Do rare events explain CDX tranche spreads?*

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Abstract

We investigate whether a model with a time-varying probability of economic disaster can explain the pricing of collateralized debt obligations, both prior to and during the 2008-2009 financial crisis. In particular, we examine the pricing of tranches on the CDX, an index of credit default swaps on large investment-grade firms. CDX senior tranches are essentially deep out-of-the money put options because they do not incur any losses until a large number of previously stable firms default. As such, CDX senior tranches provide critical information about how the market assesses the risk of rare disasters. We find that the model consistently matches the level and time series patterns of different asset classes including stocks, equity options, and CDX senior tranches, demonstrating the importance of beliefs about rare disasters even in periods of relatively high valuation.

1 Introduction

The period from 2005 to September of 2008 witnessed a more than 10-fold increase in the price of insuring against economic catastrophe. The insurance product in question derives from an index of credit default swaps, instruments that allow investors to insure against default of individual firms. Starting in the early 2000s, financial institutions began to pool these risks into an index, and then to create tranches, namely contracts that are ranked according to seniority. The so-called super-senior tranche on the CDX, as the index is named, would only be affected if the pool lost more than 30% of its value, implying a level of corporate defaults not even seen during the Great Depression. Purchasing insurance on this super-senior tranche implied that one would only receive a payoff should this event occur. The payoff distribution of senior tranches on the CDX has led researchers to refer to these assets as economic catastrophe bonds (Coval, Jurek, and Stafford (2009)).

While the price to insure the super-senior tranche was close to zero in 2005 and 2006, fluctuations began to appear in late 2007, culminating in sharply rising prices in the summer and fall of 2008. Ex post, of course, such insurance did not turn out to be necessary. In fact only a very small number of firms represented by the CDX index have defaulted. Yet, the pricing of these securities strongly suggests a substantial, and time-varying, fear of economic catastrophe.

In this paper, we investigate whether an equilibrium model with rare economic disasters, in the spirit of Barro (2006) and Rietz (1988), can explain the time series of the cost to insure the CDX index and its tranches. Unlike previous models of structured finance, firm prices in our model are derived endogenously from assumptions on investor preferences and cash flows. Firm prices embed rare disaster fears, as well as risks that are idiosyncratic. Importantly, our model, which is based on that of Seo and Wachter (2015), can explain option as well as as equity prices (the model is also consistent with the average return and

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volatility on the aggregate market). We can therefore use it to back out a time series of rare disaster probabilities from option prices alone. When we use these probabilities to calculate model-implied values for CDX index and tranche prices, we find that it can explain the low spreads on senior tranches prior to the crisis, the high spreads during the crisis, and the timing of the increase in spreads. Rather than being too low prior to the crisis and too high afterwards, our results imply that CDX spreads reflect an assessment of the risk in the economy that is consistent with other asset classes.

Our paper relates to two recent papers that also explain CDX index/tranche pricing. Coval, Jurek, and Stafford (2009) examine the pre-crisis behavior of CDX tranches (their sample ends in September 2007). Their intuitively appealing pricing technique is based on the assumption that cash flows on the CDX tranches occur at the 5-year maturity. They can then price these cash flows based on a distribution of state prices derived from 5-year option data. Assuming that the CDX index is correctly priced, they find that spreads for the senior tranches are too low in the data. They conjecture that investors were willing to provide insurance on these products, despite receiving low spreads, because of a naive interpretation of credit risk ratings. Indeed, these products were highly rated because of their low default probabilities; these ratings did not take into account that defaults would occur when they were least desirable.

These conclusions are questioned by Collin-Dufresne, Goldstein, and Yang (2012). Collin-Dufresne et al. note that the pricing of the so-called equity tranche (the most junior tranche) is sensitive to the timing of defaults and the specification of idiosyncratic risk. The implicit assumption in Coval, Jurek, and Stafford (2009) is that defaults occur at the contract maturity; however they can occur at any time. Assuming that defaults occur at the 5-year horizon makes the equity tranches look more attractive. The model spreads will thus be artificially low on the equity tranches, and, because the model is calibrated to match the

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index, the model-implied spreads on the senior tranches will be too high. Collin-Dufresne et al. also emphasize the need for fat-tails in the idiosyncratic risk of firms to capture the CDX spread at the 3-year, as well as the 5-year maturity. Introducing fat-tailed risk also raises the spread of the equity tranche in the model and lowers the spread of senior tranches.

Once payoffs occur at a horizon other than 5 years, the method of extracting state prices from options data no longer cleanly applies. Accordingly, Collin-Dufresne, Goldstein, and Yang (2012) specify a dynamic model of the pricing kernel, which they calibrate to 5-year options, and which they require to match the 3- and 5-year CDX index. They find that this model comes closer to matching the spreads on equity tranches and on senior tranches prior to the crisis.

However, the results of Collin-Dufresne, Goldstein, and Yang (2012), point to the limitations of the reduced-form approach that is the focus of both papers. Collin Dufresne et al. show that pricing tranches in the post-crisis sample (October 2007 - September 2008) requires some probability of a catastrophic jump that cannot be inferred from index or option data. Without this catastrophic risk, matching the level of the CDX indices and option prices produces model-implied spreads on the senior tranches that are too *low*, while the junior tranche spreads are too high (interestingly, this effect, while small, is also present pre-crisis). Matching the level of spreads during the crisis requires a lot of risk, and if this risk is idiosyncratic, it will lead to counterfactually high equity tranche spreads and counterfactually low senior tranche spreads. The probability of a catastrophic event cannot be determined by option prices because there are not enough options with strikes in the relevant range. In other words, CDX prices are non-redundant securities. Even a fully dynamic model cannot deliver CDX tranche prices based on options and the index alone.

These results motivate our approach. Rather than specifying a reduced-form pricing kernel and price processes for firms, we derive both in general equilibrium. As in Seo and

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Wachter (2015), we require the resulting equilibrium model to explain the equity premium and equity volatility, among other aggregate stock market facts. The equilibrium model is more restrictive than the reduced form approach; for instance, firm-level volatility, which is so crucial to explaining CDX pricing, is an equilibrium outcome based on firm cash flows and investor preferences. This endogenous firm-level volatility itself arises from rare disaster fears. Our equilibrium model implies a link from rare disaster probabilities, to equity volatility, and from there to option prices. Unlike in the reduced form approach, option prices depend strongly on rare disaster probabilities. We can thus re-examine the question posed by both Coval, Jurek, and Stafford (2009) and Collin-Dufresne, Goldstein, and Yang (2012): are option prices and CDX tranche spreads consistent? We find that they are.

2 Model

2.1 Model primitives and the state-price density

We use the two-factor stochastic disaster risk model introduced in Seo and Wachter (2015). Namely, we assume an endowment economy with complete markets and an infinitely-lived representative agent. Aggregate consumption (the endowment) solves the following stochastic differential equation:

$$\frac{dC_t}{C_{t^-}} = \mu_C dt + \sigma_C dB_{C,t} + (e^{Z_{C,t}} - 1)dN_{C,t}, \tag{1}$$

where $B_{C,t}$ is a standard Brownian motion and $N_{C,t}$ is a Poisson process. The intensity of $N_{C,t}$ is given by λ_t and assumed to be governed by the following system of equations:

$$d\lambda_t = \kappa_\lambda(\xi_t - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} \tag{2}$$

$$d\xi_t = \kappa_\xi(\bar{\xi} - \xi_t)dt + \sigma_\xi\sqrt{\xi_t}dB_{\xi,t}, \tag{3}$$

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where $B_{\lambda,t}$ and $B_{\xi,t}$ are Brownian motions (independent of each other and of $B_{C,t}$). At the parameter values we consider, λ_t is very close to the probability of a jump over, say, an annual time interval, and we thus use the terminology probability and intensity interchangeably. The system in equations (2) and (3) allows there to be a short and long-run component to the disaster probabilities, and thus to firm volatility. A substantial literature documents the presence of multiple frequencies in volatility data.

We assume a recursive generalization of power utility that allows for preferences over the timing of the resolution of uncertainty. Our formulation comes from Duffie and Epstein (1992), and we consider a special case in which the EIS is equal to one. That is, we define continuation utility V_t for the representative agent using the following recursion:

$$V_t = \mathbb{E}_t \int_t^\infty f(C_s, V_s)\, ds,$$

where

$$f(C, V) = \beta(1 - \gamma)V \left(\log C - \frac{1}{1 - \gamma} \log((1 - \gamma)V) \right).$$

The parameter β is the rate of time preference and γ is relative risk aversion. This utility function is equivalent to the continuous-time limit of the utility function defined by Epstein and Zin (1989) and Weil (1990). Assuming an EIS of one allows for closed-form solutions for equity prices up to ordinary differential equations, and facilitates the computation of options and CDX prices.

As shown in Seo and Wachter (2015), the value function has the solution

$$V_t = J(C_t, \lambda_t, \xi_t) = \frac{C_t^{1-\gamma}}{1 - \gamma} e^{a + b_\lambda \lambda_t + b_\xi \xi_t},$$

with coefficients a, b_λ and b_ξ that are related to the primitive parameters as follows:

$$a = \frac{1-\gamma}{\beta}\left(\mu - \frac{1}{2}\gamma\sigma^2\right) + \frac{b_\xi\kappa_\xi\bar{\xi}}{\beta}$$

$$b_\lambda = \frac{\kappa_\lambda + \beta}{\sigma_\lambda^2} - \sqrt{\left(\frac{\kappa_\lambda + \beta}{\sigma_\lambda^2}\right)^2 - 2\frac{E_\nu\left[e^{(1-\gamma)Z_t} - 1\right]}{\sigma_\lambda^2}}$$

$$b_\xi = \frac{\kappa_\xi + \beta}{\sigma_\xi^2} - \sqrt{\left(\frac{\kappa_\xi + \beta}{\sigma_\xi^2}\right)^2 - 2\frac{b_\lambda\kappa_\lambda}{\sigma_\xi^2}},$$

This value function implies the following state-price density

$$\pi_t = \exp\left(-\beta(a+1)t - \beta b_\lambda \int_0^t \lambda_s ds - \beta b_\xi \int_0^t \xi_s ds\right)\beta C_t^{-\gamma}e^{a+b_\lambda\lambda_t+b_\xi\xi_t}, \tag{4}$$

and risk free rate

$$r_t = \beta + \mu_C - \gamma\sigma_C^2 + \lambda_t E\left[e^{(1-\gamma)Z_{C,t}} - e^{-\gamma Z_{C,t}}\right]. \tag{5}$$

2.2 The aggregate market

The aggregate market has payoff $D_t = C_t^\phi$, with $\phi > 1$. This standard assumption on the relation between dividends and consumption captures the empirical finding that dividend payoffs are procyclical during normal times and have a greater disaster response than consumption (Longstaff and Piazzesi (2004)). By Ito's Lemma

$$\frac{dD_t}{D_{t-}} = \mu_D dt + \phi\sigma_C dB_{C,t} + (e^{\phi Z_{C,t}} - 1)dN_{C,t},$$

where $\mu_D = \phi\mu_C + \frac{1}{2}\phi(1-\phi)\sigma_C^2$. The value of the aggregate market, in equilibrium, is given by

$$F(D_t, \lambda_t, \xi_t) = E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t}D_s ds\right].$$

As shown in Seo and Wachter (2015),

$$F(D_t, \lambda_t, \xi_t) = D_t G(\lambda_t, \xi_t)$$

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for

$$G(\lambda_t, \xi_t) = \int_0^\infty \exp\left(a_\phi(\tau) + b_{\phi\lambda}(\tau)\lambda_t + b_{\phi\xi}(\tau)\xi_t\right) d\tau,$$

with coefficients solving the following system of ordinary differential equations:

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\begin{aligned}
a_\phi'(\tau) &= -\beta + (\phi - 1)\mu + \left(\tfrac{1}{2}\phi - \gamma\right)(\phi - 1)\sigma^2 + b_{\phi\xi}(\tau)\kappa_\xi\bar{\xi} \\
b_{\phi\lambda}'(\tau) &= -b_{\phi\lambda}(\tau)\kappa_\lambda + \tfrac{1}{2}b_{\phi\lambda}(\tau)^2\sigma_\lambda^2 + b_\lambda b_{\phi\lambda}(\tau)\sigma_\lambda^2 + E_\nu\left[e^{(\phi-\gamma)Z_t} - e^{(1-\gamma)Z_t}\right] \\
b_{\phi\xi}'(\tau) &= -b_{\phi\lambda}(\tau)\kappa_\lambda - b_{\phi\xi}(\tau)\kappa_\xi + \tfrac{1}{2}b_{\phi\xi}(\tau)^2\sigma_\xi^2 + b_\xi b_{\phi\xi}(\tau)\sigma_\xi^2,
\end{aligned}
$$

with boundary condition

$$a_\phi(0) = b_{\phi\lambda}(0) = b_{\phi\xi}(0) = 0.$$

2.3 Firms and default

CDX index and tranche pricing requires a model for individual firms. Let $D_{i,t}$ be the payout amount of firm i, for $i = 1, \cdots, N_f$, where N_f is the number of firms in the CDX index (the number has been 125). While we use the notation $D_{i,t}$, we intend this to mean the payout not only to the equity holders but to the bondholders as well. The firm payout is subject to three types of risk:

$$\frac{dD_{i,t}}{D_{i,t^-}} = \mu_i dt + \underbrace{\phi_i\sigma_C dB_{C,t} + (e^{\phi_i Z_{C,t}} - 1)dN_{C,t}}_{\text{aggregate risk}} + \underbrace{I_i(e^{Z_{S_i,t}} - 1)dN_{S_i,t}}_{\text{sector risk}} + \underbrace{(e^{Z_{i,t}} - 1)dN_{i,t}}_{\text{idiosyncratic risk}}. \quad (6)$$

where μ_i is defined similarly to μ_D, namely $\mu_i = \phi_i\mu_C + \frac{1}{2}\phi_i(1 - \phi_i)\sigma_C^2$. The systematic risk is standard: D_{it} has a multiplicative component that behaves like $C_t^{\phi_i}$, analogously to dividends. Firms are exposed to both normal-times aggregate risk and aggregate consumption disasters. Because financial leverage is not reflected in C_t, the value of ϕ_i will be substantially below that of ϕ for aggregate (equity) dividends above. However, we will still allow firms to have greater exposure to aggregate disasters than consumption, namely

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$\phi_i > 1$. For simplicity, we assume all idiosyncratic risk is Poisson. That is, firms are also subject to idiosyncratic negative events with constant probability λ_i. When a firm is hit by its idiosyncratic shock (which is modeled as a jump in $N_{i,t}$), the firm's payout falls by $D_{i,t^-} \times (1 - e^{Z_{i,t}})$. For parsimony, we will assume firms are homogeneous in that μ_i, σ_i, ϕ_i, Z_i and λ_i take the same value for all i. Of course, N_i will be different.

Longstaff and Rajan (2008) estimate that a substantial portion of the CDX spread is attributable to shocks that affect a subset of firms that share a common sector. We therefore allow for this sector risk in our specification (6). Let $\mathcal{S}$ denote a finite set of sectors.[1] Firm i's sector is denoted S_i which takes values in $\mathcal{S}$. The sector-wide shock is denoted N_{S_i}. When a sector shock arrives, the firm will be hit with this shock with probability p_i, namely the sector term in (6) is multiplied by I_i which takes a value 1 with probability p_i and 0 otherwise. If a firm happens to be affected by this sector shock, the firm's payout drops by $D_{i,t^-} \times (1 - e^{Z_{S_i,t}})$. Again, for parsimony, p_i and $Z_{S_i,t}$ will be the same across firms, while I_i and $N_{S_i,t}$ will differ.

Intuitively, sector risk would be (positively) correlated with aggregate consumption risk. To capture this positive correlation, we allow the intensity of $N_{S_i,t}$, $\lambda_{S_i,t}$, to depend on the state variables λ_t and ξ_t. In the Appendix, we solve a model using the specification $\lambda_{S_i,t} = w_0 + w_\lambda \lambda_t + w_\xi \xi_t$. For parsimony, we will calibrate the simpler model $\lambda_{S_i,t} = w_\xi \xi_t$.

Given this payout definition, we solve for the total value of firm i (the equity plus the debt), which we denote as $A_{i,t}$. By definition, $A_{i,t}$ is the price of the payout stream:

$$A_{i,t} = \mathbb{E}_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} D_{i,s} ds \right]. \tag{7}$$

We denote $G_i(\lambda_t, \xi_t)$ as the asset-payout ratio $(A_{i,t}/D_{i,t})$ of firm i. We show that this ratio

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[1] As described in Section 3, our sector classification comes from our data provider Markit. There are five sectors: Consumer, Energy, Financials, Industrial, Telecom, Media, and Technology. Thus $\mathcal{S} = \{C, E, F, I, T\}$.

is expressed as

$$G_i(\lambda_t, \xi_t) = \int_0^\infty \exp\left(a_i(\tau) + b_{i\lambda}(\tau)\lambda_t + b_{i\xi}(\tau)\xi_t\right) d\tau, \tag{8}$$

where $a_i(\tau)$, $b_{i\lambda}(\tau)$, and $b_{i\xi}(\tau)$ solve the system of ordinary differential equations derived in Appendix A.

Finally, we define default as the event that a firm's value falls below a threshold following Black and Cox (1976) and many subsequent studies. If we let A_B denote this boundary, then the default time can be defined as

$$\tau_i \quad = \quad \inf\{t : A_{i,t} \le A_B\}.$$

We let R_i denote the recovery rate of firm i upon default.

2.4 CDX pricing

The CDX index is a baskets of equally-weighted individual credit default swap (CDS) contracts for a set of large investment-grade firms.[2] Taking a \$1 protection sell position on the CDX index can be viewed as taking protection-sell positions with notional amounts \$$1/N_f$ on all N_f individual CDS contracts in the index.

We first develop formulas for the payoffs and pricing of the protection-sell and protection-buy positions on the index. Assume a \$1 notional amount. If firm i defaults, then the protection seller pays the protection buyer $\frac{1}{N_f}(1 - R_i)$. The cumulative loss process on the

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[2]Each CDS contract involves two parties: protection buyer and protection seller. The protection seller provides the protection buyer an insurance against a credit event of the reference entity specified in a CDS contract. Thus, the protection buyer pays a series of insurance premiums to the protection seller. In return, if the reference entity experiences a credit event, the protection seller compensates the loss of the protection buyer, either by purchasing the reference obligation from the protection buyer at its par value (physical settlement) or by directly paying the loss amount to the protection buyer (cash settlement).

index, L_t, can therefore be expressed as

$$L_t = \frac{1}{N_f} \sum_{i=1}^{N_f} 1_{\{\tau_i \leq t\}}(1 - R_i).$$

(9)

The value of the cash flows paid by the protection seller is given by

$$\mathbf{Prot}_{\mathrm{CDX}} = \mathbb{E}^Q \left[\int_0^T e^{-\int_0^t r_s ds} dL_t \right].$$

where $\mathbb{E}^Q$ denotes the expectation taken under the risk-neutral measure Q and r_t is the riskfree rate.[3]

In a CDX contract, the protection buyer makes quarterly premium payments that, over the course of a year, add up to a pre-determined spread S. It is important to note that if a firm defaults, the size of premium payments goes down because the outstanding notional amount of the CDX contract is reduced by $\frac{1}{N_f}$. Let n_t denote the fraction of firms that have defaulted as of time t:

$$n_t = \frac{1}{N_f} \sum_{i=1}^{N_f} 1_{\{\tau_i \leq t\}}.$$

(10)

For a given spread S, the expected present value of cash flows paid by the protection buyer is given by

$$\mathbf{Prem}_{\mathrm{CDX}}(S) = S\mathbb{E}^Q \left[\sum_{m=1}^M e^{-\int_0^{t_m} r_s ds}(1 - n_{t_m})\Delta_m + \int_{t_{m-1}}^{t_m} e^{-\int_0^u r_s ds}(u - t_{m-1})dn_s \right],$$

(11)

where $0 = t_0 < t_1 < \cdots < t_M$ are premium payment dates, S is the premium rate (premium payment per unit notional), and $\Delta_m = t_m - t_{m-1}$ is the m-th premium payment interval.[4] Note that if a default occurs between two premium payment dates, the protection buyer

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[3]The risk-neutral measure is implied by the model in Section 2.1. Specifically, the relationship between $\mathbb{E}$ (under the physical measure) and $\mathbb{E}^Q$ (under the risk-neutral measure) is described in Appendix B.

[4]If T is the maturity of the CDS contract (in years), the total number of premium payments M is equal to $4T$ because payments are quarterly.

must pay the accrued premium from the last premium payment date to the default date. This is taken into consideration as the second term in the expectation of equation (11).

To simplify notation, we have assumed above that the contracts originate at time 0. Because the model is stationary, this assumption is without loss of generality. The CDS spread S_{CDX} is defined as the value of the premium rate that equates the protection and premium legs (i.e. $\textbf{Prem}_{\text{CDX}}(S_{\text{CDX}}) = \textbf{Prot}_{\text{CDX}}$). That is, S_{CDX} is determined by

$$S_{\text{CDX}} = \textbf{Prot}_{\text{CDX}}/\textbf{Prem}_{\text{CDX}}(1).$$

2.5   CDX tranche pricing

Each tranche is defined by its "attachment point" and "detachment point." The attachment point refers to the level of subordination of the tranche, and the detachment point is the level at which the tranche loses its entire notional amount. For example, consider the tranche with a 10% attachment point and 15% detachment point. The protection seller of this tranche does not experience loss until the entire pool (i.e. the CDX index) accumulates more than 10% loss. After the 10% level, every loss the pool experiences "attaches" to the tranche. That is, the protection seller of the tranche should compensate the loss after 10%. If the loss of the pool reaches the detachment point of 15%, the protection seller loses the entire notional amount and no longer needs to take the loss of the pool. (i.e. further loss "detaches" from the tranche.)

The detachment of a tranche is the attachment of the next (higher) tranche. Suppose that there are J tranches based on the CDX index. We denote K_{j-1} as the attachment point and K_j as the detachment point of the j-th tranche. The attachment point of the first tranche (K_0) is equal to 0% and the detachment point of the last tranche (K_J) is equal to 100%.

To properly understand the mechanism of tranche products, it is important to understand

that a default in the pool not only affects the most junior tranche but also the most senior tranche. To illustrate this, consider a simple case where (1) the entire pool consists of 100 CDS contracts, each of which has $1 notional amount, and (2) there are only two tranches: 0-50% and 50-100% tranches based on the pool. How does an event of one firm's default in the pool affect the two tranches, assuming that the defaulted firm's recovery rate is 40%? This event affects both the 0-50% and 50-100% tranches. Since the recovery rate is 40% and each CDS contract is with $1 notional, the loss amount of the pool is $0.6. Since the 0-50% tranche is the most junior tranche, the protection seller of this tranche takes this loss. Furthermore, the difference between $1 (notional amount of the defaulted CDS contract) and $0.6 (loss amount from the defaulted CDS contract) reduces the notional amount of the 50-100% tranche exactly by $0.4.[5]

The reason why the notional amount of the most senior tranche decreases is straight-forward if we consider a case for funded CDO products: if a default occurs in the pool, the loss amount goes to the most junior tranche holder, and the recovery amount from the default goes to the most senior tranche holder as an early redemption. That is, the notional amount of the senior tranche is reduced by the redemption amount. Since CDX tranches are unfunded CDO products, which do not require initial investments, there is no actual early redemption in cash as the case for funded CDO products, but the notional amount of the most senior tranche is still reduced by the recovery amount.

In sum, a default in the pool affects both the most junior tranche (by incurring loss) and the most senior tranche (by reducing its notional amount). Keeping this in mind, we derive the expressions for the loss and recovery amount of the j-th tranche as a fraction of

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[5]This is not a loss to the protection seller for this tranche. The notional amount simply reduces to $49.6. This means that the maximum loss the protection seller can experience is $49.6. Thus, the protection seller receives insurance premium based on the reduced notional amount $49.6.

the notional amount of the tranche $(K_j - K_{j-1})$:

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\begin{aligned}
T_{j,t}^L &= \frac{\min\{L_t, K_j\} - \min\{L_t, K_{j-1}\}}{K_j - K_{j-1}} \\
T_{j,t}^R &= \frac{\min\{n_t - L_t, 1 - K_{j-1}\} - \min\{n_t - L_t, 1 - K_j\}}{K_j - K_{j-1}}.
\end{aligned}
\tag{12}
$$

Let $\mathbf{Prot}_{\text{Tran},j}$ be the protection leg of the j-th CDX tranche. The expression for $\mathbf{Prot}_{\text{Tran},j}$ is the same as the one for $\mathbf{Prot}_{\text{CDX}}$ except that the integral is with respect to the tranche loss $(T_{j,t}^L)$ rather than the entire pool loss (L_t):

$$
\mathbf{Prot}_{\text{Tran},j} = \mathbb{E}^Q \left[\int_0^T e^{-\int_0^t r_s ds} dT_{j,t}^L \right].
$$

Let $\mathbf{Prem}_{\text{Tran},j}(U, S)$ be the premium leg of the j-th CDX tranche when the protection buyer pays quantity U up front and quarterly premiums with premium rate S. Since insurance premiums accrue based on the outstanding notional amount (which can be decreased from 100% due to a tranche loss $(T_{j,t}^L)$ or a tranche recovery $(T_{j,t}^R)$), we can show that

$$
\mathbf{Prem}_{\text{Tran},j}(U, S) = U + S\mathbb{E}^Q \left[\sum_{m=1}^{Tf} \left\{ e^{-\int_0^t r_s ds} \int_{t_{m-1}}^{t_m} \left(1 - T_{j,t}^L - T_{j,t}^R\right) ds \right\} \right].
$$

In our sample period, the entire CDX pool has been partitioned into 6 tranches, which are traded as separate products: 0-3%, 3-7%, 7-10%, 10-15%, 15-30% and 30-100% tranches. The first tranche is called the equity tranche and the second tranche is called the mezzanine tranche. The four remaining tranches are called senior tranches, and, in particular, the last (most senior) tranche is called the super senior tranche.

Except for the equity tranche, tranches are traded with zero upfront payment. Thus, for each of those tranches, the CDX tranche spread $(S_{\text{Tran},j})$ is determined so that $\mathbf{Prot}_{\text{Tran},j} = \mathbf{Prem}_{\text{Tran},j}(0, S_{\text{Tran},j})$, which implies

$$
S_{\text{Tran},j} = \frac{\mathbf{Prot}_{\text{Tran},j}}{\mathbf{Prem}_{\text{Tran},j}(0, 1)} \quad \text{if } j = 2, \cdots, 6.
$$

The equity tranche is traded with the upfront payment, fixing the premium rate at 500 bps. Thus, the upfront payment for the equity tranche $(U_{\mathrm{Tran},j})$ is determined so that $\mathbf{Prot}_{\mathrm{Tran},j} = \mathbf{Prem}_{\mathrm{Tran},j}(U_{\mathrm{Tran},j}, 0.05)$, which is equivalent to

$$U_{\mathrm{Tran},j} = \mathbf{Prot}_{\mathrm{Tran},j} - \mathbf{Prem}_{\mathrm{Tran},j}(0, 0.05) \quad \text{if } j = 1.$$

2.6   Calculation of pricing formulas

As described in Section 2.4 and Section 2.5, in order to calculate the model-implied CDX and CDX tranche pricing, we need to compute the four legs ($\mathbf{Prot}_{\mathrm{CDX}}$, $\mathbf{Prem}_{\mathrm{CDX}}$, $\mathbf{Prot}_{\mathrm{Tran},j}$, $\mathbf{Prem}_{\mathrm{Tran},j}$) based on our equilibrium model. We calculate these legs as functions of the state variables λ and ξ. This implies that the model-implied CDX spread and CDX tranche spread/upfront payment are also calculated as functions of these two state variables.

Note that each of the four legs is expressed in terms of an expectation of an integral. To make the computation tractable, we discretize these integrals by assuming that defaults occur on average in the middle of premium payment dates (see, e.g., Mortensen (2006)). This discretization is fairly accurate because premium payment interval $\Delta_m = t_m - t_{m-1}$ is relatively small ($\Delta_m = 0.25$) since premium payments are quarterly. We provide further details in Appendix B.

Once discretized, the four legs can be expressed in terms of the four expectations under Q as shown in Appendix B.[6] In order to avoid deriving the risk-neutral dynamics of the model, we apply the Radon-Nikodym derivative of the physical measure with respect to the risk-neutral measure, and convert those four Q-expectations into the following four equivalent

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[6]Here we also need the expression for the default-free zero-coupon bond price. We derive this expression in Appendix D.

P-expectations:

$$
\begin{aligned}
\text{EDR}(u, t, X_0) &= \mathbb{E}\left[e^{u \cdot r_t} \frac{\pi_t}{\pi_0} n_t \right] \\
\text{ELR}(u, t, X_0) &= \mathbb{E}\left[e^{u \cdot r_t} \frac{\pi_t}{\pi_0} L_t \right] \\
\text{ETLR}_j(u, t, X_0) &= \mathbb{E}\left[e^{u \cdot r_t} \frac{\pi_t}{\pi_0} T_{j,t}^L \right] \\
\text{ETRR}_j(u, t, X_0) &= \mathbb{E}\left[e^{u \cdot r_t} \frac{\pi_t}{\pi_0} T_{j,t}^R \right].
\end{aligned}
$$

where X_0 denotes the state vector at time 0. (i.e. $X_0 = [\lambda_0, \xi_0]^\top$) While nearly all pricing formulas for credit derivatives under a reduced-form setup assume that interest rates are uncorrelated with defaults, we cannot make this simplifying assumption. This is because the systematic risk under our equilibrium model simultaneously affects both interest rates and the likelihood of defaults.

Unfortunately, it is impossible to calculate these four expectations in closed-form because there are multiple firms in the CDX pool. Thus, we calculate them based on Monte Carlo simulation. This approach is especially relevant because we are interested in multiple maturities of the CDX index and their multiple tranches. Using this simulation approach, we can price all these products together in one set of simulations, which mitigates computational burden. For each firm i, it follows that

$$
\begin{aligned}
\frac{A_{i,t+\Delta t}}{A_{i,t}} &= \frac{D_{i,t+\Delta t}}{D_{i,t}} \frac{G_i(\lambda_{t+\Delta t}, \xi_{t+\Delta t})}{G_i(\lambda_t, \xi_t)} \\
&= \exp\left[\log\left(\frac{D_{i,t+\Delta t}}{D_{i,t}} \right) \right] \frac{G_i(\lambda_{t+\Delta t}, \xi_{t+\Delta t})}{G_i(\lambda_t, \xi_t)}. \quad (13)
\end{aligned}
$$

Based on the values of $A_{i,t}$'s for all $i = 1, \cdots, N$, equations (9), (10), and (12) enable us to compute n_t, L_t, $T_{j,t}^L$, and $T_{j,t}^R$. Also note that equation (4) implies that

$$
\begin{aligned}
\frac{\pi_{t+\Delta t}}{\pi_t} \simeq \exp\Big[& \eta \Delta t - \beta b_\lambda \lambda_{t+\Delta t} \Delta t - \beta b_\xi \xi_{t+\Delta t} \Delta t \\
& - \gamma \log\left(\frac{C_{t+\Delta t}}{C_t} \right) + b_\lambda(\lambda_{t+\Delta t} - \lambda_t) + b_\xi(\xi_{t+\Delta t} - \xi_t) \Big]. \quad (14)
\end{aligned}
$$

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Equations (13) and (14) suggest that we can obtain the paths of r_t, π_t, n_t, L_t, $T_{j,t}^L$, and $T_{j,t}^R$ in the above four expectations by simulating the two state variables (λ_t, ξ_t), log consumption growth ($\log(C_{t+\Delta t}/C_t)$), and each firm's log payout growth ($\log(D_{i,t+\Delta t}/D_{i,t})$, $i = 1, \cdots, N_f$).[7] In Appendix C, we describe how we simulate these fundamental variables in detail.

We simulate the 5-year monthly paths of the above basic variables (i.e. $N_f + 3$ variables) 100,000 times. Each of the four expectations is calculated by taking the average of the values (for the expression within the expectation) over different simulation paths.

3   Data

Our analyses require the use of pricing data from options and CDX markets. First, our sample of options consists of implied volatilities on S&P 500 European put options. We collect our sample from OptionMetrics, which provides the time series and the cross section of implied volatilities on individual stocks and equity indices from January 1996 to December 2012. To construct monthly time series of implied volatility smiles, we pick observations using the data from the Wednesday of every option expiration week. Following Seo and Wachter (2015), we apply standard filters to extract contracts with meaningful trade volumes and prices. The implied volatility smile for each date can be obtained by regressing implied volatilities on a polynomial in strike price and maturity.

CDX is a family of credit default swap (CDS) indices, mainly covering firms and entities in North America. Among many different indices, we focus on the CDX North America Investment Grade (CDX NA IG) index, which is the most actively traded. From now on, we refer to the CDX NA IG index as "the CDX index." This index consists of 125 equally weighted CDX contracts on representative North American investment grade firms.

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[7]According to equation (5), the risk-free rate r_t is calculated as a function of λ_t.

Unlike equity indices, CDS indices have expiration dates because they are collections of CDS contracts with certain maturities. Three-, five-, seven-, and ten-year indices are typically traded. Since each index has an expiration date, the time-to-maturity decays. For this very reason, the CDX index rolls every 6 months in March and September. That is, every March and September, a new series of the index is introduced to the market and the previous series becomes off-the-run.

Our data set contains daily market pricing information on the five-year CDX index and its tranches from Markit. Although the first series of the index (CDX1) was traded from September 2003, Markit provides data from series 5. Among the six tranches (0-3%, 3-7%, 7-10%, 10-15%, 15-30% and 30-100%), Markit consistently provides data only on the bottom five tranches. We extract the monthly time series of on-the-run series using the same dates as our option sample. We only use on-the-run series data because the latest series are the most liquid.

In this paper, our period of interest is from October 2005 to September 2008, which corresponds to the CDX index series 5 to 10. We divide our sample period into two sub-periods: pre-crisis and post-crisis. The pre-crisis sample is from October 2005 to September 2007 (CDX5 to CDX8). The post-crisis sample (or the sample during the crisis) is from October 2007 to September 2008 (CDX9 to CDX10).[8] Note that our sample period ends before the Standard North American Contract (SNAC) was introduced.[9] Therefore we do

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[8]Our sample stops before the Lehman failure. This is because the liquidity of CDX tranches significantly shrank after CDX10, which was the last series introduced before the Lehman crisis. From series 11, these products were traded too infrequently for prices to be meaningful.

[9]Before the SNAC was introduced in April 2009, the index was quoted in terms of spread. Thus until 2009, the index data only include the time series of spreads on the index. However, after the SNAC instituted new trading conventions with fixed coupons of 100 or 500 bps, the index has been quoted in terms of upfront payment since then. These conventions also apply to tranche products - CDX tranches used to be quoted in terms of spread except for the equity tranche, but since the SNAC was introduced, all tranches are quoted

not need to consider changes in the trading convention in the market. In our entire sample, the CDX index and all tranches except for the equity tranche are quoted in terms of spreads. On the other hand, the equity tranche is quoted in terms of upfront payment with 500 bps fixed coupons.

4 Evaluating the model

A standard approach to comparing an endowment economy model with the data is to simulate population moments and compare them with data moments. In a model with rare disasters, this may not be the right approach if one is looking at a historical period that does not contain a disaster. An alternative approach is to simulate many samples from the stationary distribution implied by the model, and see if the data moments fall between the 5th and 95th percentile values simulated from the model. For this study, this approach is not ideal for two reasons. First, the short length of CDX/CDX tranche time series will likely mean that the error bars implied by the model will be very wide. Thus this test will have low power to reject the model. Second, unlike stock prices which are available in semi-closed-form, and options, which are available up to a (hard-to-compute, but nonetheless one-dimensional) integral, CDX prices must be simulated for every draw from the state variables. Thus the simulation approach is computationally infeasible.

For these reasons, we adopt a different approach. We first extract the time series of our two state variables from options data and then generate predictions for the CDX index and tranches based on this series of state variables. We are thus setting up a more stringent test than endowment economy models are usually subject to. Namely, we are asking that the model match not only moments, but the actual time series of variables of interest.

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in terms of upfront payment with fixed coupons.

4.1 Option-implied state variables

We extract the state variables from data on traded index options. Because our model is the same as in Seo and Wachter (2015), which is designed to fit options prices, we can expect that these state variables have validity given the time series of options data. We choose parameter values to be the same as those in Seo and Wachter (2015). Specifically, the presence of rare disasters allows the model to explain an equity premium with a risk aversion of 3. For tractability, we assume the EIS is equal to 1: this yields prices for equities that are available up to ordinary differential equations, and allows us to use the transform analysis of Duffie, Pan, and Singleton (2000) to obtain analytical expressions for option prices, up to a Taylor approximation. See Seo and Wachter (2015) for more details. Other parameters are described in Table 1; specifically, we note the choice of ϕ to equal 2.6 because this will be relevant to other parameter choices below.

To find the state variables, we use the time series of one-month at-the-money (ATM) and out-of-the-money (OTM) options. The OTM options have moneyness of 0.85. Figure 1 shows the time series of the extracted state variables. Our options dataset begins in January 1996 and ends in December 2012. Panel A shows the implied state variables for the full period. For most of this sample, the disaster probability λ_t and its mean ξ_t lies below 5% per annum. We see a spike in late 1998 corresponding to the rescue of Long Term Capital Management, following the Asian financial crisis and the moratorium on payments on Russian debt. There are also spikes corresponding to market declines following the NASDAQ boom in the early 2000s. These spikes come about because of increases in put option prices during these events. The sample, however, is dominated by the financial crisis of late 2008-2009. At the Lehman default, the disaster probability spiked as high as 20%. The disaster probability remains high and volatile, as compared to the prior period, until the end of the sample.

Panel B zooms in on the sample period for which we have CDX tranche data. This period

captures the first hints of a crisis in subprime mortgages in 2007, with slight increases in the level and volatility of λ and ξ. These variables become markedly higher and more volatile in 2007 and early 2008, culminating in the near-default of Bear Sterns. The CDX tranche sample ends right before the Lehman default; the last values of λ are higher than before, but nowhere near the stratospheric levels less than one month later.

4.2   Calibration of payout process

To go from state variables to CDX/CDX tranche prices requires not only dynamics for the market, but dynamics for individual firms and assumptions about what constitutes a default. For parsimony, and following standard practice in this literature, we assume firms are homogeneous. However, firms face distinct idiosyncratic shocks and potentially distinct sector-wide shocks.

To calibrate individual firm dynamics, we use results from Collin-Dufresne, Goldstein, and Yang (2012). The default boundary A_B is set to be 19.2%. This implies that if the asset value falls below 0.192 multiplied by what it was at the initiation of the contract, the firm is assumed to be in default. Collin-Dufresne et al. derive this value by calculating the average leverage ratio from firm-level data for the firms in the sample (32%). Firms are then considered to be in default if their value is 60% of their debt outstanding. We assume that the recovery rate is 40% in normal times and 20% in the event of rare disasters.[10] We choose the idiosyncratic jump size $e^{Z_{i,t}} - 1$ to equal -80%, a value sufficiently large to make default almost certain.[11]

The CDX index consists of investment-grade firms that are relatively large and stable.

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[10]Collin-Dufresne, Goldstein, and Yang (2012) do not have consumption disasters in their reduced-form model, but they do have catastrophic jumps to the firm values. In the event of one of these catastrophic jumps, the recovery rate is 20% in their model.

[11]Collin-Dufresne, Goldstein, and Yang (2012) also use a similar value ($e^{-2} - 1 = 86.47\%$).

Collin-Dufresne, Goldstein, and Yang (2012) estimate a different asset beta for each CDX series; the asset betas are between 0.5 and 0.6 for pre-crisis series and between 0.6 and 0.7 for post-crisis series. This reflects a slight increase in leverage for the firms included in the series. In our model, the asset beta will be mostly determined by the ratio ϕ_i/ϕ; however, the connection between sector-wide and aggregate risk adds an additional degree of covariance. We therefore choose $\phi_i/\phi = 0.5$ pre-crisis and $\phi_i/\phi = 0.6$ post-crisis. Because we assume a ϕ of 2.6 in Seo and Wachter (2015), this corresponds to a pre-crisis value of ϕ_i of 1.3 and a post-crisis value of 1.6.

We use the results of Longstaff and Rajan (2008) to calibrate the parameters for the sector-wide shocks. For simplicity, in the calibration we assume that the loss in the event of a sector-wide shock is sufficiently large that all firms that experience this loss go into default. Longstaff and Rajan estimate an approximately 5% loss rate on the portfolio in the event of a sector shock. Because we assume a recovery rate of 40%, the fraction of firms defaulting in the event of a sector shock is equal to $0.05/(1 - .4) = 8\%$. There are 125 total firms, so a sector shock corresponds to a default of about 10 firms. Again, for simplicity, we assume that there are 25 firms in each sector (there are 125 firms, with 5 sectors); this implies a probability of $10/25$, or 0.4, of firm default given a sector shock.[12]

We directly estimate the remaining parameters using moments of junior tranches in the data. To make this estimation computationally feasible, we impose some simplifying assumptions. Similar to the idiosyncratic jump size, we assume sector-wide jump size $\left(e^{Z_{S_i,t}} - 1\right)$ to be constant. While Collin-Dufresne, Goldstein, and Yang (2012) calibrate the idiosyncratic intensity for each series to match the term structure of CDX spreads (by assuming that the intensity for each series is a distinctive deterministic step function), we assume that this

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[12]An equivalent approach is to consider a larger number of sectors, and have firm default be certain given a sector shock. This is the approach of Longstaff and Rajan (2008). The implications for the payout process are identical.

intensity (λ_i) is truly a constant throughout the entire sample period. These simplifying assumptions leave us only three parameters to be estimated ($Z_{S_i,t}$, w_ξ, and λ_i).

Specifically, to estimate the three parameters, we use the following four moments: (1) the average of pre-crisis equity tranche upfront payments, (2) the average of post-crisis equity tranche upfront payments, (3) the average of pre-crisis mezzanine tranche spreads, and (4) the average of post-crisis mezzanine tranche spreads.[13] That is, the parameters are estimated as the values that minimize the sum of squared relative errors of the above four moments. These parameter values are reported in Table 2. Note that no data on senior tranches, nor on the index as a whole, is used to fit these parameters.

5   Results

5.1   Time series of CDX tranches

Figure 2 displays the time series of option-implied volatility surfaces in the data and in the model. Note that we calculate the time series of the two state variables to match the one-month ATM and 0.85 OTM option-implied volatilities. Therefore, the model results exactly coincide with the data counterparts for these options (first row) by construction. Furthermore, the model also generates a good fit to the data for both the three-month (second row) and six-month (third row) maturities. In other words, our model, together with the option-implied state variables, can capture the level as well as the time variation in the time series of option prices.

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[13]Note that CDX/CDX tranche pricing requires a large number of simulations. To reduce computation time during the estimation procedure, for each date (i.e. for each pair of state variables), we construct a grid of these three variables and compute CDX/CDX tranche prices on each grid. Then, we interpolate these grid points using a 3-dimensional cubic spline. We confirm that this interpolation is very accurate because each tranche spread is a monotonic function of each parameter.

Our model's ability to generate a good fit to the data is not restricted to the options market: according to Seo and Wachter (2015), the model with the option-implied state variables can match the level and the time variation of the price-dividend ratio during our sample period. This suggests that the equity market and the options market have a common source of risk, which is well-captured by changes in the probability of economic disasters.

The focus of this paper is whether the common source of risk that jointly explains the equity and options markets can also explain CDX and CDX tranche markets, which can be informative about the rare disaster mechanism. Figure 3 shows the monthly time series of five-year maturity CDX and CDX tranche spreads in the data and in the model. The blue solid line represents the data and the red dotted line represents the benchmark model. On the same figure, we also show results for the case without idiosyncratic risk, discussed further below. The top left panel presents the CDX index, and the other five panels show the five tranches of the CDX index. Here we drop the super senior tranche (30-100%) since Markit does not consistently provide the data for this tranche, especially during the pre-crisis period.[14] Note that as explained in Section 3, when the equity tranche is traded, the protection buyer makes an upfront payment to the protection seller while fixing annual premium payments to be 500 bps onwards. Thus, for the equity tranche (top right panel), we report the amount of this upfront payment.

In general, we find that the model accounts for the data surprisingly well as evident in Figure 3. To facilitate the comparison between the model and the data, we calculate the time series average of the spreads on the CDX and CDX tranches in the full sample (Panel A), pre-crisis sample (Panel B), and post-crisis sample (Panel C) of Table 3. As shown in this table, the model produces the average spreads of the CDX and CDX tranches that are

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[14]In Section 5.3, we examine our model's ability to explain the super senior tranche by comparing the model results with the data reported in Collin-Dufresne, Goldstein, and Yang (2012). See Section 5.3 for more details.

very close to their data counterparts.

In case of the CDX index, we can see that the spreads in the model and the data hover around almost the same level during the pre-crisis period (average 41 vs 42 bps). Although the model slightly undershoot the data during the post-crisis period (average 102 vs 116 bps), Figure 3 illustrates that the model and data time series fluctuate in an almost identical pattern.

We now discuss the fit to each tranche. The model has the most difficulty with the equity tranche (0-3%). As reported in Table 3, the full-sample average upfront payment is 31%, somewhat below the data value of 39%. This spread exhibits some variation in the early part of the sample, perhaps related to a credit crisis triggered by Ford and General Motors' downgrades, as well as some variation during the recent crisis, that the model is unable to explain. Nonetheless, the approximate magnitude of these spreads is well-matched, despite the fact that we have constrained idiosyncratic risk to take on the same value throughout the sample period.

The intermediate tranches including the mezzanine (3-7%) and the first senior (7-10%) tranches are also well explained by the model. Although the model values are slightly larger during the pre-crisis period and slightly smaller during the post-crisis period, the model time series for these two tranches resemble the data quite well, producing full-sample average spreads that are very close to the data values (235 vs 238 bps for the mezzanine tranche and 110 vs 102 bps for the first senior tranche). Since the attachment points of these tranches are not too high nor too low, they are most sensitive to the shocks that cause an intermediate number of defaults in the CDX index. In the model, such shocks are generated using sector-wide risk (further explained in Section 5.2). The good fit to the intermediate tranches reveals that our model's sector-wide risk component is able to capture such shocks.

The last two tranches (i.e. the 10-15% and 15-30% tranches) are the most senior tranches

reported in this figure. Before the recent financial crisis, the spreads on these tranches maintained a very low level even while the equity tranche spreads increased, suggesting these tranches are only sensitive to large systematic risk in the economy. Specifically, the pre-crisis average spread is 12 bps for the 10-15% tranche and 6 bps for the 15-30% tranche. Although the model slightly overshoot the data for the 10-15% tranche, the low level of spreads on these tranches are well captured by the model (23 bps for the 10-15% tranche, and 8 bps for the 15-30% tranche). After the recent financial crisis, the level of these tranche spreads substantially rose, averaging 136 bps for the 10-15% tranche and 69 bps for the 15-30% tranche. This phenomenon is also fully explained by the model (model values are 155 bps for the 10-15% tranche and 71 bps for the 15-30% tranche).

Out of the two most senior tranches, the 15-30% tranche is of particular interest because its attachment point is so large that the spread on this tranche is hardly affected by idiosyncratic or sector-wide risk. Instead, this tranche spread is almost entirely driven by the risk of disasters in the economy (see Section 5.2). Thus the 15-30% tranche is extremely informative about the risk of disasters. The fact that our model can explain the time series of the spreads on this tranche shows that the level and time variation of disaster risk in our model are properly specified.

5.2   How the model works - comparative statics

To better understand how our model can match the time series of the CDX and CDX tranches, we perform some comparative static analyses. In this section, we take the results of our model in 5.1 as a benchmark, and compare them with the following three special cases: (1) no idiosyncratic risk, (2) no sector-wide risk, and (3) higher disaster risk.

5.2.1 No idiosyncratic risk

The dashed line in Figure 3 shows the results from a special case of the model where firms are not subject to idiosyncratic shocks (black dashed lines). This limit case is obtained by setting either λ_i or $Z_{i,t}$ to zero.

First, we can see that the spreads of the CDX index significantly drop over the sample, reflecting lower default risks of the firms in the index (top left panel). Specifically, note that the level of CDS spreads becomes very low during the pre-crisis period when we remove idiosyncratic risk. This implies that in our model, the major portion of CDX spreads in the pre-crisis sample is due to purely idiosyncratic default risk. This is consistent with the empirical finding of Longstaff and Rajan (2008) that 65% of the CDX spread level between 2003 and 2005 is attributable to firm-specific risk.

Individual tranches respond differently to the elimination of idiosyncratic risk. As shown in Figure 3, the equity tranche is most affected by this change (top right panel): upfront payments for the equity tranche decrease so much that most of the time they stay below zero. Note that the negative amount of upfront payments means that the protection seller (not the protection buyer) pays his or her counterparty at the beginning of the contract. This happens when the equity tranche's fixed annual coupons of 500 bps are too large compared to the risk of the equity tranche that the protection seller takes.[15] In other words, removing idiosyncratic risk makes the equity tranche substantially less risky so its fair premium becomes even smaller than 500 bps.

The reason why idiosyncratic risk affects the equity tranche the most is because the equity tranche is the most junior tranche. Since its attachment point is 0%, the loss from every

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[15]In most cases, the upfront payment is positive so that the protection buyer pays the protection seller at the inception of the contract because the fixed annual coupons of 500 bps are too small compared to the risk of the equity tranche.

default in the index falls on the equity tranche until this tranche loses its entire notional amount (i.e. the loss in the index reaches 3%). Therefore, even a small number of defaults end up critically impacting the payoff of the equity tranche. For example, suppose that one of the 125 firms goes into default with a 40% recovery rate. In this case, the index experiences a loss of 1/125×(1-0.4) = 0.48%. Since this loss falls entirely on the equity tranche, the equity tranche holder (i.e. protection seller) loses 0.48/(3-0) = 16% of the tranche's notional amount. Thus, the price of the equity tranche is particularly sensitive to firm-specific shocks that generate the risk of a few defaults.

As we see in Figure 3, the effect of removing idiosyncratic risk diminishes as seniority increases. More senior tranches have higher attachment points, which makes it difficult for them to be affected by the risk of a few defaults. In fact, the spreads of the most senior tranche (15-30%) are barely affected, as can be seen on the bottom right panel. The black dashed line for the model with no idiosyncratic risk almost perfectly coincides with the red dotted line representing the benchmark model. This is not surprising: by construction, each firm's purely idiosyncratic shocks are independent of others. Unless the idiosyncratic risk is counterfactually large, it is extremely unlikely to result in defaults that are large enough in number to penetrate the 15% attachment point.

5.2.2 No sector-wide risk

Figure 4 displays a limit case of the model where firms are not subject to sector-wide shocks (black dashed lines). This limit case is achieved by setting one of the three parameters (p_i, $Z_{S_i,t}$, and w_ξ) that control sector-wide risk to zero.

Similar to the case without idiosyncratic risk, the elimination of sector-wide risk significantly reduces the spreads on the CDX index and the upfront payments for the equity tranche. Most notably, we can see that intermediate tranches including the mezzanine

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tranche (3-7%) and the first two senior tranches (7-10% and 10-15%) are most affected by the change in sector-wide risk. Especially during the post-crisis period, the model spreads become substantially low compared to the data if we remove sector-wide shocks. This suggests that we cannot capture the drastic increases in these intermediate tranches during the crisis without the sector-wide component. (In contrast, the most senior tranche is well explained even without this component.)

This is because intermediate tranches are susceptible to clustered defaults caused by the arrival of sector-wide shocks. Note that one firm's default due to its own idiosyncratic shock cannot incur any losses on intermediate tranches. In contrast, if a sector-wide shock is realized, on average, 10 firms are at the risk of default under our calibration, which is large enough to invade the attachment point of the mezzanine tranche. The larger gaps between the benchmark model and the model without sector-wide shocks during the post-crisis period confirm that sector-wide risk is indeed positively correlated with aggregate risk.

5.2.3 Higher disaster risk

The discussions in Section 5.2.1 and 5.2.2 make clear that the spreads on the most senior tranche (15-30%) are invariant to both idiosyncratic shocks and sector-wide shocks. Clustered defaults caused by either idiosyncratic shocks or sector-wide shocks are not pervasive enough to create a significant impact on the spreads of this tranche. This suggests that the level and time variation of the spreads on the most senior tranche are primarily driven by the risk of disaster. This is because a realization of a rare disaster is able to generate a large number of defaults that can potentially hurt even the most senior tranche.

To see how the CDX and CDX tranches respond to disaster risk, we uniformly increase rare disaster intensity λ_t by its mean level (i.e. 2%) over the entire sample. In Figure 5, the black dashed lines represent this scenario. Since higher disaster risk implies higher default

risk in general, the spreads on the CDX index rise as shown in the top left panel.

The higher level of disaster risk has relatively little effects on the equity tranche, which is quite intuitive. Given that a small number of defaults can wipe out the entire notional amount of the equity tranche, an extra increase in disaster risk does not really matter too much to the equity tranche. Similarly, the effects of increasing disaster risk has little impact on relatively junior tranches. On the other hand, relatively senior tranches are sensitive to the change. We can see that the spreads on the most senior tranche substantially rise, reflecting a higher chance of having losses on this tranche from a development of rare disasters.

5.3   Additional implications

In Section 5.1, we show that our model, together with the option-implied state variables, well accounts for the spreads on five-year CDX/CDX tranches (except for the super senior tranche). In this section, we investigate additional implications of the model for CDX/CDX tranche markets using the data reported in Collin-Dufresne, Goldstein, and Yang (2012). Specifically, the authors report the following data that we could not obtain from Markit: (1) average three-year CDX/CDX tranche spreads and (2) three-year and five-year super senior tranche spreads. Note that they collect the data from JP Morgan, not from Markit.

There are a few things that are worth mentioning with regard to the data from Collin-Dufresne, Goldstein, and Yang (2012). First, their pre-crisis sample is one year longer than ours. That is, their pre-crisis sample starts from CDX series 3 (until series 8) while ours starts from CDX series 5 because the data on CDX series 3 and 4 are not consistently provided in our Markit dataset. In this sense, the pre-crisis average values in Collin-Dufresne et al. do not correspond exactly to ours, so the results should be interpreted with caution. We still think that it is meaningful to make these comparisons because the variation in spreads is relatively little before the crisis. Second, their post-crisis sample exactly matches up with

ours (CDX9 to 10). However, we still need to compare the values with caution because the dates we pick to construct monthly time series are different.[16] This might generate some discrepancies because spreads are fairly volatile during the post-crisis period.

To grasp how the data in Collin-Dufresne et al. are different from ours, we compare the five-year CDX tranche spreads in both datasets. As a result, we find that the two sets of average spreads on five-year tranches exhibit fairly similar levels.

5.3.1 Three-year CDX tranche spreads

Table 4 provides the average spreads of the CDX index and its tranches for the pre-crisis (Panel A) and post-crisis (Panel B) periods. For the equity tranche, we report the average upfront payments instead.

First, the model-implied pre-crisis average CDX spread is 33 bps, which is fairly close to its data counterpart (27 bps). The model-implied post-crisis average CDX spread is 82 bps, which is much higher than the pre-crisis value. Note that we cannot compare this model value with the data because Collin-Dufresne, Goldstein, and Yang (2012) do not report the average CDX spread for the post-crisis sample.

Similar to the result from the five-year maturity, we find that the model's fit to the equity tranche is not great, mainly due to our simplifying assumption that the idiosyncratic intensity is constant. Although the model generates the pre-crisis average upfront payment that is relatively close to the data (14 vs 11%), the model's post-crisis value is much smaller than the data (24 vs 43%), implying that the idiosyncratic intensity should rise during this period to better match the data. For the same reason, we can see that the model generates relatively smaller spreads for intermediate tranches (such as the mezzanine/first senior tranches) during the post-crisis sample.

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[16]To be consistent with our options sample, we pick the Wednesday of every option expiration week. On the other hand, Collin-Dufresne, Goldstein, and Yang (2012) pick the fourth Wednesday of every month.

However, the model generates an excellent fit to relatively senior tranches, which are the main objects of our interest. For example, in the model, the average spread of the most senior tranche (15-30%) is 4 bps for the pre-crisis period and 46 bps for the post-crisis period. As shown in Table 4, these values are very close to the data where the pre-crisis average is 2 bps and the post-crisis average is 48 bps. The fact that the model can match both three-year and five-year senior tranche spreads solidifies the validity of disaster risk dynamics in the model. In other words, our calibrated model implies the term structure of disaster risk that is consistent with the term structure of senior CDX tranches.

5.3.2 Super senior tranche

As emphasized throughout the paper, senior tranches are crucial to our analyses. If tranches are senior enough (i.e. with sufficiently large attachment points), they are unlikely to be affected by a small/medium number of defaults typically caused by firm-specific or industry-specific shocks. This makes the spread of those senior tranches only responsive to large aggregate shocks. Note that if aggregate shocks are relatively small, they still cannot generate sufficiently clustered defaults that can penetrate the attachment points (or subordinate levels) of those tranches. Thus, senior tranches with large attachment points are particularly useful for studying the risk of rare disasters.

So far, we treat the 15-30% tranche as the most senior tranche because the spreads on the 30-100% tranche, so-called the super senior tranche, are not consistently available in our dataset from Markit. Here, we test the model's ability to account for this extremely senior tranche using the data reported in Collin-Dufresne, Goldstein, and Yang (2012). In Table 5, we report the pre-crisis and post-crisis average super senior tranche spreads for three-year and five-year maturities in the model and in the data. As evident in the table, the model matches the data with high precision. While the spreads on this tranche are extremely small

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during the pre-crisis period (1 bps for three-year and 4 bps for five-year), they abruptly increase to a higher level after the recent financial crisis (23 bps for three-year and 35 bps for five-year). Consistent with this observation, the model generates a low level of spreads during the pre-crisis period (2 bps for three-year and 3 bps for five-year) as well as a high level of spreads during the post-crisis period (28 bps for three-year and 34 bps for five-year).

6   Conclusion

Options have been widely used to study various questions pertaining to financial markets due to their asymmetric payoff structure. In particular, out-of-the-money put options, which serve as a hedge against price declines, are natural instruments with which to study downside risk. In principle, it is possible to use options to examine the risk of negative shocks, regardless of their size, as long as the available strike prices are sufficiently low. However, in reality, options trade with liquidity only when their strike prices are fairly close to today's price. This implies that options are not directly informative about the risk of extremely negative shocks that characterize rare disasters.

Data on CDX tranches serves as a complement to the available data on options. CDX senior tranches are analogous to extremely deep out-of-the money put options on the US economy because they do not incur any loss until a substantial portion of large investment-grade firms go into default. Due to this payoff structure, the prices of these tranches are very sensitive to the probability of large negative shocks that can potentially generate highly clustered defaults in the economy. As non-redundant assets that cannot be replicated using options in the market, CDX senior tranches additionally provide critical information about aggregate risk in the economy.

In this paper, we show that the level and time variation of the CDX and CDX tranches are well explained by the model with the risk of economic disasters. Even though the

probability of a disaster is derived purely from options data, the model accurately fits the patterns of three-year and five-year CDX senior tranche spreads. The model's success can be attributed to properly specified disaster risk because, as comparative statics confirms, senior tranche spreads vary mainly on the basis of this risk and vary little on other sources of risk. Furthermore, the result that the model consistently matches prices on different asset classes including stocks, equity options, and CDX senior tranches over the entire sample highlights the importance of beliefs about rare disasters, even in periods of high valuation.

In the data, CDX senior tranche spreads sharply rise at the outbreak of the recent subprime mortgage crisis. The level of these spreads is unattainable in a setup where the possibility of extremely bad outcomes is negligible. The finding therefore supports the view that rare disasters are a main source of the high equity premium seen in the data.

Appendix

A Individual firm value dynamics

Let $H_i(D_{i,t}, \lambda_t, \xi_t, s - t)$ denote the time-t value of firm i's payoff at time s. That is,

$$H_i(D_{i,t}, \lambda_t, \xi_t, s - t) = E_t \left[\frac{\pi_s}{\pi_t} D_{i,s} \right].$$

We conjecture that $H_i(\cdot)$ has the following functional form:

$$H_i(D_{i,t}, \lambda_t, \xi_t, \tau) = D_{i,t} \exp\left(a_i(\tau) + b_{i\lambda}(\tau)\lambda_t + b_{i\xi}(\tau)\xi_t \right). \tag{A.1}$$

To verify this conjecture, we apply Ito's Lemma to the process $\pi_t H_i(D_{i,t}, \lambda_t, \xi_t, s - t)$ and derive its conditional mean (which is the sum of its drift and jump compensator). The conditional mean of this process always equals zero because the process is a martingale.[17]

Note that by applying Ito's Lemma to equation (4), we derive the following stochastic differential equation:

$$\frac{d\pi_t}{\pi_{t^-}} = \left\{ -\beta - \mu_C + \gamma\sigma_C^2 - \lambda_t E \left[e^{(1-\gamma)Z_{C,t}} - 1 \right] \right\} dt$$
$$- \gamma\sigma_C dB_{C,t} + b_\lambda \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t} + b_\xi \sigma_\xi \sqrt{\xi_t} dB_{\xi,t} + (e^{-\gamma Z_{C,t}} - 1)dN_{C,t}. \tag{A.2}$$

Similarly, by applying Ito's Lemma to equation (A.1), it follows that

$$\frac{dH_{i,t}}{H_{i,t^-}} = \left\{ \mu_i + b_{i\lambda}(\tau)\kappa_\lambda(\bar{\xi} - \lambda_t) + \frac{1}{2}b_{i\lambda}(\tau)^2\sigma_\lambda^2\lambda_t + b_{i\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{i\xi}(\tau)^2\sigma_\xi^2\xi_t - a_i'(\tau) \right.$$
$$\left. - b_{i\lambda}'(\tau)\lambda_t - b_{i\xi}'(\tau)\xi_t \right\} dt + \phi_i\sigma_C dB_{C,t} + b_{i\lambda}(\tau)\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + b_{i\xi}(\tau)\sigma_\xi\sqrt{\xi_t}dB_{\xi,t}$$
$$+ (e^{\phi_i Z_{C,t}} - 1)dN_{C,t} + I_i(e^{Z_{i,t}} - 1)dN_{i,t} + (e^{Z_{S_i,t}} - 1)dN_{S_i,t}.$$

---

[17]Equation (A.1) implies that $\pi_t H_i(D_{i,t}, \lambda_t, \xi_t, s - t) = E_t[\pi_s D_{i,s}]$. It is straightforward that $E_t[\pi_s D_{i,s}]$ is a martingale by the law of iterative expectations.

By Ito's product rule, we can combine the SDE for $H_{i,t}$ with the one for π_t to derive the SDE for $\pi_t H_{i,t}$:

$$\frac{d(\pi_t H_{i,t})}{\pi_{t-} H_{i,t-}} = \left\{ -\beta - \mu_C + \gamma \sigma_C^2 - \lambda_t E\left[e^{(1-\gamma)Z_{C,t}} - 1\right] + \mu_i + b_{i\lambda}(\tau)\kappa_\lambda(\bar{\xi} - \lambda_t) + \frac{1}{2}b_{i\lambda}(\tau)^2 \sigma_\lambda^2 \lambda_t \right.$$

$$+ b_{i\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{i\xi}(\tau)^2 \sigma_\xi^2 \xi_t - a_i'(\tau) - b_{i\lambda}'(\tau)\lambda_t - b_{i\xi}'(\tau)\xi_t - \gamma\phi_i\sigma_C^2$$

$$\left. + b_\lambda b_{i\lambda}(\tau)\sigma_\lambda^2 \lambda_t + b_\xi b_{i\xi}(\tau)\sigma_\xi^2 \xi_t \right\} dt + (\phi_i - \gamma)\sigma_C dB_{C,t}$$

$$+ (b_\lambda + b_{i\lambda}(\tau))\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + (b_\xi + b_{i\xi}(\tau))\sigma_\xi\sqrt{\xi_t}dB_{\xi,t}$$

$$+ (e^{(\phi_i - \gamma)Z_{C,t}} - 1)dN_{C,t} + I_i(e^{Z_{i,t}} - 1)dN_{i,t} + (e^{Z_{S_i,t}} - 1)dN_{S_i,t}.$$

Since $\pi_t H_t$ is a martingale, the sum of the drift and the jump compensator of $\pi_t H_t$ equals zero. This zero mean condition provides the system of ODEs for $a_i(\tau)$, $b_{i\lambda}(\tau)$, and $b_{i\xi}(\tau)$:

$$a_i'(\tau) = -\beta - \mu_C - \gamma(\phi_i - 1)\sigma_C^2 + \mu_i + b_{i\xi}(\tau)\kappa_\xi\bar{\xi} + \lambda_i E\left[e^{Z_{i,t}} - 1\right] + p_i w_0 E\left[e^{Z_{S_i,t}} - 1\right]$$

$$b_{i\lambda}'(\tau) = -b_{i\lambda}(\tau)\kappa_\lambda + \frac{1}{2}b_{i\lambda}(\tau)^2 \sigma_\lambda^2 + b_\lambda b_{i\lambda}(\tau)\sigma_\lambda^2$$

$$+ E\left[e^{(\phi_i - \gamma)Z_{C,t}} - e^{(1-\gamma)Z_{C,t}}\right] + p_i w_\lambda E\left[e^{Z_{S_i,t}} - 1\right]$$

$$b_{i\xi}'(\tau) = b_{i\lambda}(\tau)\kappa_\lambda - b_{i\xi}(\tau)\kappa_\xi + \frac{1}{2}b_{i\xi}(\tau)^2 \sigma_\xi^2 + b_\xi b_{i\xi}(\tau)\sigma_\xi^2 + p_i w_\xi E\left[e^{Z_{S_i,t}} - 1\right].$$

This shows that H_i satisfies the conjecture (A.1). Furthermore, since $H_i(D_{i,t}, \lambda_t, \xi_t, 0) = D_{i,t}$, we obtain the following boundary conditions:

$$a_i(0) = b_{i\lambda}(0) = b_{i\xi}(0) = 0.$$

With the solution for the ODEs, equation (7) can be written as

$$A_{i,t} = \int_t^\infty H_i(D_{i,t}, \lambda_t, \xi_t, s - t)ds$$

$$= D_{i,t} \int_t^\infty \exp\left(a_i(s - t) + b_{i\lambda}(s - t)\lambda_t + b_{i\xi}(s - t)\xi_t\right) ds$$

$$= D_{i,t} \int_0^\infty \exp\left(a_i(\tau) + b_{i\lambda}(\tau)\lambda_t + b_{i\xi}(\tau)\xi_t\right) d\tau.$$

This subsequently implies that that the asset-payout ratio $(A_{i,t}/D_{i,t})$ is expressed as a function of two state variables as can be seen in equation (8).

B Discretization of pricing formulas

With the discretization assumption explained in Section 2.6, it follows that

$$
\begin{aligned}
\mathbf{Prot}_{\mathrm{CDX}} &= \mathbb{E}^Q \left[\int_0^T e^{-\int_0^t r_s ds} dL_t \right] \\
&\simeq \sum_{m=1}^M \mathbb{E}^Q \left[e^{-\int_0^{t_m - \Delta_m/2} r_s ds} (L_{t_m} - L_{t_{m-1}}) \right] \\
&= \sum_{m=1}^M \mathbb{E}^Q \left[e^{-\int_0^{t_m - \Delta_m/2} r_s ds} L_{t_m} \right] - \sum_{m=1}^M \mathbb{E}^Q \left[e^{-\int_0^{t_m - \Delta_m/2} r_s ds} L_{t_{m-1}} \right].
\end{aligned}
$$

Since Δ_m is small, the risk-free rate (which is a function of a smooth variable λ_t) is not likely to change much between time $t_m - \Delta_m/2$ and t_m. Thus, it is possible to approximate

$$
\int_{t_m - \Delta_m/2}^{t_m} r_s ds \simeq \frac{1}{2} \Delta_m r_{t_m},
$$

which allows us to further simply the protection leg ($\mathbf{Prot}_{\mathrm{CDX}}$) as

$$
\mathbf{Prot}_{\mathrm{CDX}} \simeq \sum_{m=1}^M \left\{ \mathbb{E}^Q \left[e^{\frac{1}{2}\Delta_m r_{t_m}} e^{-\int_0^{t_m} r_s ds} L_{t_m} \right] - \mathbb{E}^Q \left[e^{-\frac{1}{2}\Delta_m r_{t_{m-1}}} e^{-\int_0^{t_{m-1}} r_s ds} L_{t_{m-1}} \right] \right\}.
$$

Similarly, we can discretize the premium leg ($\mathbf{Prem}_{\mathrm{CDX}}(S)$) using the same assumption:

$$
\begin{aligned}
\mathbf{Prem}_{\mathrm{CDX}}(S) &= S\mathbb{E}^Q \left[\sum_{m=1}^M e^{-\int_0^{t_m} r_s ds} (1 - n_{t_m}) \Delta_m + \int_{t_{m-1}}^{t_m} e^{-\int_0^u r_s ds} (u - t_{m-1}) dn_s \right] \\
&\simeq S \sum_{m=1}^M \Delta_m \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} (1 - n_{t_m}) + e^{-\int_0^{t_m - \Delta_m/2} r_s ds} \frac{n_{t_m} - n_{t_{m-1}}}{2} \right].
\end{aligned}
$$

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Since $\Delta_m/2$ is small, we further approximate this expression to

$$
\begin{aligned}
\mathbf{Prem}_{\text{CDX}}(S) \;\simeq\;& S \sum_{m=1}^{M} \Delta_m \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} \left(1 - \frac{1}{2} n_{t_m} - \frac{1}{2} n_{t_{m-1}} \right) \right] \\
=\;& S \sum_{m=1}^{M} \Delta_m \left\{ H_0(t_m) - \frac{1}{2} \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} n_{t_m} \right] - \frac{1}{2} \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} n_{t_{m-1}} \right] \right\} \\
\simeq\;& S \sum_{m=1}^{M} \Delta_m \left\{ H_0(t_m) - \frac{1}{2} \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} n_{t_m} \right] - \frac{1}{2} \mathbb{E}^Q \left[e^{-\Delta_m r_{t_{m-1}}} e^{-\int_0^{t_{m-1}} r_s ds} n_{t_{m-1}} \right] \right\}.
\end{aligned}
$$

where $H_0(\tau) = H_0(\lambda, \xi, \tau)$ is the price of the default-free zero-coupon bond with maturity τ, which we derive in Appendix D.

With the same discretization approach, we also obtain the discretization of the premium leg ($\mathbf{Prot}_{\text{Tran},j}$) and the premium leg ($\mathbf{Prem}_{\text{Tran},j}(U,S)$) for CDX tranche pricing. It is straightforward to show that

$$
\begin{aligned}
\mathbf{Prot}_{\text{Tran},j} \;\simeq\;& \sum_{m=1}^{M} \mathbb{E}^Q \left[e^{-\int_0^{t_m - \Delta_m/2} r_s ds} (T^L_{j,t_m} - T^L_{j,t_{m-1}}) \right] \\
\simeq\;& \sum_{m=1}^{M} \left\{ \mathbb{E}^Q \left[e^{\frac{1}{2}\Delta_m r_{t_m}} e^{-\int_0^{t_m} r_s ds} T^L_{j,t_m} \right] - \mathbb{E}^Q \left[e^{-\frac{1}{2}\Delta_m r_{t_{m-1}}} e^{-\int_0^{t_{m-1}} r_s ds} T^L_{j,t_{m-1}} \right] \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Prem}_{\text{Tran},j}(U,S) \;\simeq\;& U + S \sum_{m=1}^{M} \Delta_m \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} \frac{(1 - T^L_{j,t_m} - T^R_{j,t_m}) + (1 - T^L_{j,t_{m-1}} - T^R_{j,t_{m-1}})}{2} \right] \\
\simeq\;& U + S \sum_{m=1}^{M} \Delta_m \left\{ H_0(t_m) - \frac{1}{2} \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} T^L_{j,t_m} \right] - \frac{1}{2} \mathbb{E}^Q \left[e^{-\int_0^{t_m} r_s ds} T^R_{j,t_m} \right] \right. \\
& \left. - \frac{1}{2} \mathbb{E}^Q \left[e^{-\Delta_m r_{t_{m-1}}} e^{-\int_0^{t_{m-1}} r_s ds} T^L_{j,t_{m-1}} \right] - \frac{1}{2} \mathbb{E}^Q \left[e^{-\Delta_m r_{t_{m-1}}} e^{-\int_0^{t_{m-1}} r_s ds} T^R_{j,t_{m-1}} \right] \right\}.
\end{aligned}
$$

Next we define the following four expectations:

$$
\begin{aligned}
\text{EDR}(u,t,X_0) \;&=\; \mathbb{E}^Q \left[e^{u \cdot r_t} e^{-\int_0^t r_s ds} n_t \right] \\
\text{ELR}(u,t,X_0) \;&=\; \mathbb{E}^Q \left[e^{u \cdot r_t} e^{-\int_0^t r_s ds} L_t \right] \\
\text{ETLR}_j(u,t,X_0) \;&=\; \mathbb{E}^Q \left[e^{u \cdot r_t} e^{-\int_0^t r_s ds} T^L_{j,t} \right] \\
\text{ETRR}_j(u,t,X_0) \;&=\; \mathbb{E}^Q \left[e^{u \cdot r_t} e^{-\int_0^t r_s ds} T^R_{j,t} \right] \qquad \forall u \in \mathbb{R}, \qquad\qquad \text{(B.1)}
\end{aligned}
$$

Then, we can re-write the pricing formulas for the CDX index and its tranches as follows:

$$\mathbf{Prot}_{\text{CDX}}(X_0, T) = \sum_{m=1}^{Tf} \left\{ \text{ELR}\left(\frac{\Delta_m}{2}, t_m, X_0\right) - \text{ELR}\left(-\frac{\Delta_m}{2}, t_{m-1}, X_0\right) \right\}$$

$$\mathbf{Prem}_{\text{CDX}}(X_0, T) = U_{\text{CDX}} + S_{\text{CDX}} \sum_{m=1}^{Tf} \Delta_m \left\{ H_0(t_m) - \frac{1}{2}\text{EDR}\left(0, t_m, X_0\right) \right.$$
$$\left. - \frac{1}{2}\text{EDR}\left(\Delta_m, t_{m-1}, X_0\right) \right\}$$

$$\mathbf{Prot}_{\text{Tran},j}(X_0, T) = \sum_{m=1}^{Tf} \left\{ \text{ETLR}_j\left(\frac{\Delta_m}{2}, t_m, X_0\right) - \text{ETLR}_j\left(-\frac{\Delta_m}{2}, t_{m-1}, X_0\right) \right\}$$

$$\mathbf{Prem}_{\text{Tran},j}(X_0, T) = U_{\text{Tran},j} + S_{\text{Tran},j} \sum_{m=1}^{Tf} \Delta_m \left\{ H_0(t_m) - \frac{[\text{ETLR}_j + \text{ETRR}_j]\left(0, t_m, X_0\right)}{2} \right.$$
$$\left. - \frac{[\text{ETLR}_j + \text{ETRR}_j]\left(\Delta_m, t_{m-1}, X_0\right)}{2} \right\}.$$

That is, the CDX index and its tranches are priced if we are able to calculate the four expectations above. If we use the measure changing technique before we use Monte Carlo simulation, computation can be even simpler and faster. Recall that the Radon-Nikodym derivative of the physical measure (P) with respect to the risk-neutral measure (Q) is

$$\frac{dP}{dQ} = e^{-\int_0^t r_s ds} \frac{\pi_0}{\pi_t},$$

which is equivalent to

$$e^{-\int_0^t r_s ds} = \frac{\pi_t}{\pi_0} \frac{dP}{dQ}. \tag{B.2}$$

By plugging (B.2) into our four expectations, we obtain

$$\text{EDR}(u, t, X_0) = \mathbb{E}^Q\left[e^{u \cdot r_t} n_t \frac{\pi_t}{\pi_0} \frac{dP}{dQ} \right] = \mathbb{E}\left[e^{u \cdot r_t} n_t \frac{\pi_t}{\pi_0} \right]$$

$$\text{ELR}(u, t, X_0) = \mathbb{E}^Q\left[e^{u \cdot r_t} L_t \frac{\pi_t}{\pi_0} \frac{dP}{dQ} \right] = \mathbb{E}\left[e^{u \cdot r_t} L_t \frac{\pi_t}{\pi_0} \right]$$

$$\text{ETLR}_j(u, t, X_0) = \mathbb{E}^Q\left[e^{u \cdot r_t} T_{j,t}^L \frac{\pi_t}{\pi_0} \frac{dP}{dQ} \right] = \mathbb{E}\left[e^{u \cdot r_t} T_{j,t}^L \frac{\pi_t}{\pi_0} \right]$$

$$\text{ETRR}_j(u, t, X_0) = \mathbb{E}^Q\left[e^{u \cdot r_t} T_{j,t}^R \frac{\pi_t}{\pi_0} \frac{dP}{dQ} \right] = \mathbb{E}\left[e^{u \cdot r_t} T_{j,t}^R \frac{\pi_t}{\pi_0} \right]. \tag{B.3}$$

There are two advantages of using equations (B.3) instead of equations (B.1) in simulation. First, we do not need to derive the risk-neutral dynamics of the model. (Note that the model is specified under the physical measure.) Second, integral expression ($e^{-\int_0^t r_s ds}$) disappears in equations (B.3) because the Radon-Nikodym derivative absorbs it when the probability measure is changed. Therefore, we do not need to use numerical integration when we implement equations (B.3).

C   Model simulation

As discussed in Section 2.6, we need to simulate the two state variables (λ_t, ξ_t), log consumption growth ($\log(C_{t+\Delta t}/C_t)$) and log payout growth ($\log(D_{i,t+\Delta t}/D_{i,t})$)) for all firms, in order to price the CDX and CDX tranches.

First, we know that one of the state variables ξ_t follows a square-root process of Cox, Ingersoll, and Ross (1985). In this case, the conditional distribution is known to follow a non-central Chi-squared distribution. Specifically, $\xi_{t+\Delta t}|\xi_t$ follows a non-central Chi-squared distribution with degree of freedom $\left(\frac{4\kappa_\xi \bar{\xi}}{\sigma_\xi^2}\right)$ and non-centrality parameter $\left(\frac{4\xi_t \kappa_\xi e^{-\kappa_\xi \Delta t}}{(1-e^{-\kappa_\xi \Delta t})\sigma_\xi^2}\right)$.

In contrast, λ_t does not exactly follow a CIR process because its long-run mean (ξ_t) is time-varying. However, since Δt is fairly small, the behavior of λ_t between time t and $t + \Delta t$ is locally well approximated as a CIR process.[18] This implies that we can draw from $\lambda_{t+\Delta t}|\lambda_t$ by assuming that it follows a non-central Chi-squared distribution with degree of freedom $\left(\frac{4\kappa_\lambda \xi_t}{\sigma_\lambda^2}\right)$ and non-centrality parameter $\left(\frac{4\lambda_t \kappa_\lambda e^{-\kappa_\lambda \Delta t}}{(1-e^{-\kappa_\lambda \Delta t})\sigma_\lambda^2}\right)$.[19]

---

[18] Moreover since we sample the discrete path of ξ here, it makes sense to assume that ξ_t is fixed between time t and $t + \Delta t$.

[19] One alternative approach is to directly apply the Euler Scheme to the definition of λ_t and ξ_t. However, in this case, the paths might contain some negative values although these processes should always remain nonnegative. We favor our approach over the Euler scheme because drawing from a non-central Chi-squared distribution always guarantees non-negativity.

Lastly, log consumption growth $(\log{(C_{t+\Delta t}/C_t)})$, and each firm's log payout growth $(\log{(D_{i,t+\Delta t}/D_{i,t})})$ can be drawn by applying the Euler scheme to the following SDEs:

$$d\log C_t = \left(\mu_C - \frac{1}{2}\sigma_C^2\right)dt + \sigma_C dB_{C,t} + Z_{C,t}N_{C,t}$$

$$d\log D_{i,t} = \left(\mu_i - \frac{1}{2}\phi_i^2\sigma_C^2\right)dt + \phi_i\sigma_C dB_{C,t} + \phi_i Z_{C,t}dN_{C,t} + I_i Z_{S_i,t}dN_{S_i,t} + Z_{i,t}dN_{i,t}.$$

Note that these SDEs are derived from applying the Ito's Lemma to equations (1) and (6).

D   Default-free zero-coupon bond price

Let $H_0(\lambda_t, \xi_t, s-t)$ denote the time-t price of the default-free zero-coupon bond maturing at time $s > t$. By the pricing relation,

$$H_0(\lambda_t, \xi_t, s-t) = E_t\left[\frac{\pi_s}{\pi_t}\right]. \tag{D.1}$$

By multiplying π_t on both sides of (D.1), we obtain a martingale:

$$\pi_t H_0(\lambda_t, \xi_t, s-t) = \underbrace{E_t\left[\pi_s\right]}_{\text{martingale}}.$$

We conjecture that

$$H_0(\lambda_t, \xi_t, \tau) = \exp\left(a_0(\tau) + b_{0\lambda}(\tau)\lambda_t + b_{0\xi}(\tau)\xi_t\right). \tag{D.2}$$

By Ito's Lemma,

$$\frac{dH_{0,t}}{H_{0,t^-}} = \left\{ b_{0\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{0\lambda}(\tau)^2\sigma_\lambda^2\lambda_t + b_{0\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{0\xi}(\tau)^2\sigma_\xi^2\xi_t \right.$$
$$\left. - a_0'(\tau) - b_{0\lambda}'(\tau)\lambda_t - b_{0\xi}'(\tau)\xi_t \right\} dt + b_{0\lambda}(\tau)\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + b_{0\xi}(\tau)\sigma_\xi\sqrt{\xi_t}dB_{\xi,t}. \tag{D.3}$$

Furthermore, we also derive the stochastic differential equation for $\pi_t H_{0,t}$ by combining equation (D.3) and (A.2) using Ito's Lemma:

$$
\begin{aligned}
\frac{d(\pi_t H_{0,t})}{\pi_{t-} H_{0,t-}} = \Bigg\{ & -\beta - \mu_C + \gamma\sigma_C^2 - \lambda_t E\left[e^{(1-\gamma)Z_{C,t}} - 1\right] \\
& + b_{0\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{0\lambda}(\tau)^2\sigma_\lambda^2\lambda_t \\
& + b_{0\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{0\xi}(\tau)^2\sigma_\xi^2\xi_t \\
& - a_0'(\tau) - b_{0\lambda}'(\tau)\lambda_t - b_{0\xi}'(\tau)\xi_t \\
& + b_\lambda b_{0\lambda}(\tau)\sigma_\lambda^2\lambda_t + b_\xi b_{0\xi}(\tau)\sigma_\xi^2\xi_t \Bigg\} dt - \gamma\sigma_C dB_{C,t} \\
& + (b_\lambda + b_{0\lambda}(\tau))\sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} + (b_\xi + b_{0\xi}(\tau))\sigma_\xi\sqrt{\xi_t}dB_{\xi,t} + (e^{-\gamma Z_{C,t}} - 1)dN_{C,t}.
\end{aligned}
$$

Since $\pi_t H_{0,t}$ is a martingale, the sum of the drift and the jump compensator of $\pi_t H_{0,t}$ equals zero. That is,

$$
\begin{aligned}
0 = & -\beta - \mu_C + \gamma\sigma_C^2 - \lambda_t E\left[e^{(1-\gamma)Z_{C,t}} - 1\right] \\
& + b_{0\lambda}(\tau)\kappa_\lambda(\xi_t - \lambda_t) + \frac{1}{2}b_{0\lambda}(\tau)^2\sigma_\lambda^2\lambda_t \\
& + b_{0\xi}(\tau)\kappa_\xi(\bar{\xi} - \xi_t) + \frac{1}{2}b_{0\xi}(\tau)^2\sigma_\xi^2\xi_t \\
& - a_0'(\tau) - b_{0\lambda}'(\tau)\lambda_t - b_{0\xi}'(\tau)\xi_t \\
& + b_\lambda b_{0\lambda}(\tau)\sigma_\lambda^2\lambda_t + b_\xi b_{0\xi}(\tau)\sigma_\xi^2\xi_t + \lambda_t E\left[e^{-\gamma Z_{C,t}} - 1\right]. \quad \text{(D.4)}
\end{aligned}
$$

By collecting terms of (D.4),

$$0 = \underbrace{\left[-\beta - \mu_C + \gamma \sigma_C^2 + b_{0\xi}(\tau)\kappa_\xi \bar{\xi} - a_0'(\tau) \right]}_{=0}$$

$$+ \lambda_t \underbrace{\left[-b_{0\lambda}(\tau)\kappa_\lambda + \frac{1}{2}b_{0\lambda}(\tau)^2 \sigma_\lambda^2 + b_\lambda b_{0\lambda}(\tau)\sigma_\lambda^2 + E\left[e^{-\gamma Z_{C,t}} - e^{(1-\gamma)Z_{C,t}} \right] - b_{0\lambda}'(\tau) \right]}_{=0}$$

$$+ \xi_t \underbrace{\left[b_{0\lambda}(\tau)\kappa_\lambda - b_{0\xi}(\tau)\kappa_\xi + \frac{1}{2}b_{0\xi}(\tau)^2 \sigma_\xi^2 + b_\xi b_{0\xi}(\tau)\sigma_\xi^2 - b_{0\xi}'(\tau) \right]}_{=0}.$$

These conditions provide a system of ODEs:

$$
\begin{aligned}
a_0'(\tau) &= -\beta - \mu_C + \gamma \sigma_C^2 + b_{0\xi}(\tau)\kappa_\xi \bar{\xi} \\
b_{0\lambda}'(\tau) &= -b_{0\lambda}(\tau)\kappa_\lambda + \frac{1}{2}b_{0\lambda}(\tau)^2 \sigma_\lambda^2 + b_\lambda b_{0\lambda}(\tau)\sigma_\lambda^2 + E\left[e^{-\gamma Z_{C,t}} - e^{(1-\gamma)Z_{C,t}} \right] \\
b_{0\xi}'(\tau) &= b_{0\lambda}(\tau)\kappa_\lambda - b_{0\xi}(\tau)\kappa_\xi + \frac{1}{2}b_{0\xi}(\tau)^2 \sigma_\xi^2 + b_\xi b_{0\xi}(\tau)\sigma_\xi^2.
\end{aligned}
\tag{D.5}
$$

This shows that H_0 satisfies the conjecture (D.2). We can obtain the boundary conditions for the system of ODEs (D.5) because $H_0(\lambda_t, \xi_t, 0) = 1$, which is equivalent to

$$a_0(0) = b_{0\lambda}(0) = b_{0\xi}(0) = 0.$$

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Figure 1: Option-implied state variables

Panel A: Option sample period
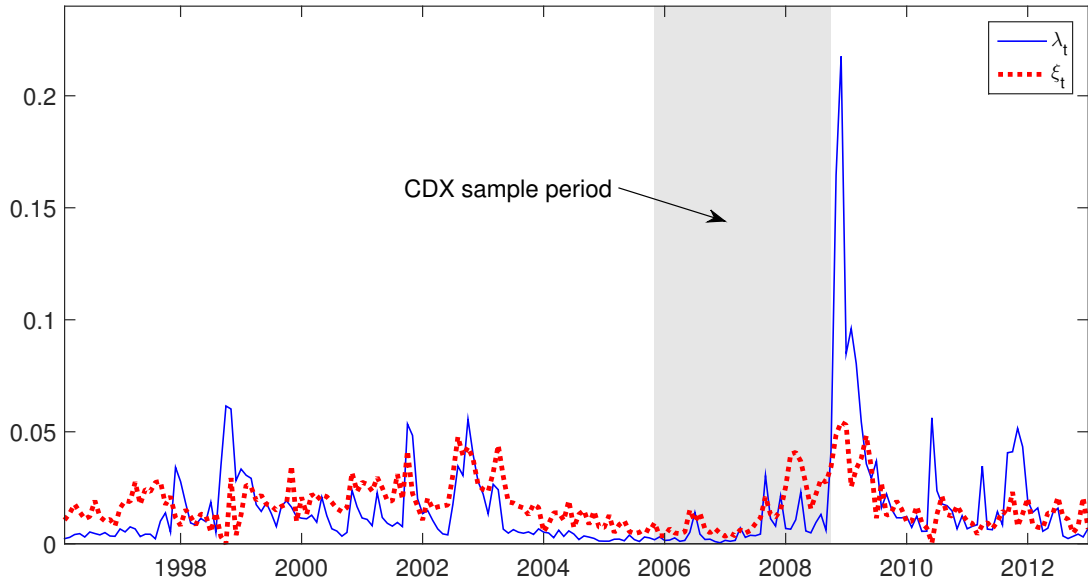
Panel B: CDX tranche sample period
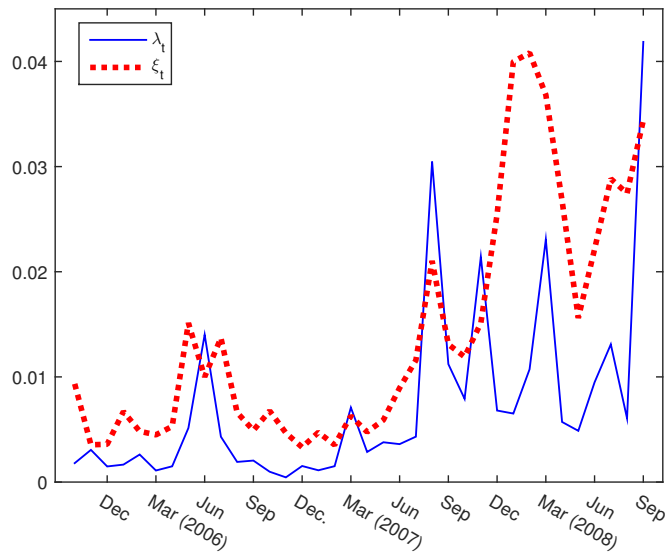
Notes: Monthly time series of the state variables λ_t (annual disaster probability) and ξ_t (long-run mean of λ_t) extracted from option prices. At each time point, the state variables are chosen to match implied volatilities of 1-month ATM and OTM (moneyness of 0.85) index put options. Panel A shows these time series for the January 1996–December 2012 period over which option data are available. The shaded area represents the period over with CDX tranche data are available. Panel B shows only this second sample period.
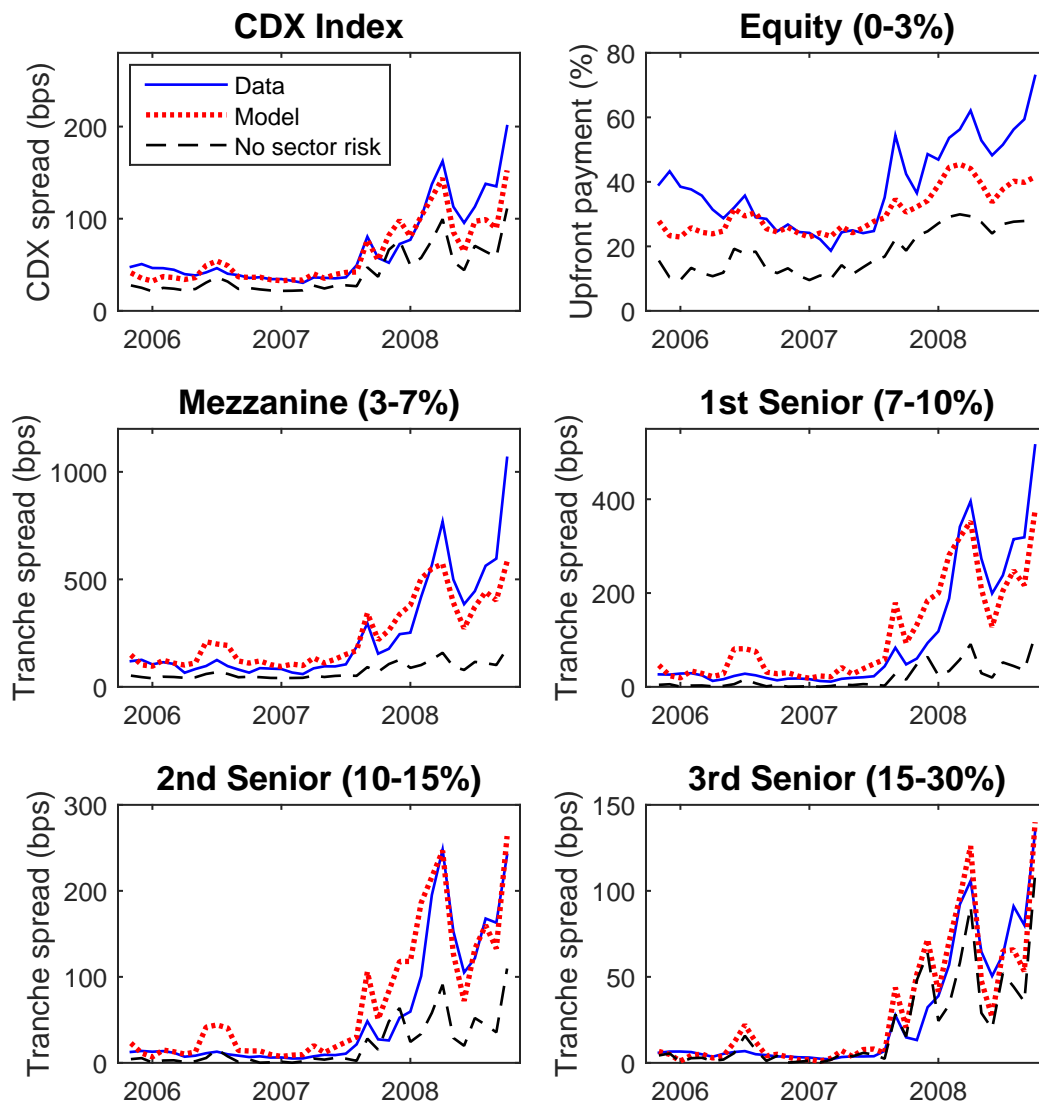
Figure 2: Time series of option-implied volatilities
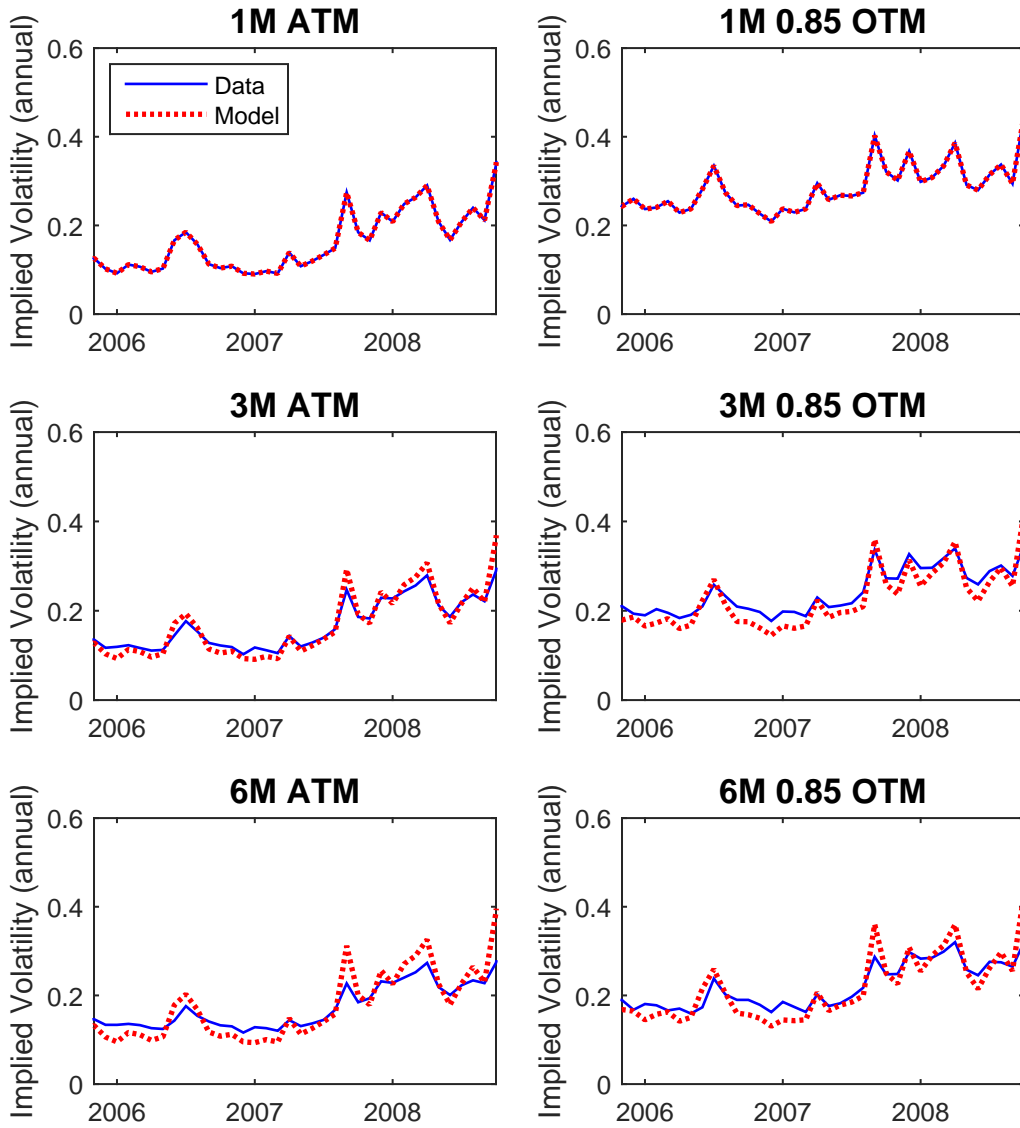
Notes: Monthly time series of option-implied volatilities in the data (blue solid lines) and in the model (red dotted lines). Results are shown for 1, 3, and 6 month options. The two state variables are computed to match the 1-month ATM and 0.85 OTM implied volatilities exactly.

Figure 3: Time series of CDX/CDX tranche spreads in the benchmark model and in a case without idiosyncratic risk
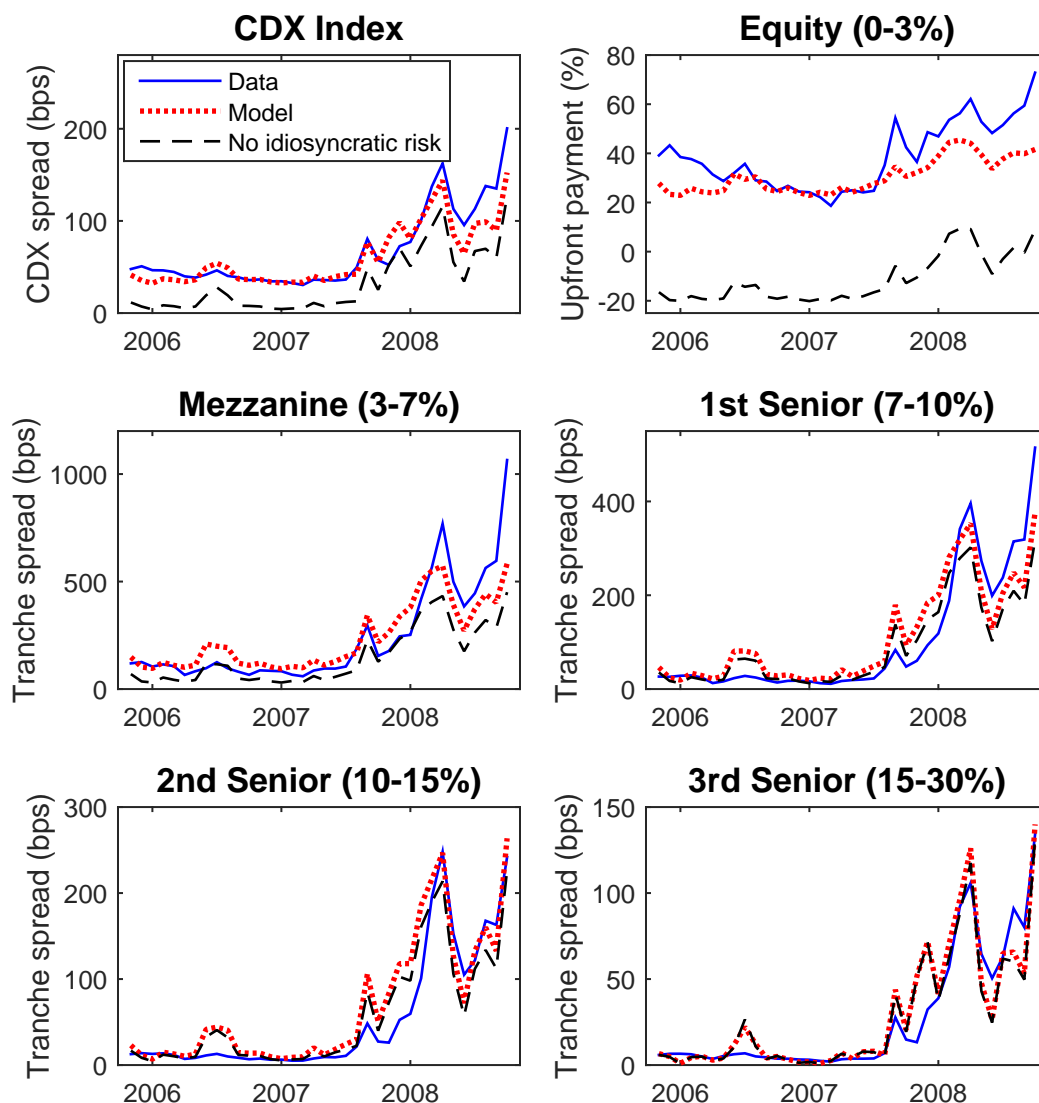
Notes: Monthly time series of 5-year CDX and CDX tranche spreads in the data (blue solid lines), in the benchmark model (red dotted lines), and in a calibration without idiosyncratic risk (black dashed lines). Spreads are reported in terms of basis points (bps). For the equity tranche, we report the upfront payment because the spread is fixed at 500 bps. The benchmark model values are computed using the option-implied state variables fit to the time series of 1-month ATM and 0.85 OTM implied volatilities. The model version without idiosyncratic risk is obtained from the benchmark by setting the probability of an idiosyncratic shock (λ_i) to zero.

Figure 4: Time series of CDX/CDX tranche spreads in the benchmark model and in a case without sector shocks
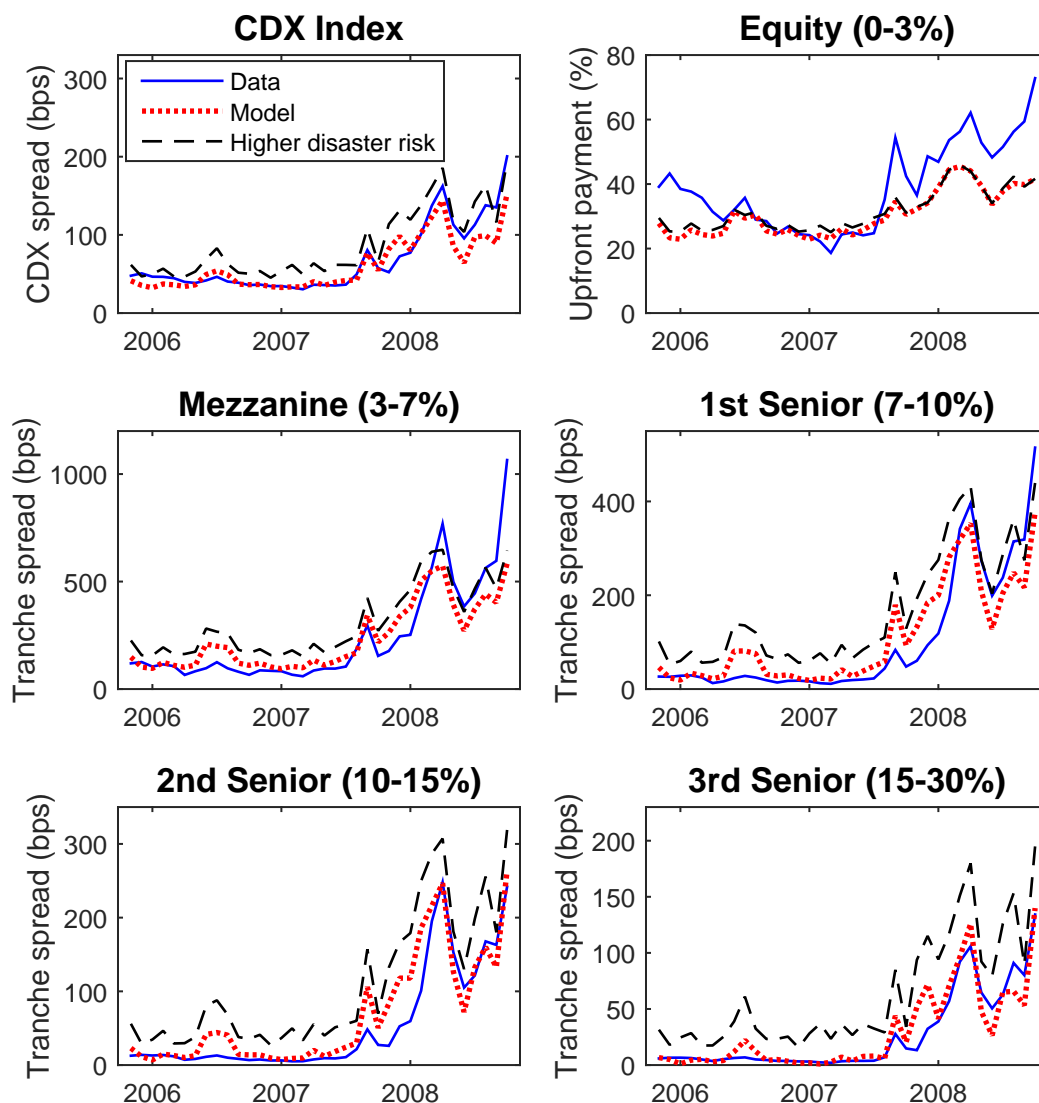

Notes: Monthly time series of 5-year CDX and CDX tranche spreads in the data (blue solid lines), in the benchmark model (red dotted lines), and in a calibration without sector shocks (black dashed lines). Spreads are reported in terms of basis points (bps). For the equity tranche, we report the upfront payment because the spread is fixed at 500 bps. The benchmark model values are computed using the option-implied state variables fit to the time series of 1-month ATM and 0.85 OTM implied volatilities. The model version without sector shocks is obtained by setting p_i to zero.

Figure 5: Time series of CDX/CDX tranche spreads in the benchmark model and in a case with higher disaster risk

Notes: Monthly time series of 5-year CDX and CDX tranche spreads in the data (blue solid lines), in the benchmark model (red dotted lines), and in a calibration with higher disaster risk. Spreads are reported in terms of basis points (bps). For the equity tranche, we report the upfront payment because the spread is fixed at 500 bps. The benchmark model values are computed using the option-implied state variables fit to the time series of 1-month ATM and 0.85 OTM implied volatilities. The calibration with higher disaster risk is obtained by uniformly increasing the disaster probability λ_t by 2 percentage points.

Table 1: Parameter values for the model

Panel A: Basic parameters	
Relative risk aversion γ	3.0
EIS ψ	1.0
Rate of time preference β	0.012
Average growth in consumption (normal times) μ_C	0.0252
Volatility of consumption growth (normal times) σ_C	0.020
Leverage ϕ	2.6
Panel B: State parameters	
Mean reversion κ_λ	0.20
Volatility parameter σ_λ	0.1576
Mean reversion κ_ξ	0.10
Volatility parameter σ_ξ	0.0606
Mean $\bar{\xi}$	0.02

Notes: All the parameter values in this table come from Seo and Wachter (2015). Panel A shows parameters for normal-times consumption and dividend processes, and for the preferences of the representative agent. Panels B shows the parameter values for λ and ξ processes:

$$
\begin{aligned}
d\lambda_t &= \kappa_\lambda(\xi_t - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} \\
d\xi_t &= \kappa_\xi(\bar{\xi} - \xi_t)dt + \sigma_\xi\sqrt{\xi_t}dB_{\xi,t}
\end{aligned}
$$

Note that $\bar{\xi}$ is the average level of the probability of a disaster. Parameter values are in annual terms.

Table 2: Parameter values for an individual firm

Default boundary A_B	19.2%
Recovery rate (normal times) R_i	40%
Recovery rate (disaster times) R_i	20%
Aggregate risk loading ϕ_i (pre-crisis)	1.3
Aggregate risk loading ϕ_i (post-crisis)	1.6
Idiosyncratic jump size $\left(e^{Z_{i,t}} - 1\right)$	-80.0%
Bernoulli parameter for sector shocks $P(I_i = 1) = p_i$	0.4
Sector-wide jump size $\left(e^{Z_{S_i,t}} - 1\right)$	-71.1%
Coefficient for sector-wide jump intensity w_ξ	1.765
Idiosyncratic jump intensity λ_i	0.0093

Notes: This table reports the parameters for the payout process on an individual firm. Note that default boundary A_B is calculated as the firm's leverage ratio (0.32) multilied by 60% following Collin-Dufresne, Goldstein, and Yang (2012). Idiosyncratic jump intensity λ_i is in annual terms.

Table 3: Average CDX and CDX tranche spreads (5-year maturity)

	Upfront (%)	Spread (bps)				
	0-3%	3-7%	7-10%	10-15%	15-30%	Index
Panel A: Full sample (Oct. 2005 - Sep. 2008)						
Data	39	238	102	54	27	67
Model	31	235	110	67	29	61
Panel B: Pre-crisis sample (Oct. 2005 - Sep. 2007)						
Data	31	108	25	12	6	42
Model	26	142	46	23	8	41
Panel C: Post-crisis sample (Oct. 2007 - Sep. 2008)						
Data	54	498	255	136	69	116
Model	39	422	238	155	71	102

Notes: Historical and model-implied average five-year CDX and CDX tranche spreads in basis points per year. In case of the equity tranche, we report the upfront payment because the spread is fixed at 500 bps. Our sample period is from October 2005 to September 2008, which corresponds to CDX series 5 to 10. We divide the sample into two sub-periods: pre-crisis and post-crisis. The pre-crisis sample is from October 2005 to September 2007 (CDX5 to CDX8). The post-crisis sample is from October 2007 to September 2008 (CDX9 to CDX10). Panel A presents the average spreads over the entire sample period. Panel B and Panel C show the average spreads for the pre-crisis sample and the post-crisis sample, respectively.

Table 4: Average CDX and CDX tranche spreads (3-year maturity)

	Upfront (%)	Spread (bps)				
	0-3%	3-7%	7-10%	10-15%	15-30%	Index
Panel A: Pre-crisis						
CGY data	11	20	8	3	2	27
Model	14	45	15	7	4	33
Panel B: Post-crisis						
CGY data	43	364	168	87	48	–
Model	24	243	130	80	46	82

Notes: This table reports historical and model-implied average 3-year CDX and CDX tranche spreads in basis points per year. In case of the equity tranche, we report the upfront payment because the spread is fixed at 500 bps. Data values are as reported in Collin-Dufresne, Goldstein, and Yang (2012). Note that their pre-crisis sample is one year longer than ours. That is, their pre-crisis sample starts from CDX series 3 (until series 8) while ours starts from CDX series 5 because the data on CDX series 3 and 4 are not consistently provided in our Markit dataset. Their post-crisis sample exactly matches up with ours (CDX9 to 10).0

Table 5: Average super-senior tranche spreads

| | Super senior tranche spread (bps) | | | |
| | Pre-crisis | | Post-crisis | |
	3-year	5-year	3-year	5-year
CGY data	1	4	23	35
Model	2	3	28	34

Notes: Historical and model-implied average 3 and 5-year super-senior tranche spreads in basis points per year. The attachment point for the super-senior tranche is 30%. Data values are as reported in Collin-Dufresne, Goldstein, and Yang (2012). Note that their pre-crisis sample is one year longer than ours. That is, their pre-crisis sample starts from CDX series 3 (until series 8) while ours starts from CDX series 5 because the data on CDX series 3 and 4 are not consistently provided in our Markit dataset. Their post-crisis sample exactly matches up with ours (CDX9 to 10).