Inequality, Financial Frictions, and Leveraged Bubbles*

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Abstract

The U.S. economy recently experienced rising income inequality, household indebtedness, and a boom-bust cycle in house prices. We link these phenomena in a rational bubble model featuring income heterogeneity and financial frictions. A high concentration of income creates excess demand for savings and depresses the interest rate. With low bubbly asset pledgeability, top earners may hold unleveraged bubbles, while with high pledgeability, bottom earners hold *leveraged* bubbles. A high pledgeability of bubbly assets enlarges the existence region of bubbly equilibria, because risk-shifting arising from limited commitment inhibits the market's ability to rule out highly risky leveraged bubbles.

Keywords: rational bubbles, financial frictions, income distribution, household debt, endogenous default.

JEL codes: E12; E24; E44; G01

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1 Introduction

The period leading to the U.S. Great Recession was characterized by several concurrent phenomena. First, there was a spectacular "bubble-like" boom-bust cycle in the housing market. The S&P/Case-Shiller U.S. National Home Price Index rose by 85% between January 2000 and July 2006 before dropping 27% below its peak value by February 2012; it has proven difficult to attribute much of these fluctuations to changes in economic fundamentals, such as demographics, construction costs, or interest rates (Shiller, 2015).[1] Second, there was a large increase in indebtedness and homeownership, particularly among low- and middle-income households. The boom was largely associated with housing debt and followed decades of financial innovations that facilitated the issuance of loans backed by housing assets (Mian and Sufi, 2011, 2014).[2] Third, there was an increase in income inequality. For example, the income share of the top 5% earners increased from 22% in 1983 to 34% in 2007 (Piketty and Saez, 2013, Kumhof et al., 2015).

Can the theory of rational bubbles provide a joint account of these three phenomena? Could financial innovations and rising inequality have played a role in facilitating the housing and credit booms? Could they have contributed to making the bubble episode more risky? Can they explain the increased participation of low- and middle-income households in the housing and mortgage markets? This paper proposes a theory that offers potential answers to these relevant questions.

Our theory embeds income inequality and financial frictions into the classic rational bubble framework of Samuelson (1958), Diamond (1965), and Tirole (1985). As in the rational bubble literature, households are willing to buy an asset (such as a house) that is priced above its fundamental value, as long as they expect to be able to resell it at a later point. Income inequality is introduced by assuming heterogenous endowments between *top earners*, who represent top-income households, and *bottom earners*, who represent low- and middle-income households. Households can borrow from each other using a standard debt contract, subject to an enforcement friction. In particular, they can pledge a fraction of their

[1] See also Case and Shiller (2003), Leamer (2007), and Mian and Sufi (2014).

[2] Household debt doubled from 49.1% of GDP in 1983 to 98% of GDP in 2007, and among the bottom 95% of the income distribution, the debt-to-income ratio more than doubled from 62.3% in 1983 to 147.3% in 2007 (Kumhof et al., 2015). Housing debt accounts for the bulk of household debt; for example, in 2011, housing debt accounted for 78% of all household debt (Gottschalck et al., 2013). Many have argued that financial innovations and changes in financial regulations contributing to the boom include the introduction of residential mortgage-backed securities in the mid-1990s, the introduction of credit default swaps on collateralized mortgage obligations in the mid-2000s, and the 1995 New Community Reinvestment Act that strengthened the role of government-sponsored enterprises, namely Fannie Mae and Freddie Mac, in buying, selling, and guaranteeing mortgage-backed securities (see, e.g., Yellen, 2009, the U.S. Financial Crisis Inquiry Commissions, 2011, and Boz and Mendoza, 2014).

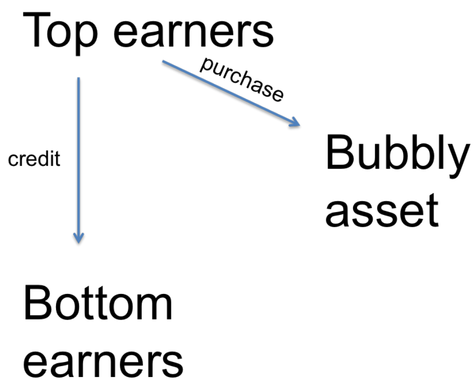
future income *and* a fraction ϕ of their holdings of bubbly assets as collateral. In the spirit of Boz and Mendoza (2014) or Caballero and Farhi (2015), we view an exogenous increase in this latter fraction ϕ as a parsimonious representation of the consequence of the financial innovations that increased households' ability to borrow against their housing wealth prior to the Great Recession. The use of a standard debt contract with the option to default may give rise to *risk-shifting*, where agents are more willing to engage in risky behaviors when they undertake projects using other people's funds (Jensen and Meckling, 1976 and Stiglitz and Weiss, 1981). As argued by Allen and Gorton (1993) and Allen and Gale (2000), such risk-shifting can boost the demand and thus the prices of risky assets.[3]

Our results point to a strong interaction between income inequality and asset pledgeability in shaping the existence and market participation characteristics of bubbly equilibria. When bubbly asset pledgeability is limited (ϕ is small), an asset bubble exists if and only if there is sufficient inequality. In equilibrium, bubble purchase is unleveraged: top earners buy the bubble using their own funds, and bubbly episodes are not associated with credit booms. High inequality facilitates bubble existence because a high income concentration increases the supply of savings and depresses the interest rate. Therefore, when inequality is high and households have limited ability to borrow against bubbly assets such as housing, then bubbly asset holding should be concentrated among high-income individuals, for whom the bubble provides a store of value. This implication is qualitatively consistent with findings about the real estate market in China (see, for example, Tomba and Tang (2008), Wang et al. (2015), and Fang et al. (2015)). Figure 1a illustrates an unleveraged bubbly equilibrium.

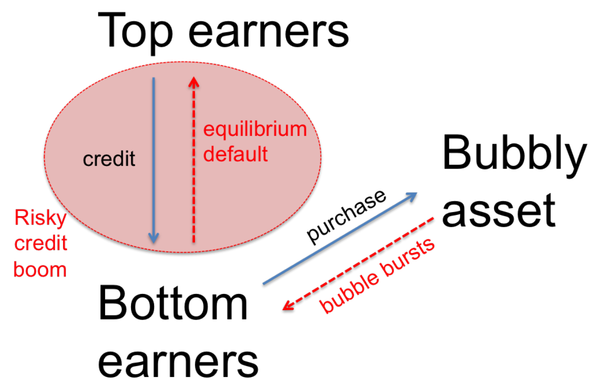
A contrasting set of results prevails when the bubbly asset is highly pledgeable (ϕ is high). In that scenario, an asset bubble still exists if and only if there is sufficient inequality. However, in equilibrium, bubble purchase is *leveraged:* bottom earners become the buyers of the bubbly asset, financing their purchase by resorting to loans backed by the bubbly asset itself. Thus, a leveraged bubbly episode is associated with a credit boom in which bottom earners borrow from top earners via collateralized debt. A leveraged bubble further comes with default risk: it is optimal for debtors to default when the bubble bursts, because then the value of their collateral drops below the face value of their debt. From an *ex ante* perspective, however, the opportunity to default when the bubble bursts allows borrowers to shift some of the downside risk of bubble investment to creditors. This makes bottom earners overvalue the bubble relative to top earners and enlarges the existence region of bubbly equilibria. Figure 1b illustrates a leveraged bubbly equilibrium.

Our theory thus predicts that the combination of limited enforcement and a high degree of bubbly asset pledgeability can facilitate the emergence of highly risky bubbles. Furthermore,

[3]See also Barlevy (2014), Doblas-Madrid and Lansing (2014), and Ikeda and Phan (forthcoming).



(a) Unleveraged bubble.



(b) Leveraged bubble.

Figure 1: Bubble market participation and credit market interactions.

easy credit, defined as lax collateral requirements, not only facilitates asset bubbles and credit booms but can also change the nature of asset bubbles from unleveraged to leveraged. These predictions are qualitatively consistent with stylized features of the U.S. economy in the early 21st century. Specifically, following decades of increased inequality *and* financial innovations that increased borrowers' ability to collateralize housing assets, the U.S. economy witnessed an unprecedented boom in borrowing and housing market participation by lower- and middle-income households (see Cooper, 2009 and Mian and Sufi, 2011, 2014). When the housing market began to falter in 2006, the boom turned into a bust characterized by widespread default and foreclosures, in particular by so-called "subprime" borrowers.[4] The equilibrium outcomes of our model are consistent with the mainstream narrative of this episode. The model also formalizes the ideas put forward by Rajan (2005, 2011) and Stiglitz (2012) that inequality and financial innovations can play crucial roles in creating a favorable environment for risky bubbles and macroeconomic instability.

From a theoretical standpoint, we make two contributions regarding the role of asset pledgeability in shaping the interaction between the bubble market and the credit market. First, we find that higher asset pledgeability relaxes the existence conditions of bubbly equilibria. Second, we find that asset pledgeability determines the characteristics of agents who participate in the market for bubbly assets. In our model, income inequality and financial frictions generate heterogeneity in the shadow value of funds across agents, and thereby heterogeneity in motives to hold the bubbly asset. While unconstrained, low shadow value investors (i.e., top earners) value the bubbly asset only for its future resell price, credit-

[4]Mian and Sufi (2009) document that not only did housing prices rise faster in the boom phase, but default rates were also higher in the bust phase in zip codes where subprime mortgages were more prevalent. Also see Barlevy and Fisher (2011).

constrained, high shadow value investors (i.e., bottom earners) additionally value the bubbly asset's collateral service. For low levels of bubbly asset pledgeability, the bubbly asset's collateral value is small; thus only low shadow value investors participate in the bubble market. But for higher levels of pledgeability, the collateral value becomes sufficiently large to draw high shadow value investors into participating. At the extreme, for very high levels of bubbly asset pledgeability, the collateral value is so high that low shadow value investors get "pushed out" of the market and only high shadow value investors participate.

Related literature: Our paper mainly relates to the literature on rational bubbles, which builds on the classic framework of Samuelson (1958), Diamond (1965), and Tirole (1985).[5] It is particularly related to a recent stream of the literature that studies rational bubbles in economies with financial frictions. We exploit the idea, also present in Caballero and Krishnamurthy (2006), Kocherlakota (2009), Hirano and Yanagawa (2010), Hirano et al. (forthcoming) , Miao and Wang (2011), Farhi and Tirole (2012), Martin and Ventura (2012, forthcoming), and Ikeda and Phan (2014, forthcoming) that bubbles can relax binding credit constraints. Our work is closest to Miao and Wang (2011) and Martin and Ventura (forthcoming), who, like us, consider the possibility that bubbly assets provide collateral value to credit-constrained agents. Unlike us, however, they do not allow for equilibrium default, which plays a key role in shaping the difference in characteristics between leveraged and unleveraged bubbles in our framework. Furthermore, while they focus on the effects of bubbles on aggregate investment and activity, we focus on the effects of inequality and financial innovations on the existence of leveraged or unleveraged bubbles and bubble market participation. Finally, our modeling of stochastic bubbles follows Weil (1987).

Other papers outside of the rational bubble literature have studied credit-fueled bubbles and emphasized the role of risk-shifting in this context (Allen and Gorton, 1993, Allen and Gale, 2000, Barlevy, 2014, and Doblas-Madrid and Lansing, 2014). A common assumption in these papers is that bubbly asset purchase is financed by credit. In contrast, in our model, whether or not bubbles are fueled by credit is an endogenous outcome of the interaction between the asset and credit markets, and crucially depends on the degree of pledgeability of the bubbly asset. This allows us to jointly analyze and compare leveraged and unleveraged bubble episodes. Furthermore, while several of these papers focus on the heterogeneity in information, we instead focus on the heterogeneity in the shadow values of funds (due to income inequality).

Our paper borrows insights from the macroeconomic literatures on financial frictions

[5] For recent surveys of the literature, see Barlevy (2012) and Miao (2014).

(Bernanke and Gertler, 1989, Kiyotaki and Moore, 1997, Aiyagari and Gertler, 1999), inequality (Galor and Zeira, 1993, Matsuyama, 2000, Krueger and Perri, 2006 and Kumhof et al., 2015), and the literature on general equilibrium with incomplete markets (Geanakoplos, 1997 and Geanakoplos and Zame, 2002). We exploit the idea, notably present in Kiyotaki and Moore (1997), that limited enforcement and collateral constraints can generate a feedback loop between credit and asset prices. Our modeling of collateral constraints with occasional default is in the spirit of Geanakoplos (1997) and Geanakoplos and Zame (2002). And like Kumhof et al. (2015), our model features loans from top earners to bottom earners.

Finally, our paper provides a leveraged bubble theory that is motivated by and is largely consistent with the empirical findings on bubbles and crises. Using historical and micro-level data, Kindleberger and Aliber (2005) and Mian and Sufi (2014) find that asset price bubbles depend on the growth of credit, and Jordà et al. (2015) find that leveraged bubbles are more likely to be associated with financial crises than unleveraged ones.

The rest of the paper is organized as follows: Section 2 lays out the environment. Section 3 provides useful benchmarks. Section 4 studies unleveraged bubbles, Section 5 studies a leveraged bubbles, and Section 6 concludes. All proofs are in the Appendix.

2 Environment

Consider a closed economy with overlapping generations. Time is discrete and infinite, denoted by $t = 0, 1, 2 \ldots.$ There is a single consumption good. Each generation consists of a continuous mass of households, each of which lives for two periods. As in Bernanke and Gertler (1989), we assume that lifetime expected utility takes the quasi-linear form:

$$u(c_{y,t}) + \beta E_t c_{o,t+1}$$

where $c_{y,t}$ and $c_{o,t+1}$ denote consumption when young and old, respectively, and $u(\cdot)$ is a strictly increasing and concave utility function satisfying the usual Inada conditions. The linearity of utility with respect to old age consumption makes households' portfolio optimization problem with risky assets particularly tractable, but is otherwise not essential for our results.

Heterogeneity: We assume that a fraction $\theta \in (0, 1)$ of each generation are "top earners" and the remaining $1 - \theta$ are "bottom earners." Top earners and bottom earners are respectively endowed with y^r and y^p units of the consumption good in young age, where $y^r > y^p$ (the superscripts stand for "rich" and "poor"). Let $\bar{y} \equiv \theta y^r + (1 - \theta) y^p$ denote the average endowment. As utility is linear in old age consumption, we assume without loss of generality that all households receive the same old age endowment T.[6] The combination of the heterogeneity in the paths of income and the concavity in the utility in young age is a simple way to induce a natural borrowing motive among bottom earners and, accordingly, a natural saving motive among top earners.[7]

Bubbly asset: Following the rational bubble literature, we model a (pure) bubbly asset as an asset in fixed unit supply, which pays no dividend (and thus has zero "fundamental value") but whose market price may be positive. Following Weil (1987), we assume that in each period, the price of the bubbly asset exogenously collapses to zero (its fundamental value) with a constant probability. This is a simple way of capturing the fragility of bubbles. Formally, we denote the price of one unit of the bubbly asset in period t by $\xi_t P_t$, where $\{P_t\}_{t=0}^{\infty}$ is the sequence of prices conditional on the bubble having not collapsed, and $\{\xi_t\}_{t=0}^{\infty}$ is a process of binary random variables representing whether the bubble persists ($\xi_t = 1$) or has collapsed ($\xi_t = 0$):

$$\begin{aligned} \Pr(\xi_{t+1} = 0 | \xi_t \xi_{t-1} \cdots \xi_0 = 1) &= p_{burst} \in [0, 1) \\ \Pr(\xi_{t+1} = 0 | \xi_t \xi_{t-1} \cdots \xi_0 = 0) &= 1. \end{aligned}$$

The first equation states that if the bubble has not collapsed up to and including period t, then it will collapse in period $t+1$ with probability p_{burst}. The second equation states that once the bubble has collapsed, households expect that it will not re-emerge.

Credit market frictions: Young households can borrow from each other using a standard debt contract. Loans have a one-period maturity and are contractually non-contingent. However, credit markets are subject to an enforcement friction. Households cannot commit to repay their debt. Let d_t^i denote the net debt position of a young household in t, where $i \in \{r, p\}$, and let R_t be the interest rate in the credit market such that a debt position d_t^i at t commits a debtor household to a repayment of $R_t d_t^i$ at $t+1$ in case of no default. Lenders are collectively able to repossess an amount D from a debtor household's old age income T, as well as a fraction $\phi \in [0, 1]$ of its bubbly asset holdings. The parameter ϕ is crucial for our analysis. It captures the pledgeability of the bubbly asset, so that high values of ϕ can be interpreted as representing a highly developed financial system in which households are able to borrow against a large share of the market value of their house. We assume that the interest rate cannot be conditioned on the size of a loan.[8] This assumption gives rise to a

[6]A possible interpretation of T is as a social security transfer.

[7]With concave utility in both young and old age, the same borrowing-lending pattern would prevail as long as income dispersion is higher in young age than in old age.

[8] This amounts to assuming that the exclusive contracts that would be required to condition the interest rate on the loan size are not feasible. Allen and Gale (2000) make the same assumption in a similar context.

problem of risk-shifting, where leveraged investors do not fully internalize the risk of their portfolios. We discuss this in more detail in Section 5.

In $t+1$, given the above stated enforcement constraint, a debtor household's default decision is characterized by:

$$\delta_{t+1}^{i} \equiv 1\left\{R_t d_t^i > D + \phi \xi_{t+1} P_{t+1} b_t^i\right\}, \tag{1}$$

where $1\{\cdot\}$ is the indicator function, $R_t d_t^i$ is the cost of repaying, and $D + \phi \xi_{t+1} P_{t+1} b_t^i$ is the cost of defaulting.[9] An implication of (1) is that if $R_t d_t^i > D + \phi P_{t+1} b_t^i$ (note the absence of bursting dummy ξ_{t+1} on the right-hand side), then a debtor household will *always* default in $t+1$. This is because even in the most favorable state of the world, where $\xi_{t+1} = 1$ (the bubble persists in $t+1$), the cost of defaulting, $D + \phi P_{t+1} b_t^i$, is strictly smaller than the cost of repaying, $R_t d_t^i$. Thus, to prevent a situation in which a household borrows so much that it always defaults, we assumed that the following credit constraint is imposed on borrowers:

$$R_t d_t^i \leq D + \phi P_{t+1} b_t^i. \tag{CC}$$

where D represents "fundamental collateral" and $\phi P_{t+1} b_t^i$ represents "bubbly collateral." Constraint (CC) guarantees that there is at least one state of the world where a borrower does not default, and by implication, that there is at most one state where the borrower defaults (when the bubble bursts).

Remark on credit constraint (CC): In the special case where $\phi = 0$, a debtor household does not lose any of its bubbly assets in default. There is thus never any default in equilibrium, even in contingencies where the bubble collapses. In that case, the credit constraint (CC) reduces to the kind of simple borrowing limit often encountered in the macroeconomic literature, as for instance in Bewley (1977), Huggett (1993), or Aiyagari (1994). A distinguishing feature of (CC) when $\phi > 0$ is that it allows for the possibility that households can borrow against bubbly collateral, i.e., $R_t d_t^i > D$. Our assumption of bubbly collateral is similar to that of Martin and Ventura (forthcoming), but differs in that we explicitly consider default.[10] If $R_t d_t^i > D$, then it follows directly from (1) that debtor households optimally default when the bubble bursts. The possibility of default is not only consistent with data

[9] We have implicitly assumed that if indifferent between defaulting and repaying, a debtor household chooses to repay.

[10] More generally, the idea that debtors can pledge a fraction of a tradable asset as collateral is similar to that in Kiyotaki and Moore (1997).

but also turns out to be important in our model, as it induces risk-shifting and facilitates the emergence of leveraged bubbles (see Section 5).[11] Throughout the paper, we focus on situations in which (CC) binds for the bottom earners.

To complete the model description, we assume that in period $t = 0$ each household of the initial old generation has one unit of the bubbly asset, the initial price of the bubble is P_0, and there is no debt to repay. In what follows, we lay out the optimization problems of households.

2.1 Bottom earners' problem

In young age, a bottom earner household chooses its borrowing and bubbly asset holding to maximize expected lifetime utility, anticipating its own default decision rule in old age as described in (1). Formally, its problem is:

$$\max_{b_t^p, d_t^p} u(c_{y,t}^p) + \beta E_t[c_{o,t+1}^p] \tag{2}$$

subject to budget constraints:

$$\begin{aligned} c_{y,t}^p + P_t b_t^p &= y_t^p + d_t^p, \\ c_{o,t+1}^p &= T + \xi_{t+1} P_{t+1} b_t^p - \underbrace{(1-\delta_{t+1})}_{\text{repay}} R_t d_t^p - \underbrace{\delta_{t+1}}_{\text{default}} (D + \phi \xi_{t+1} P_{t+1} b_t^p), \end{aligned} \tag{3}$$

a bubble no-short-selling constraint:

$$b_t^p \geq 0, \tag{4}$$

and the credit constraint (CC). Let $\lambda_{c,t}^p$, $\mu_{b,t}^p$, and $\mu_{d,t}^p$ denote the multipliers on the young age budget constraint (3), bubble no-short-selling constraint (4) and collateral constraint (CC), respectively. The first order conditions of bottom earners are:

$$\lambda_t^p = u'(c_{y,t}^p) \tag{5}$$

$$\mu_{b,t}^p = P_t \lambda_t^p - \beta P_{t+1} E_t\left[(1 - \phi\delta_{t+1})\,\xi_{t+1}\right] - \phi P_{t+1} \mu_{d,t}^p \tag{6}$$

$$R_t \mu_{d,t}^p = \lambda_t^p - \beta E_t\left[1 - \delta_{t+1}\right] R_t. \tag{7}$$

[11] A large number of papers have documented that between 2002 and 2006, when housing prices rose rapidly in the U.S., many households (especially those with a lower income or credit score) increased their expenditures by borrowing against the values of their houses; however, when housing prices collapsed in 2006-2007, default rates increased sharply (again, especially among households with a lower income or credit score). See, for instance, Greenspan and Kennedy (2008), Cooper (2009), and Mian and Sufi (2011). Adopting the widely-held view that the U.S. experienced a housing bubble in the early 2000s, this evidence is consistent with our assumption that households can borrow against bubbly assets and may default when the price of the bubbly asset collapses.

Equation (5) is standard, stating that a household's shadow value of funds in young age is equal to its marginal utility of consumption. Equation (6) states that the shadow value of the household's no-short-selling constraint on the bubbly asset $\mu_{b,t}^{p}$ is equal to the difference between its marginal cost of acquiring the bubbly asset $P_t\lambda_t^p$ and its marginal benefit of holding it, which consist of two terms. The first term is the discounted expected return on the bubble, $\beta P_{t+1}E_t\left[(1-\phi\delta_{t+1})\,\xi_{t+1}\right]$, accounting for the fact that in the event of default ($\delta_{t+1}=1$), the household loses a fraction ϕ of its bubbly asset. The second term is the collateral value of the bubbly asset: $\phi P_{t+1}\mu_{d,t}^{p}$. At an interior choice, the marginal cost and marginal benefit are equal and the shadow cost of the constraint is zero. In contrast, at a corner choice in which the household chooses not to hold the bubbly asset, the marginal cost outweighs the marginal benefit, so the shadow cost of the constraint is positive.

Finally, equation (7) states that the shadow value of the credit constraint is proportional to the difference between the household's marginal benefit of issuing debt λ_t^p and its discounted expected marginal cost of repaying it, $\beta E_t\left[1-\delta_{t+1}\right]R_t$. Again, at an interior choice, the marginal benefit and marginal cost are equal. But when the collateral constraint binds, the marginal benefit outweighs the marginal cost, so the shadow cost of the constraint is positive.

The Euler equation (6) shows that there are two motives why bottom earners may want to hold the bubbly asset. First, as in the standard rational bubble framework, bottom earners value the bubbly asset's "speculative" return. Second, if credit-constrained, they value the collateral service it provides. The second motive is new relative to the standard framework, and plays an important role in our analysis.

2.2 Top earners' problem

Since the top earners always end up lending in equilibrium, we can ignore their collateral constraint and default decision without loss of generality. In young age, a top earner household chooses its lending and bubble holding to maximize expected lifetime utility. Formally, its problem is:

$$\max_{b_t^r,d_t^r} u(c_{y,t}^r)+\beta E_t[c_{o,t+1}^r] \tag{8}$$

subject to budget constraints:

$$\begin{aligned} c_{y,t}^r+P_tb_t^r &= y_t^r+d_t^r, \\ c_{o,t+1}^r &= T+\xi_{t+1}P_{t+1}b_t^r-(1-h_{t+1})R_td_t^r, \end{aligned} \tag{9}$$

and a bubble no-short-selling constraint:

$$b_t^r \geq 0. \tag{10}$$

Here, $h_{t+1} \in [0,1]$ denotes the "haircut" (or fraction of creditor's loss) on loans. This haircut is determined in equilibrium as a result of borrowers' and lenders' aggregate choices (see equation (15) below), and is potentially stochastic, since the value of the seizable bubbly collateral depends on whether the bubble bursts. An individual lender takes the distribution of h_{t+1} as given. Let λ_t^r and $\mu_{b,t}^r$ denote the multipliers on the young age budget constraint (9) and bubble no-short-selling constraint (10), respectively. The first order conditions of top earners are:

$$\lambda_t^r = u'\left(c_{y,t}^r\right) \tag{11}$$

$$\mu_{b,t}^r = P_t \lambda_t^r - \beta P_{t+1} E_t\left[\xi_{t+1}\right] \tag{12}$$

$$\lambda_t^r = \beta E_t\left[1 - h_{t+1}\right] R_t. \tag{13}$$

Equation (11) is standard, stating that the household's shadow value of funds in young age is equal to its marginal utility of consumption. Equation (12) states that the shadow value of the household's no-short-selling constraint on the bubbly asset is equal to its marginal cost of acquiring the bubbly asset, minus its marginal benefit for holding it. This condition is analogous to the bottom earners' optimality condition for bubble holding, with the difference stemming from the fact that top earners do not use their bubble holding as collateral. Finally, equation (13) states that the marginal benefit of lending, $\beta E_t\left[1 - h_{t+1}\right] R_t$, is equal to the marginal cost of lending λ_t^r.

Combining equations (12) and (13), we obtain a no-arbitrage condition linking the expected returns on the bubbly asset and from lending:

$$\mu_{b,t}^r = \beta P_t \left(E_t\left[1 - h_{t+1}\right] R_t - E_t\left[\xi_{t+1}\right] \frac{P_{t+1}}{P_t} \right). \tag{14}$$

This condition reflects the fact that if the current bubble price is strictly positive, then the top earners' multiplier on their bubble no-short-selling constraint is proportional to the gap between the expected return on lending and the expected return on the bubbly asset. If the former is higher than the latter, then lending dominates holding the bubble, so it is not worth it for top earners to hold the bubble.

2.3 Equilibrium

In equilibrium, the haircut on loans is given by:

$$h_{t+1} = \begin{cases} 0 & \text{if } \delta_{t+1} = 0 \text{ (no default)} \\ 1 - \frac{(1-\theta)\left(D+\phi\xi_{t+1}P_{t+1}b_t^p\right)}{\theta R_t(-d_t^r)} & \text{if } \delta_{t+1} = 1 \end{cases} \tag{15}$$

When debtor households do not default on their contractual obligations, the haircut is zero. When they do default, however, the aggregate seized collateral is distributed equally among creditor households. Since in equilibrium the bottom earners borrow and the top earners lend, the haircut is determined by the ratio of the aggregate collateral, $(1-\theta)\left(D + \phi\xi_{t+1}P_{t+1}b_t^p\right)$, to the aggregate claims, $\theta R_t(-d_t^r)$ (recall that $-d_t^r$ is the top earners' loan position). This setup of haircut in the case of default is similar to that used in Kumhof et al. (2015). We define an equilibrium as follows:

Definition 1. Given the initial price of the bubbly asset $P_0 \geq 0$ and a stochastic process $\{\xi_t\}_{t=0}^{\infty}$ that determines the persistence or collapse of the bubble, a competitive equilibrium consists of portfolio choices $\{b_t^r, b_t^p, d_t^r, d_t^p\}_{t=0}^{\infty}$, default decisions $\{\delta_t\}_{t=1}^{\infty}$, haircuts $\{h_t\}_{t=1}^{\infty}$, bubble prices $\{P_t\}_{t=1}^{\infty}$ and interest rates $\{R_t\}_{t=0}^{\infty}$ such that:

1. given prices, haircuts, and the bubble bursting process, the portfolio choices and default decisions solve households' optimization problems,
2. the market for the bubbly asset clears in every period: $\theta b_t^r + (1-\theta)b_t^p = 1$,
3. the credit market clears in every period: $\theta d_t^r + (1-\theta)d_t^p = 0$,
4. the haircut is determined by equation (15).

If $P_t > 0$ for any $t > 0$, then the equilibrium is called a *bubbly equilibrium*; otherwise, it is called a *bubble-less equilibrium*. An *asymptotic bubbly equilibrium* is a bubbly equilibrium in which $\lim_{t\to\infty} P_t > 0$. A *steady state* is an equilibrium in which all quantities and prices are time-invariant.

3 Benchmarks

We now consider two useful benchmarks: the first best attainable in the absence of credit frictions and the bubble-less equilibrium prevailing under credit frictions.

3.1 First best

Consider an ideal world where households can perfectly commit to repay their debt and do not face any credit constraint. It is straightforward to see that in this world, young age consumption is equalized across households: $c^r_{y,t} = c^p_{y,t} = \bar{y}$, and the interest rate is:

$$R^{fb} = \beta^{-1} u'(\bar{y}).$$

To sustain this consumption allocation, bottom earners borrow $d^p = (\bar{y} - y^p) = \theta\,(y^r - y^p) > 0$, and top earners lend $d^r = (\bar{y} - y^r) = -\,(1-\theta)\,(y^r - y^p) < 0$.

Throughout the paper, we make the following assumption:

Assumption 1. *The frictionless economy is dynamically efficient:*

$$R^{fb} \geq 1. \tag{A1}$$

Under this assumption, the interest rate in the frictionless economy is higher than or equal to the gross growth rate rate of the economy, which is conveniently assumed to be one. As is well known, bubbles cannot emerge under this scenario (Tirole, 1985).

3.2 Bubble-less equilibrium

Next, consider a world with credit frictions but no bubble (i.e., $P_t = 0$ for all t). We impose the following additional assumption:

Assumption 2. *There is sufficient inequality or sufficiently severe financial frictions:*

$$D < \underbrace{\beta^{-1} u'\left(\bar{y}\right)}_{R^{fb}} \cdot \theta (y^r - y^p). \tag{A2}$$

This assumption guarantees that the first-best allocation cannot be achieved in the bubble-less economy. The left-hand side of (A2) is the amount of available fundamental collateral, while the right-hand side is the amount a bottom earner needs to commit to repay in old age for the first-best allocation of Section 3.1 to be sustained by a market allocation. Thus, (A2) requires that financial frictions are sufficiently severe (D is sufficiently small) or that inequality is sufficiently high ($y^r - y^p$ is sufficiently large). Assumptions (A1) and (A2) are imposed in the rest of the paper. The following Lemma then characterizes the bubble-less steady equilibrium:

Lemma 1. *There is a unique bubble-less equilibrium. In this equilibrium, the bottom earners' collateral constraint always binds: $R_t d_t^p = D \ \forall t$, and there is no equilibrium default. The bubble-less equilibrium features time-invariant allocations and prices: $d^{p,nb} = \frac{D}{R^{nb}}$, $d^{r,nb} = -\frac{1-\theta}{\theta} d^{p,nb}$, $c_y^{p,nb} = y^p + d^{p,nb}$, $c_y^{r,nb} = y^r + d^{r,nb}$, and the interest rate R^{nb} solves the top earners' Euler condition:*

$$u'\left(y^r - \frac{1-\theta}{\theta}\frac{D}{R^{nb}}\right) = \beta R^{nb}. \tag{16}$$

Proof. See Appendix A.1. □

Due to credit frictions, young bottom earners under-consume and young top earners over-consume relative to the first-best levels. The over-consumption of young top earners implies that the interest rate is below the first-best interest rate: $R^{nb} < R^{fb}$. From equation (16), it is straightforward to see that the interest rate R^{nb} is increasing in the amount of fundamental collateral D and is decreasing in the top earners' income y^r.

4 Unleveraged bubble

We now study bubbly equilibria in which bubble purchase is *unleveraged* (i.e., not financed by credit). We will see that if the bubbly asset has a sufficiently low pledgeability (ϕ is sufficiently small), then unleveraged bubbly steady states are the only ones that can prevail. Such steady states exist more generally when bubble pledgeability is low and bubbles are not too risky.

4.1 Bubble market participation

We start by arguing that bubbly steady states in which bottom earners purchase the bubble can be ruled out when the degree of bubbly asset pledgeability is sufficiently small.

Lemma 2. *For sufficiently small ϕ, there cannot exist a bubbly steady state in which bottom earners hold the bubbly asset.*

Proof. See Appendix A.2. □

Intuitively, when ϕ is small, the bubble provides little collateral value, and since bottom earners are credit constrained, their shadow value of funds in young age is strictly higher than the interest rate, which in equilibrium is equal to the expected return from investing in the bubble. Thus, for bottom earners, the cost of purchasing the bubble outweighs the benefits of holding it. As a result, in any bubbly steady state it is optimal for young bottom earners not to hold the bubble. Such steady states hence feature only top earners purchasing the bubble: $b^r = \frac{1}{\theta}$ and $b^p = 0$.

4.2 Existence of unleveraged bubbles

We derive existence conditions for unleveraged bubbles. In an equilibrium where only the top earners hold the bubbly asset, loans are default-free. As a consequence, the top earners' no-arbitrage condition between lending and investing in the bubbly asset, obtained by combining (12) and (13), reduces to $R_t = (1 - p_{burst})P_{t+1}/P_t$. Thus, in an unleveraged bubbly steady state, the interest rate must equate the expected return on investing in the bubble: $R^{ub} = 1 - p_{burst}$, where the superscript ub stands for "unleveraged bubble." The young top earners' consumption is given by their Euler equation for lending:

$$c_y^r = (u')^{(-1)} \left(\beta(1 - p_{burst})\right), \tag{17}$$

while the young bottom earners' consumption is given by their budget constraint:

$$c_y^p = y^p + \frac{D}{1 - p_{burst}}. \tag{18}$$

The credit market clearing condition implies that the steady state price of the bubble is given by the difference between young earners' aggregate income and aggregate consumption:

$$P \quad = \quad \bar{y} - \theta c_y^r - (1-\theta)c_y^p. \tag{19}$$

Hence, there is a bubbly steady state ($P > 0$) if and only if the aggregate income of the young exceeds their aggregate consumption: $\bar{y} > \theta c_y^r + (1-\theta)c_y^p$. Equations (17)-(19) allow us to establish the following necessary and sufficient conditions for the existence of a bubbly steady state:

Proposition 1 (Unleveraged bubble existence)**.** *There exists an unleveraged bubbly steady state if and only if the bubbly asset has sufficiently low collateral value relative to its probability of persisting:*

$$\phi \leq 1 - p_{burst} \tag{20}$$

and the bubble-less steady state interest rate implicitly given by (16) is sufficiently low:

$$R^{nb} < 1 - p_{burst}. \tag{21}$$

Proof. See Appendix A.3. □

An implication of condition (21) is that the riskiness of an unleveraged bubble, measured by its probability of collapsing p_{burst}, cannot be too high. Intuitively, for high probabilities of collapsing, the bubble is dominated as a savings vehicle by the risk-less loans associated with a bubble-less equilibrium, and the market efficiently selects the bubble-less equilibrium over an unleveraged bubbly equilibrium.

Another implication of Proposition 1 is that an unleveraged bubble can only exist if there is a sufficient shortage of fundamental collateral in the economy, i.e., the bottom earners' borrowing capacity D falls short of the top earners' demand for a store of value by too much.[12] This corollary parallels the classic result in Tirole (1985) that there must be an aggregate shortage of storage in order for bubbly equilibria to exist. Our model, however, has two extra ingredients relative to Tirole (1985)'s framework: inequality (or heterogeneous endowments) and financial frictions. First, due to heterogeneous endowments, a shortage of storage is only required for a *subset* of households in the economy (i.e., top earners) to sustain a bubbly equilibrium. Second, the shortage of storage is the consequence of an enforcement friction that limits bottom earners' ability to absorb top earners' savings.

Finally, it is straightforward from equation (19) to see that the steady state bubble price P is increasing in inequality (i.e., fixing $\bar{y}$, P is increasing in $\theta(y^r - y^p)$), decreasing in fundamental collateral D, and decreasing in the bursting probability p_{burst}.

To sum up, an unleveraged bubble emerges when the bubbly asset pledgeability is not too high, when the bubble is not too risky, and when there is a sufficient shortage of store of value in the economy. Under these conditions, a bubble can serve as a useful substitute savings vehicle for the top earners. As debtor households are not using bubbly collateral, it is not optimal for them to default when the bubble bursts. One can interpret these characteristics as heuristically consistent with housing bubbles in emerging economies, such as China, where households have limited ability to borrow using home equity or housing collateral, down payments for housing purchases are large, and ownership of real estate is concentrated among high-income households (see, e.g., Tomba and Tang, 2008, Wang et al., 2015, and Fang et al., 2015).

5 Leveraged bubble

We now analyze bubbly equilibria in the scenario where the bubbly asset has high pledgeability (ϕ is large). We view such a parametrization as a stylized representation of the recent U.S. economy, in which financial innovations have enabled households to borrow against increasingly large shares of their home via mortgage and home equity loans. We derive results regarding bubble market participation and then determine parameter regions for which bubbly equilibria exist.

[12] Recall from (16) that R^{nb} is increasing in the fundamental collateral D.

5.1 Bubble market participation

Our first result is that a high bubbly asset pledgeability provides incentives for bottom earners to use the bubbly asset as collateral for borrowing, potentially resulting in *leveraged* bubbly equilibria. We interpret this result as formalizing the notion that easy credit may facilitate the occurrence of a credit-fueled bubbly episode.

Proposition 2 (*Bottom earners hold bubble*). *Suppose $\phi > 1 - p_{burst}$. In any bubbly steady state, bottom earners use bubbly collateral, i.e., $b^p > 0$ and $Rd^p > D$, and are credit constrained, i.e., $\mu_d^p > 0$.*

Proof. See Appendix A.4. □

Intuitively, there are two motives for which bottom earners may want to hold the bubbly asset. First, just like top earners, bottom earners value the bubbly asset's investment return. Second, unlike unconstrained top earners, credit constrained bottom earners value the collateral service provided by the bubbly asset. When $\phi > 1 - p_{burst}$, this collateral service is so valuable to bottom earners that at prices prevailing under the conjecture that the bubbly collateral is not used, bottom earners should demand an infinite amount of debt backed by an infinite amount of bubble holding. Since the bubbly asset is in finite net supply, this cannot be an equilibrium outcome. It follows that in any bubbly steady state, if $\phi > 1 - p_{burst}$ then bottom earners must hold some bubble and use it as collateral for borrowing.

The fact that bottom earners participate in the bubble market in a bubbly equilibrium despite their relatively high shadow value of funds in young age might appear puzzling. After all, as long as the bubble cannot be fully collateralized (i.e., as long as $\phi < 1$), why would bottom earners, who are natural borrowers rather than savers, purchase the bubble at the cost of reducing their young age consumption? The answer relies on the observation that by purchasing the bubbly asset, bottom earners essentially acquire a complex security whose effective rate of return dominates that of the actual bubbly asset. The cost of this security is the price of the bubble, net of the collateral service it provides, $P\left(1 - \frac{\phi}{R}\right)$, while its expected payoff is the bubble's expected payoff, net of the expected debt repayment, $P\left(1 - p_{burst}\right)\left(1 - \phi\right)$. It is straightforward to show that the expected return on this complex security, $\frac{(1-p_{burst})(1-\phi)}{\left(1-\frac{\phi}{R}\right)}$, is higher than the expected return on the actual bubble, $1 - p_{burst}$.[13] It is this higher return that incentives bottom earners to participate in the bubble market.

A similar intuition applies to our next result, which states that when the bubbly asset's pledgeability is very high, then the demand for the asset from bottom earners is so strong that *only* bottom earners may hold it in a bubbly equilibrium.

[13] It suffices to show that $R \leq 1$. This inequality is a direct consequence of equations (34) and (36) in Appendix A.8, and equation (37) in Appendix A.9. The inequality is strict when $\phi < 1$.

Proposition 3 (*Only bottom earners hold bubble*)**.** *There exists a threshold $\hat{\phi} \in (1-p_{burst}, 1)$ such that for $\phi > \hat{\phi}$, only bottom earners hold the bubbly asset in any bubbly steady state.*

Proof. See Appendix A.5. □

This seemingly counterintuitive endogenous exclusion of top earners from the bubble market is a consequence of our limited enforcement and defaultable debt assumption. When bottom earners use bubbly collateral, they shift the bursting risk of the bubbly asset to creditors by defaulting on their debt when the bubble bursts. This *risk-shifting* behavior, leading leveraged debtors to not fully internalize the risk of their portfolio, is a classic phenomenon associated with defaultable debt. In a nutshell, the option to default generates a kink and therefore convexity in the payoff function of borrowers, thus inducing risk-loving behavior among agents with otherwise risk-neutral preferences (see, e.g., Jensen and Meckling, 1976 or Stiglitz and Weiss, 1981).[14] In most existing models of risk-shifting, investment in a risky asset is financed with defaultable debt by assumption. In our model, in contrast, whether investment is financed with debt is an endogenous outcome that depends on the extent that the bubbly asset can be used as collateral.

5.2 Existence of leveraged bubbles

Having established results regarding bottom earners' participation in the bubble market when a bubbly steady state exists, we now turn to the existence of leveraged bubbles. First, we derive an analytical existence condition for the polar case where $\phi = 1$. Then, we numerically analyze existences regions for $\phi < 1$.

The following proposition establishes a necessary and sufficient condition for the existence of a leveraged bubbly steady state in the case where the bubbly asset is fully pledgeable.

Proposition 4 (*Leveraged bubble existence with full collateralization*)**.** *Suppose $\phi = 1$. There exists a leveraged bubbly steady state if and only if the bubble-less steady state interest rate implicitly given by (16) is sufficiently low:*

$$R^{nb} < 1. \tag{22}$$

Proof. See Appendix A.6. □

When $\phi = 1$, bubble purchase is financed by debt and only bottom earners hold the bubbly asset (recall Proposition 3). The existence condition states that there is a sufficient

[14] See also Allen and Gale (2000), Barlevy (2014) and Ikeda and Phan (forthcoming) for applications in a bubble context.

shortage of store of value in a bubble-less steady state. This again has a flavor of the classic existence condition in Tirole (1985). However, unlike in the case of unleveraged bubbles in Section 4, the existence condition here does not involve the riskiness of the bubble, represented by the bursting probability p_{burst}. This indicates that very risky bubbles (i.e., ones satisfying $0 < 1 - R^{nb} < p_{burst}$) cannot exist when the pledgeability parameter ϕ is low, but do exist when ϕ is high. In other words, a high pledgeability of bubbly assets expands the existence region of bubbly equilibria. This result is confirmed numerically for high values of ϕ that are strictly smaller than 1 (see below). Intuitively, as a consequence of risk-shifting, the market does not prevent highly risky leveraged bubbles from emerging, as it would with unleveraged bubbles.[15] This result suggests that financial market development/innovation that increases asset pledgeability may facilitate the emergence of risky asset bubbles.

Finally, as was the case with unleveraged bubbles, it is straightforward to show that the steady state bubble price P is decreasing in the bursting probability p_{burst}.[16] Furthermore, P is increasing in top income y^r. Therefore, whether with unleveraged or leveraged bubbles, our model formalizes the ideas put forward by Rajan (2011) and Stiglitz (2012) that high inequality facilitates the emergence of asset bubbles.

When $\phi < 1$, existence conditions for leveraged bubbly steady states are not as straightforward to characterize analytically. We therefore proceed by exploring existence regions numerically. There are two types of leveraged bubbly steady states: *purely* leveraged ones (where only bottom earners hold the bubbly asset) and *partially* leveraged ones (where both bottom and top earners hold the bubbly asset). The conditions that need to be satisfied in a purely and partially leveraged bubbly steady state are given in Appendix A.8 and A.9, respectively. Among these conditions are the requirements that the bottom earners' credit constraint binds and that the bubble price is positive. Figure 2 displays existence regions of purely and partially leveraged bubbly steady states, along with that of an unleveraged bubbly steady state, as a function of the top income share $\theta y^r/\bar{y}$ and the bubble pledgeability parameter ϕ (left panel) or the bursting probability parameter p_{burst} (right panel).[17]

In accordance with the conditions of Proposition 1, unleveraged bubbly steady states (vertically hatched area) exist for high enough top income shares and low enough degrees of bubble pledgeability (left panel), or low enough probabilities of bubble bursting (right panel). In the latter case, as long as $\phi \leq 1 - p_{burst}$, the riskier the bubble (i.e., the higher

[15] Also see Ikeda and Phan (forthcoming) for a similar result: risk-shifting facilitates the existence of very risky bubbles.

[16] Thus, even though the risk of bursting does not affect the existence condition, it does affect the demand for bubble and thus its price.

[17] The parameters used for the calculations are $\bar{y} = 1$, $\beta = 0.7$, $D = 0.25$, $\theta = 0.1$ and $p_{burst} = 0.15$ (left panel) or $\phi = 0.5$ (right panel). These parameters satisfy Assumptions 1 and 2.

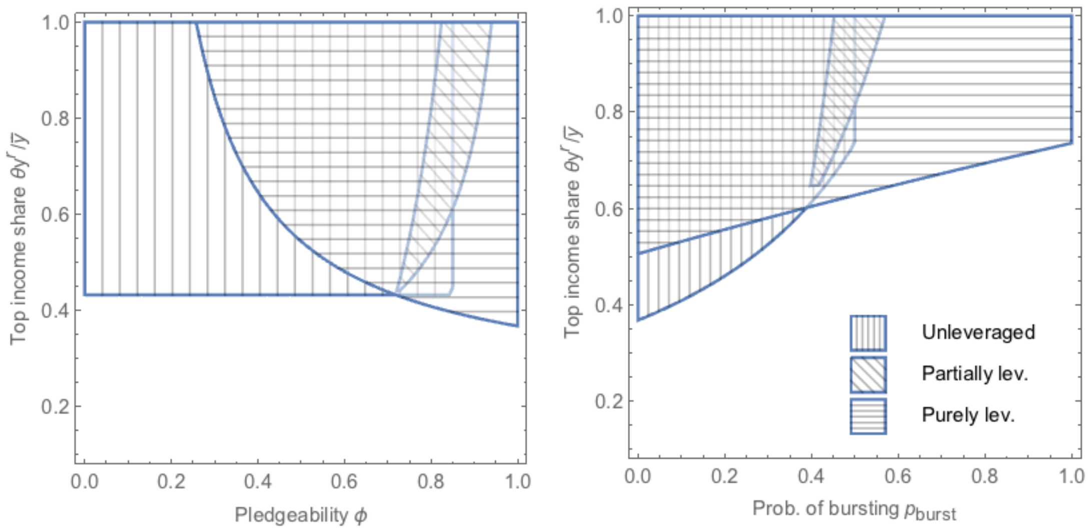


Figure 2: Existence regions of bubbly steady states.

p_{burst}), the higher the required top income share for the bubble to exist. This is because the higher p_{burst}, the higher the degree of inequality required to depress the bubble-less interest rate below $1 - p_{burst}$.

Turning to leveraged bubbly steady states, the plots indicate that the existence region (the horizontally and diagonally hatched areas) is not confined to the $\phi = 1$ polar case. Purely leveraged bubbly steady states exist for high enough top income shares and large enough degrees of pledgeability (the higher the degree of pledgeability, the lower the required top income share). Partially leveraged bubbly steady states exist for high enough top income shares and intermediate degrees of pledgeability (the higher the top income share, the larger the admissible range of pledgeability degrees). Confirming Proposition 4's intuition that a higher pledgeability parameter ϕ facilitates the existence of leveraged bubbles, the left panel of Figure 2 shows that the existence region of leveraged bubbly steady state expands when ϕ increases. Furthermore, as Proposition 3 stated, when ϕ becomes sufficiently large, only purely leveraged bubbly steady states exist.

The right panel shows that very risky bubbles can only exist in a leveraged bubbly steady state characterized by risk-shifting, as suggested by Proposition 4.[18] However, it also shows that a higher risk of bubble bursting contracts the existence region. The riskier the bubble, the higher the required top income share for a leveraged bubbly steady state to exist. The intuition is that a higher probability of bursting tames bottom earners' demand for the bubbly asset. Partially leveraged bubbly steady states exist for high enough top income

[18]In fact, purely leveraged bubbles even exist for $p_{burst} \to 1$ (arbitrarily risky bubbles), but their size converges to zero.

shares and intermediate degrees of bubble riskiness (the higher the income share, the larger the admissible range of riskiness degrees).[19]

In summary, Figure 2 illustrates the importance of income inequality, bubbly asset pledgeability, and bubble riskiness in facilitating the existence of bubbles as well as shaping their characteristics (unleveraged, partially leveraged, or purely leveraged). When bubble pledgeability ϕ is high, bubble investment is necessarily leveraged. A bubbly episode is then associated with a boom in borrowing among bottom earners, who use their bubble holding as collateral and default on their debt obligations when the bubble collapses. These characteristics are broadly consistent with the stylized features of the U.S. bubble episode of the 2000s (see Mian and Sufi, 2011, 2014).

6 Conclusion

We have presented a simple tractable general equilibrium model of rational bubbles in the presence of income inequality and financial frictions. The theory predicts that large income inequality facilitates the emergence of asset bubbles. It also suggests that financial innovations that increase the ability of agents to borrow against the values of bubbly assets render possible the emergence of *leveraged* bubble episodes, in which bottom earners finance their purchase of bubbly assets by selling collateralized debt obligations to top earners. Because of risk-shifting arising from limited commitment, top earners may not participate in the bubble market despite being the natural demanders for a store of value. Risk-shifting further eliminates the market's ability to prevent highly risky bubbles from emerging. When a leveraged bubble bursts, bottom earners find themselves "under water" and optimally choose to default. The predictions of our model are largely consistent with stylized features of the recent U.S. housing bubble.

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[19] The numerical exercise also indicates that there may be overlaps between the existence regions of leveraged and unleveraged bubbly equilibria. Despite $\phi \leq 1 - p_{burst}$ being a necessary condition for existence of an unleveraged bubbly steady state, it is apparent that $\phi > 1 - p_{burst}$ is *not* a necessary condition for leveraged bubbly steady states. Further, the existence region for a partially bubbly steady state is found to coincide with subsets of the other two types of bubbly equilibria considered. As a result, for given bursting probability p_{burst} and bubble pledgeability parameter ϕ, our model may simultaneously feature up to three types of bubbly equilibria.

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A Online Appendix

A.1 Proof of Lemma 1

Suppose on the contrary that (CC) does not bind in some period t in a bubble-less equilibrium. Then it must be that $u'(c^r_{y,t}) = \beta R_t = u'(c^p_{y,t})$. This, together with the resource constraint $\theta c^r_{y,t}+(1-\theta)c^p_{y,t} = \bar{y}$, requires $c^r_{y,t} = c^p_{y,t} = \bar{y} = c^{fb}_y$. It then follows that the interest rate is $R_t = u'(\bar{y})/\beta = R^{fb}$. From the bottom earners' budget constraint, we thus have $d^p = \theta\left(y^r - y^p\right)$. The bottom earners' collateral constraint is therefore indeed slack if $D \geq R_t d^p_t = R^{fb} \cdot \theta\left(y^r - y^p\right)$. However, this contradicts assumption (A2). Hence (A2) guarantees that (CC) is binding. The rest of the lemma is straightforward.

A.2 Proof of Lemma 2

First, we establish the following useful result:

Lemma: There is no bubbly steady state in which the following equality holds:

$$u'(c^p_y) = u'(c^r_y) = \beta(1 - p_{burst}). \tag{23}$$

Proof: We show that (23) contradicts assumption (A1). To see this, observe that (23) implies that:

$$c^p_y = c^r_y = (u')^{(-1)}(\beta(1 - p_{burst})).$$

Hence, from the aggregate resource constraint, we have:

$$P = \bar{y} - (u')^{(-1)}(\beta(1 - p_{burst})).$$

Since $P > 0$ in any bubbly steady state, it follows that $\bar{y} > (u')^{(-1)}(\beta(1 - p_{burst}))$, or equivalently $u'(\bar{y}) < \beta(1 - p_{burst})$. However, this contradicts assumption (A1) that $R^{fb} \equiv \frac{u'(\bar{y})}{\beta} \geq 1$. QED

Next, we proceed by proving Lemma 2 for $\phi = 0$ by contradiction. Suppose on the contrary that there exists a bubbly steady state in which bottom earners hold the bubble. Then the non-negativity of the multiplier on top earners' no-short-selling constraint (12) requires top earners' steady state marginal utility to satisfy:

$$u'\left(c^r_y\right) \geq \beta\left(1 - p_{burst}\right).$$

Now, for bottom earners to hold the bubble, it must be that the multiplier on their bubble no-short-selling constraint (6) is zero, or equivalently, that bottom earners' steady state marginal cost of acquiring the bubble is equal to the marginal benefit of holding it:

$$u'\left(c^p_y\right) = \beta\left(1 - p_{burst}\right).$$

Combining these two conditions, we must have $u'\left(c^p_y\right) \leq u'\left(c^r_y\right)$. On the other hand, the non-negativity of bottom earners' shadow value of credit constraint in (7) implies $u'\left(c_y^p\right) \geq \beta R = u'\left(c_y^r\right)$. For these two inequalities to hold, we must have $u'\left(c_y^p\right) = u'\left(c_y^r\right) = \beta(1-p_{burst})$. But this is a contradiction with the lemma above.

Next, we prove Lemma 2 for the case of strictly positive but small ϕ , again by contradiction. Suppose on the contrary that there exists a bubbly steady state in which bottom earners hold the bubble. Then it must be that

$$Rd^p > D. \tag{24}$$

To see why this is the case, suppose on the contrary that $Rd^p \leq D$. Then there is no default in steady state. Furthermore, since $b^p > 0$, the inequality $Rd^p \leq D$ implies that credit constraint (CC) is not binding for bottom earners. Hence $u'(c_y^p) = \beta R = u'(c_y^r)$. Furthermore, with (CC) not binding, the first order condition of bottom earners with respect to bubbles gives $u'(c_y^p) = \beta(1-p_{burst})$. Hence (23) must hold, which leads to a contradiction.

Next, note that (24) and (CC) together imply that $\lim_{\phi\to 0} Rd^p = D$ and therefore $\lim_{\phi\to 0} h = 0$. There are two possible cases, in each of which we derive a contradiction.

Case 1: only bottom earners hold the bubble. Top earners' first order condition for bond holdings is $u'(y^r - \frac{1-\theta}{\theta}\frac{D}{R}) = \beta\left(1-p_{burst}h\right)R$. Thus $\lim_{\phi\to 0} R = R^{nb}$ from equation (16), and $\lim_{\phi\to 0} c_y^r = (u')^{-1}\left(\beta R^{nb}\right) > \bar{y}$. Meanwhile, bottom earners' first order condition for bubble holdings is $u'(c_y^p) = \beta(1-p_{burst})\left(\frac{1-\phi}{1-\frac{\phi}{R}}\right)$. Thus $\lim_{\phi\to 0} c_y^p = (u')^{-1}\left(\beta(1-p_{burst})\right) > \bar{y}$, where the inequality follows from assumption (A1) that $u'\left(\bar{y}\right) \geq \beta$. Therefore, for every $\epsilon > 0$, there is a $\delta > 0$ such that $c_y^r > c_{y,nb}^r - \epsilon$ for all $\phi < \delta$. Similarly, for every $\epsilon > 0$, there is a $\delta' > 0$ such that $c_y^p > (u')^{-1}\left(\beta(1-p_{burst})\right) - \epsilon$ for all $\phi < \delta'$. Hence, for all $\phi < \min\{\delta, \delta'\}$ we have $\theta c_y^r + (1-\theta)\, c_y^p > \theta\left(c_y^r\right)^{nb} + (1-\theta)\,(u')^{-1}\left(\beta(1-p_{burst})\right) - \epsilon$. Choose $\epsilon = \theta c_y^{r,nb} + (1-\theta)\,(u')^{-1}\left(\beta(1-p_{burst})\right) - \bar{y}$. Then $\theta c_y^r + (1-\theta)\, c_y^p > \bar{y}$ and therefore $P < 0$ for all $\phi < \min\{\delta, \delta'\}$, a contradiction.

Case 2: both bottom and top earners hold the bubble. Top earners' first order condition for bubble holding and bond holding is $u'(c_y^r) = \beta\left(1-p_{burst}\right) = \beta\left(1-p_{burst}h\right)R$. Thus $c_y^r = (u')^{-1}\left(\beta(1-p_{burst})\right) > \bar{y}$, and $\lim_{\phi\to 0} R = 1-p_{burst}$. Meanwhile, bottom earners' first order condition for bubble holding is $u'(c_y^p) = \beta(1-p_{burst})\left(\frac{1-\phi}{1-\frac{\phi}{R}}\right)$. Thus $\lim_{\phi\to 0} c_y^p = (u')^{-1}\left(\beta(1-p_{burst})\right) > \bar{y}$. Therefore, for every $\epsilon > 0$, there is a $\delta' > 0$ such that $c_y^p > (u')^{-1}\left(\beta(1-p_{burst})\right) - \epsilon$ for all $\phi < \delta'$. Hence for all $\phi < \delta'$, we have $\theta c_y^r + (1-\theta)\, c_y^p > (u')^{-1}\left(\beta(1-p_{burst})\right) - (1-\theta)\,\epsilon$. Choose $\epsilon = \frac{1}{1-\theta}[(u')^{-1}\left(\beta(1-p_{burst})\right) - \bar{y}]$. Then $\theta c_y^r + (1-\theta)\, c_y^p > \bar{y}$ and therefore $P < 0$ for all $\phi < \delta'$, a contradiction.

A.3 Proof of Proposition 1

Necessity. Suppose there is an unleveraged bubbly steady state (i.e., where only top earners hold the bubble). Then we must have $b^p = 0$, $b^r = \frac{1}{\theta}$ and $\delta = 0$ and $h = 0$. Top earners' Euler equations for the bubble and bonds, (12) and (13), imply that the interest rate is given by $R = 1 - p_{burst}$. We begin by showing that bottom earners' credit constraint is binding.

Lemma: Bottom earners' credit constraint binds ($\mu_d^p > 0$).

Proof: Assume on the contrary that $\mu_d^p = 0$. Then bottom and top earners' Euler equations for bonds, (7) and (13), imply $u'(c_y^p) = u'(c_y^r) = \beta\,(1 - p_{burst})$. Given Assumption 1, we thus have

$$u'(c_y^p) = u'(c_y^r) \leq \beta \leq u'(\bar{y})$$

and therefore $c_y^p = c_y^r \geq \bar{y}$. Summing up the agents' young age budget constraints, one gets $P = \bar{y} - \theta c_y^r - (1-\theta)c_y^p \leq 0$. This contradicts the supposition of a bubbly steady state (i.e., in which $P > 0$). QED

Since $\mu_d^p > 0$, the bubbly steady state satisfies $d^p = \frac{D}{1-p_{burst}}$, $d^r = -\frac{1-\theta}{\theta}\frac{D}{1-p_{burst}}$, $c_y^p = y^p + \frac{D}{1-p_{burst}}$, $c_y^r = y^r - \frac{1-\theta}{\theta}\frac{D}{1-p_{burst}} - \frac{P}{\theta}$, $\lambda^p = u'\left(c_y^p\right)$, $\mu_d^p = \frac{\lambda^p}{1-p_{burst}} - \beta$ and

$$\mu_b^p = \left[\lambda^p - \beta(1 - p_{burst}) - \phi\mu_d^p\right] P. \tag{25}$$

Substituting the expression for μ_d^p into (25), we get $\mu_b^p = \left[(1 - p_{burst}) - \phi\right]\mu_d^p P$. Since $\mu_d^p > 0$, for μ_b^p to be non-negative, it must be that $\phi \leq 1 - p_{burst}$.

Next, we show that $R^{nb} < 1 - p_{burst}$. Top earners' Euler equation for bonds (13) implies:

$$\beta(1 - p_{burst}) = u'(y^r - \frac{1-\theta}{\theta}\frac{D}{1-p_{burst}} - \frac{P}{\theta}). \tag{26}$$

Since the right-hand side is strictly increasing in P and its maximum is positive infinity (due to the Inada condition), (26) has a positive solution if and only if the right-hand side evaluated at $P = 0$ is strictly smaller than the left-hand side:

$$\beta(1 - p_{burst}) > u'(y^r - \frac{1-\theta}{\theta}\frac{D}{1-p_{burst}}).$$

From equation (16), this inequality holds if and only if $R^{nb} < 1 - p_{burst}$, as desired.

Furthermore, as the right-hand side of (26) is strictly increasing, if there is a solution, then this solution is unique.

Sufficiency. Condition (21) guarantees that there is a unique solution to equation (26) determining the steady state bubble P. From this we construct a steady state where $R = 1 - p_{burst}$, $d^p = \frac{D}{R}$, $d^r = -\frac{1-\theta}{\theta}d^p$, $b^r = \frac{1}{\theta}$, and $b^p = 0$.

A.4 Proof of Proposition 2

We prove both parts of the statement by contradiction. With start by proving that $Rd^p > D$. Suppose instead that there is a bubbly steady state with $Rd^p \leq D$. In such a steady state, there is no bubble collateral, and there is no equilibrium default. We now argue that $R = 1 - p_{burst}$. If top earners hold the bubble, this equality is implied by their no-arbitrage condition between bonds and bubble holding. If top earners do not hold the bubble, then bottom earners must hold it, and given that they are not credit constrained, this same equality derives from their no-arbitrage condition between bonds and bubble holding. Either way, the Euler equations of bottom earners (6) and (7) under no default ($\delta = 0$) can be written as:

$$\begin{aligned} \frac{\mu_b^p}{P} &= \lambda^p - \beta(1 - p_{burst}) - \phi\mu_d^p \\ \mu_d^p &= \frac{\lambda^p - \beta(1 - p_{burst})}{(1 - p_{burst})}. \end{aligned}$$

Combining these equations yields:

$$\frac{\mu_b^p}{P} = [(1 - p_{burst}) - \phi]\, \mu_d^p$$

Since the right-hand side is negative under the maintained assumption that $\phi > 1 - p_{burst}$, for μ_b^p and μ_d^p to be both non-negative, it must be that $\mu_d^p = \mu_b^p = 0$. That is, bottom earners' credit constraint and no-short-selling constraint on bubble are both slack.

The fact that (CC) is slack for bottom earners implies that $u'(c_y^p) = u'(c_y^r) = \beta(1 - p_{burst})$. Given Assumption 1, we thus have

$$u'(c_y^p) = u'(c_y^r) \leq \beta \leq u'(\bar{y})$$

and therefore $c_y^p = c_y^r \geq \bar{y}$. Summing up the agents' young age budget constraints, one gets $P = \bar{y} - \theta c_y^r - (1-\theta)c_y^p \leq 0$. This contradicts the supposition of a bubbly steady state (i.e. in which $P > 0$). Therefore, it must be that $Rd^p > D$.

Next, we prove that $\mu_d^p > 0$. Suppose on the contrary that $\mu_d^p = 0$. Then bottom earners' Euler equation for bubble and bond holding, (6) and (7), and top earners' Euler equation for bond holding, (13), are given by

$$\begin{aligned} u'(c_y^p) &= \beta(1 - p_{burst}) && (27) \\ u'(c_y^p) &= \beta(1 - p_{burst})R && (28) \\ u'(c_y^r) &= \beta(1 - p_{burst}h)R && (29) \end{aligned}$$

Combining (27) and (28), we obtain $R = 1$. Combining (28) and (29), and making use of the fact that $h \geq 0$, we get that $u'(c_y^p) \leq u'(c_y^r)$. Given Assumption 1, we thus have

$$u'(c_y^p) \leq u'(c_y^r) < \beta \leq u'(\bar{y})$$

and therefore $c_y^p \geq c_y^r > \bar{y}$. Summing up the agents' young age budget constraints, one gets $P = \bar{y} - \theta c_y^r - (1-\theta)c_y^p < 0$. This contradicts the supposition of a bubbly steady state (i.e. in which $P > 0$). Therefore, it must be that $\mu_d^p > 0$.

A.5 Proof of Proposition 3

First, define $\hat{\phi}$ as the unique positive root of the following quadratic equation:

$$(1-p_{burst})\,\bar{y}\phi^2 + \left[(1-\theta)\,D - (1-p_{burst})\,\bar{y}\right]\phi - (1-p_{burst})\,(1-\theta)\,D = 0.$$

Note that $\phi > \hat{\phi} \in (1-p_{burst}, 1)$ if and only if $\phi \geq \gamma(\phi)$, where:

$$\gamma(\phi) \equiv \frac{1-p_{burst}}{1-p_{burst}\frac{\phi\bar{y}}{(1-\theta)D+\phi\bar{y}}}.$$

From Proposition 2, it immediately follows from $\phi > \hat{\phi} > 1-p_{burst}$ that there is no bubbly steady state in which only top earners hold the bubble, as that would imply $Rd^p \leq D$.

We now show that it is impossible for there to be a bubbly steady state in which both types of agents hold the bubble when $\phi \geq \gamma(\phi)$. Suppose on the contrary that there is one such steady state where both types hold the bubble. For this to be the case, it must be that $\mu_b^r = 0$. From Proposition 2, we know that $Rd^p > D$, and hence in this steady state, there is equilibrium default if and only if the bubble bursts, in which case the haircut is $h = \frac{\phi Pb^p}{D+\phi Pb^p}$. Hence $\delta_{t+1} = 1$ if and only if $\xi_{t+1} = 0$, so that $E_t[1-\delta_{t+1}] = E[(1-\delta_{t+1})\xi_{t+1}] = 1-p_{burst}$. The Euler equations of top earners, (12) and (13), requires that expected returns on risky bonds and the bubble are equalized: $(1-p_{burst}h)R = 1-p_{burst}$. The interest rate that makes top earners indifferent between holding bonds and the bubble is thus given by $R = (1-p_{burst})\,/\,(1-p_{burst}h)$, which is strictly increasing in the haircut h. The resource constraint puts an upper bound on the bottom earners' aggregate bubble outlays, $Pb^p \leq \frac{\bar{y}}{1-\theta}$, and thus in equilibrium the bottom earners' bubbly collateral is bounded above, $\phi Pb^p \leq \phi\frac{\bar{y}}{1-\theta}$. Hence the haircut is also bounded above: $h = \frac{\phi Pb^p}{D+\phi Pb^p} \leq \frac{\phi\bar{y}}{(1-\theta)D+\phi\bar{y}}$. As a result, the interest rate compatible with top earners holding the bubble is bounded above as well: $R \leq \gamma(\phi) < 1$.

Turning to bottom earners, combining Euler equations (6) and (7) yields:

$$\phi = R - \frac{\frac{\mu_b^p}{P} + (1-R)\,\beta\,(1-p_{burst})}{\mu_d^p}. \tag{30}$$

Since $\phi \geq \gamma(\phi)$, (30) implies:

$$\frac{\frac{\mu_b^p}{P} + (1-R)\,\beta\,(1-p_{burst})}{\mu_d^p} \leq R - \gamma(\phi) \leq 0$$

which requires a negative multiplier on the bottom earners' bubble no-short-selling constraint, $\mu_b^p < 0$, a contradiction.

A.6 Proof of Proposition 4

Necessity. Suppose there is a leveraged bubbly steady state. From Proposition 2 we know that $Rd^p > D$, $\mu_d^p > 0$, and hence in this steady state, there is equilibrium default if and only if the bubble bursts, in which case the haircut is $h = \frac{Pb^p}{D+Pb^p} > 0$. Hence $\delta_{t+1} = 1$ if and only if $\xi_{t+1} = 0$, so that $E_t[1-\delta_{t+1}] = E[(1-\delta_{t+1})\xi_{t+1}] = 1 - p_{burst}$. Next, from Proposition 3 we know that only bottom earners hold the bubble. This implies bubble holdings of $b^p = \frac{1}{1-\theta}$ and $b^r = 0$. Combining bottom earners' Euler equations for bubble and bond holdings, (6) and (7), we obtain $R = 1$. Given the obtained bubble holdings, the haircut when borrowers default after the bubble bursts is $h = \frac{\phi P/(1-\theta)}{D+\phi P/(1-\theta)} > 0$. The expected recovery rate is therefore given by $E[1-\tilde{h}] = 1 - p_{burst} \times \frac{\phi P/(1-\theta)}{D+\phi P/(1-\theta)}$. Top earners' Euler equation for bond holdings hence reduces to:

$$u'\left(y^r - \frac{1-\theta}{\theta}D - \frac{P}{\theta}\right) = \beta\left(1 - p_{burst}\frac{P}{(1-\theta)\,D + P}\right). \tag{31}$$

This equation determines the steady state P. Since u' is strictly decreasing, the left-hand side is a strictly increasing function of P. The Inada condition implies that the limit of the left-hand side as $P \to \frac{\theta}{1-\theta}y^r - D$ is $+\infty$. The right-hand side is a strictly decreasing function of P. Thus, there is a solution $P > 0$ to this equation if and only if $u'(y^r - \frac{1-\theta}{\theta}D) < \beta$, which coincides with condition (22). Also, if there is a solution, then the solution is unique. This implies that the limiting bubble must also be the steady state bubble. Thus, we have shown the necessity result.

Sufficiency. Condition (22) guarantees that there is a unique solution to (31), which gives the steady state bubble P. From this we construct a steady state where $R = 1$, $d^p = D + P/(1-\theta)$, $d^r = -\frac{1-\theta}{\theta}d^p$, $b^r = 0$, and $b^p = 1/(1-\theta)$.

Remark on convergence: In addition, as in Tirole (1985), we can show the following convergence result: If $P_0 < P$, then the equilibrium is asymptotically bubble-less, if $P_0 = P$ then the equilibrium is an asymptotic bubbly equilibrium, and there is no bubbly equilibrium if $P_0 > P$. The law of motion for bubble price is:

$$u'\left(y^r - \frac{1-\theta}{\theta}\frac{D + \phi P_{t+1}/(1-\theta)}{P_{t+1}}P_t\right) = \beta\left(1 - p_{burst}\frac{\phi P_{t+1}/(1-\theta)}{D + \phi P_{t+1}/(1-\theta)}\right)\frac{P_{t+1}}{P_t}. \tag{32}$$

The proof is as follows: The left-hand side of the law of motion equation (32) is strictly decreasing in P_{t+1}, while the right-hand side is strictly increasing in P_{t+1}. At the same time, the left-hand side of the steady state equation (31) is strictly increasing in P, while the right-hand side is strictly decreasing in P. Thus, $P_{t+1} > P_t$ if and only if $P_t > P$. This means that P is the unique saddle path stable fixed point of the dynamical system for $\{P_t\}_{t=0}^{\infty}$ defined by equation (32): if $P_0 > P$, then P_t is strictly increasing in t and thus cannot converge to the bubbly steady state; while if $P_0 < P$, then P_t is strictly decreasing in t, and can only converge to the bubble-less steady state. Only when $P_0 = P$, we have an asymptotic bubbly equilibrium, which coincides with the bubbly steady state. QED

A.7 Derivation of unleveraged bubbly steady state

We derive the system of equations characterizing an unleveraged bubbly steady state. Such a steady state, where only top earners hold the bubble, can only arise when $\phi \leq 1 - p_{burst}$ and $u'\left(y^p + \frac{D}{1-p_{burst}}\right)\frac{1}{1-p_{burst}} - \beta \geq 0$. Bubble holdings are given by $b^r = \frac{1}{\theta}$ and $b^p = 0$. It must thus be that there is no default and no haircut: $\delta = h = 0$. Top earners' Euler equations for the bubble and bonds, (12) and (13), imply that the interest rate is given by $R = 1 - p_{burst}$. Top earners' Euler equation for bonds (13) therefore implies:

$$\beta(1 - p_{burst}) = u'(y^r - \frac{1-\theta}{\theta}\frac{D}{1-p_{burst}} - \frac{P}{\theta}). \tag{33}$$

An unleveraged bubbly steady state is a bubble price P solving (33), while satisfying the conditions $P > 0$. The requirement that $u'\left(y^p + \frac{D}{1-p_{burst}}\right)\frac{1}{1-p_{burst}} - \beta \geq 0$ guarantees that $\mu_d^p \geq 0$.

A.8 Derivation of purely leveraged bubbly steady state

This appendix derives the system of equations characterizing a purely leveraged bubbly steady state for the case where $\gamma < \phi \leq 1$. From Proposition 2, we know that $Rd^p > D$, $\mu_d^p > 0$ and hence in this kind of steady state, there is equilibrium default if and only if the bubble bursts, in which case the haircut is $h = \frac{\phi P b^p}{D + \phi P b^p} > 0$. Hence $\delta_{t+1} = 1$ if and only if $\xi_{t+1} = 0$, so that $E_t[1 - \delta_{t+1}] = E_t[(1 - \phi\delta_{t+1})\xi_{t+1}] = 1 - p_{burst}$. Next, from Proposition 3 we know that only bottom earners hold the bubble. This implies bubble holdings of $b^p = \frac{1}{1-\theta}$ and $b^r = 0$. Combining bottom earners' Euler equations (6) and (7) yields an arbitrage condition:

$$u'\left(y^p + \frac{D + \phi\frac{P}{1-\theta}}{R} - \frac{P}{1-\theta}\right)\left(1 - \frac{\phi}{R}\right) = \beta(1 - p_{burst})(1 - \phi). \tag{34}$$

Given the steady state bubble holdings, the haircut when borrowers default after the bubble bursts is $h = \frac{\phi P/(1-\theta)}{D + \phi P/(1-\theta)} > 0$. The expected recovery rate is therefore given by $E[1 - \tilde{h}] = 1 - p_{burst} \times \frac{\phi P/(1-\theta)}{D + \phi P/(1-\theta)}$. Top earners' Euler equation for lending is therefore given by:

$$u'\left(y^r - \frac{1-\theta}{\theta}\frac{D + \phi\frac{P}{1-\theta}}{R}\right) = \beta\left(1 - p_{burst}\frac{\phi\frac{P}{1-\theta}}{D + \phi\frac{P}{1-\theta}}\right)R. \tag{35}$$

A purely leveraged bubbly steady state is a pair (R, P) solving the system of equations consisting of (34) and (35), while also satisfying the following conditions:

$$\begin{aligned}
P &> 0, \\
R &\geq \phi, \\
\mu_b^r &= P\left[u'\left(y^r - \frac{1-\theta}{\theta}\frac{D + \phi\frac{P}{1-\theta}}{R}\right) - \beta\left(1 - p_{burst}\right)\right] \geq 0, \\
\mu_d^p &= \frac{1}{R}u'\left(y^p + \frac{D + \phi\frac{P}{1-\theta}}{R} - \frac{P}{1-\theta}\right) - \beta\left(1 - p_{burst}\right) \geq 0.
\end{aligned} \tag{36}$$

A.9 Derivation of partially leveraged steady state

This appendix derives the system of equations characterizing a partially leveraged bubbly steady state. The holdings of the bubbly asset are given by b^p for bottom earners and $b^r = \frac{1-(1-\theta)b^p}{\theta}$ for top earners. In such a steady state, the bottom earner's collateral constraint binds, and there is equilibrium default if and only if the bubble bursts, in which case the haircut is $h = \frac{\phi P b^p}{D+\phi P b^p}$. Hence $\delta_{t+1} = 1$ if and only if $\xi_{t+1} = 0$, so that $E_t[1-\delta_{t+1}] = E[(1-\phi\delta_{t+1})\xi_{t+1}] = 1 - p_{burst}$. The top earner's Euler equations (12) and (13) yield two conditions:

$$R = \frac{1-p_{burst}}{p_{burst}\frac{D}{D+\phi P b^p} + 1 - p_{burst}} \tag{37}$$

and

$$u'\left(y^r - \frac{(1-\theta)}{\theta}\frac{D+\phi P b^p}{R} - P\frac{1-(1-\theta)\,b^p}{\theta}\right) = \beta(1-p_{burst}), \tag{38}$$

while the bottom earners' Euler equations (6) and (7) yield:

$$u'\left(y^p + \frac{D+\phi P b^p}{R} - P b^p\right)\left(1-\frac{\phi}{R}\right) = \beta(1-p_{burst})(1-\phi). \tag{39}$$

A partially leveraged bubbly steady state is a triplet (R, P, b^p) solving the system of equations consisting of (37), (38), and (39), while also satisfying conditions:

$$\begin{aligned}
P &> 0,\\
\mu_d^p &= \frac{1}{R}u'\left(y^p + \frac{D+\phi P b^p}{R} - P b^p\right) - \beta\left(1-p_{burst}\right) \geq 0,\\
b^p &> 0,\\
b^p &< \frac{1}{1-\theta}.
\end{aligned}$$