-Big) is hump-shaped over their respective distress measures although both focus only on highly distressed firms' SMB. I plan to focus on the overall shape of SMB for the entire spectrum of distress measures. Third, I construct evenly spaced distress buckets as opposed to other papers that construct unevenly spaced distress buckets. For example, Campbell, Hilscher, and Szilagyi (2008) construct five distress buckets among top and bottom 10 percentile of distress measures whereas they construct only five distress buckets among the remaining 80 percent. Unevenly-spaced buckets can wrongly imply monotonic relation between expected equity returns and distress measure. Lastly, to the best of my knowledge, I am the first author to study what hump-shaped return profile implies for the size premium.

The rest of the paper proceeds as follows. Section 2 discusses the model and derives several testable implications. Section 3 describes the construction of the data set and relates size premium to distress anomaly. Section 4 documents a cross-sectional analysis results that distress risk does explain size premium whereas distress anomaly does not. Section 5 presents an evidence for the model. In Section 6, I extend the model to explain the effect that government debt has on size premium and provide an empirical evidence. Section 7 concludes.

Related Literature: Ever since Banz (1981) has first uncovered size premium, many following papers have tried to explain it. Chan, Chen, and Hsieh (1985) show that a distress factor constructed as the difference between high- and low grade bond return can explain a large part of the size premium. Chen, Roll, and Ross (1986) and Fama and French (1992) document the power of a similarly defined distress factor in explaining the size premium. Chan and Chen (1991) justify the role of distress risk by arguing that the size premium is primarily driven by "marginal firms," that is, firms with low market value, cash-flow problems, and high leverage that are more sensitive to adverse economic fluctuations. More recently, Vassalou and Xing (2004) argue that distress risk is positively priced in the market and is associated with size premium.

A parallel literature has focused specifically on distress anomaly. Using both Altman (1968) Z-score and Ohlson (1980) O-score, Dichev (1998) document a negative relationship between stock return and distress risk. Griffin and Lemmon (2002) find that this pattern is stronger for firms with low book-to-market ratios. More recently, Campbell, Hilscher, and Szilagyi (2008) confirms this pattern by using a hazard model to predict distress risk and employ market mispricing to explain it. Garlappi, Shu, and Yan (2008) shows that shareholder's bargaining power increases as the firm becomes more distressed and employ it to explain distress anomaly. In this paper, I rely on market mispricing story to explain distress anomaly.

Lastly, Fama and French (2015) recently proposes a five-factor asset pricing model to capture size, value, profitability, and investment patterns in average stock returns. However, as admitted by authors, it fails to capture the low average returns on small stocks whose returns behave like those of firms that invest a lot despite low profitability. I argue that such firms are distressed thus distress anomaly risk decreases their expected returns. I hypothesize that a five-factor asset pricing model would perform better if the sample is restricted to financially sound firms, which do not suffer from distress anomaly.

2 Model

I follow Arbitrage Pricing Theory to write firm i's expected returns as

$$E[\tilde{r}_{i,t}] = \beta_i^D \lambda_D + \vec{B}_i \cdot \vec{\Gamma}$$

Here, β_i^D measures risk exposures to distress risk and $\lambda_D > 0$ is the implied market price of risk. In the rest of this document, I explore excess returns with respect to $\vec{B}_i \cdot \vec{\Gamma}$:

$$E[r_{i,t}] = \beta_i^D \lambda_D$$

In an empirical exercise, I set $\vec{B}_i \cdot \vec{\Gamma}$ to market premium and focus on CAPM-α. I use Z_i to measure a firm i's financial soundness, i.e. larger Z_i implies better financial position. If so, Fama and French (1995) argues that $\frac{\partial \beta_i^D}{\partial Z_i} < 0$ and Campbell, Hilscher, and Szilagyi

(2008) argues that $\frac{\partial \beta_i^D}{\partial Z_i} > 0$. In this paper, I reconcile these two results by arguing that $\frac{\partial \beta_i^D}{\partial Z_i} < 0$ when firms are financially sound whereas $\frac{\partial \beta_i^D}{\partial Z_i} < 0$ when firms are financially distressed. Then, I extend this to explain conditional variation of size premium.

2.1 Distress Risk and Distress Anomaly

Distress risk measures the probability that a firm may default in the future, hence measures the firm's future performance. However, because distress risk is an unobservable firm characteristic, it can be tricky to measure; pure long list of these measures attests to it. Altman (1968) (Z-score), Ohlson (1980) (O-score) Vassalou and Xing (2004) (default likelihood indicators) Chava and Jarrow (2004) Shumway (2001) Campbell, Hilscher, and Szilagyi (2008) (logit-model), Crosbie and Bohn (2001) (Moody's KMV) and Garlappi, Shu, and Yan (2008) Garlappi and Yan (2011) (expected default frequency) have come up with their own measures for distress risk.

Despite the vast difference in how these measures are constructed, they all share two characteristics in measuring distress risk. First, they all use financial book ratios or versions of it, among which the notable one is Z-score. Such a measure is essentially backward-looking because it aims to report a firm's past performance to estimate the distress risk. Nonetheless, Z-score and the like are still relevant to the extent that the past performance contains information about a firm's future performance. On the contrary to Z-score, firm size (calculated as market prices) reflects investors' expectations about a firm's future performance. As a result, they contain forward-looking information, which is well suited for estimating the distress risk.

Taken together, I write β_i^D in terms of firm i's Z-score (Z_i) and size (C_i) as

$$\beta_i^D = \beta^0 + \beta^Z(Z_i - Z^*) + \beta^C C_i$$

Here, firms with $Z_i > Z^*$ are considered financially sound and firms with $Z_i < Z^*$ are considered financially distressed. For simplicity, I assume that all firms are subject to the same Z^*. Now, I list propositions to put more structures on the above representation.

Proposition 1 *Regardless of whether firms are financially sound or constrained, larger size implies lower exposure to distress risk ($\beta^C < 0$)*

As argued above, firm size is an important explanatory variable for distress risk. Because the market equity of firms that are close to bankruptcy is typically discounted by traders, keeping everything equal, big firms are subject to lower distress risk than than small firms. Chava and Jarrow (2004) Shumway (2001) and Campbell, Hilscher, and Szilagyi (2008) empirically found a firm size to be a very important bankruptcy predicting variable. Thus, this justifies $\beta^C < 0$.

Proposition 2 *When firms are financially sound, larger Z-score implies lower risk exposure to distress risk ($\beta^Z < 0$). When firms are financially distressed, larger Z-score implies higher risk exposure to distress risk ($\beta^Z > 0$)*

Many papers following Altman (1968) have used Z as a predictor for firms' distress risk. Because larger Z implies lower distress risk and larger β_i^D implies larger distress risk, this justifies $\beta^Z < 0$. However, this is empirically true only for financially sound firms. In order to accommodate distress anomaly, I set $\beta^Z > 0$ for financially distressed firms. Taken together, I slightly rewrite β_i^D's as:

$$\beta_i^D = \beta^0 + \beta^C C_i + \beta^{Z+}(Z_i - Z^*)\mathbb{1}(Z_i > Z^*) + \beta^{Z-}(Z_i - Z^*)\mathbb{1}(Z_i < Z^*)$$

where $\beta^{Z+} < 0$ and $\beta^{Z-} > 0$. This particular form ensures that β_i^D is continuous at $Z_i = Z^*$. Moreover, I naturally obtain my objective: $\frac{\partial \beta_i^D}{\partial Z_i} < 0$ when firms are financially sound and $\frac{\partial \beta_i^D}{\partial Z_i} > 0$ when firms are financially distressed. Based on the above, I can rewrite $E[r_{i,t}]$ as

$$E[r_{i,t}] = [\beta^0 + \beta^C C_i + \beta^{Z+}(Z_i - Z^*)\mathbb{1}(Z_i > Z^*) + \beta^{Z-}(Z_i - Z^*)\mathbb{1}(Z_i < Z^*)]\lambda_D$$

Figure 1 graphically shows $E[r_{i,t}]$ over Z. When firms are financially sound, $E[r_t]$ decreases over Z-score. However, when firms are financially distressed, $E[r_t]$ increases over Z-score. The kink at $z = Z^*$ is driven by **Proposition** 2. Lastly, although the plot may look as if $E[r_{i,t}|Z_i = z]$ is positive for most of z, it does not have to be the case. For instance, when β^0 is sufficiently negative, $E[r_{i,t}|Z_i = z]$ is negative for all z.

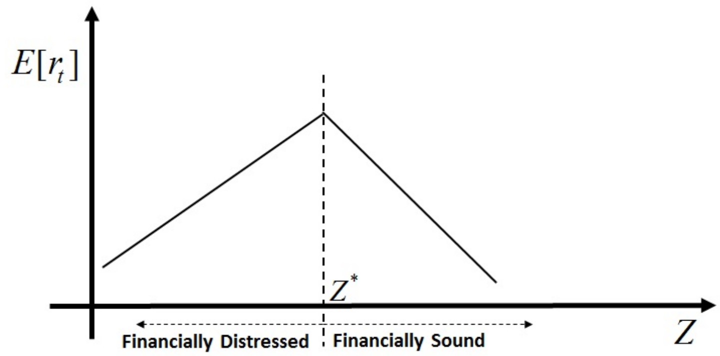

Figure 1: Theoretical relation between Z-score and portfolio returns. Implication of **Proposition** 1 and 2

This naturally leads to two testable implications.

Implication 1 *When firms are financially distressed, $E[r_{i,t}|Z_i = z]$ increases in z*

Implication 2 *When firms are financially sound, $E[r_{i,t}|Z_i = z]$ decreases in z*

2.2 Size Premium and Z-score

Although **Proposition** 1 and **Proposition** 2 explain both Fama-French-type distress risk and Campbell-Hilscher-Szilagyi-type distress anomaly unconditionally, it does not capture possible variation of size premium over Z-score. In order to capture this variation, I consider second-order effects: interaction terms between Z_i and C_i. This captures sensitivity of distress risk to C_i and Z_i. To that end, I slightly modify β_i^D by adding $\beta^{ZC}(Z_i - Z^*)C_i$:

$$\beta_i^D = \beta^0 + \beta^C C_i + \beta^{Z+}(Z_i - Z^*)\mathbb{1}(Z_i > Z^*) + \beta^{Z-}(Z_i - Z^*)\mathbb{1}(Z_i < Z^*) + \beta^{ZC}(Z_i - Z^*)C_i$$

Then, similar to **Proposition** 2, I introduce a new proposition regarding β^{ZC}.

Proposition 3 *When firms are financially sound, firm size's exposure to distress risk is negative and becomes less negative over Z-score. ($\beta^{ZC} > 0$) When firms are financially distressed, firm size's expsoure to distress risk is negative and becomes more negative over Z-score. ($\beta^{ZC} < 0$)*

When firms are financially sound, firm size's sensitivity to distress risk is negative and becomes less negative over Z-score. In other words, it is less beneficial to be large when firms already have good Z-score. This is reasonable if marginal benefit of the firm size decreases as Z-score increases. This sensitivity is measured as $\frac{\partial^2 \beta_i^D}{\partial Z_i \partial C_i} = \beta^{ZC}$ and implies $\beta^{ZC} > 0$.

When firms are financially distressed, firm size's sensitivity to distress risk is negative and becomes more negative over Z-score. In other words, it is more beneficial to be large when firms already have good Z-score. This can be explained by extending market mispricing explanation. Because it is easier to acquire information about big firms than small firms, big firms are subject to less mispricing than small firms are. Similarly, this sensitivity is measured as $\frac{\partial^2 \beta_i^D}{\partial Z_i \partial C_i} = \beta^{ZC}$ and implies $\beta^{ZC} < 0$.

Taken together, I rewrite β_i^D and $E[r_{i,t}]$ as:

$$\beta_i^D = \begin{bmatrix} \beta^0 + \beta^C C_i + (\beta^{Z+} + \beta^{ZC+} C_i)(Z_i - Z^*)\mathbb{1}(Z_i > Z^*) \\ +(\beta^{Z-} + \beta^{ZC-} C_i)(Z_i - Z^*)\mathbb{1}(Z_i < Z^*) \end{bmatrix} \tag{1}$$

$$E[r_{i,t}] = \begin{bmatrix} \beta^0 + \beta^C C_i + (\beta^{Z+} + \beta^{ZC+} C_i)(Z_i - Z^*)\mathbb{1}(Z_i > Z^*) \\ +(\beta^{Z-} + \beta^{ZC-} C_i)(Z_i - Z^*)\mathbb{1}(Z_i < Z^*) \end{bmatrix} \lambda_D \tag{2}$$

where $\beta^{Z+} < 0$, $\beta^{ZC+} > 0$, $\beta^{Z-} > 0$ and $\beta^{ZC-} < 0$. It is crucial to check how our additional term, β^{ZC+} and β^{ZC-}, would change the model's implication about Fama-French-type distress risk and Campbell-Hilscher-Szilagyi-type distress anomaly.

$$\frac{\partial \beta_i^D}{\partial Z_i} = \begin{cases} \beta^{Z+} + \beta^{ZC+} C_i & \text{if } Z_i > Z^* \\ \beta^{Z-} + \beta^{ZC-} C_i & \text{if } Z_i < Z^* \end{cases}$$

If β^{Z+} is sufficiently negative and β^{Z-} is sufficiently positive, I still obtain my objective: $\frac{\partial \beta_i^D}{\partial Z_i} < 0$ when firms are financially sound and $\frac{\partial \beta_i^D}{\partial Z_i} > 0$ when firms are financially distressed. Thus, this modified setup still reconciles both distress risk and distress anomaly.

Furthermore, as the firm becomes larger in size, $\left|\frac{\partial \beta_i^D}{\partial Z_i}\right|$ decreases in magnitude. This pattern is clearly demonstrated in Figure 2 where big firms' plot is flatter than small firms' plot. In other words, **Proposition** 3 allows big firm to have smaller sensitivity of Z-score to expected returns compared to small firms.

Now, I analyze how $E[r_{i,t}|Z_i = z]$ changes for different firm sizes. For small firm $i = 0$, I set $C_0 = S$:

$$E[r_{0,t}] = [\beta^0+\beta^C S+(\beta^{Z+}+\beta^{ZC+}S)(Z_0-Z^*)\mathbb{1}(Z_0 > Z^*)+(\beta^{Z-}+\beta^{ZC-}S)(Z_0-Z^*)\mathbb{1}(Z_0 < Z^*)]\lambda_D$$

For big firm $i = 1$, I set $C_1 = B > S$:

$$E[r_{1,t}] = [\beta^0+\beta^C B+(\beta^{Z+}+\beta^{ZC+}B)(Z_1-Z^*)\mathbb{1}(Z_1 > Z^*)+(\beta^{Z-}+\beta^{ZC-}B)(Z_1-Z^*)\mathbb{1}(Z_1 < Z^*)]\lambda_D$$

Naturally, I can write size premium in reduced form as $SMB = E[r_{0,t}] - E[r_{1,t}]$. This allows us to study SMB as a function of Z-score:

$$\begin{aligned} SMB(Z_i = z) &= E[r_{0,t}|Z_0 = z] - E[r_{1,t}|Z_1 = z] \\ &= (\beta^C + \beta^{ZC+}z \cdot \mathbb{1}(z > Z^*) + \beta^{ZC-}z \cdot \mathbb{1}(z < Z^*))(S - B)\lambda_D \end{aligned}$$

The above model has two testable implications.

Implication 3 *When firms are financially distressed, $SMB(Z_i = z)$ increases in z*

Implication 4 *When firms are financially sound, $SMB(Z_i = z)$ decreases in z*

These two implications are better demonstrated in Figure 2. Dotted line represents $E[r_{0,t}]$ and solid line represents $E[r_{1,t}]$ hence SMB is the gap between these two lines. This figure clearly demonstrates a non-monotonic relation between size premium and Z-score. When firms are distressed ($Z < Z^*$), SMB increases. On the contrary, when firms are sound ($Z > Z^*$), SMB decreases.

In the figure, SMB is positive when $\underline{Z} < Z < \overline{Z}$ and SMB is negative when $Z < \underline{Z}$ or $Z > \overline{Z}$. However, this does not need to be the case. In fact, a particular parameterizations can shift big firms' plot upward to the point that SMB is negative for all Z's.

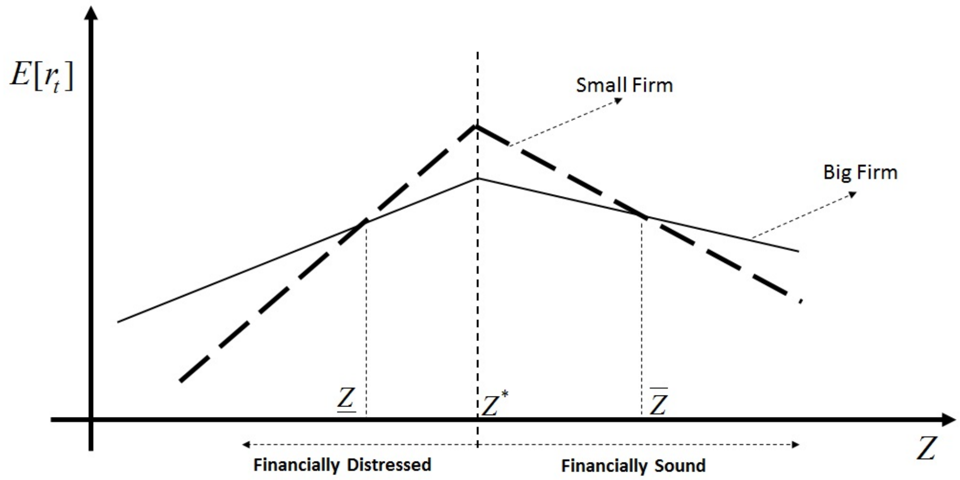

Figure 2: Theoretical relation between Z-score and portfolio returns for small and big firms. Implication of **Proposition** 1, 2, and 3

I now illustrate how unevenly-spaced distress buckets fail to capture hump-shaped size premium. For an illustrative purpose, in the spirit of Campbell, Hilscher, and Szilagyi (2008), let us create 9 distress-buckets when $Z < \underline{Z}$ and only 1 distress-bucket when $Z = \overline{Z}$. If so, this unevenly-spaced buckets falsely imply an monotonic relationship between Z-score and expected returns for both small and large firms, hence a monotonic relation between size premium and Z-score as opposed to true hump-shaped relationship. This illustrates why it is important to construct evenly-spaced buckets.

Lastly, **Implication** 3 implies that size premium is low and even negative among financially distressed firms ($Z < \underline{Z}$). This is not consistent with Vassalou and Xing (2004)'s finding that size premium are concentrated in distressed firms. I reconcile this seemingly contradiction in the next subsection.

2.3 Size Premium and Other Measures

So far, I have studied how size premium varies over Z-score. In order to reconcile Vassalou and Xing (2004)'s results, I study how size premium varies over another distress risk measure that does take size effect into consideration. For simplicity, I write another measure for distress risk as:

$$\hat{Z}_i = \frac{Z_i}{1+\kappa C_i}$$

where $\kappa > 0$. In this particular form, $\hat{Z}_i$ takes size effect into consideration in addition to the original Z-score. Again, it is important to check $\frac{\partial \beta_i^D}{\partial \hat{Z}_i}$ (note that we are now interested in measuring the sensitivity of β_i^D with respect to $\hat{Z}_i$, not Z_i.)

$$\begin{aligned}
\frac{\partial \beta_i^D}{\partial \hat{Z}_i} =& \frac{\partial \beta_i^D}{\partial Z_i}\frac{\partial Z_i}{\partial \hat{Z}_i} + \frac{\partial \beta_i^D}{\partial C_i}\frac{\partial C_i}{\partial \hat{Z}_i} \\
=& \begin{cases} (\beta^{Z+} + \beta^{ZC+}C_i)(1+\kappa C_i) - \frac{\beta^{ZC+}}{\kappa}(1+\kappa C_i)^2 - \frac{1+\kappa C_i}{\kappa \hat{Z}_i}(\beta^C - \beta^{ZC+}Z^*) & \text{if } \hat{Z}_i > \hat{Z}^* \\ (\beta^{Z-} + \beta^{ZC-}C_i)(1+\kappa C_i) - \frac{\beta^{ZC-}}{\kappa}(1+\kappa C_i)^2 - \frac{1+\kappa C_i}{\kappa \hat{Z}_i}(\beta^C - \beta^{ZC-}Z^*) & \text{if } \hat{Z}_i < \hat{Z}^* \end{cases}
\end{aligned}$$

If β^{Z+} is sufficiently negative and β^{Z-} is sufficiently positive, I still obtain: $\frac{\partial \beta_i^D}{\partial \hat{Z}_i} < 0$ when firms are financially sound and $\frac{\partial \beta_i^D}{\partial \hat{Z}_i} > 0$ when firms are financially distressed. I want to point out one key implication that sets this analysis apart from the previous subsection. Under a particular parameterization[2], as the firm becomes larger in size, $\left|\frac{\partial \beta_i^D}{\partial \hat{Z}_i}\right|$ increases in magnitude. This pattern is clearly demonstrated in Figure 3.

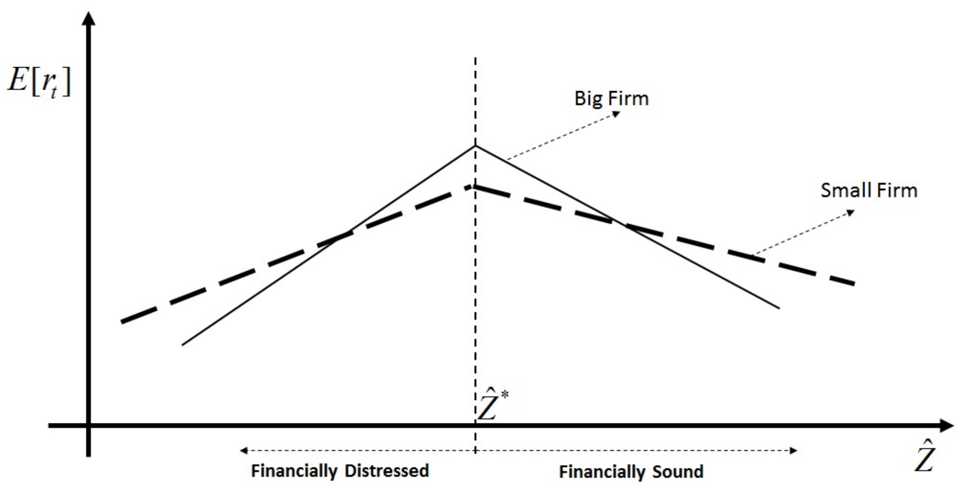

Figure 3: Theoretical relation between $\hat{Z}$ and portfolio returns for small and big firms. Contrary to Z-score, $\hat{Z}$ significantly accounts for firm size

[2]so that the last term $(\frac{1+\kappa C_i}{\kappa \hat{Z}_i}(\beta^C - \beta^{ZC+}Z^*)$, $\frac{1+\kappa C_i}{\kappa \hat{Z}_i}(\beta^C - \beta^{ZC-}Z^*))$ are small relative to the first two terms

Again, Figure 3 differs from Figure 2 on the x-axis variable and demonstrates that big firms' plot is steeper than small firms' plot. When firms are financially distressed, i.e. $\hat{Z} < \hat{Z}^*$, small firms' equity returns are larger than big firms. In other words, when firms are distressed, SMB is particularly strong. Because their distress risk measure does account for firm size, our model's implication is consistent with Vassalou and Xing (2004)'s results.

Moreover, this modified setup helps to reconcile another seemingly contradicting result of Garlappi, Shu, and Yan (2008) and Garlappi and Yan (2011). Garlappi et al. argue that among financially distressed firms, big firms' shareholder advantage is greater than small firms.' Thus, they imply that big firms' expected returns should be more sensitive to distress risk than small firms'. Because Garlappi et al. use a measure that is similar to $\hat{Z}_i$, this modified setup's implication is consistent with Garlappi et al's.

In the rest of this document, I focus on Z_i (Z-score), which does not take firm size into consideration, and not on $\hat{Z}_i$.

3 Data

3.1 Summary Statistics

My sample of firm-level data is from the Center for Research in Security Prices (CRSP) monthly stock file and annual Compustat industrial files. I select the sample by first deleting any firm-year observations with missing data for total assets, book value per equity or long term debt, all of which are needed to calculate Z-score. Regulated firms (whose standard industrial classification code is between 4900 and 4999) or financial firms (whose standard industrial classification code is between 6000 and 6999) are omitted because it is likely that distress risk would impact these firms differently. My sample period spans from 1960 to 2015. Lastly, I winsorize sample based on Altman-Zscore at 0.3% level.

	All	Small	size2	size3	size4	Big
Num of Obs	164449	96536	24689	16824	13613	12787
$\ln(size)$	4.74	3.35	5.58	6.39	7.28	8.72
	(2.23)	(1.42)	(1.03)	(1.06)	(1.11)	(1.43)
Z	4.20	3.79	4.92	4.79	4.59	4.63
	(4.22)	(4.21)	(4.48)	(4.20)	(3.71)	(3.98)

Table 1: Summary Stats of firm size ($\ln(size)$) and Z-score (Z). Numbers are equal-weighted averages and numbers in parenthesis are standard deviations.

Table 1 shows summary statistics. Notably, I want to note that the variation of Z-score within each size bucket are close to each other. This assures that differential expected returns between small and big size bucket is *not* driven by differential variation of Z-score between small and big size bucket.

3.2 Size Premium and Distress Anomaly

Dichev (1998) shows that size premium, strong in the 1960s and 1970s, has virtually disappeared since 1980. Indeed, time-series average of size premium[3] between 1964 and 1979 is 7.31% whereas that between 1980 and 2015 is -0.93%. However, I argue that this has to do with "distress anomaly." If I exclude financially distressed firms, time-series average of size premium between 1964 and 1979 is 9.79% whereas that between 1980 and 2015 is 3.25%. Although the size premium has decreased over period, the magnitude is still sizable, hence worth studying.

I slightly modify Figure 2 to to explain this rather interesting phenomena. As implied in Figure 4, size premium is negative when $Z < 2.7$ [4]. This cutoff point of 2.7 is reasonable

[3]Here, for simplicity, I focus on size premium without accounting for market premium.

[4]Following Altman (1968), I estimate Z-score as

$$Z - Score = 3.3\frac{EBIT}{AT} + 0.99\frac{SALE}{AT} + 0.6\frac{ME}{LT} + 1.2\frac{WCAP}{AT} + 1.4\frac{RE}{AT}$$

Firms with $Z > 3$ is considered "safe", firms with $3 > Z > 2.7$ is considered "on alert", firms with $2.7 > Z > 1.8$ is considered to have "good chances of going bankrupt within 2 years" and firms with $1.8 > Z$ has very high financial distress.

because Altman (1968) empirically shows that firms with $Z < 2.7$ have good chances of going bankrupt within 2 years or less. Given that the distribution of firms with $Z < 2.7$ is not degenerate, the model implies that

$$E[r_t] < E[r_t|Z > 2.7]$$

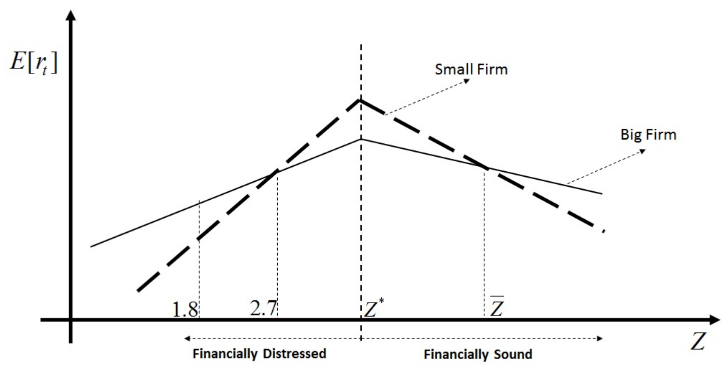

Figure 4: Theoretical relation between Z-score and portfolio returns for small and big firms.

Now, I examine how the above implication holds in the data. I follow standard procedure to divide firms into 10 buckets based on their size and calculate value-weighted average of equity returns for each size bucket. I perform this analysis on four different samples: all firms (any Z), financially distressed firms ($Z > 1.8$), financially less-distressed firms ($Z > 2.7$) and the most financially-sound firms ($Z > 3$). I estimate size premium for each sample and summarize results in Table 2.

		small	s2	s3	s4	s5	s6	s7	s8	s9	big	SMB
(1)	$r(\%)$	12.85	14.41	15.53	12.22	14.51	13.89	13.78	13.15	12.94	11.25	1.60
	s	5.12	6.04	6.40	6.87	7.18	7.42	7.77	8.24	8.65	10.10	4.99
(2)	$r(\%)$	14.50	15.78	15.93	15.34	16.04	14.71	14.92	13.85	13.57	11.78	2.72
	s	5.27	6.02	6.38	6.79	7.02	7.44	7.73	8.23	8.66	10.10	4.82
(3)	$r(\%)$	17.50	15.70	17.13	17.38	16.81	15.63	16.26	14.68	14.45	12.35	5.16
	s	5.13	5.86	6.05	6.61	6.76	7.41	7.75	8.16	8.67	10.16	5.03
(4)	$r(\%)$	17.83	16.45	17.16	18.37	16.48	15.84	16.31	15.14	14.41	12.57	5.26
	s	5.15	5.86	5.88	6.57	6.72	7.30	7.71	8.15	8.62	10.16	5.01

Table 2: Value-Weighted Equity Returns for Different Size Decile. s stands for log of firm size. $s1 - s10$ corresponds to size premium. (1) corresponds to the unrestricted sample (any Z). (2) corresponds to financially distressed firms ($1.8 < Z$) (3) corresponds to financially less-distressed firms ($2.7 < Z$) (4) corresponds to the most financially-sound firms ($3 < Z$).

Let us compare (1) and (4). Despite the fact that the size differential between small and big are almost equivalent at 4.99/5.01 for (1)/(4), the size premium (5.26%) at (4) is more than three times of the size premium (1.60%) at (1). This large difference in size premium is mainly driven by increase in (4)'s small firms' returns compared to (1)'s. Compared to (1), (4)'s small's returns increase by $17.83\% - 12.85\% = 4.98\%$ whereas (4)'s large's returns increase by only $12.57\% - 11.25\% = 1.32\%$. This implies that distress anomaly keeps size premium small by keeping small firms' returns depressed, and without it, size premium is sizable. This is consistent with **Implication** 3 and 4.

4 Cross-Sectional Analysis: Distress and Distress Anomaly

In this section, I document an empirical evidence that typical distress risk explains size premium quite well. Moreover, I document an evidence that distress anomaly risk offsets distress-risk-induced risk premium. Thus, I argue that true relation between size premium and distress risk can be uncovered only when we condition on distress anomaly risk.

Conventionally, researchers have constructed factors by subtracting one-extreme portfolio from the other-extreme portfolios. In order to capture distress risk and distress anomaly, I slightly deviate from the conventional way. I construct DC by subtracting high Z-score portfolio from medium Z-score portfolio. Similarly, I construct DAC by subtracting medium Z-score portfolio from low Z-score portfolio. I generate DC and DAC return time series the same way that Fama and French (1992) create the factors as follows:

$$DC = \frac{1}{2}(SM + BM) - \frac{1}{2}(SH + BH)$$
$$DAC = \frac{1}{2}(SL + BL) - \frac{1}{2}(SM + BM)$$

where SH, SM, SL, BH, BM, and BL are defined in Table 3. Here, breakpoints for Z-score and size use only NYSE stocks.

	High Z-score	Med Z-score	Low Z-score
Small Size	NYSE-SH	NYSE-SM	NYSE-SL
Med Size	NYSE-MH	NYSE-MM	NYSE-ML
Big Size	NYSE-BH	NYSE-BM	NYSE-BL

Table 3: Factor Construction

Then, I construct 25 testing portfolios based on Fama-French-size and Altman-Z-score. In Table 4, I document time-series average of equities returns, CAPM-α's, Z-score and size for 25 testing portfolios. As shown, portfolio #1 represents a bucket that corresponds to the worst Z-score and the smallest size and portfolio #25 represents a bucket that corresponds to the best Z-score and the biggest size.

#	Z-bucket	Size-bucket	Average Returns (%)	CAPM-α(%)	Zscore	ln(Size)
1	Bad	Small	2.96	-8.72	-0.80	4.82
2	Bad	S2	5.11	-6.02	0.37	6.30
3	Bad	S3	3.93	-6.38	0.60	7.06
4	Bad	S4	4.22	-5.27	0.80	7.93
5	Bad	Big	10.36	2.72	1.20	10.33
6	Z2	Small	4.05	-5.83	1.53	5.02
7	Z2	S2	4.15	-5.91	1.51	6.48
8	Z2	S3	4.79	-4.90	1.59	7.26
9	Z2	S4	5.46	-3.61	1.61	8.15
10	Z2	Big	8.18	1.89	1.61	10.48
11	Z3	Small	9.82	0.58	2.88	5.02
12	Z3	S2	7.74	-0.47	2.82	6.47
13	Z3	S3	9.34	1.30	2.78	7.33
14	Z3	S4	8.52	0.36	2.78	8.26
15	Z3	Big	8.72	2.29	2.83	10.41
16	Z4	Small	10.41	1.64	4.62	5.07
17	Z4	S2	10.04	1.41	4.57	6.44
18	Z4	S3	10.59	1.64	4.50	7.25
19	Z4	S4	11.09	3.67	4.53	8.24
20	Z4	Big	8.52	3.20	4.53	10.87
21	Good	Small	5.61	-2.69	12.07	5.13
22	Good	S2	7.07	-1.33	11.84	6.40
23	Good	S3	8.18	1.76	11.57	7.20
24	Good	S4	9.22	2.72	11.24	8.14
25	Good	Big	5.49	3.52	11.48	10.81

Table 4: Summary Statistics on the 25 Size and Z-Score Sorted Portfolios. Time-Series averages of average returns, CAPM-α, Z-score and ln(size)

Then, I run Fama-MacBeth regressions on 25 testing portfolios.

$$R_{i,t} - RF_t = \lambda_0 + \lambda_M \hat{\beta}_i^M + \lambda_{DC} \hat{\beta}_i^{DC} + \lambda_{DAC} \hat{\beta}_i^{DAC} + \nu_i$$

where $R_{i,t}$ is the return on Fama-French-size/Altman-Zscore portfolio ($i = 1, 2, \cdots, 25$) in each year (t= 1964 to 2015). Then, I compare risk premium attributed to distress risk and risk premium attributed to distress anomaly. More specifically, for testing portfolio $\#i$, I capture the former by $\lambda_{DC}\hat{\beta}_i^{DC}$ and the latter by $\lambda_{DAC}\hat{\beta}_i^{DAC}$. I plot these two components in Figure 5 over portfolio # on X-axis.

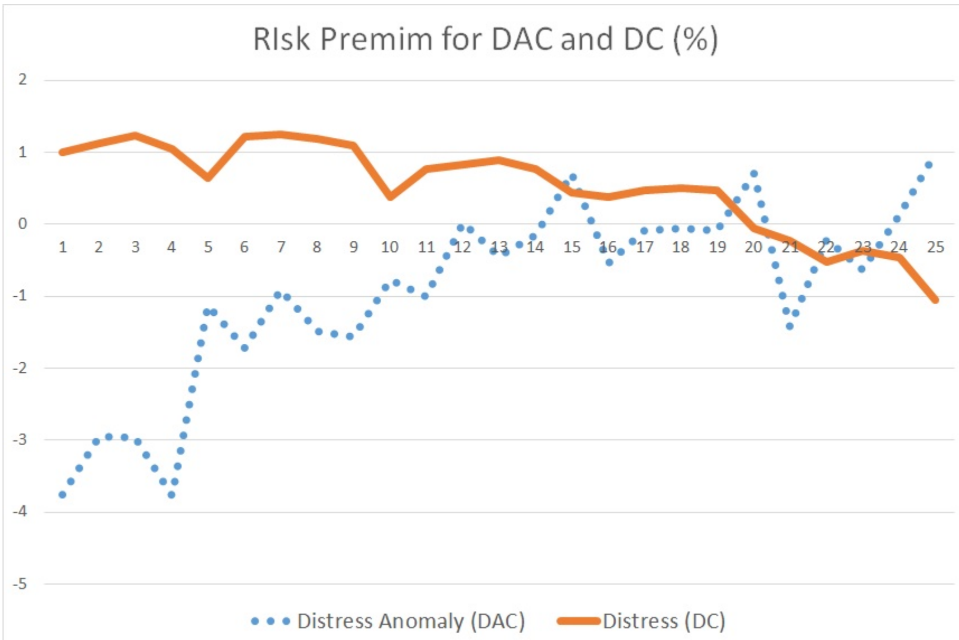

Figure 5: Risk premium ($\lambda_{DC}\hat{\beta}_i^{DC}$) contributed by DC (distress factor) in orange-solid line for all $i = 1, 2, \cdots, 25$ testing portfolios. Risk premium ($\lambda_{DAC}\hat{\beta}_i^{DAC}$) contributed by DAC (distress anomaly factor) in blue-dotted line for all $i = 1, 2, \cdots, 25$ testing portfolios.

By construction (see Table 4), higher portfolio # is associated with good Z-score. Thus, I expect low#-portfolios to have larger exposure to DC factor and high#-portfolios to have smaller exposure to DC factor. Downward-sloping orange-solid line in Figure 5 attests to it. More interestingly, I expect portfolios with bad Z-score to have more exposure to DAC factor and I expect the associated risk premium to be negative. Negative and upward-sloping blue-dotted line attests to my intuition. In other words, when firms' expected equity returns have low exposure to DAC, their expected returns are best explained by DC. However, when firms' expected returns have high exposure to DAC, DC-induced expected-returns are partially offset by DAC-induced expected-returns.

We can also observe the size-related patterns on DC and DAC curves. For example, among testing portfolios (#1 through #5) that fall into the worst Z-score bucket, small firms' equities returns (portfolio #1) has larger equities returns than large firms (portfolio #5). This pattern repeats for other Z-score buckets (#6 vs #10; #11 vs. #15; #16 vs. #20; #21 vs. #25). Yet, size-related patterns for DAC are pronounced for the worst Z-score bucket (#1 vs. #5) and not as much for other Z-score buckets (#6 vs #10; #11 vs. #15; #16 vs. #20; #21 vs. #25).

Therefore, in order to properly test the relation between distress risk and size premium, we need to restrict the sample to financially sound firms (i.e. #6 through #25). If we wish to look at size premium among financially distressed firms (#1 through #5), it is important to condition on DAC-induced expected returns. This justifies the intuition behind **Proposition** 2 and 3 that differently set parameters based on whether firms have high exposure to DAC or not.

5 Empirical Test of Model

5.1 Preliminary Evidence

I follow standard procedure to double-sort stocks into 5×10 size/Z-score buckets. The size and Z-score breakpoints use only NYSE stocks, but the sample is all NYSE, AMEX, and NASDAQ stocks on both CRSP and Compustat. Stocks are first sorted into five quintiles according to their Fama-French size. Subsequently, the stocks within each size quintile are sorted into ten Z portfolios based on their Z-score. This procedure produces 50 portfolios in total. In what follows, I examine how size premium varies over different Z-scores.

Table 5 summarizes value-weighted average of equity returns for each bucket under **CAPM-**α panel.[5] For each bucket, panel **ln(ME)** shows equal-weighted average of ln(ME) and panel **Z-score** shows equal-weighted average of Z-score.

Small firms' CAPM-α difference between z5 and z2 bucket is $5.00\% - (-0.90\%) = 5.90\%$ whereas big firms' is $6.34\% - 5.73\% = 0.61\%$. CAPM-α difference between small firms'

[5]Based on French's firm size breakpoints, I construct breakpoints for Z-score using NYSE-listed firms.

and big firm's is not driven by firm size because small firms' size difference between z5 and z2 bucket ($3.33 - 3.03 = 0.3$) is very close to big firms' ($8.73 - 8.58 = 0.15$). This difference is not entirely driven by Z-score because small firms' Z-score difference between z5 and z2 bucket ($2.67 - 1.3 = 1.37$) is very close to big firms' ($3.18 - 2.00 = 1.18$). The similar argument can be made for CAPM-α difference between z5 and z9 bucket. I argue that this CAPM-α difference is mainly driven by differential sensitivity between small and big firms and this is captured by **Proposition** 3.

	Bad	z2	z3	z4	z5	z6	z7	z8	z9	Good
					CAPM-α					
Small	-1.33	-0.90	1.49	4.20	5.00	6.75	5.91	6.60	4.87	1.33
size2	1.44	1.09	3.88	3.36	4.14	5.18	5.82	6.06	4.28	2.18
size3	0.65	3.96	3.86	4.85	7.37	5.55	4.96	6.22	4.74	3.76
size4	1.74	1.26	5.24	4.77	4.30	4.90	6.05	5.00	5.41	5.04
Big	4.88	5.73	4.44	6.59	6.34	3.16	5.55	5.44	6.09	4.69
Whole Sample	1.47	2.23	3.78	4.75	5.43	5.11	5.66	5.86	5.08	3.40
SMB	-6.21	-6.63	-2.95	-2.39	-1.34	3.59	0.36	1.16	-1.21	-3.37
t-stat (SMB)	-7.91	-10.68	-8.77	-4.89	-2.81	7.97	0.85	2.43	-2.23	-5.77
					ln(ME)					
Small	2.94	3.03	3.14	3.19	3.33	3.38	3.46	3.52	3.60	3.70
size2	5.49	5.44	5.49	5.55	5.53	5.53	5.59	5.65	5.64	5.71
size3	6.32	6.33	6.31	6.35	6.35	6.35	6.37	6.43	6.44	6.52
size4	7.21	7.21	7.19	7.12	7.26	7.23	7.27	7.32	7.38	7.50
Big	8.55	8.58	8.61	8.61	8.73	8.70	8.66	8.70	8.85	9.07
Whole Sample	4.22	4.54	4.70	4.68	4.79	4.85	4.91	4.91	4.90	4.89
					Z-score					
Small	-1.08	1.30	1.83	2.27	2.67	3.07	3.55	4.17	5.21	10.31
size2	0.78	1.70	2.22	2.66	3.08	3.52	4.05	4.78	6.01	12.13
size3	0.97	1.79	2.30	2.70	3.11	3.60	4.15	4.91	6.31	12.53
size4	1.16	1.92	2.34	2.78	3.14	3.66	4.26	5.05	6.39	11.79
Big	1.32	2.00	2.41	2.82	3.18	3.64	4.26	5.13	6.56	11.95
Whole Sample	-0.37	1.52	2.04	2.46	2.86	3.30	3.82	4.50	5.64	11.02

Table 5: Value-Weighted CAPM-α (%) Size-ZScore Double Sort. Sample: 1960-2015. Average number of observations per bucket is 3289.

Implication of **Proposition** 3 can be better demonstrated in the empirical counterpart of Figure 2. For small firms and big firms, I fit a quadratic polynomial to average CAPM-α vs Z-score. I show the fitted curve in Figure 6.

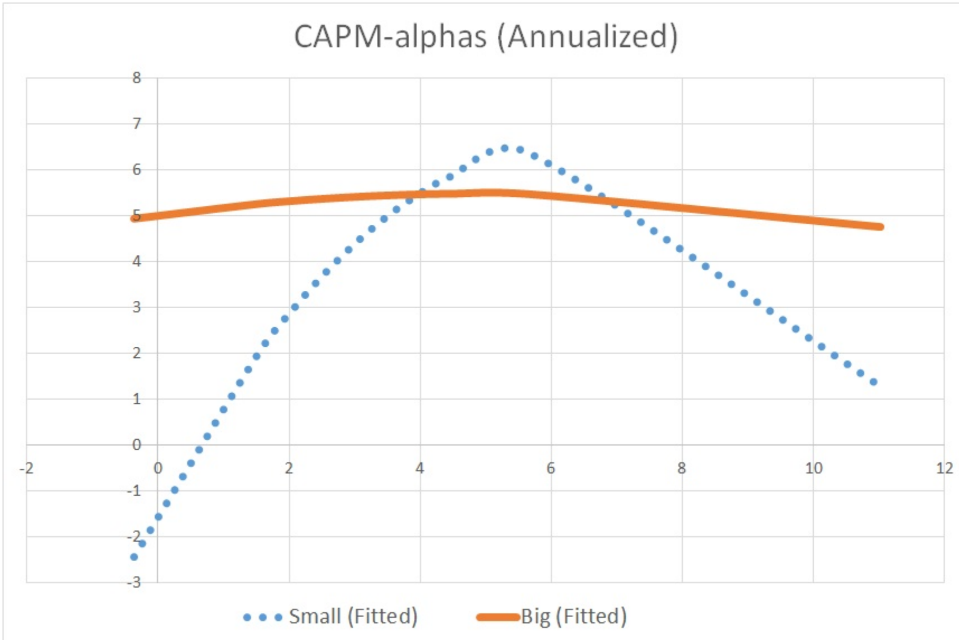

Figure 6: Fitted quadratic polynomial curve between Z-score and average CAPM-α

Here, I note a striking resemblance between Figure 2 and 6. This plot implies that time-series average of SMB (the gap between orange-solid and blue-dotted curves) is positive when Z is between 4 and 7 and negative when Z is less than 4 or greater than 7. As demonstrated in the next section, I show that macro economic conditions, such as government bond level, can parallel-shift both curves, hence alter the value of SMB with respect to Z.

Before I move onto the next section, I would like to relate our empirical finding to the model's implications. When firms are financially distressed (Z is small), both big and small firms' equities returns increase in Z-score. When firms are financially sound (Z is large), both big and small firms' equities returns decrease in Z-score. These two patterns are consistent with **Implication** 1 and 2. Moreover, I observe that SMB increases in Z when firms are financially distressed (Z is small) and SMB decreases in Z when firms are financially sound. These two patterns are exactly consistent with the **Implication** 3 and 4.

5.2 Time-Series Model Estimation

I closely follow Vassalou and Xing (2004) to estimate the model. I am interested in a component in Z-score that's orthogonal to size. Thus, I render Z-score orthogonal to size. I divide the sample into small and big firms. For each subsample, I run the following regression:

$$\begin{aligned}\text{CAPM-}\alpha_{it+1} =& \lambda^D(\beta^C \ln(ME_{it}) + (\beta^{Z+} Z_{it} + \beta^{ZC+} Z_{it} \cdot \ln(ME)) \cdot \mathbb{1}(Z_{it} > Z^*) \\ & +(\beta^{Z-} Z_{it} + \beta^{ZC-} Z_{it} \cdot \ln(ME_{it})) \cdot \mathbb{1}(Z_{it} < Z^*)) \end{aligned} \tag{3}$$

I run Fama-MacBeth regression to estimate the coefficients and calculate Newey-West standard errors (with 3 lags).

	$\hat{\beta}^C$	$\hat{\beta}^{Z+}$	$\hat{\beta}^{ZC+}$	$\hat{\beta}^{Z-}$	$\hat{\beta}^{ZC-}$
			Small		
coeff	-9.65	-6.03	1.93	41.51	-10.21
t-stat	-2.93	-1.71	1.59	0.86	-0.71
			Big		
coeff	-0.38	-1.34	0.12	11.72	-1.16
t-stat	-0.50	-1.02	0.72	1.75	-1.50

Table 6: Model Estimation

More precisely, each estimate above is not normalized by $\hat{\lambda}^D$ because time-series regression cannot identify $\hat{\lambda}^D$. However, if we reasonably assume that λ^D is positive, we can make a few statements. First, consistent with **Proposition** 1, $\hat{\beta}^C$'s are negative for both small and big firms. Consistent with **Proposition** 2, $\hat{\beta}^{Z+} < 0$ and $\hat{\beta}^{Z-} > 0$. Lastly, consistent with **Proposition** 3, $\hat{\beta}^{ZC+} > 0$ and $\hat{\beta}^{ZC-} < 0$.

6 Extension: Time-Series Variation in Size Premium

There are conflicting results on whether distress risk is systematic or not. Opler and Titman (1994) and Asquith, Gertner, and Sharfstein (1994) find that distress risk is related to idiosyncratic factors and therefore does not represent systematic risk. Denis and Denis

(1995), on the other hand, shows that distress risk is related to macroeconomic factors and that it varies with the business cycle. In this section, I plan to offer an evidence that weighs on the latter and demonstrate how I can update my model to accommodate a feature that distress risk is systematic. My extended model implies that shift in government bond level alters relative distress risk between big and small firm, thus naturally leading to different size premium. In this section, I provide not only description of extended model but also an empirical evidence for it.

6.1 Model Extension

As argued before, big firms face smaller distress risk than small firms. Thus, in the spirit of Friedman (1978) and Greenwood, Hanson, and Stein (2010), government bond and big firms' bond are similar in nature (in terms of distress risk) and the bond market is segmented in a way that increasing supply of the former increases the cost of bond for the latter. As government issues bond, big firms cannot enjoy the benefit of being big as much and are forced to reduce their investment/capital, which is empirically confirmed by Graham, Leary, and Roberts (2014). To make matter worse for big firms, government debt issuance tends to coincide with economic downturn. This naturally leads to higher distress cost for big firms, implying higher expected equity returns. However, small firms' bonds are not as substitutable for government bonds thus do not face the same effect. In sum, larger level of government debt decreases big firms' expected equity returns yet leaving small firms' relatively unchanged. In other words, larger level of government debt leads to smaller size premium as big firms' increased-return eats away size premium that would have existed with lower government debt.

Implication 5 *Government debt level and size premium are negatively correlated.*

In addition, because big firms face government-debt-induced effect and small firms do not as much, I state the second implication as:

Implication 6 *Big firms' returns are more positively correlated with government debt and its statistical significance is larger than small firms'. Furthermore, government debt can better explain big firms' equity returns than small firms.'*

In my model in Section 2, the benefit of being large was captured by β^C. I capture **Implication** 6 by equating lower β^C in magnitude to larger government debt. As graphically illustrated in Figure 7, larger government debt forces big firms' $E[r_t]$ curve to move up from A to B, hence the gap between big firms' curve and small firms' diminishes, leading to smaller size premium.

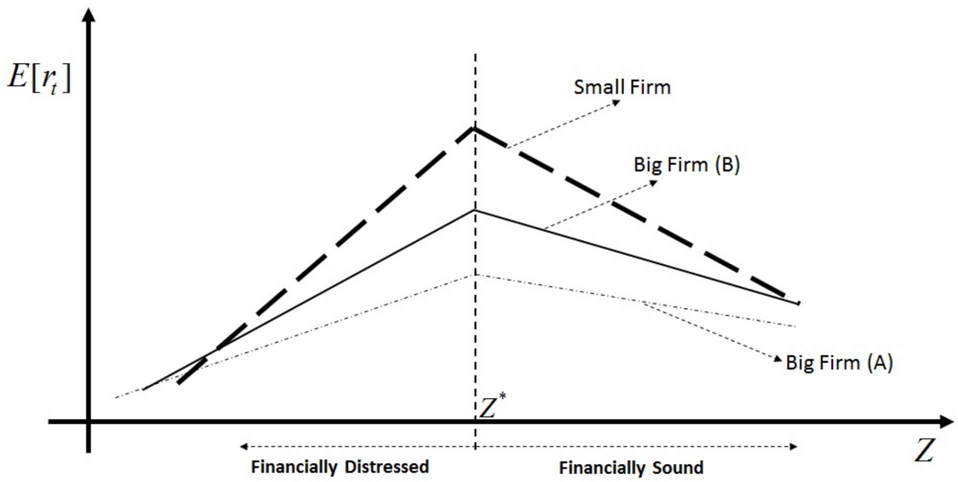

Figure 7: Theoretical relation between Z-score and portfolio returns for small and big firms

However, I would like to note that there is potentially a force that might offset the above effect. Larger level of government debt can decrease the leverage of big firms relative to small firms and this can potentially decrease big firms' distress risk relative to small firms. If this leverage-induced effect is strong enough, it can reduce big firms' distress risk, hence reduce expected equities returns, leading to larger size premium. This implies positive relation between government debt level and size premium. However, as documented below, we empirically observe statistically significant negative relation between government debt level and size premium thus, I argue that leverage effect is not strong enough to cancel out the aforementioned effect. Moreover, this relation is consistent with Gomes and Schmid (2010)'s result that shows that low leverage does not always lead to low risk especially when high risk forces a firm to optimally lower their leverage.

6.2 Empirical Test of Model Extension

6.2.1 Fitted Polynomial Curve

I test **Implication** 5. I relate government's bond level (normalized by GDP) to size premium. As demonstrated in the scatter plots (Figure 8), these two variables are negatively statistically significantly correlated.

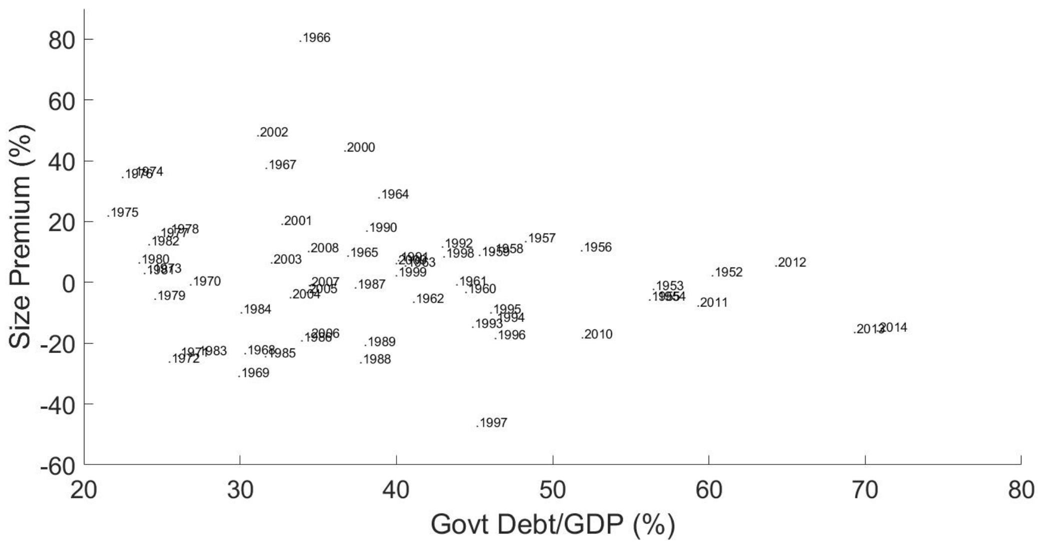

Figure 8: x-axis: federal debt issuance held by the public; y-axis: market-weighted average return (between t and $t+1$) spread between low (bottom 10%) and high (top 10%) size portfolios. correlation is -0.2417 (p-value=0.0544). Sample: Annual data from 1952 to 2015

In year 1975, which sits on the left-most-end of the plot, when the government's bond level is very low at 22% of GDP, the annualized size premium is 21%. On the other extreme of the plot, in year 2014, the government's bond level is very high at 72% of GDP and the annualized size premium is -18%. All the other points roughly sit in-between and the correlation is -0.2417 and statistically significant at 5%. This is consistent with **Implication** 5. In the rest of this section, I offer a list of evidence to support that this negative correlation between government's bond level and size premium is *economically* significant.

6.2.2 Panel Regressions

First, I empirically test **Implication** 6 by running the following panel regression for all firms that fall into each of 5×10 size/Z-score buckets.

$$
\begin{aligned}
r_{i,t\to t+1} = {} & \lambda^{CAPM}\beta_{it}^{CAPM} + \beta^{SIZE}\log(ME_{i,t}) + \beta^{Lev}\log\frac{AT_{it}}{BE_{it}} + \beta^{BM}\log\frac{BE_{it}}{ME_{it}} + \beta^{ZScore}Z_{i,t} \\
& +\beta^{G}\log\frac{GovtDebt_t}{GDP_t} + \beta^{GDP}GDPGrowth_{t-1\to t} + FE_i \qquad (4)
\end{aligned}
$$

Conditioned on various firm characteristics, macro variables and firm fixed effects, I test how much additional explanatory power $\log\frac{GovtDebt_t}{GDP_t}$ contributes. To this end, I first estimate adj-R^2 in Equation (4). Next, I exclude $\log\frac{GovtDebt_t}{GDP_t}$ and estimate adj-R^2. Table 7 reports a difference between these two adj-R^2.

	Bad	z2	z3	z4	z5	z6	z7	z8	z9	Good
Small	0.03	0.01	0.10	0.00	0.05	0.21	0.38	0.13	0.19	0.27
size2	0.00	0.03	0.16	0.51	0.66	0.10	1.45	0.58	1.24	0.48
size3	0.26	0.01	0.13	0.28	0.25	0.44	1.51	2.08	0.60	0.71
size4	0.67	0.00	1.33	0.46	1.81	2.35	1.40	4.83	1.58	0.51
Big	1.65	0.13	0.72	1.54	3.59	0.48	1.87	1.70	3.62	2.46

Table 7: Adj-R^2 (%) difference between Regression Equation (4) and Regression Equation (4) with $\log\frac{GovtDebt_t}{GDP_t}$ term dropped. Average sample size for each bucket is 3289 firm-year observation. Sample spans from 1961 to 2015.

As shown above, bigger firms' expected returns are better explained by $\log\frac{GovtDebt_t}{GDP_t}$ than small firms for all different Z-score buckets. In the worst Z-score bucket, $\log\frac{GovtDebt_t}{GDP_t}$ increases adj-R^2 by only 0.03% for small firms whereas it does by 1.65% for big firms. In the best Z-score bucket, $\log\frac{GovtDebt_t}{GDP_t}$ increases adj-R^2 by only 0.27% for small firms whereas it does by 2.46% for big firms. This is consistent with **Implication** 6.

	Bad	z2	z3	z4	z5	z6	z7	z8	z9	Good
Small	-0.12	0.06	0.14	0.00	-0.09	0.16	0.19	0.11	0.13	0.19
	(-1.64)	(0.84)	(2.00)	(-0.04)	(-1.52)	(2.96)	(4.17)	(2.61)	(3.56)	(5.71)
size2	0.02	0.06	0.11	0.20	0.18	0.07	0.24	0.16	0.25	0.20
	(0.23)	(0.59)	(1.20)	(2.17)	(2.57)	(1.01)	(3.84)	(2.61)	(4.27)	(3.59)
size3	0.16	-0.03	-0.08	0.13	0.11	0.13	0.23	0.32	0.18	0.22
	(1.60)	(-0.28)	(-1.02)	(1.44)	(1.34)	(1.84)	(3.61)	(4.29)	(2.52)	(3.27)
size4	0.20	-0.01	0.23	0.13	0.26	0.27	0.21	0.40	0.24	0.16
	(2.53)	(-0.13)	(3.18)	(1.92)	(3.76)	(4.32)	(3.39)	(6.58)	(3.97)	(2.53)
Big	0.24	0.06	0.12	0.17	0.32	0.11	0.21	0.20	0.31	0.30
	(4.27)	(1.10)	(2.56)	(3.63)	(5.61)	(2.06)	(4.22)	(4.10)	(6.37)	(5.90)

Table 8: Regression estimates of β^G in Equation (4). Numbers in parenthesis are t-stat's. Average sample size for each bucket is 3289 firm-year observation. Sample spans from 1961 to 2015.

I now examine regression coefficients. As shown in Table 8, big firms' $\hat{\beta}^G$ tends to be larger and more statistically significant than small firms' regardless of which Z-score bucket they are in. In the worst Z-score bucket, small firms' $\hat{\beta}^G$ is -0.12 (1.64) whereas big firms' is 0.24 (4.27). In the best Z-score bucket, small firms' $\hat{\beta}^G$ is 0.19 (5.71) whereas big firm's 0.30 (5.90). These results are consistent with **Implication** 6.

6.2.3 Fitted Polynomial Curve

In order to test how my extended model performs, I run subsample analysis and follow a procedure similar to Section 5.1. I divide sample into ones that have experienced low government debt (below 30-percentile, which is equivalent to 31.6% relative to GDP) and ones that have experienced high government debt (above 70-percentile, which is equivalent to 43.5% relative to GDP).

For each subsample, I follow standard procedure to double-sort stocks into 5×10 size/Z-score buckets. Stocks are first sorted into five quintiles according to their market size. Subsequently, the stocks within each size quintile are sorted into ten Z portfolios based on their Z-score. This procedure produces 50 portfolios in total.

First, I look at years with low government debt. Average number of observations per bucket is 1077. Years that are included in the sample are 1967, 1968, 1969, 1970, 1971, 1972, 1973, 1974, 1975, 1976, 1977, 1978, 1979, 1980, 1981, 1982, 1983, 1984, 1985 and 2002. I report a fitted quadratic polynomial curve of average CAPM-α vs Z-score for small firms and big firms (Figure 9).

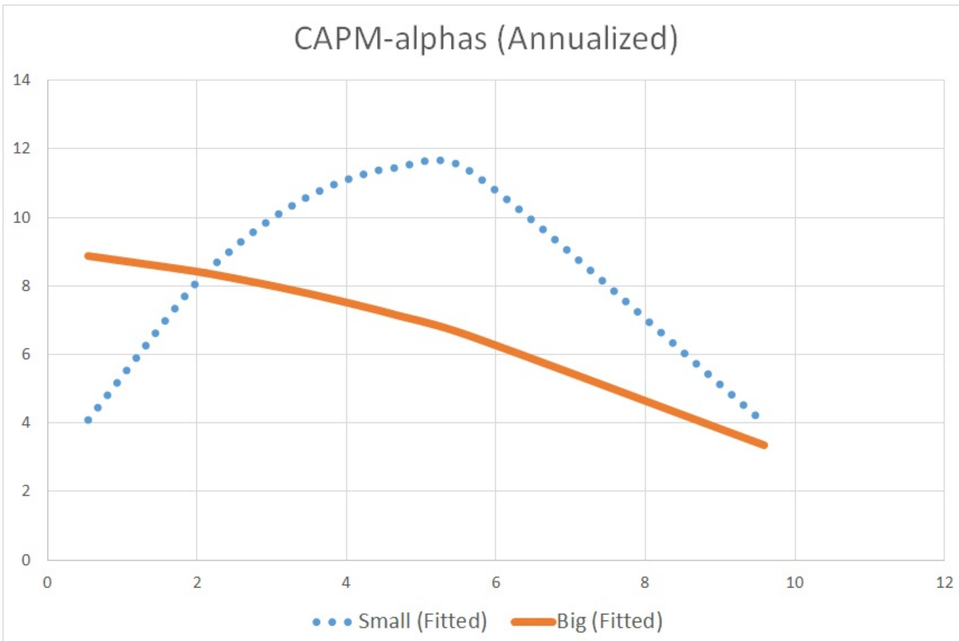

Figure 9: Fitted quadratic polynomial curve between Z-score and average CAPM-α. Sample is restricted to years with low government debt (below 30-percentile, which is equivalent to 31.6% relative to GDP). Years that are included in the sample are 1967, 1968, 1969, 1970, 1971, 1972, 1973, 1974, 1975, 1976, 1977, 1978, 1979, 1980, 1981, 1982, 1983, 1984, 1985 and 2002.

Any inference about Figure 9 is more meaningful if I compare it to the counterpart for high government debt. Average number of observations per bucket is 784. Years that are included in the sample are 1954, 1955, 1956, 1957, 1958, 1959, 1960, 1961, 1993, 1994, 1995, 1996, 1997, 2010, 2011, 2012, 2013, 2014 and 2015. I show the fitted quadratic polynomial curve in Figure 10.

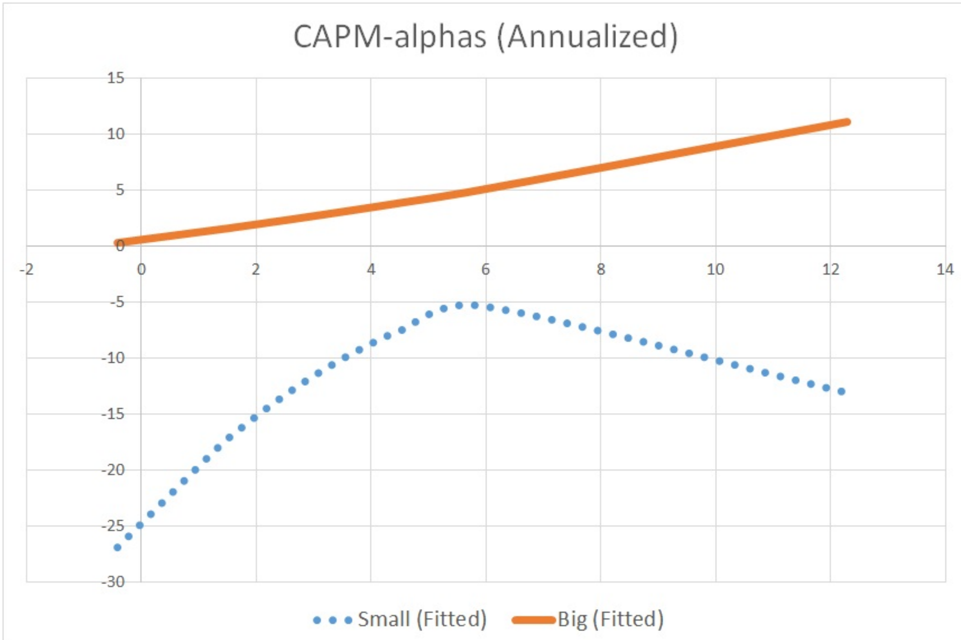

Figure 10: Fitted quadratic polynomial curve between Z-score and average CAPM-α. Sample is restricted to years with high government debt (above 70-percentile, which is equivalent to 43.5% relative to GDP). Years that are included in the sample are 1954, 1955, 1956, 1957, 1958, 1959, 1960, 1961, 1993, 1994, 1995, 1996, 1997, 2010, 2011, 2012, 2013, 2014 and 2015.

Comparison between Figure 9 and 10 yields two results. First, size premium is positive for the majority of relevant Z-score when government debt is low whereas size premium is negative for all Z-score's. Second, relative position of big firm's (orange solid curve) have shifted upward which implies that β^C have decreased in magnitude. These two results are consistent with the extended model's implications.

I note that the level of CAPM-α is much lower during high-government-debt regime compared to low-government-debt regime. I suspect that this happens due to mismatch between market return and government debt. In estimating CAPM-α from $t = 0$ to 1 (annual), I use market return from $t = 0$ to 1. However, I use the level of government debt at $t = -1$. When government debt is low at $t = -1$, because low market return tends to follow low government spending, CAPM-α's during low-government-bond regime tend to be inflated. Similarly, when government debt is high at $t = -1$, because high market return tends to follow high government spending, CAPM-α's during high-government-bond regime tend to be deflated. However, there is no reason to believe that this macro-level bias will affect small and big firms differently. Because my focus is only on the gap between big firms' curve and small firm's, this potential bias does not concern me at this point.

6.2.4 Time-Series Model Estimation

Similar to Section 5.2, I closely follow Vassalou and Xing (2004) to estimate the model. I am interested in a component in Z-score that's orthogonal to size. Thus, I render Z-score orthogonal to size. I divide the sample into small and big firms. For each subsample, I run the following regression:

$$\begin{aligned}\text{CAPM-}\alpha_{it+1} =& \lambda^D(\beta^C \ln(ME_{it}) + (\beta^{Z+} Z_{it} + \beta^{ZC+} Z_{it} \cdot \ln(ME)) \cdot \mathbb{1}(Z_{it} > Z^*) \\ & + (\beta^{Z-} Z_{it} + \beta^{ZC-} Z_{it} \cdot \ln(ME_{it})) \cdot \mathbb{1}(Z_{it} < Z^*))\end{aligned}$$

I run Fama-MacBeth regression to estimate the coefficients. Because the sample does not cover consecutive years of data, I cannot calculate Newey-West standard errors. Instead, I follow standard procedure to estimate standard errors.

	Low Govt Debt Level					High Govt Debt Level				
	$\hat{\beta}^C$	$\hat{\beta}^{Z+}$	$\hat{\beta}^{ZC+}$	$\hat{\beta}^{Z-}$	$\hat{\beta}^{ZC-}$	$\hat{\beta}^C$	$\hat{\beta}^{Z+}$	$\hat{\beta}^{ZC+}$	$\hat{\beta}^{Z-}$	$\hat{\beta}^{ZC-}$
	Small					Small				
coeff	-4.66	-0.74	0.05	0.39	-0.43	-4.60	0.02	-0.06	-4.05	2.20
t-stat	-3.85	-1.11	0.22	0.01	-0.05	-2.88	0.03	-0.36	-0.76	1.71
	Big					Big				
coeff	-0.93	-1.38	0.06	16.24	-1.86	3.98	-6.25	0.67	7.47	-0.53
t-stat	-0.95	-0.53	0.17	1.54	-1.46	3.11	-3.61	4.03	0.74	-0.52

Table 9: Model Estimation for low and high level of government debt.

Table 9 summarizes estimation results. More precisely, each estimate above is not normalized by $\hat{\lambda}^D$ because time-series regressions cannot identify $\hat{\lambda}^D$. However, if we reasonably assume that λ^D is positive, we can make a few several statements. First, regardless of whether the level of government debt is low or high, small firms' $\hat{\beta}^C$'s are almost the same and both statistically significant. Second, however, larger government debt level increases big firms' $\hat{\beta}^C$ from -0.93 to 3.98. In other words, big size marginally decreases (-0.93 (-0.95)) distress risk when government debt level is low yet big size increases (3.98 (3.11)) distress risk when government debt level is high. This is consistent with stated implications of the extended model above.

7 Conclusion

In this paper, I attempt to reconcile "distress risk" finding and "distress anomaly" finding. By reconciling these two findings, I argue that size premium compensates for distress risk. To that end, I provide a model that generates a hump-shaped return profile with respect to Z-score. Using this model and empirical findings, I show that the size premium's behavior depends on whether firms are financially sound or distressed and show that size premium is also hump-shaped over Z-score. Lastly, I extend the model to show that size premium depends on government debt: government debt level and size premium are negatively correlated.

I have not provided an equilibrium model for **Proposition** 2 nor 3. Because these propositions are key drivers of my result, a partial or general equilibrium model will certainly strengthen my results, thus I will leave it for a possible future research topic. Nonetheless, I would like to mention that this is the first paper to attempt 1) to provide an unified framework to simultaneously capture both "distress risk" and "distress anomaly" and 2) to extend the framework to study the size premium. At the least, I hope that my research helps to alleviate some of the skepticism towards a role of distress risk in expected equity returns, hence the size premium.

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