Dynamic Noisy Rational Expectations Equilibria with Anticipative Information[1]

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Abstract

This paper studies a dynamic continuous time economy with discrete dividend payment dates and anticipative private information about future dividends. The economy is populated by informed and uninformed investors as well as active unskilled investors. Both competitive and monopolistic informed behaviors are examined. The existence of noisy rational expectations equilibria is demonstrated. Equilibria are derived in closed form and their properties analyzed. Weak-form efficiency is shown to fail. Informed trading is found to reduce price volatility, hence to stabilize the market. Conditions for Pareto efficiency of equilibria with private information are derived.

Keywords: Intertemporal noisy rational expectations equilibria, anticipative information, competitive behavior, monopolistic behavior, noise trading model, price volatility, market price of risk, premium, welfare, Pareto optimality, multiple dividend cycles.

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1 Introduction

The question of informational efficiency of the stock market has been hotly debated for decades. On the one hand, a more efficient financial market seems desirable to the extent that it transmits more information to uninformed investors and therefore helps to improve the allocation of resources. On the other hand, it can be argued that allowing informed trading is inherently unfair, because it favors those with private information. Moreover, the presence of informed trading could reduce participation, leading to a decrease in liquidity (adverse selection problem). Regulation restricting informed trading has therefore been enacted in several countries. Regulation is meant to offer some protection to small investors and ensure the smooth operation of the financial market.

This article examines issues surrounding the informational and allocational efficiency of the stock market, in a dynamic framework with private information about future dividends. Closed form solutions for dynamic equilibria with competitive as well as monopolistic informed behavior are derived. Price, volatility and risk premium properties are studied. In both types of equilibria, weak form efficiency fails. The price is not a sufficient statistic for public information. The informational efficiency of the economy, however, is shown to increase and the stock price volatility to decrease, relative to an equilibrium without private information. Strong horizon effects are found. The price reactions to the underlying fundamental and to the endogenous noisy signal revealed vary through the dividend cycle. Equilibria are compared. Monopolistic behavior reduces informational efficiency and increases volatility, hence destabilizes the market, relative to the competitive outcome.

Classical studies pertaining to informational efficiency are based on static models. Seminal articles, identifying the determinants of efficiency in competitive markets, are those of Grossman (1976, 1978) and Grossman and Stiglitz (1980). They demonstrate the possibility, as well as the limits, of informationally efficient markets. Issues related to non-competitive behavior are examined by Hellwig (1980), Kyle (1989) and Leland (1992). The first study argues that informed investors who are aware of their price impact should not behave competitively. It resolves this apparent inconsistency, dubbed the "schizophrenia" problem, by showing that agents can no longer affect the price, in the limit competitive equilibrium, as the number of informed investors becomes large. The second study considers informed investors who explicitly account for the impact of their demands on the equilibrium price. It shows that imperfect competition resolves the schizophrenia problem. It also finds that prices are less informative with imperfect competition. The third study focuses more specifically on insider trading and on properties of equilibrium in a static model with production and monopolistic insider behavior. Among other results, it finds that private information trading increases the average stock price, decreases the stock return's expectation and variance for the uninformed, reduces the liquidity of the market and can increase or decrease welfare.

Dynamic models with asymmetric information and competitive behavior were pioneered by Wang (1993, 1994). In these models, the stock is an infinitely-lived asset that pays dividends continuously through time. Informed investors observe the state variable driving the expected future dividend. Uninformed investors do not, but they learn through dividends and prices. Noise trading injects supply uncertainty and prevents full revelation. Wang (1993) derives a competitive noisy rational expectations equilibrium (NREE). This equilibrium is stationary as the coefficients of the price process are constant. Asymmetric information is shown to increase the stock's long run risk premium. It can also increase the price volatility and enhance negative serial correlation. Asymmetric information can therefore have a destabilizing effect. Wang (1994) focuses on issues pertaining to the volume of trade in a similar, but not identical, setting. The article highlights the relation between volume and price changes. The effects of imperfect competition and asymmetric information on the dynamic properties of prices and liquidity are examined in Vayanos and Wang (2012). Their analysis is cast in a model with three periods. They show, in particular, that asymmetric information and imperfect competition can have opposite effects on ex-ante expected returns.[3]

Albuquerque and Miao (2014) extend the competitive model of Wang (1994) by allowing for private advance information about future dividends. They also allow for a private investment opportunity. In their model, time is discrete and advance information pertains to the temporary component of the dividend paid at the next date. Agents derive utility over next period wealth. They solve for the stationary equilibrium by conjecturing a state space and a pricing rule. The stationary solution is obtained up to a system of non-linear equations. The paper shows that good advance information increases the stock price and the risk premium. It also shows that informed

[3] A vast microstructure literature also deals with non-competitive informed trading. Fundamental contributions are in Kyle (1985) and Glosten and Milgrom (1985). In these models, risk neutral market makers extract private information from the aggregate order flow and set the price so as to break even on average. This pricing rule does not account for the endogenous interactions between risk, price appreciation and price level. The absence of diversification benefits implies that trading is purely informational. Moreover, the price evolution is typically determined by the exogenous noise trading behavior and is locally orthogonal to fundamental risk.

(resp. uninformed) investors behave as trend chasers (resp. contrarian).

The model developed in this article builds on both Wang (1993, 1994) and Albuquerque and Miao (2014). It differs in several respects. The first difference is that the analysis is not restricted to stationary equilibria. Both competitive and monopolistic non-stationary equilibria are derived and studied. The second is that equilibria are obtained in closed form. All coefficients are explicit functions of time, reflecting the time left to the next dividend payment date. Strong timing effects are identified. The third difference is that a new solution method is introduced. The approach relies on the construction of the private information price of risk (PIPR) in the equilibrium under consideration.[4] The PIPR isolates the effects of private information. Its properties hint at the structure of equilibrium and can be used to formulate natural conjectures about the informational content of the stock price. The fourth difference is the nature of private information that pertains to the dividend level at the future payment date and is therefore long-lived. More precisely, information has value throughout a dividend cycle and will be used continuously for trading. The value of information, reflected in the PIPR, nevertheless changes in light of fundamental news that accumulate.

Another difference with the literature is that noise trading takes a more elaborate form in our setting. Noise traders are utility maximizing agents with bounded rationality. They hold correct beliefs conditional on the realization of the signal, but evaluate these beliefs based on unfounded rumors as opposed to factual private information. They can be viewed as unskilled active traders. Ultimately, their optimal demand behavior mimics the demand behavior of the informed, but based on conditional beliefs evaluated at irrelevant noise. The behavioral noise trading model postulated enables us to endogeneize the noise trading demand function and conduct a meaningful welfare analysis. The dynamic welfare results obtained extend the static analysis in Leland (1992).

The following insights emerge from the analysis. First, weak form efficiency fails. That is, the stock price alone is not a sufficient statistic for the endogenous noisy signal embedded in the aggregate demand function. The pair composed of the price and the fundamental is needed to recover public information. Second, private information trading is found to stabilize the market. Private information reduces the volatility of the stock price, in both the competitive and monopolistic models. The reduction is more significant in the competitive setting. Third, the volatility of the

[4]The notion of PIPR is introduced in Detemple and Rindisbacher (2013) in the context of a portfolio selection problem with private information.

stock price increases through the dividend cycle and converges to the volatility of the fundamental as the next dividend date approaches. Paradoxically, fundamental information, which accumulates through the dividend cycle and eventually announces the dividend payment, is the source of this risk increase. Fourth, the Sharpe ratio varies stochastically through the dividend cycle. Its volatility also increases over time. Hence, the covariance between the change in the stock price and the change in the Sharpe ratio increases within a dividend cycle. Last, conditions for Pareto dominance of equilibria where regulation permits private information trades, are identified. Low or high risk tolerances are conducive to welfare improvements across agents. With multiple dividend cycles, informational efficiency gains can be reinforced. The stock price can become substantially less sensitive to fundamental shocks, relative to the economy where private information trades are not permitted. Enhanced price stabilization and welfare gains follow.

Section 2 describes the model. Section 3 derives and studies the noisy rational expectations equilibrium with competitive informed behavior. Section 4 studies the case of monopolistic informed behavior. An extension of the model to multiple dividend cycles is in Section 5. Conclusions follow. Appendix A provides welfare results for the multi-cycle model. Proofs are collected in Appendix B.

2 The Economy

This section describes the structure of the model. The financial market is presented in Section 2.1, agents and their information sets in Section 2.2 and candidate stock price processes in Section 2.3. Preferences and optimal stock demands are in Sections 2.4 and 2.5. Equilibrium is defined in Section 2.6.

2.1 Assets and Markets

There are two types of assets in the economy, a riskless asset and a risky stock. The riskless asset is a money market account paying interest at the instantaneous rate r. In the absence of intertemporal consumption, which will be assumed, the interest rate can be set at zero ($r = 0$). The risky stock pays a liquidating dividend D_T at the terminal date T. The dividend payment is the terminal value of the process,

$$dD_t = \mu^D dt + \sigma^D dW_t^D, \qquad t \ \in [0,T]$$

where μ^D is a constant drift coefficient and σ^D is a constant and positive volatility coefficient. W^D is a Brownian motion process with filtration $\mathcal{F}^D_{(\cdot)}$, defined on a probability space $\left(\Omega, \mathcal{F}^D, P\right)$. The process D can be viewed as a fundamental factor that eventually determines the terminal dividend.

The stock trades at an endogenously determined price S. Trading takes place in continuous time. There are no restrictions on stock holdings or borrowing.

2.2 Agents, Noise and Information Signal

Three groups of investors operate in the financial market, informed, uninformed and noise traders. The respective fractions of the three groups in the population are ω^i, ω^u and ω^n, with $\omega^i + \omega^u + \omega^n = 1$. Each group is treated as a homogeneous entity with a representative individual.

The (representative) informed investor is a skilled individual, able to extract information about the future stock payoff D_T. Information extraction is carried out at the initial date $t = 0$ and generates the noisy signal $G = D_T + \zeta$, where $\zeta \sim \mathcal{N}\left(0, \left(\sigma^\zeta\right)^2\right)$. Skill is measured by the precision $v_\zeta = \left(\sigma^\zeta\right)^{-2}$ of the signal. When $\left(\sigma^\zeta\right)^2$ increases, precision falls and the informational content of the signal decreases. Thus, skill decreases. In the limit, when $\left(\sigma^\zeta\right)^2 \to \infty$, the signal becomes pure noise and skill vanishes. The "informed" investor effectively becomes unskilled (uninformed).

The uninformed investor does not have extraction ability. He/she observes prices and other quantities that are in the public information set. Let $\mathcal{F}^m_{(\cdot)}$ be the public information filtration.

The noise trader is a mimicking agent. He/she tries to emulate the demand behavior of the informed agent, but on the basis of irrelevant noise as opposed to factual private information. The noise trader's demand depends on an independent random variable ϕ. A precise description is provided below.

2.3 Stock Price and Information Sets

The opportunity set of investors depends on the stock price structure. In this environment, there are two sources of uncertainty, W^D associated with fundamental information and ϕ with noise trading behavior. Standard arguments can be invoked to write any candidate price process as,

$$dS_t = \mu^S_t dt + \sigma^S_t dW^S_t, \qquad S_T = D_T. \tag{1}$$

In this structure W^S is a Brownian motion relative to the public information filtration $\mathcal{F}^m_{(\cdot)}$. It is endogenous and, ultimately, relates to the underlying source of fundamental uncertainty W^D. The coefficients $\left(\mu^S, \sigma^S\right)$ of the price process are also endogenous and adapted to $\mathcal{F}^m_{(\cdot)}$. The uninformed observes the stock price, hence can retrieve the volatility coefficient from its quadratic variation. The Brownian motion $dW^S_t = \left(\sigma^S_t\right)^{-1}\left(dS_t - \mu^S_t dt\right)$ is an innovation process in their filtration. The information filtration $\mathcal{F}^S_{(\cdot)}$ generated by S is in the public information flow $\mathcal{F}^m_{(\cdot)}$. That is, $\mathcal{F}^S_{(\cdot)} \subseteq \mathcal{F}^m_{(\cdot)}$.

The information set of the informed is augmented by the private signal G. Private information is carried by the enlarged filtration $\mathcal{F}^G_{(\cdot)} \equiv \mathcal{F}^m_{(\cdot)} \vee \sigma(G)$. As private information modifies the perception of the risk-reward trade-off, the fundamental source of risk W^D is no longer Brownian motion relative to the enlarged filtration. Instead, the translated process,

$$dW^G_t = dW^S_t - \theta^{G|m}_t(G)\,dt \qquad \text{where} \qquad \theta^{G|m}_t(G)\,dt \equiv E\left[dW^S_t \middle| \mathcal{F}^G_t\right]$$

becomes a Brownian motion. The translation factor $\theta^{G|m}_t(G)$ is the private information price of risk (PIPR), which is a function of the private signal G.[5] Relative to private information, the stock price evolution is $dS_t = \left(\mu^S_t + \sigma^S_t\theta^{G|m}_t(G)\right)dt + \sigma^S_t dW^G_t$. The superior information is reflected in the private information premium $\sigma^S_t\theta^{G|m}_t(G)$. Given that public information $\mathcal{F}^m_{(\cdot)}$ is endogenous, the private information premium is endogenous as well.

2.4 Informed and Uninformed Preferences and Optimal Stock Demands

Throughout the paper, superscripts i and u are used to distinguish the informed (i) from the uninformed (u) investor. Let X^j_t denote the wealth of investor j at time t, $j \in \{i, u\}$. Conditional preferences have the mean-variance structure,

$$\mathcal{U}^j\left(\mathcal{F}^j_0\right) = \begin{cases} E\left[X^i_T - \frac{1}{2\Gamma}\int_0^T d\left[X^i\right]_s \middle| \mathcal{F}^G_0\right] & \text{for } j = i \\ E\left[X^u_T - \frac{1}{2\Gamma}\int_0^T d\left[X^u\right]_s \middle| \mathcal{F}^m_0\right] & \text{for } j = u \end{cases} \tag{2}$$

[5]The PIPR is invariant with respect to strictly monotonic transformations of G. Indeed, the private information generated by G coincides with the information generated by $G^* = h(G)$, if $h(\cdot)$ is strictly monotone. The PIPR for G^* is $\theta^{G|m}_t\left(h^{-1}(G^*)\right) = \theta^{G|m}_t(G)$. As any signal with a strictly monotonic continuous distribution is a strictly monotonic transformation of a Gaussian signal, the equilibrium analysis in this paper applies for signals of this type.

where $[X]$ denotes the quadratic variation (realized variance) of X and Γ (resp. $1/\Gamma$) is a common absolute risk tolerance (resp. risk aversion) parameter. Preferences of the informed (resp. uninformed) are conditional on private (resp. public) information. The conditional utility functional (2) shows that investors care about terminal wealth X_T, but also dislike the risk $[X]_T = \int_0^T d[X]_s$, i.e., the realized variance, associated with it. The utility function depends on these two attributes.[6] Foundations for multiattribute preferences are in Keeney and Raiffa (1976). The ex-ante utility is $\mathcal{U}^j = E\left[\mathcal{U}^j\left(\mathcal{F}_0^j\right)\right]$ where the expectation is taken relative to the information signals in the sets $\mathcal{F}_0^j, j \in \{i, u\}$.

Investors maximize utility (2) subject to the dynamics of wealth,

$$dX_t^j = \begin{cases} N_t^i\left(\left(\mu_t^S + \sigma_t^S \theta_t^{G|m}(G)\right) dt + \sigma_t^S dW_t^G\right) & \text{for } j = i \\ N_t^u\left(\mu_t^S dt + \sigma_t^S dW_t^S\right) & \text{for } j = u \end{cases} \tag{3}$$

and the informational constraint mandating that N^j be adapted to $\mathcal{F}_{(\cdot)}^j$ for $j \in \{i, u\}$. The policy N^j represents the number of shares held. Proposition 1 describes the optimal demands.

Proposition 1 *The optimal number of shares held by the informed and uninformed investors are,*

$$N_t^u = \Gamma \frac{\mu_t^S}{\left(\sigma_t^S\right)^2} = \Gamma \frac{\sigma_t^S \theta_t^m}{\left(\sigma_t^S\right)^2} \qquad \text{and} \qquad N_t^i = \Gamma \frac{\mu_t^S + \sigma_t^S \theta_t^{G|m}(G)}{\left(\sigma_t^S\right)^2} = \Gamma \frac{\sigma_t^S\left(\theta_t^m + \theta_t^{G|m}(G)\right)}{\left(\sigma_t^S\right)^2}$$

for $t \in [0, T]$, where θ^m is the price of risk for the uninformed. The informed holds more (resp. less) shares than the uninformed if and only if the private information premium $\sigma_t^S \theta_t^{G|m}(G)$ is positive (resp. negative).

Optimal stock demands have a mean-variance structure. The difference between the two investors resides in their evaluation of the expected stock return. The informed evaluates the return on the basis of private information as well as public information. The resulting expected return has two components. The first one, $\mu_t^S = \sigma_t^{S,D} \theta_t^m$, is the expected return based on public information. The second one, $\sigma_t^S \theta_t^{G|m}(G)$, is the additional premium calculated on the basis of private information. This premium is affine in the PIPR $\theta_t^{G|m}(G)$, i.e., the private information price of risk (see Detemple and Rindisbacher (2013)). The PIPR is the incremental price of risk assessed in light of information

[6] The preferences in (2) are linear in probabilities, hence time-consistent.

that is not revealed by public information sources. It represents the private information price of risk conditional on public information. Thus, the informed has an allocational demand, $\Gamma\theta_t^m/\sigma_t^S$, and an informational demand, $\Gamma\theta_t^{G|m}(G)/\sigma_t^S$. The uninformed has a pure allocational demand, $\Gamma\theta_t^m/\sigma_t^S$.

2.5 Mimicking Noise Trading, Bounded Rationality and Optimal Stock Demand

The noise trader is an agent with bounded rationality, who ultimately replicates the demand of the informed, but without the benefit of observing the private signal. Instead, this investor believes in rumors, blogs and other reports that are unrelated to fundamentals underlying the stock price. Specifically, conditional beliefs are $dP^n = a_T^n(\phi)\,dP \equiv \exp\left(\int_0^T \theta_t^{G|m}(\phi)\,dW_t^S - \frac{1}{2}\int_0^T \theta_t^{G|m}(\phi)^2\,dt\right)dP$ where ϕ is the realization of an independent, normally distributed random variable with mean μ^ϕ and variance $\left(\sigma^\phi\right)^2$. The function $a_T^n(\phi)$ is a beliefs distortion capturing the departure from rationality, conditional on the realization ϕ. It corresponds to the density of the private signal, $a_T^n(\phi) = \frac{P(G\in dx|\mathcal{F}_T^m)}{P(G\in dx|\mathcal{F}_0^m)}_{|x=\phi}$, but evaluated at the noise ϕ. The informed has the same beliefs distortion, but evaluated at the private signal G. The noise trader's information is the public information filtration $\mathcal{F}_{(\cdot)}^m$.

The noise trader conditional preferences are $\mathcal{U}^n(\phi) = E^n\left[X_T^n - \frac{1}{2\Gamma}\int_0^T d\left[X^n\right]_s \middle| \mathcal{F}_0^m\right]$ where the expectation is under the beliefs P^n, Γ is an absolute risk tolerance parameter and wealth satisfies $dX_t^n = N_t^n\left(\mu_t^S dt + \sigma_t^S dW_t^S\right)$. Equivalently, conditional preferences can be written as $\mathcal{U}^n(\phi) = E\left[X_T^n - \frac{1}{2\Gamma}\int_0^T d\left[X^n\right]_s \middle| \mathcal{F}_0^G\right]_{|G=\phi}$ where the expectation is under P and information is $\mathcal{F}_0^G = \mathcal{F}_0^m \vee \sigma(G)$ evaluated at $G=\phi$. In the beliefs P^n (resp. information $\mathcal{F}_0^G$ evaluated at ϕ) the stock price evolves according to $dS_t = \left(\mu_t^S + \sigma_t^S\theta_t^{G|m}(\phi)\right)dt + \sigma_t^S dW_t^\phi$ where W^ϕ is a P^n-Brownian motion (resp. $\mathcal{F}_0^{G|G=\phi}$-Brownian motion). The stock price of risk is believed to be $\theta_t^\phi \equiv \theta_t^m + \theta_t^{G|m}(\phi)$. Ex-ante utility is $\mathcal{U}^n = E\left[\mathcal{U}^n(\phi)\right]$ where the expectation is over the random variable ϕ.

Proposition 2 *The optimal number of shares held by the noise trader is,*

$$N_t^n = \Gamma\frac{\sigma_t^S\left(\theta_t^m + \theta_t^{G|m}(\phi)\right)}{\left(\sigma_t^S\right)^2} \tag{4}$$

for $t\in[0,T]$, where θ^m is the uninformed price of risk and $\theta_t^{G|m}(\phi)$ is a speculative premium/discount reflecting the departure from rationality. The noise trader holds more (resp. less) shares than the *uninformed if and only if the speculative premium $\sigma_t^S \theta_t^{G|m}(\phi)$ is positive (resp. negative).*

The optimal noise trading demand has two parts. The first part, $\Gamma\theta_t^m / \sigma_t^S$, is the usual mean-variance demand of an uninformed rational agent. This part reflects an allocational trading motive. The second part, $\Gamma\theta_t^{G|m}(\phi) / \sigma_t^S$, is a speculative demand associated with an informational signal consisting of pure noise. In the end, the noise trader demand mimics the demand of the informed. It effectively corresponds to the demand of an investor with randomized beliefs, i.e., an unskilled active investor.

Remark 3 *The combined demand of the informed and the noise trader, called the complementary demand, is,*

$$N_t \equiv \omega^i N_t^i + \omega^n N_t^n = \Gamma \frac{\omega\mu_t^S + \sigma_t^S \left(\omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi)\right)}{\left(\sigma_t^S\right)^2}$$

where $\omega = \omega^i + \omega^n$. The complementary demand is an affine function of the weighted average price of risk (WAPR) $\Theta_t\left(G, \phi; \omega^i, \omega^n\right) \equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi)$. If the PIPR is also an affine function, the complementary demand depends on $\Theta_t\left(G, \phi; \omega^i, \omega^n\right) = \Theta_t(Z; \omega)$, which is a function of the signal $Z \equiv \omega^i G + \omega^n \phi$ and is parametrized by the combined population weight $\omega = \omega^i + \omega^n$.

2.6 Equilibrium

A rational expectations equilibrium (REE) for the economy under consideration is a triplet of demands $\left(N^u, N^i, N^n\right)$ and a price process $dS_t = \mu_t^S dt + \sigma_t^S dW_t^S$, $S_T = D_T$, such that (i) Individual rationality: N^j is optimal for agent $j \in \{u, i, n\}$, and (ii) Market clearing: $\omega^u N^u + \omega^i N^i + \omega^n N^n = 1$.

The REE is noisy (NREE) if the informed and uninformed filtrations differ, $\mathcal{F}_{(\cdot)}^u \subset \mathcal{F}_{(\cdot)}^i$. The equilibrium is a competitive NREE if all agents take the price process as given when expressing their optimal demands. It is a monopolistic NREE if the informed agent takes the price impact of his/her trades into account when calculating the optimal demand function.

3 Competitive Noisy Rational Expectations Equilibrium

The competitive NREE is described in Section 3.1. Properties of the PIPR and the WAPR are examined in Section 3.2. Price and return properties are discussed in Section 3.3. Properties of the market depth measure and stock holdings are outlined in Section 3.4. A welfare analysis is performed in Section 3.5.

3.1 Competitive Equilibrium Structure

In order to present the main result, define the combined share of the informed and the noise trader $\omega = \omega^i + \omega^n$ and the functions of time,

$$\alpha(t) = \frac{1-\kappa_t\omega}{H(t)}\sigma^D, \qquad \beta(t) = -\omega\frac{1-\kappa_t\omega^i}{H(t)}\sigma^D, \qquad \kappa_t = \frac{\omega^i H(t)}{M(t)} \tag{5}$$

$$\gamma(t) = -\omega\frac{\left(1-\kappa_t\omega^i\right)\mu^D(T-t) - \omega^n\kappa_t\mu^\phi}{H(t)}\sigma^D, \qquad \lambda(t,s) = \frac{\omega^i\left(\sigma^D\right)^2(s-t)}{M(t)}, \ s\in[t,T] \tag{6}$$

$$H(t) = \left(\sigma^D\right)^2(T-t) + \left(\sigma^\zeta\right)^2, \qquad M(t) = \left(\omega^i\right)^2 H(t) + (\omega^n)^2\left(\sigma^\phi\right)^2. \tag{7}$$

The function $H(t) = Var\left(G|\mathcal{F}_t^D\right)$ is the conditional variance of the private signal G given fundamental information at time t. The function $M(t) = Var\left(Z|\mathcal{F}_t^D\right)$ is the conditional variance of an endogenous signal $Z \equiv \omega^i G + \omega^n\phi$ given fundamental information at time t. The coefficients $\kappa_t = \frac{COV(G,Z|\mathcal{F}_t^D)}{VAR(Z|\mathcal{F}_t^D)}$ and $\lambda(t,s) = \frac{COV(D_s,Z|\mathcal{F}_t^D)}{VAR(Z|\mathcal{F}_t^D)}$ are regression coefficients. The next proposition presents the NREE.

Proposition 4 *A competitive NREE exists. The equilibrium stock price is,*

$$S_t = A(t)D_t + B(t)Z + F(t) \qquad \textit{where} \qquad Z = \omega^i G + \omega^n\phi \tag{8}$$

and,

$$A(t) = \left(\frac{H(T)}{H(t)}\right)^\omega\left(\frac{M(T)}{M(t)}\right)^{1-\omega} \tag{9}$$

$$B(t) = \lambda(t,T) + \sigma^D\left(\int_t^T A(s)\left(\alpha(s) + \beta(s)\lambda(t,s)\right)ds\right) \tag{10}$$

$$F(t) = A(t)\mu^D(T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma}\int_t^T A(s)^2 ds + \sigma^D\int_t^T A(s)\gamma(s)ds - \omega^n I(t)\mu^\phi \tag{11}$$

$$I(t) = \lambda(t,T) + \sigma^D\int_t^T A(s)\beta(s)\lambda(t,s)ds \tag{12}$$

with $(\alpha, \beta, \gamma, \lambda)$ as defined in (5)-(7). The coefficients of the equilibrium stock price process (1) are,

$$\mu_t^S = \frac{\left(\sigma_t^S\right)^2}{\Gamma} - \sigma_t^S \Theta_t\left(Z; \omega\right), \qquad \sigma_t^S = A\left(t\right) \sigma^D \tag{13}$$

$$\Theta_t\left(Z; \omega\right) = \alpha\left(t\right) Z + \beta\left(t\right) D_t + \gamma\left(t\right) \tag{14}$$

where $\Theta_t\left(Z; \omega\right) \equiv \omega^i \theta_t^{G|m}\left(G\right) + \omega^n \theta_t^{G|m}\left(\phi\right)$ is the endogenous WAPR. Innovations in the uninformed filtration are $dW_t^S = dW_t^D - \theta_t^{D|m} dt$ with,

$$\theta_t^{D|m} = \frac{E\left[dW_t^D | \mathcal{F}_t^m\right]}{dt} = \frac{\omega^i \sigma^D}{M\left(t\right)} \left(Z - \omega^i \left(D_t + \mu^D\left(T - t\right)\right) - \omega^n \mu^\phi\right). \tag{15}$$

The evolution of the stock price in the public information is given by (1) where W^S is an $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,Z}$-Brownian motion.

The competitive equilibrium price in (8) is an affine function of the fundamental D and of the random variable Z. This random variable is a noisy translation of the private information signal G. It provides anticipative information about the terminal dividend, but is less informative than the private signal. Both the price S and the fundamental D are in the public information set $\mathcal{F}_{(\cdot)}^m$. It follows that Z is publicly observed as well. Thus, $Z \in \mathcal{F}_{(\cdot)}^m$ and $\mathcal{F}_{(\cdot)}^{D,Z} \subseteq \mathcal{F}_{(\cdot)}^{D,S} \subseteq \mathcal{F}_{(\cdot)}^m$. Conversely, the pair (D, Z) reveals the price S, i.e., $\mathcal{F}_{(\cdot)}^S \subseteq \mathcal{F}_{(\cdot)}^{D,Z}$. Thus, $\mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z} \subseteq \mathcal{F}_{(\cdot)}^m$.

In equilibrium, the uninformed extracts the noisy signal Z from the pair (D, S). The uninformed also observes the complementary aggregate demand function $\omega^i N_t^i + \omega^n N_t^n$, described in Remark 3. At equilibrium, the complementary demand is also affine in D and Z. If therefore fails to reveal any information beyond what is already contained in (D, S). In the end, the equilibrium public information set consists of the pair (D, Z). That is, $\mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z} = \mathcal{F}_{(\cdot)}^m$. The equilibrium uninformed filtration is $\mathcal{F}_{(\cdot)}^u = \mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{D,S} = \mathcal{F}_{(\cdot)}^{D,Z}$. The equilibrium informed filtration is strictly more informative, $\mathcal{F}_{(\cdot)}^i = \mathcal{F}_{(\cdot)}^G = \mathcal{F}_{(\cdot)}^m \vee \sigma\left(G\right) \supset \mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^u$. The equilibrium is a noisy rational expectations equilibrium.

In this competitive NREE, the stock price S_t is not a sufficient statistic for public information. In fact, $\sigma\left(S_t\right) \subset \mathcal{F}_t^m = \mathcal{F}_t^{D,Z}$, where the inclusion is strict. Weak form efficiency therefore fails. The pairs (D, Z) or (D, S) are needed to summarize the public information set. Fundamental information plays a crucial role for the evaluation of future opportunities and the determination of optimal demands.

Remark 5 *(Limit economy with small informed) Consider the limit economy with an infinitesimal informed population ($\omega^i \to 0$ and $\omega^u \to 1-\omega^n$). The limit equilibrium is,*

$$S_t^{si} = A^{si}(t)D_t + B^{si}(t)Z^{si} + F^{si}(t), \qquad Z^{si} = \omega^n \phi$$

$$\mu_t^{S,si} = \frac{\left(\sigma_t^{S,si}\right)^2}{\Gamma} - \sigma_t^{S,si}\Theta_t^{si}\left(Z^{si};\omega^n\right), \qquad \sigma_t^{S,si} = A^{si}(t)\,\sigma^D$$

$$\Theta_t^{si}\left(Z^{si};\omega^n\right) = \alpha^{si}(t)\,Z^{si} + \beta^{si}(t)\,D_t + \gamma^{si}(t)$$

where $\left(A^{si}, B^{si}, F^{si}, \alpha^{si}, \beta^{si}, \gamma^{si}\right)$ are defined in (68)-(70). The limit WAPR is $\Theta_t^{si}\left(Z^{si};\omega^n\right) = \omega^n \theta_t^{G|m,si}(\phi)$. Innovations in the uninformed filtration vanish $dW_t^S = dW_t^D$ because $\theta_t^{D|m} \to 0$ when $\omega^i \to 0$. The limit equilibrium fails to reveal any private information. If, in addition, there is no mimicking investor ($\omega^i, \omega^n \to 0$), the equilibrium collapses to a no-trade equilibrium where,

$$S_t^{si,0} = D_t + \mu^D(T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma}(T-t), \qquad \sigma_t^{S,si,0} = \sigma^D, \qquad \mu_t^{S,si,0} = \frac{\left(\sigma^D\right)^2}{\Gamma}.$$

Stock price volatilities in the economy of Proposition 4 and the two limit economies rank as $\sigma_t^S < \sigma_t^{S,si} < \sigma_t^{S,si,0} = \sigma^D$ for $t < T$. As the payment date approaches the volatilities converge, $lim_{t\to T}\sigma_t^S = lim_{t\to T}\sigma_t^{S,si} = lim_{t\to T}\sigma_t^{S,si,0} = \sigma^D$. Informed trading increases the informational efficiency of the market. It also stabilizes the market by reducing the stock's exposure to fundamental shocks and the associated price volatility.

Remark 6 *(Limit economy with small uninformed) Consider the limit economy with an infinitesimal uninformed population ($\omega^i \to 1-\omega^n$ and $\omega^u \to 0$). The limit equilibrium is,*

$$S_t^{su} = A^{su}(t)\,D_t + B^{su}(t)\,Z^{su} + F^{su}(t), \qquad Z^{su} = (1-\omega^n)\,G + \omega^n\phi$$

$$\mu_t^{S,su} = \frac{\left(\sigma_t^{S,su}\right)^2}{\Gamma} - \sigma_t^{S,su}\Theta_t^{su}\left(Z^{su};1\right), \qquad \sigma_t^{S,su} = A^{su}(t)\,\sigma^D$$

$$\Theta_t^{su}\left(Z^{su};1\right) = \alpha^{su}(t)\,Z^{su} + \beta^{su}(t)\,D_t + \gamma^{su}(t)$$

where the functions $(A^{su}, B^{su}, F^{su}, \alpha^{su}, \beta^{su}, \gamma^{su})$ *are defined in (71)-(75). If, in addition, there is no mimicking investor* ($\omega^i \to 1, (\omega^u, \omega^n) \to 0$)*, the equilibrium collapses to a no-trade equilibrium where*

$$S_t^{su,0} = A^{su}(t) D_t + B^{su,0}(t) G + F^{su,0}(t), \qquad Z^{su} = G$$

$$\mu_t^{S,su,0} = \frac{\left(\sigma_t^{S,su,0}\right)^2}{\Gamma}, \qquad \sigma_t^{S,su,0} = \sigma_t^{S,su} = A^{su}(t) \sigma^D, \qquad \Theta_t^{su}(Z^{su}; 1) = 0$$

with $\left(A^{su}, B^{su,0}, F^{su,0}\right)$ *as defined in (76)-(77). The pair* $\left(D, S^{su,0}\right)$*, in the limit economy, is fully revealing. Stock price volatilities in the three equilibria rank as* $\sigma_t^{S,su} = \sigma_t^{S,su,0} < \sigma_t^S < \sigma^D$ *for* $t < T$*. As the payment date approaches,* $lim_{t\to T}\sigma_t^{S,su} = lim_{t\to T}\sigma_t^{S,su,0} = lim_{t\to T}\sigma_t^S = \sigma^D$*. Equilibrium prices in economies with small uninformed (large informed) populations are less sensitive to fundamental shocks and have lower volatility.*

3.2 PIPR and WAPR Properties

To provide further insights about the structure of equilibrium, it is instructive to start with the PIPR. The PIPR is the (negative of the) instantaneous volatility of the growth rate of the conditional density of the private information signal given public information. In equilibrium, with $\mathcal{F}_t^m = \mathcal{F}_t^{D,Z}$,

$$\theta_t^{G|m}(G) = vol\left(\frac{dp_t^G(G)}{p_t^G(x)}\right) = \frac{G - \mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2} vol\left(\mu_t^{G|D,Z}\right) = \frac{G - \mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2}\left(1 - \kappa_t \omega^i\right)\sigma^D.$$

In the model under consideration, given the linearity of the endogenous signal Z revealed, the conditional density is normal. The conditional mean alone depends on the dividend. The conditional variance is a function of time. The PIPR therefore reduces to the volatility of the conditional mean suitably normalized. It is affine in the private signal. As noted in Remark 3, it follows that the WAPR becomes $\Theta_t(G, \phi; \omega) \equiv \Theta_t(Z; \omega)$ and that the complementary demand is an affine function of $\Theta_t(Z;\omega)$. The equilibrium risk premium inherits this affine structure. Moreover, the equilibrium complementary demand, being affine in $\Theta_t(Z;\omega)$, also reveals the signal $Z = \omega^i G + \omega^n \phi$.

The next corollary describes the behavior of the endogenous PIPR.

Corollary 7 *The equilibrium PIPR is,*

$$\theta_t^{G|m}(G) = \frac{G - \mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z}\right)^2}\left(1 - \kappa_t \omega^i\right)\sigma^D = \alpha_1(t)\,G + \alpha_2(t)\,Z + \beta_0(t)\,D_t + \gamma_0(t)$$

$$\alpha_1(t) \equiv \frac{\sigma^D}{H(t)}, \qquad \alpha_2(t) \equiv -\frac{\kappa_t \sigma^D}{H(t)} = -\frac{\omega^i \sigma^D}{M(t)}, \qquad \beta_0(t) = \frac{\beta(t)}{\omega}, \qquad \gamma_0(t) = \frac{\gamma(t)}{\omega}$$

where $\omega = \omega^i + \omega^n$ *and* $\beta(t), \gamma(t)$ *are defined in (5)-(7). The coefficients* $\alpha_1(t), \alpha_2(t)$ *and* $\beta(t)$ *are the sensitivities with respect to the private signal* G*, the endogenous public signal* Z *and the fundamental* D_t*. The coefficient* $\gamma(t)$ *is a translation factor. The following properties hold,*

(i) Sensitivity to information: $\alpha_1(t) > 0, \alpha_2(t) < 0, \beta(t) < 0$.

(ii) Dynamic behavior:

(ii-1) $\frac{\partial \alpha_1(t)}{\partial t} > 0$, $\frac{\partial \alpha_2(t)}{\partial t} < 0$, $\frac{\partial \beta(t)}{\partial t} < 0$

(ii-2) $\frac{\partial \gamma(t)}{\partial t} > 0$ *if and only if* $H(t) < H^+$ *with* H^+ *as defined in (78)-(80).*

(iii) Population effects (informed to noise trader ratio): Fix ω *and let* $s = \omega^i/\omega^n$ *vary. Then,*

(iii-1) $\frac{\partial \alpha_1(t)}{\partial s} = 0$, $\frac{\partial \beta(t)}{\partial s} > 0$

(iii-2) $\frac{\partial \alpha_2(t)}{\partial s} > 0$ *if and only if* $H(t) > \frac{2s+1}{s^2}\left(\sigma^\phi\right)^2$

(iii-3) $\frac{\partial \gamma(t)}{\partial s} > 0$ *if and only if* $2s\left(\sigma^\phi\right)^2 \mu^D (T-t) > \left(-s^2 H(t) + \left(\sigma^\phi\right)^2\right)\mu^\phi$.

(iv) Bias effects: $\frac{\partial \alpha_1(t)}{\partial \mu^\phi} = \frac{\partial \alpha_2(t)}{\partial \mu^\phi} = \frac{\partial \beta(t)}{\partial \mu^\phi} = 0$, $\frac{\partial \gamma(t)}{\partial \mu^\phi} < 0$.

The reaction of the equilibrium PIPR to news is intuitive. Indeed, a larger private signal indicates a greater terminal dividend, thus provides more valuable information. In contrast public information, be it endogenous or exogenous, reduces the local value of private information.

The evolution of these sensitivities over time is also intuitive. The reaction to private information $\alpha_1(t)$ is tamed by the unconditional variance of the signal $H(t)$ in the denominator. Over time, the informed observes the fundamental and updates the content of the private signal. Effectively, the residual private information is $G - D_t$. This residual signal becomes more informative over time, as uncertainty resolves, thereby enhancing the value of information. For the same reason, the precision of the endogenous public signal increases. This reduces the (negative) sensitivity of the PIPR to the endogenous signal, which decreases the value of private information. The reaction to fundamental information reflects the same effect. Its decrease contributes to a further reduction in the value of information.

Population effects can be traced to the informational content of the endogenous public signal which depends on the relative fraction s of informed to noise trader. When s increases, endogenous information becomes more precise. This decreases both sensitivities,

$$\alpha_2(t) \equiv -\frac{\kappa_t}{H(t)}\sigma^D = -\frac{\omega^i}{M(t)}\sigma^D = -\frac{1}{\omega}\frac{1+\frac{1}{s}}{H(t)+\frac{1}{s^2}\left(\sigma^\phi\right)^2}\sigma^D$$

$$\beta_0(t) \equiv -\frac{1-\kappa_t\omega^i}{H(t)}\sigma^D = -\frac{\frac{1}{s^2}\left(\sigma^\phi\right)^2}{H(t)\left(H(t)+\frac{1}{s^2}\left(\sigma^\phi\right)^2\right)}\sigma^D$$

(denominator effects), which become more negative. At the same time, the covariance between the endogenous signal and private information decreases (numerator effect), which increases the sensitivities. In the case of $\alpha_2(t)$, the second effect dominates under the condition stated. For $\beta_0(t)$, it always dominates.

The impact of the bias is through the conditional mean of the private signal. A higher bias increases the conditional mean, leading to a reduction in the PIPR.

The WAPR is closely related to the PIPR and inherits most of its properties.

Corollary 8 *The equilibrium WAPR is given by (14). The coefficients $\alpha(t)$ and $\beta(t)$ are the sensitivities with respect to the endogenous public signal and the fundamental information. The coefficient $\gamma(t)$ is a translation factor. The properties of $(\beta(t),\gamma(t))$ are the same as those of $(\beta_0(t),\gamma_0(t))$ in Corollary 7. The behavior of $\alpha(t)$ differs in the following respects,*
(i) Sensitivity to information: $\alpha(t)>0$ if and only if $\left(\sigma^\phi\right)^2 > sH(t)$.
(ii) Dynamic behavior: $\alpha(t)$ increases with time if and only if $\kappa_t^2 < 1/\omega^i\omega$.
(iii) Population effects: $\alpha(t)$ increases with s if and only if $H(t) > \frac{2s+1}{s^2}\left(\sigma^\phi\right)^2$.

The behavior of $\alpha(t) = \alpha_1(t) + \alpha_2(t)\omega$ is more intricate because $\alpha_1(t), \alpha_2(t)$ have different, sometimes opposite properties. The evolution of $\alpha(t)$ over time is especially noteworthy. If $\omega^i\omega\kappa_0^2 < 1$, the coefficient increases over time. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 < 1$, it initially decreases, then increases. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 > 1$, it decreases throughout. The possibility of a U-shaped pattern reflects conflicting effects on $\alpha_1(t)$ and $\alpha_2(t)$. Under the conditions stated, the decrease in $\alpha_2(t)$ dominates early on, then is overtaken by the increase in $\alpha_1(t)$. An illustration is in Figure 1.

3.3 Price and Return Properties

Fundamental information accumulates with the passage of time, providing more precise estimates of the next dividend payment. Information accumulation affects the properties of equilibrium. The next corollary describes the dynamic behavior of the price and the return components.

Corollary 9 *The stock price sensitivity to the fundamental (resp. the endogenous public signal) increases (resp. decreases) over time. The volatility of the stock price, $\sigma_t^S = A(t)\,\sigma^D$, increases over time. The minimal and maximal volatility values are obtained at the initial and terminal dates,*

$$\sigma_0^S = A(0)\,\sigma^D = \left(\frac{H(T)}{H(0)}\right)^{\omega}\left(\frac{M(T)}{M(0)}\right)^{1-\omega}\sigma^D, \qquad \lim_{t\to T}\sigma_t^S = A(T)\,\sigma^D = \sigma^D.$$

The stock's price of risk $\mu_t^S/\sigma_t^S = A(t)\,\sigma^D/\Gamma - (\alpha(t)\,Z + \beta(t)\,D_t + \gamma(t))$ becomes more sensitive to the fundamental over time (i.e., $-\beta(t) > 0$ increases for all $t \in [0,T]$). Its sensitivity with respect to the endogenous public signal increases at date t if and only if $\omega^i\omega\kappa_t^2 < 1$ (i.e., $-\alpha(t)$ increases if $\omega^i\omega\kappa_t^2 < 1$).

At the initial date, the uninformed extracts the noisy signal Z from the price. This information is most valuable when there is no other source of information, i.e., at the initial date. In the early stages of the economy, the price is heavily influenced by this initial information and, for this reason, does not react significantly to fundamental information. Over time, fundamental information accumulates, reducing the usefulness of the initial piece of information extracted. The impact of fundamental information (resp. the endogenous noisy signal) on the stock price grows (resp. decreases), thereby increasing the stock's volatility.

The behavior of the price of risk is more intricate. As for the stock price, the sensitivity to fundamental information increases. The volatility of the price of risk therefore increases over time. The sensitivity with respect to the endogenous public signal can exhibit three types of patterns. If $\omega^i\omega\kappa_0^2 < 1$, it decreases over time. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 < 1$, it initially increases, then decreases. If $\omega^i\omega\kappa_0^2 > 1$ and $\omega^i\omega\kappa_T^2 > 1$, it increases throughout. The possibility of an $\cap$-shaped pattern reflects the U-shaped behavior of the WAPR. Figure 2 illustrates the price of risk and volatility behaviors.

The impact of risk attitudes is outlined next.

Corollary 10 *The stock price is a increasing function of risk tolerance, but its sensitivity coefficients with respect to fundamental information and to the noisy signal do not depend on it. Likewise, the volatility of the stock price is not affected by risk tolerance. The stock's risk premium is a decreasing function of risk tolerance.*

An increase in risk tolerance promotes an increase in the demand for the stock, which increases value. As shown by expression (11) for $F(t)$, risk tolerance effectively acts on the risk discount embedded in the stock price. When risk tolerance increases, the willingness to bear risk increases, reducing the price discount required to hold the asset. The absence of an impact on the coefficients $(A(t), B(t))$ capturing the price sensitivity to the information sources (D_t, Z), follows from the mean-variance structure of the demand functions and the assumption of common risk attitudes across investors. Under these circumstances, the aggregate demand function is an affine function of the WAPR, that carries information and is unrelated to risk attitudes. The stock price inherits this behavior. It depends on information through the WAPR, unaffected by risk attitudes. Moreover, the absence of an impact on the sensitivity $A(t)$ with respect to the fundamental implies that the volatility of the stock price is not affected by risk attitudes either.

Because aggregate demand has a mean-variance form, the risk premium is also linear in the PIPR. The risk premium is determined by the return variance per unit risk tolerance adjusted by a discount related to the WAPR. Given that the variance of the stock price and the WAPR do not depend on risk attitudes the result stated follows.

The last corollary in this section reports the effects of variations in the population of investors and in the noise trading bias.

Corollary 11 *Suppose that the ratio of informed to noise trader, $s \equiv \omega^i/\omega^n$, increases, but that their combined fraction in the population, $\omega = \omega^i + \omega^n$, stays the same. Under this scenario, the sensitivity of the stock price with respect to fundamental information and its volatility both decrease. The stock's risk premium can increase or decrease. If the noise trading bias $\mu^\phi - E[G]$ (i.e., μ^ϕ) increases, the stock price decreases. The stock volatility is not affected. The stock's risk premium increases.*

When the fraction of informed to noise trader s increases, the information extracted from the price becomes more precise. This tames the response to other sources of information such as the fundamental. The volatility of the stock price, which is entirely driven by the volatility of the fundamental, inherits this behavior. In contrast, the stock risk premium can increase or decrease because of the conflicting effects on the coefficients of the WARP. The effects of noise trading bias are straightforward and follow from the behavior of the non-stochastic component of the WARP.

3.4 Market Depth and Investor Strategies

3.4.1 Market Depth Properties

Market depth seeks to capture the impact of trading on the price. It is typically measured by the inverse of the coefficient of the regression of the stock price on the complementary demand function (Kyle (1985)). Properties of market depth are described next,

Corollary 12 *Market depth m is given by,*

$$m(t) \equiv \left(\frac{d[S_t, N_t]}{d[N_t, N_t]}\right)^{-1} = \frac{\frac{\omega^u \Gamma}{\sigma^D}\frac{\beta(t)}{A(t)}\sigma_D \times \frac{\omega^u \Gamma}{\sigma^D}\frac{\beta(t)}{A(t)}\sigma_D}{\frac{\omega^u \Gamma}{\sigma^D}\frac{\beta(t)}{A(t)}\sigma_D \times A(t)\,\sigma_D} = \frac{\omega^u \Gamma \beta(t)}{A^2(t)\,\sigma_D}. \tag{16}$$

Market depth is negative, and increases over time if and only if $\omega > 1/2$. Under this condition, its minimal and maximal values are reached at the initial and terminal dates,

$$m(0) = -\frac{\omega^u \Gamma \omega (\omega^n)^2 (\sigma^\phi)^2}{H(0)\,M(0)}\left(\frac{H(0)}{H(T)}\right)^{2\omega}\left(\frac{M(0)}{M(T)}\right)^{2(1-\omega)}, \qquad m(T) = -\frac{\omega^u \Gamma \omega (\omega^n)^2 (\sigma^\phi)^2}{H(T)\,M(T)}.$$

It also decreases with risk tolerance Γ and increases with the fraction s of informed. Market depth is not related to the bias component μ^ϕ.

Market depth is negative because the covariance between the price change and the change in the combined demand of the informed and the noise trader is negative. The passage of time has two effects on depth. On the one hand, it increases the volatility of the stock price, which increases the covariance between the stock price change and the demand change. On the other hand, it has a negative effect on the volatility of the complementary demand through the coefficient $\beta(t)$, which becomes more negative. The trade-off between these two opposite effects is determined by the fraction of informed and noise trader in the total population. When this fraction is greater than half, the first effect dominates, leading to an increasing market depth over time, i.e., a market depth that becomes less negative. When the informed and noise trader are a majority, the price effect is dominated by the demand effect. When the informed and the noise trader form a minority, the price impact is sufficiently important to offset the demand effect.

The behavior with respect to the other quantities such as risk tolerance and the informed-to-noise trader ratio is monotone. The latter increases, because the volatility of the stock price decreases while the volatility of the complementary demand increases. Both effects contribute to an increase in market depth.

Remark 13 *Collin-Dufresne and Fos (2013) generate time-varying market depth by extending Kyle (1985) to more general processes for exogenous noise trading. Their time-varying measures of liquidity are supported by their empirical findings (Collin-Dufresne and Fos (2014)) that, in contrast to the predictions of standard microstructure models, market depth can increase with more informed trading. Market depth in the present model is tied to the underlying fundamental. It is time-varying and can also increase with informed trading.*

3.4.2 Momentum and Reversal Strategies

The next corollary describes the investment strategies of the three groups of agents.

Corollary 14 *Let* $N_t^{i,G} = \Gamma \theta_t^{G|m}(G)/\sigma_t^S$ *be the private information component of the informed demand. The optimal portfolio policy of the informed (resp. uninformed) is a contrarian (resp. momentum) strategy,*

$$\frac{d[N^u, S]_t}{dt} = \frac{\Gamma}{\sigma^D} \frac{d[\theta^m, D]_t}{dt} = -\frac{\Gamma}{\sigma^D} \frac{d[\Theta_t(Z;\omega), D]_t}{dt} = -\Gamma \beta(t) > 0$$

$$\frac{d\left[N^{i,G}, S\right]_t}{dt} = \frac{\Gamma}{\sigma^D} \frac{d\left[\theta^{G|m}(G), D\right]_t}{dt} = -\frac{\Gamma \sigma^D}{H(t)} \frac{d\left[\mu^{G|D,Z}, W^D\right]_t}{dt} = \frac{\Gamma}{\omega} \beta(t) < 0$$

$$\frac{d\left[N^i, S\right]_t}{dt} = \frac{d\left[N^u + N^{i,G}, S\right]_t}{dt} = \Gamma\left(-1 + \frac{1}{\omega}\right) \beta(t) < 0.$$

The mimicking noise trader pursues a contrarian strategy, $d[N^m, S] = -d\left[\omega^i N^i + \omega^u N^u, S\right]_t = \frac{\omega^n}{\omega}(1-\omega)\Gamma A(t)\beta(t) < 0$. *Momentum, for the uninformed strategy, is a decreasing function of the conditional variance* $H(t)$ *and the weight* ω^n *of mimicking noise traders. It increases over the dividend cycle. The informed strategy is contrarian because the contrarian private information* *component $N_t^{i,G}$ dominates the overall portfolio behavior. Reversal decreases (i.e., becomes more pronounced) with respect to $H(t)$ and t. The contrarian strategy of the mimicking noise trader is the counterpart of the informed and uninformed strategies.*

The uninformed behaves as a trend-chaser because the endogenous market price of risk is positively related to the fundamental. A positive shock to the fundamental induces an increase in the market price of risk, prompting an increase in the uninformed portfolio demand. The behavior of the informed is the opposite. The reason is because the local value of private information, the PIPR, is negatively correlated with the fundamental. The informed acts as a contrarian. Market clearing ensures that the noise trader adopts a contrarian strategy.

Over time, these strategies grow. Contrarians (resp. trend-chasers) become more intense contrarian (resp. trend-chasers). This follows from the fact that fundamental information becomes more important as the dividend date approaches. The market price of risk and the PIPR both become more sensitive to fundamental news over time, prompting investors to amplify their reactions to fundamental news. See Figure 3 for illustration.

Remark 15 *These findings differ from those in Albuquerque and Miao (2013) and Wang (1993). In Wang's model, the uninformed can be a trend chaser or a contrarian. The informed is a contrarian. The uninformed is a momentum trader if the positive covariance associated with fundamental information dominates the covariance related to the endogenous signal. In Albuquerque and Miao, the uninformed (resp. informed) is a contrarian (resp. trend chaser). This pattern is attributable to the agents' information structures and the properties of the private investment opportunity available to the informed. In this model, the informed invests in the private opportunity and hedges the associated exposure to risk with the stock. The hedging component of the stock demand creates the condition for trend chasing behavior.*

3.5 Welfare Analysis

3.5.1 Welfare of Informed and Uninformed Agents

The ex-ante value of private information is an important component of the welfare of the informed. It captures the ex-ante utility gains associated with the use of private information for trading.

The local value of private information is the PIPR $\theta_t^{G|m}(G)$. Mean-variance trades based on the PIPR generate the local trading gain $\Gamma\theta_t^{G|m}(G)^2$. The expected gains from trade over the cycle are $\Gamma E\left[\int_0^T \theta_t^{G|m}(G)^2\,dt\right]$. The ex-ante value of information combines this trading gain with the loss associated with aversion to realized variance. It equals,

$$\mathcal{I}^i = \frac{\Gamma}{2} E\left[\int_0^T \theta_t^{G|m}(G)^2\,dt\right]. \tag{17}$$

When risk tolerance is $\Gamma = 1$, the ex-ante value of private information corresponds to the relative entropy of the private signal. In the NREE, the value of private information quantifies the difference between the welfare or the informed and that of the uninformed investor.

In order to compare welfare across economies with and without private information, the equilibrium public information structure matters. In the economy without private information, information is homogeneous and generated by the fundamental. In the NREE, the uninformed extracts the endogenous signal Z from equilibrium. The ex-ante value of this signal to the uninformed consists of associated trading gains and realized variance losses. It is a quadratic function of the WAPR,

$$\mathcal{I}^u = \frac{\Gamma}{2} E\left[\int_0^T \Theta_t(Z;\omega)^2\,dt\right] = \left(\omega^i\right)^2 \mathcal{I}^i(G) + (\omega^n)^2 \frac{\Gamma}{2} E\left[\int_0^T \theta_t^{G|m}(\phi)^2\,dt\right]. \tag{18}$$

The weighted average structure of the WAPR implies that it can also be written as a linear function of the ex-ante value of private information for the informed. Moreover, both (17) and (18) are linear increasing functions of risk tolerance.

Proposition 16 *Let $E\left[\mathcal{U}^j\right]$ be the ex-ante utility of investor $j \in \{u, i\}$. In the NREE, ex-ante utilities differ by the ex-ante value of private information, $E\left[\mathcal{U}^i\right] = E\left[\mathcal{U}^u\right] + \mathcal{I}^i$, where $\mathcal{I}^i$ is given by (17) and,*

$$E\left[\mathcal{U}^u\right] = E\left[N_0^u S_0\right] + \frac{1}{2\Gamma}\int_0^T \left(\sigma_t^S\right)^2 dt - \omega^n \int_0^T \sigma_t^S \theta_t^{G|m}\left(\mu^\phi \,\middle|\, E\left[D_t\right], E\left[Z\right]\right) + \mathcal{I}^u \tag{19}$$

$$E\left[N_0^u S_0\right] = \left(1 - \Gamma\delta_0\right) E\left[S_0\right] - \Gamma K_0, \qquad \delta_0 \equiv \frac{\beta(0) D_0 + \gamma(0)}{A(0)\,\sigma^D}, \qquad K_0 \equiv \frac{\alpha(0)}{\sigma_0^S} E\left[Z S_0\right] \tag{20}$$

$$E\left[Z S_0\right] = \left(A(0) D_0 + F(0)\right) E\left[Z\right] + B(0) E\left[Z^2\right] \tag{21}$$

$$E\left[S_0\right] = A\left(0\right) D_0 + B\left(0\right) E\left[Z\right] + F\left(0\right), \qquad \sigma_t^S = A\left(t\right)\sigma^D \tag{22}$$

$$\theta_t^{G|m}\left(x|\, d, z\right) \equiv \frac{x - E\left[G|\, D_t = d, Z = z\right]}{H\left(t\right)}\sigma^D. \tag{23}$$

In this expression $\theta_t^{G|m}\left(x|\, d, z\right)$ *is the PIPR evaluated at the constants* $\left(D_t, Z\right) = \left(d, z\right)$. *In the equilibrium without private information, all agents are uninformed. The ex-ante utilities are equal,*

$$E\left[\mathcal{U}^{j,ni}\right] = E\left[S_0^{ni}\right] + \frac{1}{2\Gamma}\int_0^T \left(\sigma_t^{S,ni}\right)^2 dt, \qquad j \in \{u, i\} \tag{24}$$

$$E\left[S_0^{ni}\right] = D_0 + \mu^D T - \frac{\left(\sigma^D\right)^2}{\Gamma}T, \qquad \sigma_t^{S,ni} = \sigma^D. \tag{25}$$

Initial share allocations are also equal, $N_0^i = N_0^u = N_0^n = 1$.[7]

Ex-ante utilities, in the NREE, can be decomposed as,

$$E\left[\mathcal{U}^j\right] = E\left[N_0^j S_0\right] + \frac{\Gamma}{2}\int_0^T E\left[\left(\theta_s^m + 1_{\{j=i\}}\theta_s^{G|m}\left(G\right)\right)^2\right] ds, \qquad \text{for } j = \{i, u\}.$$

The first term, in each decomposition, is the ex-ante value of the initial stock holdings. These ex-ante values are the same, $E\left[N_0^i S_0\right] = E\left[N_0^u S_0\right]$, because the informed demand differs from the uninformed demand by a mean-preserving spread ($E\left[N_0^i|\mathcal{F}_0^m\right] = N_0^u$). In effect, the uninformed demand is an unbiased estimate of the informed demand. At the root of this property is the fact that the expectation of the PIPR given public information is null ($E\left[\theta_s^{G|m}\left(G\right)|\mathcal{F}_0^m\right] = 0$). In the public information, the PIPR is a mean-preserving perturbation of the market price of risk. The second term, which is proportional to the expected gains from trade, captures all the benefits of trading and the costs associated with aversion to realized variance. The expected gains from trade for the informed,

$$\Gamma\int_0^T E\left[\left(\theta_s^m + \theta_s^{G|m}\left(G\right)\right)^2\right] ds = \Gamma\int_0^T E\left[\left(\theta_s^m\right)^2\right] ds + \mathcal{I}^i$$

differ from those for the uninformed by the value of private information. The underlying reason is again the mean-preserving property of the PIPR ($E\left[\theta_s^{G|m}\left(G\right)|\mathcal{F}_0^m\right] = 0$). Combining these

[7] In each equilibrium, the initial endowments of shares across agents are assumed to be equal to the optimal holdings at date 0. See Remark 22 for further discussion.

two properties, shows that the ex-ante utility difference reduces to the ex-ante value of private information. The positive value of the latter implies that the informed is always better off than the uninformed.

It is also instructive to examine the constituents of the gains from trade for the uninformed. As shown by (19), the normalized gains from trade can be split in three parts,

$$\frac{\Gamma}{2}\int_0^T E\left[(\theta_t^m)^2\right]dt = \frac{1}{2\Gamma}\int_0^T \left(\sigma_t^S\right)^2 dt - \omega^n \int_0^T \sigma_t^S \theta_t^{G|m}\left(\mu^\phi \middle| E\left[D_t\right], E\left[Z\right]\right)dt + \mathcal{I}^u.$$

To shed light on this expression, note that $(\theta_t^m)^2 = \left(\sigma_t^S/\Gamma\right)^2 - 2\left(\sigma_t^S/\Gamma\right)\Theta_t\left(Z;\omega\right)^2 + \Theta_t\left(Z;\omega\right)^2$. The first term, $\frac{1}{2\Gamma}\int_0^T\left(\sigma_t^S\right)^2 dt$, therefore reflects the variance impact on the endogenous market price of risk. This term is positive because a greater riskiness induces an increased equilibrium price of risk. The last term, $\mathcal{I}^u = \frac{\Gamma}{2}\int_0^T E\left[\Theta_t\left(Z;\omega\right)^2\right]dt$, is the value of the endogenous information signal extracted from equilibrium. It is also positive, because information improves the efficiency of the pricing of risk and the resulting gains from trade. The middle term, $-\omega^n\int_0^T \sigma_t^S\theta_t^{G|m}\left(\mu^\phi\middle| E\left[D_t\right], E\left[Z\right]\right)dt$, captures the interaction between the risk and information components of the price of risk. This specific form, which represents a risk premium, emerges because the unconditional expectation of the PIPR is null and the stock volatility is deterministic. The expected WAPR therefore reduces to the expected noise trading part, $E\left[\Theta_t\left(Z;\omega\right)\right] = E\left[\omega^n\theta_t^{G|m}\left(\phi\right)\right] = \omega^n\theta_t^{G|m}\left(\mu^\phi\middle| E\left[D_t\right], E\left[Z\right]\right)$. It is null if the noise trader has unbiased beliefs, $\mu^\phi = E\left[G\right]$. In the equilibrium without private information, investors are symmetric in all respects. The market price of risk is then entirely determined by the riskiness of the stock. The expected gains from trade are completely driven by the stock's variance.

The relation between informed and uninformed utilities in the NREE simplifies welfare comparisons across equilibria. Both agents are better off if the welfare of the uninformed improves.

Proposition 17 *The uninformed is better off in the NREE if and only if* $\Delta^u \equiv E\left[\mathcal{U}^u\right] - E\left[\mathcal{U}^{u,ni}\right] = \Delta P^u + \Delta N^u + \Delta T^u > 0$, *where* $\Delta P^u = E\left[S_0 - S_0^{ni}\right]$ *is the gain/loss from the valuation of initial holdings (price impact),* $\Delta N^u = E\left[\left(N_0^u - 1\right)S_0\right]$ *is the gain/loss from the change in the initial allocation of shares across equilibria (allocation impact) and* ΔT^u *is the gain/loss from dynamic trading (trading impact). The allocation impact is* $\Delta N^u = -\Gamma\left(\delta_0 E\left[S_0\right] + K_0\right)$. *The price and* *trading impacts are,*

$$\Delta P^u = (A(0) - 1) E[D_T] + B(0) E[Z] - \frac{\Delta V}{\Gamma} + \sigma^D \int_0^T A(s) \gamma(s) ds - \omega^n I(0) \mu^\phi \tag{26}$$

$$\Delta T^u = \frac{\Delta V}{2\Gamma} - \omega^n \sigma^D \int_0^T A(t) \theta_t^{G|m} \left(\mu^\phi; E[D_t], E[Z]\right) dt + \mathcal{I}^u \tag{27}$$

where $\Delta V = \left(\sigma^D\right)^2 \int_0^T \left(A(t)^2 - 1\right) dt$ *is the change in the realized variance of the price. The welfare of the uninformed improves, in particular, if risk tolerance is sufficiently small (*$\lim_{\Gamma \to 0} \Delta^u = +\infty$*). For sufficiently large risk tolerance, it improves (*$\lim_{\Gamma \to +\infty} \Delta^u = +\infty$*) if and only if,*

$$\frac{1}{2} E\left[\int_0^T \left(\left(\omega^i\right)^2 \theta_t^{G|m} (G)^2 + (\omega^n)^2 \theta_t^{G|m} (\phi)^2\right) dt\right] \geq \lim_{\Gamma \to \infty} \left(\delta_0 E[S_0] + K_0\right). \tag{28}$$

A sufficient condition is $\delta_0 \lim_{\Gamma \to \infty} E[S_0] + \lim_{\Gamma \to \infty} K_0 \leq 0$.

Proposition 17 identifies the sources of welfare gains and losses for the uninformed when private information trades are allowed. The first effect, ΔP^u, captures the price impact on the initial stock holdings of the uninformed. This price impact can be positive or negative depending on parameter values. It is positive if the risk reduction associated with the partial dissemination of private information in the NREE is sufficiently important. The second effect, ΔN^u, captures the impact of the change in initial holdings. This term can also take either sign. It depends in particular on the covariance between the signal extracted and the initial stock price. The last effect, ΔT^u, captures the change in the gains from trade. This component splits into three parts, a riskiness effect ($\Delta V / 2\Gamma$), an informational efficiency effect ($\mathcal{I}^u$) and a noise trading effect ($-\omega^n \sigma^D \int_0^T A(t) \theta_t^{G|m} \left(\mu^\phi; E[D_t], E[Z]\right) dt$). Allowing private information trading reduces the volatility of the stock, hence the market price of risk, which decreases welfare. The first part is therefore negative. It also disseminates private information and increases the informational efficiency of the market. Better information improves investment allocations and leads to welfare gains. The second part is positive. Finally, permitting the use of private information will prompt the emergence of mimicking noise traders, whose activity limits efficiency gains. The bias induced by their activities can be a source of welfare gains or losses. The third part can be positive or negative. It is null when noise trading beliefs are unbiased, i.e., $\mu^\phi = E[G]$. Overall, when risk tolerance is large, the positive informational efficiency effect dominates if initial holdings per unit risk tolerance don't grow too fast. The welfare of the uninformed improves. When risk tolerance is small, the riskiness effect dominates. It increases the stock price, implying a positive price impact. It also reduces the price of risk, generating a negative trading impact. The price impact dominates leading to an overall welfare gain.

The next corollary provides further insights. To simplify notation, define the relative entropy of the private signal $\mathcal{E}^u \equiv \mathcal{I}^u/\Gamma$ and the coefficients,

$$C \equiv C_P + C_T, \qquad C_T \equiv -\omega^n \sigma^D \int_0^T A(t)\, \theta_t^{G|m}\left(\mu^\phi; E[D_t], E[Z]\right) dt$$

$$C_P \equiv (A(0) - 1)\, E[D_T] + B(0)\, E[Z] + \sigma^D \int_0^T A(s)\, \gamma(s)\, ds - \omega^n I(0)\, \mu^\phi$$

$$J_0 \equiv \delta_0 E[S_0] + K_0, \qquad J \equiv C^2 + 2(\mathcal{E}^u - J_0)\,\Delta V, \qquad \Gamma_\pm \equiv \frac{-C \pm \sqrt{J}}{2(\mathcal{E}^u - J_0)}.$$

With this notation,

Corollary 18 *The uninformed is as well off in the NREE as in the equilibrium without private information under the following conditions,*
(i) uniformly in Γ *if* $J < 0$ *or if* $J \geq 0$ *and* $\Gamma_+ \leq 0$
(ii) for $\Gamma \in [0, \Gamma_+]$*, if* $J \geq 0$*,* $\Gamma_- < 0$ *and* $\Gamma_+ > 0$
(iii) for $\Gamma \in [0, \Gamma_-] \bigcup [\Gamma_+, +\infty)$*, if* $J > 0$ *and* $\Gamma_- \geq 0$*.*

The corollary identifies parameter regions for which banning the use of private information reduces the welfare of the uninformed. Figure 4 illustrates the various configurations.

3.5.2 Welfare of Mimicking Noise Trader

Let $f_{\phi|Z}(x|\, Z)$ (resp. $f_{G|Z}(x|\, Z)$ be the Gaussian density of ϕ (resp. G) conditional on Z. The likelihood ratio $\mathcal{L}_{\phi,G}(x|\, Z) \equiv f_{\phi|Z}(x|\, Z) / f_{G|Z}(x|\, Z)$ captures the beliefs divergence between the noise trader and the informed. Explicit formulas are in the proof of Corollary 20 in Appendix B.

Corollary 19 *The ex-ante utility of the mimicking noise trader is* $E[\mathcal{U}^n(\phi)] = E[\mathcal{U}^u(Z)] + \mathcal{I}^n +$

ΔT^n+ ΔN^n *where,*

$$\mathcal{I}^n \equiv \frac{\Gamma}{2}\int_0^T E\left[\theta_v^{G|m}(G)^2\,\mathcal{L}_{\phi,G}(G|\,Z)\right]dv \tag{29}$$

$$\Delta T^n \equiv \Gamma\int_0^T\left(E\left[\theta_v^{G|m}(G)\,\theta_v^m\mathcal{L}_{\phi,G}(G|\,Z)\right]+\frac{1}{2}E\left[COV\left(\mathcal{L}_{\phi,G}(G|\,Z),(\theta_v^m)^2\right|Z\right)\right]\right)dv \tag{30}$$

$$\Delta N^n \equiv \Gamma\frac{\sigma^D}{H(0)\,\sigma_0^S}\left(\left(\mu^\phi - E[G]\right)E[S_0]+B(0)\left(\omega^n\left(\sigma^\phi\right)^2-\omega^i H(0)\right)\right). \tag{31}$$

The term ΔN^n *is the differential allocation impact. The sum* $\mathcal{I}^n+\Delta T^n$ *is the differential trading impact. The noise trader is as well off as the uninformed if and only if the combined differential allocation and trading impacts are nonnegative,* $\mathcal{I}^n+\Delta T^n+\Delta N^n\geq 0$*.*

The corollary expresses the welfare of the noise trader relative to the welfare of the uninformed. The ex-ante utility differential stems from the difference in the value of initial holdings (ΔN^n) and in the grains from trade ($\mathcal{I}^n+\Delta T^n$). The differential allocation impact, $\Delta N^n = E\left[\left(a_0(\phi)\,N_0^n - N_0^u\right)S_0\right]$, is the cross-moment between the difference in initial allocations and the stock price. It depends, in particular, on the beliefs bias $\mu^\phi - E[G]$. The differential trading impact, $\mathcal{I}^n+\Delta T^n = \frac{1}{2\Gamma}\int_0^T E\left[a_v(\phi)\left(N_v^n\sigma_v^S\right)^2-\left(N_v^u\sigma_v^S\right)^2\right]dv$, reflects the difference in expected portfolio variances, where expectations are taken under the relevant beliefs. It has three constituents because $a_v(\phi)\left(N_v^n\sigma_v^S\right)^2-\left(N_v^u\sigma_v^S\right)^2 = a_v(\phi)\,\theta_v^{G|m}(G)^2+2a_v(\phi)\,\theta_v^{G|m}(G)\,\theta_v^m+(a_v(\phi)-1)\,(\theta_v^m)^2$. These constituents lead to the three terms in $\mathcal{I}^n$ and ΔT^n. They depend on the likelihood ratio $\mathcal{L}_{\phi,G}(x|\,Z)$ due to the beliefs distortion $a_v(\phi)$. The constituent of $\mathcal{I}^n$ is the perceived value of information under the distorted beliefs.

It the noise trader happens to have the same conditional beliefs as the informed, i.e., if the conditional distributions of ϕ and G given public information coincide, then $\mathcal{L}_{\phi,G}(G|\,Z)=1$. In this case, the conditional covariance vanishes and $\Delta T^n \equiv \Gamma\int_0^T E\left[E\left[\theta_v^{G|m}(G)\,|\mathcal{F}_v^m\right]\theta_v^m\right]dv = 0$ (the PIPR is a mean-preserving spread in the public information filtration). Moreover, $\mathcal{L}_{\phi,G}(G|\,Z)=1$ implies that beliefs are unbiased ($\mu^\phi = E[G]$) and that the noise trader and informed have the same weight ($\omega^n=\omega^i$). It follows that the differential allocation impact $\Delta N^n = \Gamma\left(\sigma^D/\sigma_0^S\right)B(0)\left(\omega^n-\omega^i\right)=0$.

3.5.3 Pareto Optimal NREE

The relations between ex-ante utilities in the NREE simplify welfare comparisons across equilibria. Pareto dominance of the NREE over an equilibrium where investors are symmetric is ensured if the welfare of the uninformed and the noise trader improves.

Corollary 20 *The NREE is (weakly) Pareto optimal if and only if* $0 \leq \Delta^u$ *and* $0 \leq \Delta^u + \mathcal{I}^n + \Delta N^n + \Delta T^n$*. Suppose that the noise trader has unbiased beliefs* $\mu^\phi = E[G]$*. Then,* $\lim_{\Gamma \downarrow 0} E[\mathcal{U}^n(\phi)] = E[\mathcal{U}^u(Z)]$ *and the NREE is Pareto optimal for sufficiently small levels of risk tolerance. If, in addition,* $\omega^i = \omega^n$ *and* $VAR[\phi] = VAR[G]$*, then the ex-ante utility of the mimicking noise trader and the informed are identical,* $E[\mathcal{U}^n(\phi)] = E\left[\mathcal{U}^i(G)\right]$*. In this case, the NREE is (weakly) Pareto optimal under the conditions of Corollary 18.*

The conditions for weak Pareto optimality ensure that all agents are as well off. When risk tolerance converges to zero, the uninformed utility eventually becomes at least as large in the NREE because of the price impact (Proposition 17). At the same time, if beliefs are unbiased, the differential allocation and trading impacts vanish, ensuring that the noise trader attains the same ex-ante utility as the uninformed. The NREE becomes Pareto optimal. If the noise trader happens to have the same beliefs as the informed, he/she reaches the same ex-ante utility. The NREE is then Pareto optimal under the conditions ensuring that the uninformed agent is as well off.

Corollary 20 has ramifications for market regulation. Permitting private information trades is Pareto efficient when risk tolerance is sufficiently low or sufficiently large if conditions (28) and $0 \leq \Delta^u + \mathcal{I}^n + \Delta N^n + \Delta T^n$ hold. In those cases, either the informational efficiency gains or the decrease in the riskiness of the stock market dominate, leading to a welfare improvement. Scope for regulation exists in intermediate cases. In these cases, factors such as the behavior of mimicking investors, the properties of dividends and the weights of the various investor populations matter, and have to be evaluated to determine the relevance of regulatory constraints.

Corollary 21 *Suppose that conditions (i), (ii) or (iii) of Corollary 18 hold. If* $\Delta^u + \mathcal{I}^n + \Delta N^n + \Delta T^n \geq 0$*, then the NREE is (weakly) Pareto optimal.*

Under the conditions of the corollary, regulation banning the use of private information is welfare reducing. Figure 4 illustrates the various configurations.

Remark 22 *If initial share endowments are the same across equilibria (i.e., $N_0^j = 1$ for $j = i, u, n$), the allocation impact components vanish, $\Delta N^u = \Delta N^n = 0$. The characterization of uninformed welfare improvements in Corollary 18 continues to hold with $J_0 = 0$. Likewise, the welfare comparison in Corollary 19 holds if and only if $\mathcal{I}^n + \Delta T^n \geq 0$. Finally, the conditions for weak Pareto optimality in Corollary 20 become $0 \leq \Delta^u = \Delta P^u + \Delta T^u$ and $0 \leq \Delta^u + \mathcal{I}^n + \Delta T^n$.*

Remark 23 *The results above extend Leland's (1992) analysis to a dynamic competitive setting. In a static framework, the dynamic trading components $(\Delta T^n, \Delta T^u)$ are absent. So are intertemporal aspects of the price impacts $(\Delta P^n, \Delta P^u)$, such as volatility (ΔV). As conjectured by Leland, some dynamic effects, e.g., trading effects, can dampen the price impacts.*[8] *However, for sufficiently low risk tolerance, the price impacts continue to dominate. As will become clear from Section 4, similar insights apply with monopolistic informed behavior.*

4 Monopolistic Noisy Rational Expectations Equilibrium

Informed investors have market power and will therefore seek to exploit their informational advantage by behaving non-competitively (Hellwig (1980)). This section derives equilibrium and examines its properties under monopolistic informed trading. The monopolistic demand function is described in Section 4.1. Equilibrium is presented in Section 4.2. Properties of equilibrium are examined in Section 4.3.

4.1 Monopolistic Demand

The optimal monopolistic demand is described next,

Proposition 24 *The optimal number of shares held by the monopolistic informed investor is,*

$$N_t^i = \frac{1}{1+\omega^i}\left(1 - \omega^n \Gamma \frac{\theta_t^{G|m}(\phi)}{\sigma_t^S}\right) + \frac{\omega^u + \omega^n}{1+\omega^i}\Gamma\frac{\theta_t^{G|m}(G)}{\sigma_t^S} = \left(1-\omega^i\right)\Gamma\frac{\theta_t^m + \theta_t^{G|m}(G)}{\sigma_t^S}$$

for $t \in [0, T]$. The monopolistic informed investor reduces his/her overall demand for the stock by a fraction ω^i.

[8] "Insider trading "moves up" the resolution of uncertainty. This one time benefit may be relatively more important in a two-period model than in a multiperiod model. If so, my results may overestimate the benefits from insider trading. But we must await the development of multiperiod rational expectations models to answer this question definitively." (Leland (1992), p. 885).

The monopolistic investor takes account of the endogenous price impact when formulating his/her optimal stock demand. Given the demands of the uninformed and the mimicking noise trader, the market clearing price of risk,

$$\theta_t^m = \frac{\sigma_t^S}{\Gamma\left(\omega^u + \omega^n\right)}\left(1 - \omega^i N_t^i - \omega^n \Gamma \frac{\theta_t^{G|m}\left(\phi\right)}{\sigma_t^S}\right)$$

is a decreasing, affine function of the informed demand. A greater informed demand therefore reduces the reward for risk-taking. The monopolistic investor responds by optimally reducing his/her stock demand. For a given price of risk, demand is reduced by the weight of the informed in the population, ω^i.

Remark 25 *Recall that $\omega = \omega^i + \omega^n$. The complementary demand function,*

$$N_t \equiv \omega^i N_t^i + \omega^n N_t^n = \Gamma \frac{\left(\omega - \left(\omega^i\right)^2\right)\mu_t^S + \sigma_t^S\left(\omega^i\left(1-\omega^i\right)\theta_t^{G|m}\left(G\right) + \omega^n \theta_t^{G|m}\left(\phi\right)\right)}{\left(\sigma_t^S\right)^2}$$

is an affine function of the WARP $\Theta_t^{se}\left(G, \phi; \omega^i\left(1-\omega^i\right), \omega^n\right) = \omega^i\left(1-\omega^i\right)\theta_t^{G|m}\left(G\right) + \omega^n \theta_t^{G|m}\left(\phi\right)$. If the PIPR is also affine, the complementary demand depends on $\Theta_t^{se}\left(G, \phi; \omega^i\left(1-\omega^i\right), \omega^n\right) = \Theta_t^{se}\left(Z^{se}; \omega^{se}\right)$, which is a function of the signal $Z^{se} \equiv \omega^i\left(1-\omega^i\right)G + \omega^n\phi$, calculated with weights $\omega^i\left(1-\omega^i\right), \omega^n$ and parametrized by $\omega^{se} = \omega^i\left(1-\omega^i\right) + \omega^n$.

4.2 Monopolistic Equilibrium Structure

The monopolistic noisy rational expectations equilibrium is as follows,

Proposition 26 *Define the adjusted population weights $\omega^{i,se} = \omega^i\left(1-\omega^i\right)$ and $\omega^{se} = \omega^{i,se} + \omega^n$. Also define the functions $(\alpha^{se}, \beta^{se}, \gamma^{se}, \lambda^{se}, M^{se})$ as in (5)-(7), but with $\left(\omega^{i,se}, \omega^{se}\right)$ in place of $\left(\omega^i, \omega\right)$ and the functions $(A^{se}, B^{se}, F^{se}, I^{se})$ as in (9)-(12) but with $\left(\alpha^{se}, \beta^{se}, \gamma^{se}, \lambda^{se}, M^{se}, \omega^{i,se}, \omega^{se}\right)$ in place of $\left(\alpha, \beta, \gamma, \lambda, M, \omega^i, \omega\right)$. A monopolistic NREE exists. The equilibrium stock price and the coefficients of the price process are,*

$$S_t^{se} = A^{se}(t) D_t + B^{se}(t) Z^{se} + F^{se}(t), \qquad Z^{se} \equiv \omega^{i,se} G + \omega^n \phi. \tag{32}$$

$$\mu_t^S = \frac{1}{1-(\omega^i)^2}\left(\frac{(\sigma_t^S)^2}{\Gamma} - \sigma_t^S \Theta_t^{se}(Z^{se};\omega^{se})\right), \qquad \sigma_t^S = A^{se}(t)\,\sigma^D \tag{33}$$

$$\Theta_t^{se}(Z^{se};\omega^{se}) = \alpha^{se}(t)\,Z^{se} + \beta^{se}(t)\,D_t + \gamma^{se}(t) \tag{34}$$

where $\Theta_t^{se}(Z^{se};\omega^{se}) \equiv \omega^{i,se}\theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi)$ *is the endogenous monopolistic WAPR. Innovations in the uninformed filtration are* $dW_t^S = dW_t^D - \theta_t^{D|m}dt$ *with,*

$$\theta_t^{D|m} = \frac{E\left[dW_t^D|\mathcal{F}_t^m\right]}{dt} = -\frac{\omega^{i,se}\sigma^D}{M^{se}(t)}\left(Z^{se} - \omega^{i,se}\left(D_t + \mu^D(T-t)\right) - \omega^n\mu^\phi\right). \tag{35}$$

The evolution of the stock price in the public information is given by (1) where W^S *is an* $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{Z^{se},D}$*-Brownian motion.*

The monopolistic equilibrium price (32) has the same structure as the competitive equilibrium price. Differences appear within this structure, as the relevant endogenous weights become $\omega^{i,se} = \omega^i\left(1-\omega^i\right)$ and $\omega^{se} = \omega^{i,se} + \omega^n$ instead of ω^i and ω. These differences reflect the behavior of the monopolistic investor. As shown by Proposition 24, the monopolistic stock demand is reduced by a fraction ω^i relative to the competitive demand. The aggregate demand function and the equilibrium stock price inherit this adjustment. So does the information revealed by the stock price.

The endogenous signal conveyed by the pair composed of the equilibrium price and the fundamental can be written as $Z^{se} \equiv \omega^{i,se}\left(G + \frac{\omega^n}{\omega^{i,se}}\phi\right)$. The informational content of this signal is reduced (i.e., information is less precise) relative to the competitive signal. By reducing demand, the monopolistic investor controls the leakage taking place through the price, hence protects his/her privileged information. Equilibrium public information in the presence of a monopolistic informed is more diffuse than in the presence of a competitive informed.

4.3 Monopolistic Equilibrium Properties

The equilibrium with monopolistic informed behavior has the same structural form as the competitive equilibrium with the substitution of the adjusted weights $\left(\omega^{i,se},\omega^{se}\right)$ in place of $\left(\omega^i,\omega\right)$. The properties described in Corollaries 7-12 therefore apply, with appropriate weight adjustments in the relevant conditions. The remainder of this section compares the monopolistic equilibrium to the competitive equilibrium.

Corollary 27 *The PIPR in the monopolistic equilibrium is* $\theta_t^{G|m}(G) = \alpha_1(t)G + \alpha_2^{se}(t)Z^{se} + \beta_0^{se}(t)D_t + \gamma_0^{se}(t)$ *where,*

$$\alpha_1(t) \equiv \frac{\sigma^D}{H(t)}, \qquad \alpha_2^{se}(t) = -\frac{\omega^{i,se}\sigma^D}{M^{se}(t)}, \qquad \beta_0^{se}(t) = \frac{\beta^{se}(t)}{\omega^{se}}, \qquad \gamma_0^{se}(t) = \frac{\gamma^{se}(t)}{\omega^{se}}.$$

The relations between the PIPR coefficients in the monopolistic and competitive equilibria are,

(i) $\alpha_2^{se}(t) < \alpha_2(t) \Longleftrightarrow \omega^{i,se}\omega^i H(t) > (\omega^n)^2\left(\sigma^\phi\right)^2$

(ii) $\beta_0^{se}(t) < \beta_0(t)$

(iii) $\gamma_0^{se}(t) > \gamma_0(t) \Longleftrightarrow \mu^\phi\omega^i\omega^{i,se}H(t) > \omega^n\left(\sigma^\phi\right)^2\left(\omega^i\mu^D(T-t)\left(2-\omega^i\right)+\omega^n\mu^\phi\right)$

The reduction in the stock demand by a fraction ω^i reduces the informational content of the price, which affects the conditional distribution of the private signal given public information. The conditional mean becomes,

$$\mu_t^{G|D,Z} = \left(1-\kappa_t^{se}\omega^{i,se}\right)\left(D_t+\mu^D(T-t)\right)+\kappa_t^{se}Z^{se}-\kappa_t^{se}\omega^n\mu^\phi$$

where $\kappa_t^{se} = \omega^{i,se}H(t)/M^{se}(t)$ captures the sensitivity to public information. The smaller weight $\omega^{i,se}$ has two effects. It reduces the conditional covariance between the private signal and the endogenous public signal given fundamental information, $\omega^{i,se}H(t) = Cov\left(G,\omega^{i,se}G+\omega^n\phi|\mathcal{F}_t^D\right)$. It also reduces the conditional variance given fundamental information, $M^{se}(t) = Var\left(\omega^{i,se}G+\omega^n\phi|\mathcal{F}_t^D\right)$. The sensitivity coefficient κ_t^{se} experiences conflicting effects. When $\omega^{i,se}\omega^i H(t) > (\omega^n)^2\left(\sigma^\phi\right)^2$, the second effect dominates, leading to an increase in κ_t^{se}. The negative relation between the PIPR and the conditional mean, implies that the sensitivity $\alpha_2^{se}(t)$ of the PIPR with respect to the endogenous signal Z will then decrease.

Several patterns emerge when time is factored in. When $\omega^{i,se}\omega^i H(t) > (\omega^n)^2\left(\sigma^\phi\right)^2$ for all $t \in [0,T]$, the sensitivity of the PIPR to Z is systematically lower in the monopolistic equilibrium. When $\omega^{i,se}\omega^i H(t) < (\omega^n)^2\left(\sigma^\phi\right)^2$ for all $t \in [0,T]$, it is systematically higher. Finally, when $\omega^{i,se}\omega^i H(0) > (\omega^n)^2\left(\sigma^\phi\right)^2$ and $\omega^{i,se}\omega^i H(T) < (\omega^n)^2\left(\sigma^\phi\right)^2$, the sensitivity coefficient $\alpha_2^{se}(t)$ is initially lower, then greater in the monopolistic equilibrium.

The effect on the sensitivity coefficient $\beta_0^{se}(t)$ is uniform through time. The decrease in the equilibrium weight $\omega^{i,se}$ of the monopolistic insider reduces the (negative) response of the PIPR to fundamental information. In effect, the reduced weight $\omega^{i,se}$ increases the sensitivity $1-\kappa_t^{se}\omega^{i,se}$ of the conditional mean $\mu_t^{G|D,Z}$ to fundamental information. The PIPR's response follows from its negative relation to the conditional mean.

The translation factor is driven by the component $\left(1-\kappa_t^{se}\omega^{i,se}\right)\mu^D\left(T-t\right)-\kappa_t^{se}\omega^n\mu^\phi$ in the conditional mean $\mu_t^{G|D,Z}$. It decreases because of the positive impact of monopolistic behavior on $1-\kappa_t^{se}\omega^{i,se}$. But it also experiences the conflicting effects on κ_t^{se} in the bias term $-\kappa_t^{se}\omega^n\mu^\phi$. The overall impact is positive under the condition indicated. Over time, the translation factor can be greater or smaller than in the competitive equilibrium depending on the prevailing condition.

Corollary 28 *The WAPR in the monopolistic equilibrium is given by (34). The relations between the WAPR coefficients in the monopolistic and competitive equilibria are,*

(i) $\alpha^{se}\left(t\right)<\alpha\left(t\right)\Longleftrightarrow\omega^{i,se}\omega^iH\left(t\right)>\left(\omega^n\right)^2\left(\sigma^\phi\right)^2$

(ii) $\beta^{se}\left(t\right)<\beta\left(t\right)\Longleftrightarrow\omega^i\left(\omega^i\left(1-\omega\right)+2\omega^n\right)H\left(t\right)>\left(\omega^n\right)^2\left(\sigma^\phi\right)^2$

(iii) $\gamma^{se}\left(t\right)>\gamma\left(t\right)\Longleftrightarrow H\left(t\right)>h^+\left(t\right)$ *where* $h^+\left(t\right)$ *is defined in the Appendix.*

The impact of imperfect competition on the coefficients of the WAPR reflects the impact on the weighted average of PIPRs. As $\alpha^{se}\left(t\right)=\alpha_1\left(t\right)+\alpha_2^{se}\left(t\right)$ where $\alpha_1\left(t\right)$ is the same as the competitive equilibrium, the change in $\alpha^{se}\left(t\right)$ is driven by the change in $\alpha_2^{se}\left(t\right)$, described in Corollary 27(i). For $\beta^{se}\left(t\right)=\omega^{se}\beta_0^{se}\left(t\right)$, the negative impact of monopolistic behavior on $\beta_0^{se}\left(t\right)$ in Corollary 27(ii) is counter-balanced by the reduced weight ω^{se}. The first effect dominates under the condition stated. Likewise, the impact on $\gamma^{se}\left(t\right)=\omega^{se}\gamma_0^{se}\left(t\right)$ reflects the combination of effects on $\gamma_0^{se}\left(t\right)$ and ω^{se}. The overall effect is positive under the condition indicated.

The presence of multiple effects on the components of the WAPR creates conditions for intricate dynamic relations between these component in the two equilibria. Figure 5 illustrates some of the patterns that can materialize.

Corollary 29 *The volatility of the stock price is greater in the monopolistic equilibrium than in the competitive equilibrium. monopolistic behavior destabilizes the market.*

Paradoxically, even though the demand function of the informed is tamer, monopolistic behavior increases the sensitivity of the stock price with respect to fundamental information, which leads to an increase in volatility. Underlying this phenomenon is the fact that monopolistic behavior reduces the information content of the price, which provides an incentive for increased reliance on publicly available information sources such as fundamental information. The stock price inherits this increased sensitivity to the fundamental. monopolistic behavior reduces the informational efficiency of the market. It also destabilizes the market.

5 Multiple Dividend Cycles

This section develops a framework with multiple dividend cycles. Section 5.1 presents the model. Section 5.2 describes the structure of equilibria. Section 5.3 examines their properties.

5.1 Economic Model

Consider a setting where the previous model is replicated over N consecutive periods (cycles). The economy has finite horizon $[0, T]$. Dividends are paid at the discrete dates $T_1, \ldots, T_N$ where $T_N = T$. The payment at T_n, denoted by D_{T_n}, is the terminal value of the process $dD_t = \mu_n^D dt + \sigma_n^D dW_t^D$ where μ_n^D is a constant expected growth rate and σ_n^D is a constant and positive volatility coefficient. W^D is a Brownian motion process with filtration $\mathcal{F}_{(\cdot)}^D$. The process D is the fundamental factor underlying the payments. Note that the characteristics $\left(\mu_n^D, \sigma_n^D\right)$ of the fundamental can change from cycle to cycle.

The economy is populated by informed and uniformed investors as well as mimicking noise traders. Population weights vary across cycles. The distribution is $\left(\omega_n^i, \omega_n^u, \omega_n^n\right)$ during the n^{th} cycle $[T_{n-1}, T_n)$, where $\omega_n^i + \omega_n^u + \omega_n^n = 1$ for $n = 1, ..., N$. Each population group is treated as a representative individual.

The informed investor receives a new private signal at the beginning of each cycle. The signal received at T_{n-1} is $G_n \equiv D_{T_n} + \zeta_n$ where $\zeta_n \sim \mathcal{N}\left(0, \left(\sigma_n^\zeta\right)^2\right)$. It conveys noisy information about the dividend D_{T_n} paid at T_n. The sequence of private signals constitutes a stochastic process $G \equiv \{G_n : n = 1, ..., N\}$. The associated filtration is $\mathcal{F}_{(\cdot)}^G$. The informed filtration $\mathcal{F}_{(\cdot)}^i \equiv \mathcal{F}_{(\cdot)}^{G,m}$ is generated by private and public information $\mathcal{F}_{(\cdot)}^G \vee \mathcal{F}_{(\cdot)}^m$. The informed maximizes $\mathcal{U}^i$ in (2) with respect to the number of shares N^i. The optimal demand function is described in Proposition 1 where $G = G_n$ in period $n = 1, ..., N$.

The noise trader is a mimicking agent who duplicates the demand of the informed, but on the basis of pure noise. Let $\phi \equiv \{\phi_n : n = 1, ..., N\}$ be the noise process where ϕ_n applies to cycle n. The random variable ϕ_n is normally distributed with mean μ_n^ϕ and variance $\left(\sigma_n^\phi\right)^2$. There is no serial dependence in ϕ (i.e., ϕ_n independent of ϕ_j for $n \neq j$). Moreover, ϕ, ζ, D are independent, where $\zeta \equiv \{\zeta_n : n = 1, ..., N\}$. The noise trading demand is (4) evaluated at $\phi = \phi_n$ in cycle n.

The uninformed investor observes public information $\mathcal{F}^m_{(\cdot)}$ and maximizes $\mathcal{U}^u$ in (2). The optimal uninformed demand function is given in Proposition 1.

5.2 Equilibria Structures

The functions arising in equilibrium are, for $n = 1, \ldots, N$, amended as follows,

$$\alpha_n(t) = \frac{1 - \kappa_n(t)\,\omega_n}{H_n(t)}\sigma_n^D, \qquad \beta_n(t) = -\omega_n \frac{1 - \kappa_n(t)\,\omega_n^i}{H_n(t)}\sigma_n^D \tag{36}$$

$$\gamma_n(t) = -\omega_n \frac{\left(1 - \kappa_n(t)\,\omega_n^i\right)\mu_n^D (T_n - t) - \omega_n^n \kappa_n(t)\,\mu_n^\phi}{H_n(t)}\sigma_n^D, \qquad \kappa_n(t) = \frac{\omega_n^i H_n(t)}{M_n(t)} \tag{37}$$

$$\lambda_n(t,s) = \frac{\omega_n^i \left(\sigma_n^D\right)^2 (s-t)}{M_n(t)}, \quad A_n^1(t,s) \equiv \left(\frac{H_n(s)}{H_n(t)}\right)^{\omega_n}\left(\frac{M_n(s)}{M_n(t)}\right)^{1-\omega_n}, \quad s \in [t, T_n] \tag{38}$$

$$H_n(t) = \left(\sigma_n^D\right)^2 (T_n - t) + \left(\sigma_n^\zeta\right)^2, \qquad M_n(t) = \left(\omega_n^i\right)^2 H_n(t) + \left(\omega_n^n\right)^2 \left(\sigma_n^\phi\right)^2 \tag{39}$$

$$B_n^D(t) \equiv \lambda_n(t, T_n), \quad A_n^D(t) \equiv 1 - \omega_n^i B_n^D(t), \quad F_n^D(t) \equiv A_n^D(t)\,\mu_n^D (T_n - t) - \omega_n^n B_n^D(t)\,\mu_n^\phi \tag{40}$$

with $\omega_n = \omega_n^i + \omega_n^n$. The function $A_n^1(t,s)$ is the stock price response to dividend shocks in a one-cycle model with parameters $\sigma_n^D, \sigma_n^\zeta, \sigma_n^\phi, \mu_n^D, \mu_n^\phi, \omega_n^i, \omega_n^n$. Equilibria are described next.

Proposition 30 *A competitive and monopolistic NREE exist. The stock price in the competitive equilibrium with N dividend cycles is a right continuous left limit process given by,*

$$S_t = A_n^N(t)\,D_t + B_n^N(t)\,Z_n + F_n^N(t) \qquad \text{where} \qquad Z_n = \omega_n^i G_n + \omega_n^n \phi_n \tag{41}$$

for $t \in [T_{n-1}, T_n)$ and $n = 1, \ldots, N$, with,

$$\begin{bmatrix} A_n^N(t) \\ B_n^N(t) \\ F_n^N(t) \end{bmatrix} = \begin{bmatrix} A_n(t) \\ B_n(t) \\ F_n(t) \end{bmatrix} + \begin{bmatrix} A_n^S(t) \\ B_n^S(t) \\ F_n^S(t) \end{bmatrix} 1_{n \leq N-1}, \qquad \begin{bmatrix} A_n^S(t) \\ B_n^S(t) \\ F_n^S(t) \end{bmatrix} = L_n^N \begin{bmatrix} A_n^D(t) \\ B_n^D(t) \\ F_n^D(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_n^N \end{bmatrix} \quad (42)$$

$$L_n^N \equiv A_{n+1}^N(T_n) + \omega_{n+1}^i B_{n+1}^N(T_n) \quad (43)$$

$$K_n^N \equiv F_{n+1}^N(T_n) + B_{n+1}^N(T_n)\left(\omega_{n+1}^i \mu^D (T_{n+1} - T_n) + \omega_{n+1}^n \mu_{n+1}^\phi\right) \quad (44)$$

$$A_n(t) = A_n^1(t, T_n) + \sigma_n^D \left(\int_t^{T_n} A_n^1(t,s) A_n^S(s) \beta_n(s) ds\right) 1_{n \leq N-1} \quad (45)$$

$$B_n(t) = \lambda_n(t, T_n) + \sigma_n^D \left(\int_t^{T_n} A_n^N(v)(\alpha_n(v) + \beta_n(v)\lambda_n(t,v)) dv\right) \quad (46)$$

$$F_n(t) = A_n(t)\mu_n^D(T_n - t) - \frac{\left(\sigma_n^D\right)^2}{\Gamma} \int_t^{T_n} A_n^N(v)^2 dv + \sigma_n^D \int_t^{T_n} A_n^N(v)\gamma_n(v) dv - \omega_n^n I_n(t)\mu_n^\phi \quad (47)$$

$$I_n(t) = \lambda_n(t, T_n) + \sigma_n^D \int_t^{T_n} A_n^N(v)\beta_n(v)\lambda_n(t,v) dv \quad (48)$$

and $\alpha_n, \beta_n, \gamma_n, \kappa_n, \lambda_n, A_n^1, H_n, M_n, A_n^D, B_n^D, F_n^D$ as defined in (36)-(40). The equilibrium stock price coefficients are, for $t \in [T_{n-1}, T_n)$,

$$\mu_t^S = \frac{\left(\sigma_t^S\right)^2}{\Gamma} - \sigma_t^{S,N}\Theta_t(Z_n; \omega_n), \qquad \sigma_t^S = A_n^N(t)\sigma_n^D \quad (49)$$

$$\Theta_t(Z_n; \omega_n) = \alpha_n(t) Z_n + \beta_n(t) D_t + \gamma_n(t) \quad (50)$$

where $\Theta_t(Z_n; \omega_n) \equiv \omega_n^i \theta_t^{G|m}(G_n) + \omega_n^n \theta_t^{G|m}(\phi_n)$ is the endogenous WAPR. Innovations in the uninformed filtration are $dW_t^S = dW_t^D - \theta_t^{D|m} dt$ with,

$$\theta_t^{D|m} = \frac{E\left[dW_t^D | \mathcal{F}_t^m\right]}{dt} = \frac{\omega_n^i \sigma^D}{M_n(t)}\left(Z_n - \omega_n^i\left(D_t + \mu_n^D(T - t)\right) - \omega_n^n \mu_n^\phi\right). \quad (51)$$

The evolution of the stock price in the public information is given by (1) where W^S is an $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^{Z,D}$-Brownian motion with $\mathcal{F}_t^{Z,D} = \left\{\mathcal{F}_t^{Z_n,D} : t \in [T_{n-1}, T_n), n = 1, \ldots, N\right\}$. The monopolistic NREE is obtained by replacing ω_n^i by $\omega_n^{i,se} \equiv \omega_n^i\left(1 - \omega_n^i\right)$ in the functions above.

The overall structures of equilibria remain the same. The stock price has two parts,

$$S_t = E\left[D_{T_n} - \int_t^{T_n} \mu_v^S dv \middle| \mathcal{F}_t^m \right] + E\left[S_{T_n} \middle| \mathcal{F}_t^m \right].$$

The first component captures the value at t of the dividend payment at the end of the current cycle net of the mean price appreciation. This component is the same, structurally, as in a one-cycle model. The second component represents the value at t of the stock at the beginning of the next dividend cycle. The equilibrium coefficients also have two parts reflecting these two components, $A_n^N(t) = A_n(t) + A_n^S(t)$, $B_n^N(t) = B_n(t) + B_n^S(t)$ and $F_n^N(t) = F_n(t) + F_n^S(t)$, for $n \leq N-1$. In the competitive equilibrium, the informational content of the stock price, given fundamental information, is $Z_n = \omega_n^i G_n + \omega_n^n \phi_n$ in period n. It depends on the prevailing distribution of investors and the relevant private signal and noise. At the beginning of each dividend cycle, a new signal materializes. The information structure then jumps, leading to a jump in the stock price and in the endogenous signal conveyed. Within each cycle, the sensitivity of the stock price to fundamental information $A_n^N(t)$ and the associated stock price volatility $\sigma_t^S = A_n^N(t)\sigma_n^D$ reflect the impact of the future stock prices.

5.3 Equilibria Properties

The PIPR and WAPR capture local properties of returns and are therefore not sensitive to the horizon. Their values and behavior are determined by the specifics of the information-uncertainty structure within each cycle, i.e., the coefficients $\sigma_n^D, \sigma_n^\zeta, \sigma_n^\phi, \mu_n^D, \mu_n^\phi, \omega_n^i, \omega_n^n$. The stock price and its coefficients, in contrast, display intricate intra-cycle and horizon effects. This section provides an analysis of their behavior and numerical illustrations.

5.3.1 Stock Price Sensitivity to Fundamental and Volatility

As indicated above, the price sensitivity to the fundamental has two parts, respectively related to the expected net dividend and to the expected future price, $A_n^N(t) = A_n(t) + A_n^S(t)$.

The expected net dividend sensitivity, $A_n(t)$, further decomposes in two pieces, as shown by (45). The first (resp. second) is a myopic (resp. dynamic) component attributable to the current cycle (resp. future cycles). The myopic part, $A_n^1(t, T_n)$, is the same as in a one-cycle model with coefficients $\sigma_n^D, \sigma_n^\zeta, \sigma_n^\phi, \omega_n^i, \omega_n^n$. It has the monotonicity property outlined in Corollary 9. The dynamic part, $A_n^\sigma(t, T_n) \equiv \sigma_n^D \left(\int_t^{T_n} A_n^1(t,s) A_n^S(s) \beta_n(s) ds\right) 1_{n \le N-1}$, stems from the dependence of the stock volatility, in the mean price appreciation, on future cycles. It vanishes throughout the last cycle $n = N$. Prior to that, it exhibits intricate behavior. During cycle $n < N$, it is continuous with respect to time and can increase, decrease or exhibit hump-shaped behavior. It converges to zero at the end of cycle n. It then jumps to a new value at the beginning of cycle $n+1$ as long as $n+1 < N$.

The expected future price sensitivity, $A_n^S(t)$, also represents a dynamic component. It vanishes throughout the last dividend cycle $n = N$. In prior cycles, it is continuously increasing or decreasing. It can be positive or negative at the end of each cycle, and jumps at the beginning of the next cycle.

The overall behavior of A_n^N exhibits rich patterns. Within a cycle, it can have zero, one or multiple humps and display concave or convex structure. It can be positive or negative at the end of each cycle. It can also jump up or down at the beginning of the next cycle. The top row of Figure 6 displays a possible pattern for A_n^N and its components A_n^S, A_n. In this numerical illustration, A_n^N and A_n^S increase during each cycle, then jump down at the onset of the next cycle. The coefficient A_n has the opposite (resp. same) behavior in the competitive (resp. monopolistic) equilibrium. The general pattern is one where the price becomes progressively more sensitive to the fundamental over the initial cycles, reaches a peak sensitivity during an intermediate cycle, then becomes less responsive during later cycles. This general pattern is more pronounced in the monopolistic equilibrium. The top row of Figure 7 (left panel) shows that the volatility $\sigma_t^S = A_n^N(t) \sigma_n^D$ displays the same pattern.

Further inspection of the structure of A_n^N reveals that the sensitivity to the fundamental is the combination of a low frequency and high frequency components. The low frequency component becomes prominent at the end of each cycle. It corresponds to $L_n^N = A_{n+1}^N(T_n) + \omega_{n+1}^i B_{n+1}^N(T_n)$ in the limit $\lim_{t \uparrow T_n} A_n^N(t) = \lim_{t \uparrow T_n} A_n(t) + \lim_{t \uparrow T_n} A_n^S(t) = 1 + L_n^N$. The process $\left\{L_n^N : n = 1, ..., N\right\}$ determines the behavior at cycle ends $\{t_n : n = 1, ..., N\}$. It encapsulates the feedback effect from the future: if $L_n^N = 0$, the sensitivity collapses to the myopic term $A_n(t) = A_n^1(t, T_n)$. During the n^{th} cycle, L_n^N pegs the level of the sensitivity to the fundamental. Indeed, the dynamic component $A_n^S(t) = L_n^N A_n^D(t)$ converges to L_n^N as $t \to T_n$. It also determines the size of the (high frequency) component $A_n^\sigma(t, T_n)$, through $\left\{A^S(s) : s \in [t, T_n]\right\}$. The high frequency components correspond to $A_n^1(t, T_n)$, $A_n^\sigma(t, T_n)$ and $A^D(t)$. Each of these measures a specific aspect of the price responsiveness to the fundamental within cycle n. The low frequency component determines the overall shape of $A_n^N(t)$ and σ_t^S over the period $[0, T]$, i.e. across cycles. The high frequency components pin down the intra-cycle behavior.

5.3.2 Stock Price Sensitivity to Endogenous Information

The stock price sensitivity with respect to the endogenous signal also decomposes into an expected net signal response $B_n(t)$ and an expected future price response $B_n^S(t)$. Both vanish at the payment dates T_n, so that $\lim_{t \uparrow T_n} B_n^N(t) = 0$. The structure of $B_n(t)$ is the same as in the one-cycle model except for $A(t)$, which is replaced by $A_n^N(t)$ and is no longer a monotone function of time. The coefficient $B_n^S(t)$ captures the feedback from future cycles, namely the fact that endogenous information revealed at the beginning of cycle n helps to assess the fundamental, hence the stock price, at the beginning of cycle $n+1$. The level of $B_n^S(t)$ is determined by the low frequency component L_n^N. It vanishes at T_n because D_{T_n} becomes publicly known.

Both components of $B_n^N(t)$ can take positive or negative values. Both can be increasing or decreasing in time. Possible patterns are shown in the middle row of Figure 6. In this illustration, the intra-cycle peak sensitivity decreases over the first few dividend cycles, increases mildly over a couple of intermediate cycles, then decreases over the remaining cycles. At the end of each cycle, the sensitivity vanishes. Intra-cycle peaks are larger in the monopolistic NREE due to the reduced informational leakage.

5.3.3 Dynamic Stock Price Behavior

The stock price path is right continuous with left limits. It jumps at times T_n when dividends are paid and new information signals are received. Straightforward calculations show that the jump size,

$$\Delta S_{T_n} = -D_{T_n} + B_{n+1}^N(T_n)\left(Z_{n+1} - \left(\omega_{n+1}^i D_{T_n} + \omega_{n+1}^i \mu^D (T_{n+1} - T_n) + \omega_{n+1}^n \mu_{n+1}^\phi\right)\right)$$

is determined by the dividend payment $-D_{T_n}$ and the surprise in the endogenous signal, $Z_{n+1} - E\left[Z_{n+1} \middle| \mathcal{F}_{T_n-}^m\right]$. The coefficient $B_{n+1}^N(T_n)$ accounts for the sensitivity of the price to the endogenous signal. Within each cycle the price behavior is determined by the evolution of the fundamental and of the high frequency coefficients. The behavior across cycles reflects the properties of the low frequency coefficients K_n^N and L_n^N.

The inter-cycle trend in the stock price can be positive or negative because of the non-monotonicity of the coefficients with respect to time. Intra-cycle patterns can also vary across cycles. An illustration is provided in the bottom row of Figure 7. In this example, the stock price shows a mild positive (resp. strong negative) inter-cycle trend during the initial (resp. later) cycles. This pattern is influenced by the behavior of $F_n^S(t)$, and more specifically its low frequency component K_n^N, as seen in the bottom row of Figure 6. The intra-cycle trends are mostly negative.

5.3.4 Expected Price Change and Optimal Portfolios

The expected change in the stock price (the stock premium) depends on the volatility and the WAPR. The WAPR is determined locally and is therefore not subject to feedback effects across dividend cycles. As discussed above, the stock volatility has myopic and dynamic components. It increases within each cycle and can increase or decrease across cycles. These patterns are inherited by the Sharpe ratio of the stock, $\theta_t^m = \sigma_t^S/\Gamma - \Theta_t(Z_n;\omega_n)$. They are also amplified (resp. dampened) in the stock premium if the WAPR is positive (resp. negative) and varies in the opposite (resp. same) direction. The top row of Figure 7 (middle and right panels) illustrates the behavior of the premium and the Sharpe ratio. It shows that anticipative information can be the source of a sizeable positive or negative premium.

Equilibrium holdings reflect similar considerations. The optimal portfolio of the uninformed, a mean-variance portfolio, is proportional to the Sharpe ratio of the stock normalized by its volatility. It is therefore related to the inverse of the stock volatility. If the WAPR is negative (resp. positive), the behavior of the uninformed portfolio is the opposite of (resp. same as) the stock volatility behavior. Given that the WAPR changes stochastically over time, the uninformed portfolio can amplify, dampen or even reverse the inverse volatility patterns depending on economic conditions. The optimal portfolio of the informed has similar properties. But the response to the inverse volatility is now driven by the WAPR net of the PIPR.[9] The resulting positions are more extreme when the PIPR and WAPR have opposite signs. The middle row of Figure 7 shows the optimal

[9]The informed portfolio can be written as $N_t^i = 1 - \Gamma\left(\Theta_t(Z_n;\omega_n) - \theta_t^{G|m}(G_n)\right)/\sigma_t^S$

portfolios of the public (left panel), the informed (middle panel) and the noise trader (right panel) in the numerical example displayed. The overall pattern of the uninformed holdings portfolio is similar to the behavior of the market price of risk. Volatility scaling implies that the market price of risk levels and variations are amplified toward the beginning and end of the period $[0, T]$, relative to the middle of the period. The intra-cycle holdings display more aggressive behavior at the beginning of each cycle when volatility is lower. Informed stock holdings display similar patterns, but modulated by the PIPR. Jump sizes at private information arrival dates are generally greater than for uninformed holdings, due to the jumps in the PIPR. Overall, the informed portfolio experiences smaller variations intra-cycle, but greater variations at jump times.

5.3.5 Information, Stabilization and Welfare

Should trading based on private information be permitted? The one-cycle model of the previous sections shows that the private information leakage through the price reduces volatility, thereby stabilizing the financial market. It also shows that private information trades can be Pareto optimal under certain circumstances. A relevant question is whether these conclusions are artifacts of the myopic nature of the model or whether they have deeper roots.

In order to assess the volatility impact of private information in the multi-cycle model, consider the equilibria in the economies without and with private information. The volatility differential, $\Delta\sigma_t^S \equiv \sigma_t^{S,ni} - \sigma_t^S$, can be decomposed as,

$$\frac{\Delta\sigma_t^S}{\sigma_n^D} = 1 - A_n^1(t, T_n) + L_n^{N,ni} 1_{n \leq N-1} - L_n^N \left(A_n^D(t) + \sigma_n^D \left(\int_t^{T_n} A_n^1(t, s) A_n^D(s) \beta_n(s) \, ds \right) 1_{n \leq N-1} \right)$$

where $L_n^{N,ni} \equiv A_{n+1}^{N,ni}(T_n)$ and $L_n^N \equiv A_{n+1}^N(T_n) + \omega_{n+1}^i B_{n+1}^N(T_n)$. The term $1 - A_n^1(t, T_n) \geq 0$ captures the positive myopic effect associated with the on-going dividend cycle. The remaining terms reflect the feedback effects of future cycles through the end-of-cycle stock price. These effects can be positive or negative, depending on parameter values. In the absence of private information, the feedback term $L_n^{N,ni}$ captures the volatility of expected future dividends. With private information, the feedback terms reflect the volatility associated with expected future dividends as well as the volatility related to expected future endogenous signals. The size of the latter can be substantial leading to an overall increase in volatility relative to the economy without private information.

At the end of each cycle, the volatility differential becomes,

$$\frac{\Delta\sigma_t^S}{\sigma_n^D} = \left(L_n^{N,ni} - L_n^N\right) 1_{n\leq N-1}, \quad \text{where } L_n^{N,ni} = 1 + L_{n+1}^{N,ni}, \quad L_n^N = \varrho_{0,n+1} + \varrho_{1,n+1} L_{n+1}^N$$

and the coefficients $\varrho_{0,n+1}, \varrho_{1,n+1}$ are convolutions of parameters defined in (93)-(96). A sufficient condition for a positive volatility differential at cycle ends is $1 \geq \varrho_{0,n+1}, \varrho_{1,n+1} \geq 0$ for $n = 1, ..., N-1$. Under this condition, the information leakage through the price system is sufficiently strong to ensure that investors, and ultimately prices, are less responsive to fundamental shocks, at cycle ends, in the NREE.

Figure 8 illustrates possible volatility configurations for the two economies, when G and ϕ are identically distributed. The plots are for different levels of skills. In these numerical examples, volatility is initially lower in the NREE, then eventually exceeds volatility in the economy without private information. The effect of skill is notable. When skill increases, the NREE volatility decreases throughout $[0, T]$ and falls significantly below the no-information volatility. When skill reaches a sufficiently high level, the NREE volatility falls below the no-information volatility except during the next-to last cycle. The intuition is straightforward. As skill increases, the quality of the endogenous information revealed in equilibrium increases, prompting investors to tame their responses to shocks. The equilibrium price also becomes more resilient to fundamental fluctuations.

The reduction in price volatility in the NREE, has important ramifications for welfare. As shown in Appendix A, the welfare gain for the uninformed $\Delta^u \equiv E\left[\mathcal{U}^u\right] - E\left[\mathcal{U}^{u,ni}\right]$, relative to the benchmark economy without private information, has the decomposition $\Delta^u = \Delta P^u + \Delta N^u + \Delta T^u$. The terms involved correspond to the price impact (ΔP^u), the allocation impact (ΔN^u) and the trading impact (ΔT^u). The reduced volatility implies that the trading impact is negative for low risk tolerance and increases with the tolerance level. The trading impact eventually becomes positive due to the positive value of the endogenous signal. A reduced volatility also implies that the price impact is positive for low risk tolerance and decreases hyperbolically with the tolerance level. Lastly, it has an ambiguous effect on the linear relation between the allocation impact and the risk tolerance parameter. The overall impact on the uninformed welfare is the sum of these effects. When initial volatility falls significantly below the volatility in the benchmark economy, the welfare gains for the uninformed are substantial. Figure 8 shows the uninformed welfare gains for three levels of skill. In this example, the beliefs of the noise trader and the informed coincide, implying that Pareto optimality is determined by the welfare of the uninformed. When skill is high, the information revealed through the price is more reliable, leading to tame portfolio adjustments in response to fundamental shocks and to low price fluctuations. Welfare gains are substantial at low risk tolerance levels. The lower threshold risk tolerance for indifference between allowing or prohibiting trades based on private information increases. Moreover, in this particular example, the trading (informational efficiency) impact, which dominates the overall welfare comparison when risk tolerance becomes sufficiently large, is positive and significant. The upper threshold for indifference decreases. Overall, the region of risk tolerance values over which the NREE Pareto dominates the benchmark equilibrium expands to the whole space. This striking conclusion is also valid for the lower skill levels examined. The underlying reason is the behavior of the allocation and price impacts as risk tolerance increases. The allocation impact is immaterial because $\delta_0^N, \alpha_1(0)$ are very small. The price impact remains positive because the adverse drop in the sensitivity to the fundamental is not strong enough to offset the positive effect of endogenous information on the price level. Thus, in all three cases the NREE is Pareto optimal, for all possible values of risk tolerance.

6 Conclusion

This paper examines the structure and properties of non-stationary noisy rational expectations equilibria in models with continuous trading and discrete dividend payment dates. Equilibrium prices fail to be weak-form efficient. Public information is carried by the price-fundamental pair. Informed trading has a stabilizing effect, as it reduces the volatility of the stock price, and it can be Pareto optimal. Over the dividend cycle, the stock price volatility, the price of risk and the covariance between the stock price and the price of risk all increase. Monopolistic behavior leads to a tamer optimal demand function. The resulting information available to the public is more diffuse and the equilibrium stock price volatility increases relative to the competitive setting. Potential welfare gains are weakened.

The dynamic model developed in this paper is tractable and produces closed form solutions, in the competitive as well as the monopolistic case. It therefore offers a useful platform to examine complex issues related to information asymmetry in financial markets. For instance, it provides a natural setting to study policy questions. Should trades based on private information be banned? The results in this paper suggest that an outright ban may not be best for society. Should they be restricted in less stringent ways? If so, what are appropriate regulations? What are the efficiency and liquidity effects of various types of restrictions? These questions are of fundamental importance for the smooth functioning of financial markets and the welfare of market participants. Their analysis requires extensions of the model incorporating the relevant regulatory constraints under consideration and is therefore beyond the scope of the present study. Issues such as these could be interesting avenues for future research.

7 Appendix

7.1 Appendix A: Welfare in the Multi-cycle Model

Let $\mathbf{G} = (G_1, \ldots, G_N)$, $\mathbf{Z} = (Z_1, \ldots, Z_N)$ and $\Phi = (\phi_1, \ldots, \phi_N)$. The components of welfare are,

$$\mathcal{I}^i = \sum_{n=1}^{N} \frac{\Gamma}{2} E\left[\int_{T_{n-1}}^{T_n} \theta_t^{G_n|m} (G_n)^2 \, dt\right] \tag{52}$$

$$\mathcal{I}^u = \frac{\Gamma}{2} \sum_{n=1}^{N} \left(\left(\omega_n^i\right)^2 E\left[\int_{T_{n-1}}^{T_n} \theta_t^{G_n|m} (G_n)^2 \, dt\right] + \left(\omega_n^n\right)^2 E\left[\int_0^T \theta_t^{G|m} (\phi_n)^2 \, dt\right] \right). \tag{53}$$

$$\mathcal{I}^n \equiv \frac{\Gamma}{2} \sum_{n=1}^{N} \int_{T_{n-1}}^{T_n} E\left[\theta_v^{G_n|m} (G_n)^2 \mathcal{L}_{\phi_n, G_n} \left(G_n | Z_n\right)\right] dv \tag{54}$$

$$\Delta T^n \equiv \Gamma \sum_{n=1}^{N} \int_{T_{n-1}}^{T_n} \left(E\left[\theta_v^{G_n|m} (G_n) \theta_v^m \mathcal{L}_{\phi_n, G_n} \left(G_n | Z_n\right)\right] + \frac{1}{2} E\left[COV\left(\mathcal{L}_{\phi_n, G_n} \left(G_n | Z_n\right), \left(\theta_v^m\right)^2 \middle| Z_n\right)\right] \right) dv \tag{55}$$

$$\Delta N^n \equiv \Gamma \frac{\sigma_1^D}{H_1(0)\, \sigma_0^S} \left(\left(\mu_1^\phi - E\left[G_1\right]\right) E\left[S_0^N\right] + B_1^N(0) \left(\omega_1^n \left(\sigma_1^\phi\right)^2 - \omega_1^i H_1(0) \right) \right). \tag{56}$$

Ex-ante expected utilities are, respectively, $E\left[\mathcal{U}^i(\mathbf{G})\right], E\left[\mathcal{U}^u(\mathbf{Z})\right]$ and $E\left[\mathcal{U}^n(\mathbf{\Phi})\right]$.

Proposition 31 *In the NREE, ex-ante utilities satisfy the relations,* $E\left[\mathcal{U}^i(\mathbf{G})\right] = E\left[\mathcal{U}^u(\mathbf{Z})\right] + \mathcal{I}^i$ *and* $E\left[\mathcal{U}^n(\mathbf{\Phi})\right] = E\left[\mathcal{U}^u(\mathbf{Z})\right] + \mathcal{I}^n + \Delta T^n + \Delta N^n$ *where the components are defined in (52)-(56) and,*

$$E\left[\mathcal{U}^u(\mathbf{Z})\right] = E\left[N_0^u S_0\right] + \frac{1}{2\Gamma} \sum_{n=1}^{N} \left(\int_{T_{n-1}}^{T_n} \left(\sigma_t^S\right)^2 dt - \omega_n^n \int_{T_{n-1}}^{T_n} \sigma_t^S f_t^n dt \right) + \mathcal{I}^u \tag{57}$$

$$E\left[N_0^u S_0\right] = \left(1 - \Gamma \delta_0^N\right) E\left[S_0\right] - \Gamma K_0^N, \qquad \delta_0^N \equiv \frac{\beta_1(0)\, D_0 + \gamma_1(0)}{A_1^N(0)\, \sigma_1^D} \tag{58}$$

$$K_0^N \equiv \frac{\alpha_1(0)}{\sigma_0^{N,S}} E\left[Z_1 S_0\right] = \frac{\alpha_1(0)}{A_1^N(0)\, \sigma_1^D} \left(\left(A_1^N(0)\, D_0 + F_1^N(0)\right) E\left[Z_1\right] + B_1^N(0)\, E\left[Z_1^2\right] \right) \tag{59}$$

$$E\left[S_0\right] = A_1^N(0)\, D_0 + B_1^N(0)\, E\left[Z_1\right] + F_1^N(0), \qquad \sigma_t^S = A_n^N(t)\, \sigma_n^D 1_{t \in [T_{n-1}, T_n)} \tag{60}$$

$$f_t^n = \theta_t^{G_n|m} \left(\mu_n^\phi \middle| E\left[D_t\right], E\left[Z_n\right] \right), \quad \theta_t^{G|m}(x | d, z) \equiv \frac{x - E\left[G_n | D_t = d, Z_n = z\right]}{H_n(t)} \sigma_n^D 1_{t \in [T_{n-1}, T_n]}. \tag{61}$$

The function $\theta_t^{G|m}(x | d, z)$ *is the PIPR evaluated at* $(D_t, Z_n) = (d, z)$ *for* $t \in [T_{n-1}, T_n)$. *In the* *equilibrium without private information, all agents are uninformed. The ex-ante utilities,*

$$E\left[\mathcal{U}^{j,ni}\right] = E\left[S_0^{ni}\right] + \frac{1}{2\Gamma}\sum_{n=1}^{N}\int_{T_{n-1}}^{T_n}\left(\sigma_t^{ni}\right)^2 dt, \qquad j \in \{u, i\} \tag{62}$$

$$E\left[S_0^{ni}\right] = S_0^{ni} = A_1^{N,ni}(0)\,D_0 + F_1^{N,ni}(0)\,, \qquad \sigma_t^{ni} = A_n^{N,ni}(t)\,\sigma_n^D 1_{t\in[T_{n-1},T_n)} \tag{63}$$

are equal. Initial share allocations are also equal, $N_0^{i,ni} = N_0^{u,ni} = N_0^{n,ni} = 1$.

Proposition 32 *The NREE Pareto dominates the equilibrium without private information if and only if $\Delta^u \equiv E\left[\mathcal{U}^u(\mathbf{Z})\right] - E\left[\mathcal{U}^{u,ni}\right] = \Delta P^u + \Delta N^u + \Delta T^u > \max\{0, -\mathcal{I}^n - \Delta T^n - \Delta N^n\}$, where $\Delta P^u = E\left[S_0 - S_0^{ni}\right]$ is the price impact, $\Delta N^u = E\left[\left(N_0^u - 1\right)S_0\right]$ the allocation impact and ΔT^u the trading impact. The three components are $\Delta N^u = -\Gamma\left(\delta_0^N E\left[S_0\right] + K_0^N\right)$ and,*

$$\Delta P^u = \left(A_1^N(0) - A_1^{N,ni}(0)\right)D_0 + B_1^N(0)\,E\left[Z_1\right] - \frac{\Delta V}{\Gamma} + R \tag{64}$$

$$\Delta T^u = \frac{\Delta V}{2\Gamma} - \sum_{n=1}^{N}\omega_n^n\sigma_n^D\int_{T_{n-1}}^{T_n} A_n^N(t)\,\theta_t^{G|m}\left(\mu_n^\phi; E\left[D_t\right], E\left[Z_n\right]\right)dt + \mathcal{I}^u \tag{65}$$

where $\Delta V = \sum_{n=1}^{N}\left(\sigma_n^D\right)^2\int_{T_{n-1}}^{T_n}\left(A_n^N(t)^2 - A_n^{N,ni}(t)^2\right)dt$ is the change in the realized variance of the price and where $R \equiv F_0^N(0) - F_0^{N,ni}(0) - \Delta V/\Gamma$ denotes the terms of $F_0^N(0) - F_0^{N,ni}(0)$ independent of the tolerance Γ. If $\Delta V < 0$, the NREE Pareto dominates the equilibrium without private information if risk tolerance is sufficiently small ($\lim_{\Gamma\to 0}\Delta^u = +\infty$). If $\Delta V > 0$, there is no variance stabilization and the symmetric REE without private information Pareto dominates the NREE for small risk tolerance($\lim_{\Gamma\to 0}\Delta^u = +\infty$). For large risk tolerance, the NREE Pareto dominates ($\lim_{\Gamma\to+\infty}\Delta^u = +\infty$) if and only if,

$$\frac{1}{2}\sum_{n=1}^{N}\left(E\left[\int_{T_{n-1}}^{T_n}\left(\left(\omega_n^i\right)^2\theta_t^{G|m}\left(G_n\right)^2 + \left(\omega_n^n\right)^2\theta_t^{G|m}\left(\phi_n\right)^2\right)dt\right]\right) \geq \lim_{\Gamma\to\infty}\left(\delta_0^N E\left[S_0^N\right] + K_0^N\right). \tag{66}$$

A sufficient condition is $\delta_0^N \lim_{\Gamma\to\infty} E\left[S_0^N\right] + \lim_{\Gamma\to\infty} K_0^N \leq 0$.

The next result provides further insights on the scope for regulation. Define the relative entropy $\mathcal{E}^u \equiv \mathcal{I}^u/\Gamma$ and the coefficients,

$$C \equiv C_P + C_T, \qquad C_T \equiv -\sum_{n=1}^{N} \omega_n^n \sigma_n^D \int_{T_{n-1}}^{T_n} A_n^N(t)\, \theta_t^{G|m}\left(\mu_n^\phi; E[D_t], E[Z_n]\right) dt$$

$$C_P \equiv \left(A_1^N(0) - A_1^{N,ni}(0)\right) D_0 + B_1^N(0)\, E[Z_1] + R$$

$$J_0 \equiv \delta_0^N E\left[S_0^N\right] + K_0^N, \quad J \equiv C^2 + 2\left(\mathcal{E}^u - J_0\right)\Delta V, \quad \Gamma_\pm \equiv \frac{-C \pm \sqrt{J}}{2\left(\mathcal{E}^u - J_0\right)}.$$

With this notation, the following multi-cycle version of Corollary (18) holds.

Corollary 33 *The uninformed is as well off in the NREE under the following alternative conditions, (i) uniformly in Γ if $J < 0$ or if $J \geq 0$ and $\Gamma_+ \leq 0$, or (ii) for $\Gamma \in [0, \Gamma_+]$, if $J \geq 0$, $\Gamma_- < 0$ and $\Gamma_+ > 0$, or (iii) for $\Gamma \in [0, \Gamma_-] \bigcup [\Gamma_+, +\infty)$, if $J > 0$ and $\Gamma_- \geq 0$. Suppose that (i), (ii) or (iii) holds. If, in addition, $\mathcal{I}^n + \Delta T^n + \Delta N^n \geq 0$, the NREE Pareto dominates (weakly) the equilibrium without private information.*

If $\mathcal{I}^n + \Delta T^n + \Delta N^n \geq 0$, the mimicking noise trader is as well off as the uninformed. Under this condition, banning the use of private information is welfare reducing (case (i)), or justified for risk tolerance above Γ_+ (case (ii)), or justified for the risk tolerance range (Γ_-, Γ_+) (case (iii)). The sufficient condition $\mathcal{I}^n + \Delta T^n + \Delta N^n \geq 0$ is satisfied in symmetric equilibria where $\omega_n^i = \omega_n^n$ and $\phi_n \sim G_n$ (identically distributed) for all $n = 1, \ldots, N$. It also holds if $\Gamma \to 0$.

7.2 Appendix B: Proofs

Proof of Proposition 4. The aggregate demand function $N_t^a \equiv \omega^u N_t^u + \omega^i N_t^i + \omega^n N_t^n$ is,

$$N_t^a = \omega^u \Gamma \frac{\sigma_t^S \theta_t^m}{\left(\sigma_t^S\right)^2} + \omega^i \Gamma \frac{\sigma_t^S \left(\theta_t^m + \theta_t^{G|m}(G)\right)}{\left(\sigma_t^S\right)^2} + \omega^n \Gamma \frac{\sigma_t^S \left(\theta_t^m + \theta_t^{G|m}(\phi)\right)}{\left(\sigma_t^S\right)^2}$$

where the function $\theta_t^{G|m}(x)$ is endogenous. Conjecture that $\theta_t^{G|m}(x)$ is an affine function of x and let $\Theta_t(z;\omega) \equiv \omega^i \theta_t^{G|m}(x_1) + \omega^n \theta_t^{G|m}(x_2)$ where $z = \omega^i x_1 + \omega^n x_2$ and $\omega = \omega^i + \omega^n$. Under this conjecture the aggregate demand function becomes $N_t^a = \Gamma\left(\theta_t^m + \Theta_t(Z;\omega)\right)/\sigma_t^S$ and, at equilibrium, $N_t^a = 1$, $\sigma_t^S \theta_t^m = \left(\sigma_t^S\right)^2/\Gamma - \sigma_t^S \Theta_t(Z;\omega)$. Information revealed in equilibrium includes the noisy translation of the private signal $Z = \omega^i G + \omega^n \phi$. Thus, $\mathcal{F}_{(\cdot)}^m \supseteq \mathcal{F}_{(\cdot)}^{D,Z}$.

Suppose that $\mathcal{F}^m_{(\cdot)} = \mathcal{F}^{D,Z}_{(\cdot)}$. Given that,

$$G = D_T + \zeta = D_t + \mu^D (T-t) + \int_t^T \sigma^D dW_s^D + \zeta$$

$$Z = \omega^i G + \omega^n \phi = \omega^i \left(D_t + \mu^D (T-t) \right) + \omega^i \left(\int_t^T \sigma^D dW_s^D + \zeta \right) + \omega^n \phi$$

the conditional density at time t of the signal is $p_t^G(x) = \frac{1}{\sigma_t^{G|D,Z}} n\left(\frac{x-\mu_t^{G|D,Z}}{\sigma_t^{G|D,Z}} \right)$ where,

$$\mu_t^{G|D,Z} = D_t + \mu^D (T-t) + \kappa_t \left[Z - \omega^i \left(D_t + \mu^D (T-t) \right) - \omega^n \mu^\phi \right]$$

$$\left(\sigma_t^{G|D,Z} \right)^2 = \left(\left(\sigma^D \right)^2 (T-t) + \left(\sigma^\zeta \right)^2 \right) \left(1 - \kappa_t \omega^i \right) \equiv H(t) \left(1 - \kappa_t \omega^i \right)$$

$$\kappa_t = \frac{\omega^i \left(\left(\sigma^D \right)^2 (T-t) + \left(\sigma^\zeta \right)^2 \right)}{M(t)} = \frac{\omega^i H(t)}{M(t)}$$

$$M(t) = \left(\omega^i \right)^2 \left(\left(\sigma^D \right)^2 (T-t) + \left(\sigma^\zeta \right)^2 \right) + \left(\omega^n \right)^2 \left(\sigma^\phi \right)^2 = \left(\omega^i \right)^2 H(t) + \left(\omega^n \right)^2 \left(\sigma^\phi \right)^2$$

($M(t)$ is the variance of $Z - \omega^i \left(D_t + \mu^D (T-t) \right) = \omega^i \left(\int_t^T \sigma^D dW_s^D + \zeta \right) + \omega^n \phi$). Ito's lemma gives the PIPR,

$$\theta_t^{G|m}(x) = \frac{x - \mu_t^{G|D,Z}}{\left(\sigma_t^{G|D,Z} \right)^2} \left(1 - \kappa_t \omega^i \right) \sigma^D = \frac{x - \mu_t^{G|D,Z}}{H(t)} \sigma^D.$$

The PIPR for dividend risk, $\theta_t^{G|m}(x)$, is affine in x, as conjectured.

The information revealed in equilibrium is contained in,

$$\begin{aligned}
\Theta_t(Z;\omega) &\equiv \omega^i \theta_t^{G|m}(G) + \omega^n \theta_t^{G|m}(\phi) = \frac{Z - \left(\omega^i + \omega^n \right) \mu_t^{G|D,Z}}{H(t)} \sigma^D \equiv \frac{Z - \omega \mu_t^{G|D,Z}}{H(t)} \sigma^D \\
&= \frac{Z - \omega \left(\left(1 - \kappa_t \omega^i \right) \left(D_t + \mu^D (T-t) \right) + \kappa_t \left(Z - \omega^n \mu^\phi \right) \right)}{H(t)} \sigma^D \\
&= \left(\frac{1 - \kappa_t \omega}{H(t)} Z - \omega \frac{1 - \kappa_t \omega^i}{H(t)} D_t - \omega \frac{\left(1 - \kappa_t \omega^i \right) \mu^D (T-t) - \omega^n \kappa_t \mu^\phi}{H(t)} \right) \sigma^D \\
&\equiv \alpha(t) Z + \beta(t) D_t + \gamma(t)
\end{aligned}$$

where $\omega = \omega^i + \omega^n$, and is indeed equivalent to Z provided $\alpha(t) \neq 0$, i.e., $1 - \kappa_t \omega \neq 0$ for t in a neighborhood of 0. At $t = 0$, the condition is equivalent to, $1 - \kappa_0 \omega \neq 0 \iff \omega^n \left(\sigma^\phi \right) - \omega^i \left(\left(\sigma^D \right)^2 T + \left(\sigma^\zeta \right)^2 \right) \neq 0$. If the condition fails at $t = 0$, it holds at $t = 0_+$, so Z is immediately revealed in this case as well. Moreover, the pair (Z, D) is a sufficient statistic, in equilibrium, for the PIPR and the conditional density of the signal. This suggests that the pair is a sufficient statistic for the rest of the equilibrium as well. This still needs to be verified.

Suppose that uninformed agents use $\mathcal{F}^{D,Z}_{(\cdot)}$ to forecast the future dividend and assess the price of risk θ^m. In equilibrium, $\mu_t^S = \sigma_t^S \theta_t^m = \left(\sigma_t^S \right)^2 / \Gamma - \sigma_t^S \Theta_t(Z;\omega)$, which is affine with respect to the pair (Z, D). The volatility structure remains to be identified. Assuming volatility coefficients are functions of time and simplifying yields,

$$\begin{aligned}
S_t &= \mathbb{E}\left[D_T - \int_t^T \mu_s^S ds \middle| \mathcal{F}_t^m\right] = D_t + \mu^D(T-t) + \sigma^D \mathbb{E}\left[W_T^D - W_t^D \middle| \mathcal{F}_t^{Z,D}\right] \\
&\quad -\frac{1}{\Gamma}\int_t^T \left(\sigma_s^S\right)^2 ds + \int_t^T \mathbb{E}\left[\sigma_s^S\left(\alpha(s)Z + \beta(s)D_s + \gamma(s)\right)\middle| \mathcal{F}_t^{Z,D}\right] \\
&= D_t + \mu^D(T-t) - \frac{1}{\Gamma}\int_t^T \left(\sigma_s^S\right)^2 ds + \int_t^T \sigma_s^S \gamma(s) ds + \left(\int_t^T \sigma_s^S \alpha(s) ds\right) Z \\
&\quad +\sigma^D \mathbb{E}\left[W_T^D - W_t^D \middle| \mathcal{F}_t^{Z,D}\right] + \int_t^T \sigma_s^S \beta(s) \mathbb{E}\left[D_s \middle| \mathcal{F}_t^{Z,D}\right] ds \\
&\equiv D_t + G_0(t,T) Z + \widehat{F}\left(\sigma^S, t\right) + \sigma^D \mathbb{E}\left[W_T^D - W_t^D \middle| \mathcal{F}_t^{Z,D}\right] + \int_t^T \sigma_s^S \beta(s) \mathbb{E}\left[D_s | \mathcal{F}_t^{Z,D}\right] ds
\end{aligned}$$

where $G_0(t,T) = \int_t^T \sigma_s^S \alpha(s) ds$ and $\widehat{F}\left(\sigma^S, t\right) = \mu^D(T-t) - (1/\Gamma)\int_t^T \left(\sigma_s^S\right)^2 ds + \int_t^T \sigma_s^S \gamma(s) ds$. Moreover,

$$\begin{aligned}
\mathbb{E}\left[\int_t^T \sigma^D dW_s^D \middle| \mathcal{F}_t^{Z,D}\right] &= \lambda(t,T)\left(Z - \omega^i\left(D_t + \mu^D(T-t)\right) - \omega^n \mu^\phi\right) \\
&= \lambda(t,T) Z - \omega^i \lambda(t,T) D_t - \lambda(t,T)\left(\omega^i \mu^D(T-t) + \omega^n \mu^\phi\right)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[D_s | \mathcal{F}_t^{Z,D}\right] &= D_t + \mu^D(T-t) + \lambda(t,s)\left(Z - \omega^i\left(D_t + \mu^D(T-t)\right) - \omega^n \mu^\phi\right) \\
&= \left(D_t + \mu^D(T-t)\right)\left(1 - \omega^i \lambda(t,s)\right) + \lambda(t,s) Z - \omega^n \lambda(t,s) \mu^\phi
\end{aligned}$$

where $\lambda(t,s) = \frac{\omega^i \left(\sigma^D\right)^2 (s-t)}{M(t)}$, so that,

$$\int_t^T \sigma_s^S \beta(s) \mathbb{E}\left[D_s | \mathcal{F}_t^{Z,D}\right] ds = G_1(t,T)\left(D_t + \mu^D(T-t)\right) + G_2(t,T)\left(Z - \omega^n \mu^\phi\right)$$

$$G_1(t,T) = \int_t^T \sigma_s^S \beta(s)\left(1 - \omega^i \lambda(t,s)\right) ds, \qquad G_2(t,T) = \int_t^T \sigma_s^S \beta(s) \lambda(t,s) ds.$$

Hence,

$$\begin{aligned}
S_t &= D_t + G_0(t,T) Z + \widehat{F}\left(\sigma^S, t\right) + \lambda(t,T) Z - \omega^i \lambda(t,T) D_t \\
&\quad -\lambda(t,T)\left(\omega^i \mu^D(T-t) + \omega^n \mu^\phi\right) + G_1(t,T)\left(D_t + \mu^D(T-t)\right) + G_2(t,T)\left(Z - \omega^n \mu^\phi\right) \\
&= \left(1 - \omega^i \lambda(t,T) + G_1(t,T)\right) D_t + \left(G_0(t,T) + \lambda(t,T) + G_2(t,T)\right) Z + \widehat{F}\left(\sigma^S, t\right) \\
&\quad + \left(-\omega^i \lambda(t,T) + G_1(t,T)\right) \mu^D(T-t) - \left(\lambda(t,T) + G_2(t,T)\right) \omega^n \mu^\phi \\
&\equiv A(t) D_t + B(t) Z + F(t)
\end{aligned}$$

where,

$$A(t) = 1 - \omega^i \lambda(t,T) + G_1(t,T) = 1 - \omega^i \lambda(t,T) + \int_t^T \sigma_s^S \beta(s)\left(1 - \omega^i \lambda(t,s)\right) ds$$

$$B(t) = G_0(t,T) + \lambda(t,T) + G_2(t,T) = \lambda(t,T) + \int_t^T \sigma_s^S\left(\alpha(s) + \beta(s)\lambda(t,s)\right) ds$$

$$F(t) = \widehat{F}\left(\sigma^S, t\right) + \left(A(t) - 1\right) \mu^D(T-t) - \left(B(t) - \int_t^T \sigma_s^S \alpha(s) ds\right) \omega^n \mu^\phi.$$

An application of Ito's lemma shows that $\sigma_t^S = A(t)\,\sigma^D$. The volatility coefficient is deterministic as conjectured. This validates the construction of the equilibrium stock price to this stage. Substituting in the coefficients above gives,

$$A(t) = 1 - \omega^i \lambda(t,T) + \sigma^D \left(\int_t^T A(s)\,\beta(s) \left(1 - \omega^i \lambda(t,s)\right) ds \right)$$

$$B(t) = \lambda(t,T) + \sigma^D \left(\int_t^T A(s)\,(\alpha(s) + \beta(s)\lambda(t,s))\,ds \right)$$

$$F(t) = \widehat{F}\left(A(t)\,\sigma^D, t\right) + (A(t) - 1)\,\mu^D\,(T - t) - I(t)\,\omega^n \mu^\phi$$

$$I(t) = B(t) - \sigma^D \int_t^T A(s)\,\alpha(s)\,ds.$$

Inserting $\widehat{F}\left(A(t)\,\sigma^D, t\right) = \mu^D (T-t) - \frac{1}{\Gamma}\left(\sigma^D\right)^2 \int_t^T A(s)^2\,ds + \sigma^D \int_t^T A(s)\,\gamma(s)\,ds$ in the last coefficient and collecting terms leads to,

$$F(t) = A(t)\,\mu^D\,(T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma} \int_t^T A(s)^2\,ds + \sigma^D \int_t^T A(s)\,\gamma(s)\,ds - I(t)\,\omega^n \mu^\phi$$

$$I(t) = \lambda(t,T) + \sigma^D \int_t^T A(s)\,\beta(s)\,\lambda(t,s)\,ds.$$

In these expressions, with $\omega = \omega^i + \omega^n$,

$$\alpha(t) = \frac{1 - \kappa_t \omega}{H(t)}\sigma^D, \qquad \beta(t) = -\omega \frac{1 - \kappa_t \omega^i}{H(t)}\sigma^D$$

$$\gamma(t) = -\omega \frac{\left(1 - \kappa_t \omega^i\right)\mu^D\,(T-t) - \omega^n \kappa_t \mu^\phi}{H(t)}\sigma^D, \quad \lambda(t,s) = \frac{\omega^i \left(\sigma^D\right)^2 (s-t)}{M(t)}.$$

Equilibrium exists if the backward Volterra equation,

$$A(t) = 1 - \omega^i \lambda(t,T) + \sigma^D \left(\int_t^T A(s)\,\beta(s) \left(1 - \omega^i \lambda(t,s)\right) ds \right), \quad A(T) = 1 \tag{67}$$

for the coefficient $A(\cdot)$ has a solution. This issue is addressed in the next lemma. ■

Lemma 34 *The unique solution of the backward Volterra equation is* $A(t) = \left(\frac{H(T)}{H(t)}\right)^{\omega} \left(\frac{M(T)}{M(t)}\right)^{1-\omega}$ *with* $M(t) = \left(\omega^i\right)^2 H(t) + \left(\omega^n\right)^2 \left(\sigma^\phi\right)^2$ *and* $\omega = \omega^i + \omega^n$. *Moreover,* $A(t) > 0$ *for* $t \in [0,T]$.

Proof of Lemma 34. With $M(t) = \left(\omega^i\right)^2 H(t) + \left(\omega^n\right)^2 \left(\sigma^\phi\right)^2$, note that,

$$1 - \omega^i \lambda(t,T) = 1 - \frac{\left(\omega^i\right)^2 \left(\sigma^D\right)^2 (T-t)}{M(t)} = \frac{\left(\omega^i\right)^2 \left(\sigma^\zeta\right)^2 + \left(\omega^n\right)^2 \left(\sigma^\phi\right)^2}{M(t)} \equiv \frac{M(T)}{M(t)}$$

$$1 - \omega^i \lambda(t,s) = \frac{\left(\omega^i\right)^2 \left(\left(\sigma^D\right)^2 (T-s) + \left(\sigma^\zeta\right)^2\right) + \left(\omega^n\right)^2 \left(\sigma^\phi\right)^2}{M(t)} = \frac{M(s)}{M(t)}.$$

Substituting in (67) and using the change of variables $C(t) = A(t)M(t)$ leads to,

$$\begin{aligned} A(t) &= 1-\omega^i\lambda(t,T)+\sigma^D\left(\int_t^T A(s)\,\beta(s)\left(1-\omega^i\lambda(t,s)\right)ds\right) \\ &= \frac{M(T)}{M(t)}+\sigma^D\left(\int_t^T A(s)\,\beta(s)\frac{M(s)}{M(t)}ds\right) \\ \iff & A(t)M(t) = M(T)+\sigma^D\left(\int_t^T A(s)\,\beta(s)M(s)\,ds\right) \\ \iff & C(t) = M(T)+\sigma^D\left(\int_t^T C(s)\,\beta(s)ds\right) \end{aligned}$$

subject to the boundary condition $C(T)M(T)$. Equivalently, $dC(t) = -\sigma^D C(t)\,\beta(t)dt$. The solution is $C(t) = M(T)\exp\left(\sigma^D\int_t^T\beta(s)ds\right)$. Substituting,

$$\beta(t) = -\frac{\omega}{H(t)}\left(1-\kappa_t\omega^i\right)\sigma^D = -\frac{\omega}{H(t)}\left(1-\frac{\left(\omega^i\right)^2H(t)}{M(t)}\right)\sigma^D = -\omega\sigma^D\left(\frac{1}{H(t)}-\frac{\left(\omega^i\right)^2}{M(t)}\right)$$

and performing the integration,

$$C(t) = M(T)\exp\left(\omega\left(\log\left(\frac{H(T)}{H(t)}\right)-\log\left(\frac{M(T)}{M(t)}\right)\right)\right) = M(T)\left(\frac{H(T)}{H(t)}\right)^{\omega}\left(\frac{M(T)}{M(t)}\right)^{-\omega}.$$

Substituting $C(t) = A(t)M(t)$ and rearranging leads to the formula stated. ■

Proof of Remark 5. Fix ω^n and let $\omega^i\to 0$ and $\omega^u\to 1-\omega^n$. This yields $\kappa_t^{si} = \lambda^{si}(t,s) = 0$, $M^{si}(t) = \left(\omega^n\right)^2\left(\sigma^\phi\right)^2$ and,

$$A^{si}(t) = \left(\frac{H(T)}{H(t)}\right)^{\omega^n},\qquad B^{si}(t) = \sigma^D\left(\int_t^T A^{si}(s)\,\alpha^{si}(s)\,ds\right) \tag{68}$$

$$F^{si}(t) = A^{si}(t)\,\mu^D(T-t)-\frac{\left(\sigma^D\right)^2}{\Gamma}\int_t^T A^{si}(s)^2\,ds+\sigma^D\int_t^T A^{si}(s)\,\gamma^{si}(s)\,ds \tag{69}$$

$$\alpha^{si}(t) = \frac{\sigma^D}{H(t)},\quad \beta^{si}(t) = -\omega^n\frac{\sigma^D}{H(t)},\quad \gamma^{si}(t) = -\omega^n\frac{\mu^D(T-t)}{H(t)}\sigma^D. \tag{70}$$

The formulas stated follow. If, in addition, $\omega^n\to 0$, then $\beta^{si}(t) = \gamma^{si}(t) = M^{si}(t) = Z^{si} = 0$ and $A^{si}(t) = 1$. The stock price and return components announced follow. Note that $A(t) < A^{si}(t) < 1$ for $t < T$ and $A(T) = A^{si}(T) = 1$. Therefore, $\sigma_t^S < \sigma_t^{S,si} < \sigma_t^{S,si,0} = \sigma^D$ for $t < T$. In the limit, $lim_{t\to T}\sigma_t^S = lim_{t\to T}\sigma_t^{S,si} = lim_{t\to T}\sigma_t^{S,si,0} = \sigma^D$. ■

Proof of Remark 6. Fix ω^n and let $\omega^i\to 1-\omega^n$ and $\omega^u\to 0$. This yields

$$A^{su}(t) = \frac{H(T)}{H(t)},\quad B^{su}(t) = \lambda^{su}(t,T)+\sigma^D\left(\int_t^T A^{su}(s)\left(\alpha^{su}(s)+\beta^{su}(s)\lambda^{su}(t,s)\right)ds\right) \tag{71}$$

$$F^{su}(t) = A^{su}(t)\,\mu^D(T-t)-\frac{\left(\sigma^D\right)^2}{\Gamma}\int_t^T A^{su}(s)^2\,ds+\sigma^D\int_t^T A^{su}(s)\,\gamma^{su}(s)\,ds-\omega^nI^{su}(t)\,\mu^\phi \tag{72}$$

$$I^{su}(t) = \lambda^{su}(t,T)+\sigma^D\int_t^T A^{su}(s)\,\beta^{su}(s)\,\lambda^{su}(t,s)\,ds \tag{73}$$

$$\alpha^{su}(t) = \frac{1-\kappa_t^{su}}{H(t)}\sigma^D,\qquad \beta^{su}(t) = -\frac{1-\kappa_t^{su}\left(1-\omega^n\right)}{H(t)}\sigma^D,\qquad \kappa_t^{su} = \frac{\left(1-\omega^n\right)H(t)}{M^{su}(t)} \tag{74}$$

$$\gamma^{su}(t) = -\frac{\left(1-\kappa_t^{su}\left(1-\omega^n\right)\right)\mu^D(T-t) - \omega^n \kappa_t^{su}\mu^\phi}{H(t)}\sigma^D, \quad \lambda^{su}(t,s) = \frac{\left(1-\omega^n\right)\left(\sigma^D\right)^2(s-t)}{M^{su}(t)} \tag{75}$$

and $M^{su}(t) = (1-\omega^n)^2 H(t) + (\omega^n)^2 \left(\sigma^\phi\right)^2$. The formulas stated follow. If, in addition, $\omega^n \to 0$, then $\alpha^{su}(t) = \beta^{su}(t) = \gamma^{su}(t) = 0$, $\kappa_t^{su} = 1$ and,

$$A^{su,0}(t) = A^{su}(t) = \frac{H(T)}{H(t)}, \qquad B^{su,0}(t) = I^{su,0}(t) = \lambda^{su,0}(t,T) = \frac{\left(\sigma^D\right)^2(s-t)}{H(t)} \tag{76}$$

$$F^{su,0}(t) = A^{su}(t)\mu^D(T-t) - \frac{\left(\sigma^D\right)^2}{\Gamma}\int_t^T A^{su}(s)^2\,ds \tag{77}$$

and $M^{su,0}(t) = H(t)$. This gives the formulas announced. Volatility rankings follows from $A^{su,0}(t) = A^{su}(t) < A(t)$ for all $t < T$ and $A(T) = A^{su}(T) = A^{su,0}(T) = 1$. ■

Proof of Corollary 7. The proof follows from Lemmas 35 and 36. ■

Proof of Corollary 8. The proof follows from Corollary 7 and Lemmas 35 and 36. ■

Proof of Corollary 9. The proof follows from Corollary 8 and Lemma 35. ■

Proof of Corollary 10. Differentiating with respect to the risk tolerance parameter gives the results. ■

Proof of Corollary 11. The results regarding the impact of s follows from Corollary 8 and Lemma 36. The results about μ^ϕ follow from the structure of the coefficient $\gamma(t)$. ■

The next auxiliary lemmas are used to derive comparative statics results. Proofs are straightforward, but long and tedious. They are in a companion Technical Appendix.

Lemma 35 *The following holds,*

$$\frac{\partial H(t)}{\partial t} = -\left(\sigma^D\right)^2 < 0, \qquad \frac{\partial M(t)}{\partial t} = \left(\omega^i\right)^2 \frac{\partial H(t)}{\partial t} < 0$$

$$\frac{\partial \kappa_t}{\partial t} = \frac{\omega^i(\omega^n)^2\left(\sigma^\phi\right)^2}{M(t)^2}\frac{\partial H(t)}{\partial t} < 0, \qquad \frac{\partial \lambda(t,s)}{\partial t} = -\omega^i\left(\sigma^D\right)^2\frac{M(s)}{M(t)^2} < 0$$

$$\frac{\partial A(t)}{\partial t} = -A(t)\left(\frac{\omega}{H(t)} + \frac{(1-\omega)\left(\omega^i\right)^2}{M(t)}\right)\frac{\partial H(t)}{\partial t} > 0$$

$$\frac{\partial \alpha(t)}{\partial t} = -\frac{(\omega^n)^2\left(\sigma^\phi\right)^2\omega\kappa_t + (1-\kappa_t\omega)M(t)}{M(t)H(t)^2}\frac{\partial H(t)}{\partial t}\sigma^D \gtrless 0 \Longleftrightarrow \kappa_t^2 \lessgtr \frac{1}{\omega^i\omega}$$

$$\frac{\partial \beta(t)}{\partial t} = \omega\frac{(\omega^n)^4\left(\sigma^\phi\right)^4 + 2\left(\omega^i\right)^2 H(t)(\omega^n)^2\left(\sigma^\phi\right)^2}{M(t)^2 H(t)^2}\frac{\partial H(t)}{\partial t}\sigma^D < 0$$

$$\partial\gamma(t) = \frac{\omega\sigma^D}{H(t)^2}\left(\frac{\partial\kappa_t}{\partial t}\left(\omega^i\mu^D(T-t) - \omega^n\mu^\phi\right)H(t) - \omega^n\kappa_t\mu^\phi\left(\sigma^D\right)^2 + \left(1-\kappa_t\omega^i\right)\mu^D\left(\sigma^\zeta\right)^2\right)$$

$$\begin{cases} \partial\gamma(t) > 0 \Longleftrightarrow 0 \leq H(t) < H(t)^+ \\ \partial\gamma(t) < 0 \Longleftrightarrow H^+ < H(t) \end{cases}, \qquad H^+ = \frac{-b+\sqrt{b^2-4ac}}{2a} \tag{78}$$

$$a=s^2\left(s\left(\sigma^D\right)^2\mu^\phi+\left(\sigma^\phi\right)^2\mu^D\right),\qquad b=-2s^2\left(\sigma^\phi\right)^2\mu^D\left(\sigma^\zeta\right)^2 \tag{79}$$

$$c=-\left(\sigma^\phi\right)^2\left(\sigma^\phi\right)^2\mu^D\left(\sigma^\zeta\right)^2,\quad s=\frac{\omega^i}{\omega^n} \tag{80}$$

$$\frac{\partial B(t)}{\partial t}=-\left(\frac{\omega^i\left(\sigma^D\right)^2}{M(t)}+\sigma^D\alpha(t)\right)A(t)<0.$$

Lemma 36 *Let $s=\omega^i/\omega^n$ and $E(t)=s^2H(t)+\left(\sigma^\phi\right)^2$. The following holds,*

$$\frac{\partial A(t)}{\partial s}=-(1-\omega)\left(\frac{H(T)}{H(t)}\right)^\omega\left(\frac{E(T)}{E(t)}\right)^{-\omega}\frac{2s\left(\sigma^\phi\right)^2\left(\sigma^D\right)^2(T-t)}{E(t)^2}<0$$

$$\frac{\partial\alpha(t)}{\partial s}=-\frac{-s^2H(t)+(2s+1)\left(\sigma^\phi\right)^2}{E(t)^2}\sigma^D\gtrless 0\Longleftrightarrow H(t)\gtrless\frac{2s+1}{s^2}\left(\sigma^\phi\right)^2$$

$$\frac{\partial\beta(t)}{\partial s}=\omega\frac{2s\left(\sigma^\phi\right)^2}{E(t)^2}\sigma^D>0$$

$$\begin{aligned}\frac{\partial\gamma(t)}{\partial s} &= \omega\frac{2s\left(\sigma^\phi\right)^2\mu^D(T-t)-\left(-s^2H(t)+\left(\sigma^\phi\right)^2\right)\mu^\phi}{E(t)^2}\sigma^D\gtrless 0\\ \Longleftrightarrow &\quad 2s\left(\sigma^\phi\right)^2\mu^D(T-t)\gtrless\left(-s^2H(t)+\left(\sigma^\phi\right)^2\right)\mu^\phi.\end{aligned}$$

$$\frac{\partial A(t)}{\partial\mu^\phi}=\frac{\partial B(t)}{\partial\mu^\phi}=\frac{\partial\alpha(t)}{\partial\mu^\phi}=\frac{\partial\beta(t)}{\partial\mu^\phi}=0,\qquad\frac{\partial\gamma(t)}{\partial\mu^\phi}=-\omega\frac{\omega^n\kappa_t}{H(t)}\sigma^D$$

$$\frac{\partial F(t)}{\partial\mu^\phi}=-\omega\left(\sigma^D\right)^2\int_t^T s\frac{A(s)}{E(s)}ds-\omega^nI(t).$$

Proof of Corollary 12. Straightforward, but lengthy derivations lead to,

$$\frac{\partial m(t)}{\partial t}=\frac{2A(t)\,\partial A(t)\,\beta(t)-A(t)^2\,\partial\beta(t)}{\beta(t)^2}\frac{\sigma_D}{\omega^u\Gamma}=\frac{A(t)^2}{\beta(t)}\left(2\frac{\partial_tA(t)}{A(t)}-\frac{\partial\beta(t)}{\beta(t)}\right)\frac{\sigma_D}{\omega^u\Gamma}$$

$$\frac{\partial m(t)}{\partial s}=\frac{A(t)^2}{\beta(t)}\left(2\frac{\partial_sA(t)}{A(t)}-\frac{\partial_s\beta(t)}{\beta(t)}\right)\frac{\sigma_D}{\omega^u\Gamma}$$

$$2\frac{\partial_tA(t)}{A(t)}-\frac{\partial\beta(t)}{\beta(t)}=-2\left(\omega-\frac{1}{2}\right)\frac{(\omega^n)^2\left(\sigma^\phi\right)^2}{M(t)\,H(t)}\frac{\partial H(t)}{\partial t}$$

$$2\frac{\partial_sA(t)}{A(t)}-\frac{\partial_s\beta(t)}{\beta(t)}=2s\left(\sigma^\phi\right)^2\frac{M(t)}{E(t)^2}\frac{(\omega^n)^2\left(\sigma^\phi\right)^2\Pi(t)+\left(\omega^i\right)^2\left(\sigma^\zeta\right)^2H(t)}{M(T)\,(\omega^n)^2\left(\sigma^\phi\right)^2}>0$$

where $\Pi(t)=\left(\sigma^\zeta\right)^2+\omega\left(\sigma^D\right)^2(T-t)$. ■

Proof of Proposition 16. Substituting the optimal strategy in the ex-ante utility gives,

$$E\left[\mathcal{U}^u\right]=E\left[N_0^uS_0\right]+\frac{\Gamma}{2}\int_0^T E\left[\left(\theta_s^m\right)^2\right]ds$$

$$E\left[\mathcal{U}^i\right]=E\left[N_0^iS_0\right]+\frac{\Gamma}{2}\int_0^T E\left[\left(\theta_s^m+\theta_s^{G|m}(G)\right)^2\right]ds=E\left[\mathcal{U}^u\right]+\frac{\Gamma}{2}\int_0^T E\left[\left(\theta_s^{G|m}(G)\right)^2\right]ds.$$

The final expression for $E\left[\mathcal{U}^i\right]$ uses $E\left[\theta_t^{G|m}(G)\middle|\mathcal{F}_t^m\right]=0$ and $E\left[\theta_t^{G|m}(G)\,\theta_t^m\right]=E\left[E\left[\theta_t^{G|m}(G)\middle|\mathcal{F}_t^m\right]\theta_t^m\right]=0$. It also uses $E\left[N_0^u S_0\right]=E\left[N_0^i S_0\right]$, which follows by substituting optimal demands and using $E\left[\theta_0^{G|m}(G)\,S_0\right]=E\left[E\left[\theta_0^{G|m}(G)\middle|\mathcal{F}_0^m\right]S_0\right]=0$. This shows $E\left[\mathcal{U}^i\right]=E\left[\mathcal{U}^u\right]+\mathcal{I}^i$ where $\mathcal{I}^i=(\Gamma/2)\int_0^T E\left[\left(\theta_s^{G|m}(G)\right)^2\right]ds$ is the ex-ante value of private information. The informed agent is always better off than the uninformed agent. In the economy without private information, the ex-ante utility is,

$$E\left[\mathcal{U}^{j,ni}\right]=N_0^i E\left[S_0^{ni}\right]+\frac{\Gamma}{2}\int_0^T\left(\sigma_t^{S,ni}\right)^2dt,\qquad j\in\{u,i\}$$

where $N_0^i=N_0^u=\frac{\Gamma}{\sigma^D}\frac{\sigma^D}{\Gamma}=1$, $S_0^{ni}=E\left[D_T|\mathcal{F}_0^m\right]=E\left[D_T\right]$ and $\sigma_t^{S,ni}=\sigma^D$. ■

Proof of Proposition 17. Recall that $N_0^u=\frac{\Gamma}{A(0)\sigma^D}\left(\frac{A(0)\sigma^D}{\Gamma}-\left(\alpha(0)\,Z+\beta(0)\,D_0+\gamma(0)\right)\right)$ in the NREE. For $j=u$, the utility gain/loss across equilibria is $E\left[\mathcal{U}^u\right]-E\left[\mathcal{U}^{u,ni}\right]=(1-\Gamma\delta_0)\,\Delta P^u-\Gamma K_0+\Delta T^u$ where,

$$\Delta P^u=E\left[S_0\right]-E\left[S_0^{ni}\right],\qquad\qquad \Delta T^u=\frac{\Gamma}{2}\int_0^T\left(E\left[\left(\theta_t^m\right)^2\right]-\left(\sigma_t^{S,ni}\right)^2\right)dt$$

$$\Delta P^u=\left(A(0)-1\right)E\left[D_T\right]+B(0)\,E\left[Z\right]-\frac{\Delta V}{\Gamma}+\sigma^D\int_0^T A(s)\,\gamma(s)\,ds-\omega^n I(0)\,\mu^\phi$$

with $\Delta V=\left(\sigma^D\right)^2\int_t^T\left(A(s)^2-1\right)ds$. As $\theta_t^m=\sigma_t^S/\Gamma-\Theta_t(Z;\omega)$ and $E\left[\theta_t^{G|m}(G)\middle|\mathcal{F}_t^m\right]=0$, it follows that,

$$E\left[\Theta_t(Z;\omega)\right]=\omega^i E\left[\theta_t^{G|m}(G)\right]+\omega^n E\left[\theta_t^{G|m}(\phi)\right]=\omega^n E\left[\theta_t^{G|m}(\phi)\right]$$

$$\begin{aligned}E\left[\left(\theta_t^m\right)^2\right] &= E\left[\left(\frac{\sigma_t^S}{\Gamma}\right)^2-2\frac{\sigma_t^S}{\Gamma}\Theta_t(Z;\omega)+\Theta_t(Z;\omega)^2\right]\\ &= \left(\frac{A(t)\,\sigma^D}{\Gamma}\right)^2-2\frac{A(t)\,\sigma^D}{\Gamma}\omega^n E\left[\theta_t^{G|m}(\phi)\right]+E\left[\Theta_t(Z;\omega)^2\right]\end{aligned}$$

where $E\left[\theta_t^{G|m}(\phi)\right]=E\left[\theta_t^{G|m}\left(\mu^\phi\right)\right]=\theta_t^{G|m}\left(\mu^\phi;E\left[D_t\right],E\left[Z\right]\right)$ with $\theta^{G|m}(x;d,z)=\sigma^D\left(x-\mu_t^{G|D_t=d,Z=z}\right)/H(t)$, because $E\left[\mu_t^{G|D_t,Z}\right]=\mu_t^{G|E[D_t],E[Z]}$. Thus,

$$E\left[\left(\theta_t^m\right)^2\right]-\left(\sigma_t^{S,ni}\right)^2=\left(\frac{\sigma^D}{\Gamma}\right)^2\left(A(t)^2-1\right)-2\frac{A(t)\,\sigma^D}{\Gamma}\omega^n E\left[\theta_t^{G|m}(\phi)\right]+E\left[\Theta_t(Z;\omega)^2\right]$$

$$\Delta T^u=\frac{\left(\sigma^D\right)^2}{2\Gamma}\int_0^T\left(A(t)^2-1\right)dt-\omega^n\sigma^D\int_0^T A(t)\,\theta_t^{G|m}\left(\mu^\phi;E\left[D_t\right],E\left[Z\right]\right)dt+\mathcal{I}^u$$

where $\mathcal{I}^u\equiv(\Gamma/2)\int_0^T E\left[\Theta_t(Z;\omega)^2\right]dt$.

If $\Gamma\to\infty$, then $\Delta T^u\to\infty$ and, under condition (28), $E\left[\mathcal{U}^u\right]-E\left[\mathcal{U}^{u,ni}\right]\to\infty$. If $\Gamma\to0$, then $\Delta T^u\approx\frac{\left(\sigma^D\right)^2}{2\Gamma}\int_0^T\left(A(t)^2-1\right)dt$ and $\Delta P^u\approx-\frac{\left(\sigma^D\right)^2}{\Gamma}\int_0^T\left(A(s)^2-1\right)ds$, so that $E\left[\mathcal{U}^u\right]-E\left[\mathcal{U}^{u,ni}\right]\approx-\frac{\left(\sigma^D\right)^2}{2\Gamma}\int_0^T\left(A(s)^2-1\right)ds\to\infty$. This completes the proof. ■

Proof of Corollary 18. Note that $\Delta N^u = -\Gamma\left(\delta_0 E\left[S_0\right] + K_0\right) \equiv -\Gamma J_0$,

$$\Delta P^u = C_P - \frac{\Delta V}{\Gamma}, \quad \Delta T^u = C_T + \frac{\Delta V}{2\Gamma} + \Gamma\mathcal{E}^u, \quad \mathcal{I}^u \equiv \Gamma\mathcal{E}^u.$$

Thus, $\Delta^u = C_P + C_T - \frac{\Delta V}{2\Gamma} + \Gamma\left(\mathcal{E}^u - J_0\right)$. The equation $\Delta^u = 0$ in Γ has either zero roots (case (i)), or one positive root Γ_+ (case (ii)), or two positive roots $\Gamma_\pm$ (case (iii)). ■

Proof of Corollary 19. Let $B_v^\phi \equiv E\left[\phi - G|\mathcal{F}_v^m\right] = E\left[\phi - G|D_v, Z\right]$ be the conditional forecast bias and note that, because $\phi - G, Z$ conditional on D_v is bivariate Gaussian,

$$\begin{aligned} B_v^\phi &= E\left[\phi - G|\, D_v\right] + \frac{COV\left(\phi - G, Z|\, D_v\right)}{VAR\left[Z|\, D_v\right]}\left(Z - E\left[Z|\, D_v\right]\right) \\ &= \mu^\phi - E\left[G|\, D_v\right] + \frac{\omega^n\left(\sigma^\phi\right)^2 - \omega^i H\left(v\right)}{M\left(v\right)}\left(Z - E\left[Z|\, D_v\right]\right) \\ &= E\left[B_v^\phi\right] - \left(D_v - E\left[D_v\right]\right) + \delta^\phi\left(v\right)\left(Z - E\left[Z|\, D_v\right]\right) \end{aligned}$$

where $\delta^\phi\left(v\right) \equiv \left(\omega^n\left(\sigma^\phi\right)^2 - \omega^i H\left(v\right)\right)/M\left(v\right)$. Thus, $E\left[B_v^\phi Z\right] = E\left[B_v^\phi\right]E\left[Z\right] - COV\left(D_v, Z\right) + \delta^\phi\left(v\right)M\left(v\right)$. Hence, as $S_0 = A\left(0\right)Z + B\left(0\right)D_0 + F\left(0\right)$, $E\left[B_0^\phi S_0\right] = B\left(0\right)E\left[B_0^\phi Z\right] + \left(A\left(0\right)D_0 + F\left(0\right)\right)E\left[B_0^\phi\right] = E\left[B_0^\phi\right]E\left[S_0\right] + B\left(0\right)\delta^\phi\left(0\right)M\left(0\right)$.

Under the noise trader beliefs, for $\phi = x$, $dS_v = \left(\theta_v^m + \theta_v^{G|m}\left(x\right)\right)dv + dW_v^x$ and $N_v^n\left(x\right) \equiv \Gamma\left(\theta_v^m + \theta_v^{G|m}\left(x\right)\right)/\sigma_v^S$. It follows that,

$$\begin{aligned} \mathcal{U}^n\left(\phi\right) &= E\left[N_v^n\left(x\right)S_0|\, \mathcal{F}_0^m, G = x\right]_{|x=\phi} + E\left[\int_0^T N_v^n\left(x\right)dS_v\right|\left.\mathcal{F}_0^m, G = x\right]_{|x=\phi} \\ &\quad -\frac{1}{2\Gamma}E\left[\int_0^T\left(N_v^n\left(x\right)\sigma_v^S\right)^2 dv\right|\left.\mathcal{F}_0^m, G = x\right]_{|x=\phi} \end{aligned}$$

where,

$$E\left[N_v^n\left(x\right)S_0|\, \mathcal{F}_0^m, G = x\right]_{|x=\phi} = \Gamma\frac{\left(\theta_0 + \theta_0^{G|m}\left(\phi\right)\right)S_0}{\sigma_0^S}$$

$$\begin{aligned} E\left[N_v^n\left(x\right)dS_v - \frac{\left(N_v^n\left(x\right)\sigma_v^S\right)^2}{2\Gamma}\right|\left.\mathcal{F}_0^m, G = x\right]_{|x=\phi} &= \frac{\Gamma}{2}E\left[\left(\theta_v^m + \theta_v^{G|m}\left(x\right)\right)^2\right|\left.\mathcal{F}_0^m, G = x\right]_{|x=\phi} \\ &= \frac{\Gamma}{2}\int_0^T E\left[a_v\left(x\right)\left(\theta_v^m + \theta_v^{G|m}\left(x\right)\right)^2\right|\left.\mathcal{F}_0^m\right]_{|x=\phi}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{U}^n\left(\phi\right) &= \mathcal{U}^u\left(Z\right) + \Gamma\frac{\left(\theta_0^{G|m}\left(\phi\right)\right)S_0}{\sigma_0^S} + \frac{\Gamma}{2}\int_0^T E\left[\left(a_v\left(x\right) - 1\right)\left(\theta_v^m\right)^2\right|\left.\mathcal{F}_0^m\right]_{|x=\phi}dv \\ &\quad +\Gamma\int_0^T E\left[a_v\left(x\right)\theta_v^m\theta_v^{G|m}\left(x\right)\right|\left.\mathcal{F}_0^m\right]_{|x=\phi}dv + \frac{\Gamma}{2}\int_0^T E\left[a_v\left(x\right)\theta_v^{G|m}\left(x\right)^2\right|\left.\mathcal{F}_0^m\right]_{|x=\phi}dv. \end{aligned}$$

Ex ante utility is then $E\left[\mathcal{U}^{n}(\phi)\right]=E\left[\mathcal{U}^{u}(Z)\right]+\Delta N^{n}+\mathcal{I}^{n}+\Delta T^{n}$ where,

$$\Delta N^{n}\equiv\Gamma\frac{E\left[\theta_{0}^{G|m}(\phi)\,S_{0}\right]}{\sigma_{0}^{S}}=\frac{\Gamma\sigma^{D}}{H(0)\,\sigma_{0}^{S}}E\left[B_{0}^{\phi}S_{0}\right],\qquad\mathcal{I}^{n}\equiv\frac{\Gamma}{2}\int_{0}^{T}\mathcal{I}_{1}^{\phi}(v)\,dv$$

$$\Delta T^{n}\equiv\Gamma\int_{0}^{T}\Delta T_{1}^{n}(v)\,dv+\frac{\Gamma}{2}\int_{0}^{T}\Delta T_{2}^{n}(v)\,dv$$

with $E\left[B_{0}^{\phi}S_{0}\right]=E\left[B_{0}^{\phi}\right]E\left[S_{0}\right]+B(0)\,\delta^{\phi}(0)\,M(0)$ and,

$$\begin{aligned}
\mathcal{I}_{1}^{n}(v) &\equiv E\left[E\left[a_{v}(x)\,\theta_{v}^{G|m}(x)^{2}\middle|\mathcal{F}_{0}^{m}\right]_{|x=\phi}\right]=E\left[\int_{\mathbb{R}}a_{v}(x)\,\theta_{v}^{G|m}(x)^{2}f_{\phi|Z}(x|\,Z)\,dx\right]\\
&= E\left[\int_{\mathbb{R}}\theta_{v}^{G|m}(x)^{2}\frac{f_{G|Z,D_{v}}(x|\,Z,D_{v})}{f_{G|Z}(x|\,Z)}f_{\phi|Z}(x|\,Z)\,dx\right]\\
&= \frac{\left(\sigma^{D}\right)^{2}}{H(v)^{2}}E\left[\int_{\mathbb{R}}(x-E\left[G|\,Z,D_{v}\right])^{2}f_{G|Z,D_{v}}(x|\,Z,D_{v})\,\mathcal{L}_{\phi,G}(x|\,Z)\,dx\right]\\
&= \frac{\left(\sigma^{D}\right)^{2}}{H(v)^{2}}\left(VAR\left[G|\,Z,D_{v}\right]+E\left[\int_{\mathbb{R}}(x-E\left[G|\,Z,D_{v}\right])^{2}\left(\mathcal{L}_{\phi,G}(x|\,Z)-1\right)f_{G|Z,D_{v}}(x|\,Z,D_{v})\,dx\right]\right)\\
&= \frac{\left(\sigma^{D}\omega^{n}\sigma^{\phi}\right)^{2}}{M(v)\,H(v)}+\frac{\left(\sigma^{D}\right)^{2}}{H(v)^{2}}E\left[\int_{\mathbb{R}}(x-E\left[G|\,Z,D_{v}\right])^{2}\left(\mathcal{L}_{\phi,G}(x|\,Z)-1\right)f_{G|Z,D_{v}}(x|\,Z,D_{v})\,dx\right]
\end{aligned}$$

$$\begin{aligned}
\Delta T_{1}^{n}(v) &\equiv E\left[E\left[a_{v}(x)\,\theta_{v}^{m}\theta_{v}^{G|m}(x)\middle|\mathcal{F}_{0}^{m}\right]_{|x=\phi}\right]=E\left[\int_{\mathbb{R}}\theta_{v}^{G|m}(x)\,a_{v}(x)\,f_{\phi|Z}(x|\,Z)\,dx\theta_{v}^{m}\right]\\
&= E\left[\int_{\mathbb{R}}\theta_{v}^{G|m}(x)\,\frac{f_{G|Z,D_{v}}(x|\,Z,D_{v})}{f_{G|Z}(x|\,Z)}f_{\phi|Z}(x|\,Z)\,dx\theta_{v}^{m}\right]\\
&= \frac{\sigma^{D}}{H(v)}E\left[\int_{\mathbb{R}}(x-E\left[G|\,Z,D_{v}\right])\,f_{G|Z,D_{v}}(x|\,Z,D_{v})\,\mathcal{L}_{\phi,G}(x|\,Z)\,dx\theta_{v}^{m}\right]
\end{aligned}$$

$$\begin{aligned}
\Delta T_{2}^{n}(v) &\equiv E\left[E\left[(a_{v}(x)-1)\,(\theta_{v}^{m})^{2}\middle|\mathcal{F}_{0}^{m}\right]_{|x=\phi}\right]=E\left[\int_{\mathbb{R}}(a_{v}(x)-1)\,f_{\phi|Z}(x|\,Z)\,dx\,(\theta_{v}^{m})^{2}\right]\\
&= E\left[\int_{\mathbb{R}}\left(\frac{f_{G|Z,D_{v}}(x|\,Z,D_{v})}{f_{G|Z}(x|\,Z)}-1\right)f_{\phi|Z}(x|\,Z)\,dx\,(\theta_{v}^{m})^{2}\right]\\
&= E\left[\left(\int_{\mathbb{R}}\mathcal{L}_{\phi,G}(x|\,Z)\,f_{G|Z,D_{v}}(x|\,Z,D_{v})\,dx-1\right)(\theta_{v}^{m})^{2}\right].
\end{aligned}$$

These derivations use $VAR\left[G|\,Z,D_{v}\right]=\left(\omega^{n}\sigma^{\phi}\right)^{2}H(v)\,/M(v)$ and the relation $P\left(G\in dx|\,\mathcal{F}_{v}^{m}\right)=P\left(G\in dx|\,\mathcal{F}_{0}^{m}\right)+\int_{0}^{v}P\left(G\in dx|\,\mathcal{F}_{t}^{m}\right)\theta_{t}^{G|m}(x)\,dW_{t}^{S}$, so that $f_{G|Z,D_{v}}(x|\,Z,D_{v})\,/f_{G|Z}(x|\,Z)=P\left(G\in dx|\,\mathcal{F}_{v}^{m}\right)/P\left(G\in x|\,\mathcal{F}_{0}^{m}\right)=a_{v}(x)$.

■

Proof of Corollary 20. The first statement follows directly from Corollary 19.

To prove the second statement assume $\mu^{\phi}=E\left[G\right]$ and note that $\lim_{\Gamma\to0}\Gamma\theta_{v}^{m}=\sigma_{v}^{S}=A(v)\,\sigma^{D}$. Then,

$$\lim_{\Gamma\to0}\Delta N^{n}=\lim_{\Gamma\to0}\Gamma\frac{\sigma^{D}}{H(0)\,\sigma_{0}^{S}}B(0)\left(\omega^{n}\left(\sigma^{\phi}\right)^{2}-\omega^{i}H(0)\right)=0$$

$$\lim_{\Gamma\to0}\mathcal{I}^{n}=\lim_{\Gamma\to0}\frac{\Gamma}{2}\int_{0}^{T}E\left[\theta_{v}^{G|m}(G)^{2}\,\mathcal{L}_{\phi,G}(G|\,Z)\right]dv=0$$

$$\lim_{\Gamma\to 0}\Delta T^{n}=\int_{0}^{T}\sigma_{v}^{S}E\left[\theta_{v}^{G|m}\left(G\right)\mathcal{L}_{\phi,G}\left(G|\,Z\right)\right]dv-\int_{0}^{T}\sigma_{v}^{S}E\left[COV\left(\mathcal{L}_{\phi,G}\left(G|\,Z\right),\Theta\left(Z;\omega\right)|\,Z\right)\right]dv$$

where $COV\left(\mathcal{L}_{\phi,G}\left(G|\,Z\right),\left(\theta_{v}^{m}\right)^{2}\middle|\,Z\right)=-2\left(\sigma_{v}^{S}/\Gamma\right)COV\left(\mathcal{L}_{\phi,G}\left(G|\,Z\right),\Theta\left(Z;\omega\right)|\,Z\right)+COV\left(\mathcal{L}_{\phi,G}\left(G|\,Z\right),\Theta\left(Z;\omega\right)^{2}\middle|\,Z\right)$ is used in the last limit. Straightforward computations show,

$$\begin{aligned}E\left[\theta_{v}^{G|m}\left(G\right)\mathcal{L}_{\phi,G}\left(G|\,Z\right)\middle|\,Z\right] &= \frac{\sigma^{D}}{H\left(v\right)}E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)\left(G-E\left[G|\,Z,D_{v}\right]\right)|\,Z\right]\\ &= \frac{\sigma^{D}}{H\left(v\right)}E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)\left(G-E\left[E\left[G|\,Z,D_{v}\right]|\,G,Z\right]\right)|\,Z\right].\end{aligned}$$

As $E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)G|\,Z\right]=E\left[\phi|\,Z\right]$ and (G,Z) is bivariate Gaussian, it follows that $E\left[E\left[G|\,Z,D_{v}\right]|\,G,Z\right]=E\left[G|\,Z\right]+k\left(v\right)\left(G-E\left[G|\,Z\right]\right)$ where $k\left(v\right)\equiv COV\left(E\left[G|\,Z,D_{v}\right],G|\,Z\right)/VAR\left[G|\,Z\right]$ is deterministic. Thus,

$$E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)\left(G-E\left[E\left[G|\,Z,D_{v}\right]|\,G,Z\right]\right)|\,Z\right]=E\left[\phi|\,Z\right]-E\left[G|\,Z\right]-k\left(v\right)\left(E\left[\phi|\,Z\right]-E\left[G|\,Z\right]\right)$$

and $E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)\left(G-E\left[E\left[G|\,Z,D_{v}\right]|\,G,Z\right]\right)\right]=\left(1-k\left(v\right)\right)\left(E\left[\phi\right]-E\left[G\right]\right)$. Unbiasedness implies that the right hand side is null, so that $E\left[\theta_{v}^{G|m}\left(G\right)\mathcal{L}_{\phi,G}\left(G|\,Z\right)\right]=0$. To show $E\left[COV\left(\mathcal{L}_{\phi,G}\left(G|\,Z\right),\Theta\left(Z;\omega\right)|\,Z\right)\right]=0$, note that,

$$\begin{aligned}COV\left(\mathcal{L}_{\phi,G}\left(G|\,Z\right),\Theta\left(Z;\omega\right)|\,Z\right) &= E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)\left(\Theta\left(Z;\omega\right)-E\left[\Theta\left(Z;\omega\right)|\,Z\right]\right)|\,Z\right]\\ &= E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)\left(E\left[\Theta\left(Z;\omega\right)|\,G,Z\right]-E\left[\Theta\left(Z;\omega\right)|\,Z\right]\right)|\,Z\right]\\ &= \beta\left(v\right)E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)\left(E\left[D_{v}|\,G,Z\right]-E\left[D_{v}|\,Z\right]\right)|\,Z\right]\\ &= \rho\left(v\right)\left(E\left[\mathcal{L}_{\phi,G}\left(G|\,Z\right)G|\,Z\right]-E\left[G|\,Z\right]\right)=\rho\left(v\right)\left(E\left[\phi|\,Z\right]-E\left[G|\,Z\right]\right)\end{aligned}$$

where $\rho\left(v\right)\equiv\beta\left(v\right)COV\left(D_{v},G|\,Z\right)/VAR\left[G|\,Z\right]$ is deterministic. Thus, $E\left[COV\left(\mathcal{L}_{\phi,G}\left(G|\,Z\right),\Theta\left(Z;\omega\right)|\,Z\right)\right]=\rho\left(v\right)$ $\left(E\left[\phi\right]-E\left[G\right]\right)=0$ if $E\left[\phi\right]=E\left[G\right]$.

To prove the last statement, note that $\mathcal{L}_{\phi,G}\left(x|\,z\right)=h_{1}\exp\left(-h_{2}\right)$ where $h_{1}=\omega^{n}/\omega^{i}$ and,

$$h_{2}=\begin{cases}\frac{1}{2}\frac{\left(\omega^{i}\right)^{2}\left(\omega^{n}\right)^{2}H(0)\left(\sigma^{\phi}\right)^{2}}{M(0)\left(\left(\omega^{n}\right)^{2}-\left(\omega^{i}\right)^{2}\right)}\left(x-\frac{\left(\omega^{n}\right)^{2}\mu^{\phi}-\left(\omega^{i}\right)^{2}E[G]+\frac{\left(\omega^{n}\right)^{3}\left(\sigma^{\phi}\right)^{2}-\left(\omega^{i}\right)^{3}H(0)}{M(0)}\left(z-E[Z]\right)}{\left(\omega^{n}\right)^{2}-\left(\omega^{i}\right)^{2}}\right)^{2} & \text{if } \omega^{i}\neq\omega^{n}\\ \frac{M(0)\left(\mu^{\phi}-E[G]+\frac{\bar{\omega}\left(\left(\sigma^{\phi}\right)^{2}-H(0)\right)}{M(0)}\left(z-E[Z]\right)\right)\left(\mu^{\phi}+E[G]+\frac{\bar{\omega}\left(\left(\sigma^{\phi}\right)^{2}+H(0)\right)}{M(0)}\left(z-E[Z]\right)-2x\right)}{2\left(\bar{\omega}\sigma^{\phi}\right)^{2}H(0)} & \text{if } \omega^{i}=\omega^{n}\equiv\bar{\omega}.\end{cases}$$

If $\omega^{i}=\omega^{n}$, $\mu^{\phi}=E\left[G\right]$ and $\left(\sigma^{\phi}\right)^{2}=VAR\left[G\right]$, then $\mathcal{L}_{\phi,G}\left(x|\,z\right)=1$ and $\omega^{n}\left(\sigma^{\phi}\right)^{2}=\omega^{i}H\left(0\right)$. Thus $\Delta N^{\phi}=0$ and,

$$\mathcal{I}_{1}^{n}\left(v\right)=\frac{\left(\sigma^{D}\omega^{n}\sigma^{\phi}\right)^{2}}{M\left(v\right)H\left(v\right)},\qquad\mathcal{I}^{n}=\frac{\Gamma}{2}\int_{0}^{T}\mathcal{I}_{1}^{n}\left(v\right)dv=\frac{\Gamma}{2}\int_{0}^{T}\frac{\left(\sigma^{D}\omega^{n}\sigma^{\phi}\right)^{2}}{M\left(v\right)H\left(v\right)}dv$$

$$\Delta T_{1}^{n}\left(v\right)=\frac{\sigma^{D}}{H\left(v\right)}E\left[\int_{\mathbb{R}}\left(x-E\left[G|\,Z,D_{v}\right]\right)f_{G|Z,D_{v}}\left(x|\,Z,D_{v}\right)dx\,\theta_{v}^{m}\right]=0$$

$$\Delta T_{2}^{n}\left(v\right)=E\left[\left(\int_{\mathbb{R}}\mathcal{L}_{\phi,G}\left(x|\,Z\right)f_{G|Z,D_{v}}\left(x|\,Z,D_{v}\right)dx-1\right)\left(\theta_{v}^{m}\right)^{2}\right]=0.$$

It follows that $\Delta T^n \equiv \Gamma \int_0^T \Delta T_1^n(v)\,dv + \frac{\Gamma}{2}\int_0^T \Delta T_2^n(v)\,dv = 0$ and $E[\mathcal{U}^n(\phi)] = E[\mathcal{U}^i(G)]$. In this case, the ex-ante utilities of the mimicking noise trader and of the informed are identical and the NREE is a Pareto improvement under the conditions of Corollary 18. ■

Proof of Proposition 24. The preferences of the informed are,

$$\begin{aligned}
\mathcal{U}^i &= E\left[X_T^i - \frac{1}{2\Gamma}\int_0^T d\left[X^i\right]_s \middle| \mathcal{F}_0^G\right] = E\left[\int_0^T N_v^i\left(\mu_v^S + \sigma_v^S\theta_v^{G|m}(G)\right)dv - \frac{1}{2\Gamma}\int_0^T \left(N_v^i\sigma_v^S\right)^2 dv \middle| \mathcal{F}_0^G\right] \\
&= E\left[\int_0^T \left(N_v^i\sigma_v^S\left(\theta_v^m + \theta_v^{G|m}(G)\right) - \frac{1}{2\Gamma}\left(N_v^i\sigma_v^S\right)^2\right)dv \middle| \mathcal{F}_0^G\right] \equiv E\left[\int_0^T f_v\left(N_v^i\right)dv \middle| \mathcal{F}_0^G\right].
\end{aligned}$$

The market clearing condition gives,

$$1 = \omega^i N^i + \omega^u \Gamma \frac{\theta_t^m}{\sigma_t^S} + \omega^n \Gamma \frac{\theta_t^m + \theta_t^{G|m}(\phi)}{\sigma_t^S} \Longleftrightarrow \theta_t^m = \frac{\sigma_t^S}{\Gamma(\omega^u + \omega^n)}\left(1 - \omega^i N_t^i - \omega^n \Gamma \frac{\theta_t^{G|m}(\phi)}{\sigma_t^S}\right).$$

Substituting in the integrand and rearranging leads to,

$$f_v\left(N_v^i\right) = N_v^i\sigma_v^S\left(-\frac{\omega^i N_v^i\sigma_v^S}{\Gamma(\omega^u+\omega^n)} + \frac{\sigma_v^S}{\Gamma(\omega^u+\omega^n)}\left(1 - \omega^n\Gamma\frac{\theta_v^{G|m}(\phi)}{\sigma_v^S}\right) + \theta_v^{G|m}(G)\right) - \frac{\left(N_v^i\sigma_v^S\right)^2}{2\Gamma}$$

Maximizing $\mathcal{U}^i$ with respect to N^i is equivalent to maximizing $f_v\left(N_v^i\right)$ for each $v \in [0,T]$. The first order condition for this optimization problem is,

$$\begin{aligned}
0 &= -2\frac{\omega^i N_v^i \sigma_v^S}{\Gamma(\omega^u+\omega^n)} + \frac{\sigma_v^S}{\Gamma(\omega^u+\omega^n)}\left(1 - \omega^n\Gamma\frac{\theta_v^{G|m}(\phi)}{\sigma_v^S}\right) + \theta_v^{G|m}(G) - \frac{1}{\Gamma}N_v^i\sigma_v^S \\
\Longleftrightarrow \quad & N_v^i\sigma_v^S = \frac{\sigma_v^S}{1+\omega^i}\left(1 - \omega^n\Gamma\frac{\theta_v^{G|m}(\phi)}{\sigma_v^S}\right) + \frac{(\omega^u+\omega^n)}{1+\omega^i}\Gamma\theta_v^{G|m}(G).
\end{aligned}$$

As $f_v\left(N_v^i\right)$ is concave in N_v^i, this condition is necessary and sufficient for a maximum. Substituting,

$$\theta_v^m = \frac{\sigma_v^S}{\Gamma(\omega^u+\omega^n)}\left(1 - \omega^i N_v^i - \omega^n\Gamma\frac{\theta_v^{G|m}(\phi)}{\sigma_v^S}\right) \Longleftrightarrow \frac{\Gamma(\omega^u+\omega^n)}{\sigma_v^S}\theta_v^m + \omega^i N_v^i = 1 - \omega^n\Gamma\frac{\theta_v^{G|m}(\phi)}{\sigma_v^S}$$

back in the demand function, gives $N_v^i = (\omega^u+\omega^n)\Gamma\left(\theta_v^m + \theta_v^{G|m}(G)\right)/\sigma_v^S$. ■

Proof of Proposition 26. The aggregate demand function with the monopolistic informed trader $N_t^a \equiv \omega^u N_t^u + \omega^i N_t^i + \omega^n N_t^n$ is,

$$N_t^a = \Gamma\frac{\omega^u\sigma_t^S\theta_t^m}{\left(\sigma_t^S\right)^2} + \Gamma\frac{\left(\omega - \left(\omega^i\right)^2\right)\sigma_t^S\theta_t^m + \sigma_t^S\left(\omega^i\left(1-\omega^i\right)\theta_t^{G|m}(G) + \omega^n\theta_t^{G|m}(\phi)\right)}{\left(\sigma_t^S\right)^2}.$$

Its structure is the same as in the competitive model, except that ω^i is replaced by $\omega^{i,se} = \omega^i\left(1-\omega^i\right)$. Proceeding as in the competitive case leads to the formulas stated. ■

Proof of Corollary 27. As $\alpha_2^{se}(t) = -\omega^{i,se}\sigma^D/M^{se}(t)$ and $\alpha_2(t) = -\omega^i\sigma^D/M(t)$,

$$\begin{aligned}
\alpha_2^{se}(t) &< \alpha_2(t) \Longleftrightarrow \frac{\omega^{i,se}}{\omega^i}M(t) > M^{se}(t) \\
&\Longleftrightarrow \left(1-\omega^i\right)\left(\left(\omega^i\right)^2 H(t) + (\omega^n)^2\left(\sigma^\phi\right)^2\right) > \left(\omega^{i,se}\right)^2 H(t) + (\omega^n)^2\left(\sigma^\phi\right)^2 \\
&\Longleftrightarrow \omega^{i,se}\left(\omega^i\right)^2 H(t) - \omega^i(\omega^n)^2\left(\sigma^\phi\right)^2 > 0 \Longleftrightarrow \omega^{i,se}\omega^i H(t) > (\omega^n)^2\left(\sigma^\phi\right)^2.
\end{aligned}$$

This proves Property (i). To show (ii), use $\beta_0^{se}(t) = \beta^{se}(t)/\omega^{se}$, $\beta_0(t) = \beta(t)/\omega$ and,

$$\beta_0^{se}(t) \equiv -\frac{1-\kappa_t^{se}\omega^{i,se}}{H(t)}\sigma^D = -\frac{\left(\sigma^\phi\right)^2}{H(t)\left(s^2H(t)+(\sigma^\phi)^2\right)}\sigma^D, \qquad s = \frac{\omega^{i,se}}{\omega^n}$$

to establish,

$$\begin{aligned}
\beta_0^{se}(t) &< \beta_0(t) \Longleftrightarrow -\frac{\left(\sigma^\phi\right)^2}{H(t)\left((s^{se})^2H(t)+(\sigma^\phi)^2\right)}\sigma^D < -\frac{\left(\sigma^\phi\right)^2}{H(t)\left(s^2H(t)+(\sigma^\phi)^2\right)}\sigma^D \\
&\Longleftrightarrow s^2H(t) + \left(\sigma^\phi\right)^2 > (s^{se})^2H(t) + \left(\sigma^\phi\right)^2 \Longleftrightarrow s^2 > (s^{se})^2.
\end{aligned}$$

The result follows because $s > s^{se}$.

For Property (iii), use $\gamma_0^{se}(t) = \gamma^{se}(t)/\omega^{se}$ and $\gamma_0(t) = \gamma(t)/\omega$ to obtain,

$$\gamma_0^{se}(t) = -\frac{\left(1-\kappa_t^{se}\omega^{i,se}\right)\mu^D(T-t) - \omega^n\kappa_t^{se}\mu^\phi}{H(t)}\sigma^D = -\frac{(\omega^n)^2\left(\sigma^\phi\right)^2\mu^D(T-t) - \omega^n\omega^{i,se}H(t)\mu^\phi}{H(t)M^{se}(t)}\sigma^D$$

$$\begin{aligned}
\gamma_0^{se}(t) &> \gamma_0(t) \Longleftrightarrow -\frac{(\omega^n)^2\left(\sigma^\phi\right)^2\mu^D(T-t) - \omega^n\omega^{i,se}H(t)\mu^\phi}{H(t)M^{se}(t)}\sigma^D \\
&> -\frac{(\omega^n)^2\left(\sigma^\phi\right)^2\mu^D(T-t) - \omega^n\omega^iH(t)\mu^\phi}{H(t)M(t)}\sigma^D \\
&\Longleftrightarrow \left((\omega^n)^2\left(\sigma^\phi\right)^2\mu^D(T-t) - \omega^n\omega^{i,se}H(t)\mu^\phi\right)M(t) \\
&< \left((\omega^n)^2\left(\sigma^\phi\right)^2\mu^D(T-t) - \omega^n\omega^iH(t)\mu^\phi\right)M^{se}(t) \\
&\Longleftrightarrow (\omega^n)^2\left(\sigma^\phi\right)^2\mu^D(T-t)\left(M(t)-M^{se}(t)\right) < \omega^nH(t)\mu^\phi\left(\omega^{i,se}M(t) - \omega^iM^{se}(t)\right).
\end{aligned}$$

Substituting (81)-(82) from Lemma 37 leads to,

$$\begin{aligned}
\gamma_0^{se}(t) &> \gamma_0(t) \Longleftrightarrow \omega^n\left(\sigma^\phi\right)^2\mu^D(T-t)\left(\omega^i\right)^3\left(2-\omega^i\right)H(t) \\
&< H(t)\mu^\phi\left(\omega^i\right)^2\left(\omega^i\omega^{i,se}H(t) - (\omega^n)^2\left(\sigma^\phi\right)^2\right) \\
&\Longleftrightarrow \omega^i\omega^n\left(\sigma^\phi\right)^2\mu^D(T-t)\left(2-\omega^i\right) < \mu^\phi\left(\omega^i\omega^{i,se}H(t) - (\omega^n)^2\left(\sigma^\phi\right)^2\right) \\
&\Longleftrightarrow \left(\omega^i\mu^D(T-t)\left(2-\omega^i\right) + \omega^n\mu^\phi\right)\omega^n\left(\sigma^\phi\right)^2 < \mu^\phi\omega^i\omega^{i,se}H(t).
\end{aligned}$$

This completes the proof. ■

Proof of Corollary 28. Using $\alpha^{se}(t) = \alpha_1(t) + \alpha_2^{se}(t)$ and $\alpha(t) = \alpha_1(t) + \alpha_2(t)$ where $\alpha_1(t)$ is the same in the two equilibria, leads to $\alpha^{se}(t) > \alpha(t) \Longleftrightarrow \alpha_2^{se}(t) > \alpha_2(t)$. The first statement then follows from Corollary 27.

To obtain the second result, recall that $\beta^{se}(t) = \omega^{se}\beta_0^{se}(t)$ and $\beta(t) = \omega\beta_0(t)$. Then,

$$\begin{aligned}
\beta^{se}(t) &< \beta(t) \Longleftrightarrow -\frac{\omega^{se}\left(\sigma^{\phi}\right)^2}{H(t)\left(\left(s^{se}\right)^2 H(t) + \left(\sigma^{\phi}\right)^2\right)}\sigma^D < -\frac{\omega\left(\sigma^{\phi}\right)^2}{H(t)\left(s^2 H(t) + \left(\sigma^{\phi}\right)^2\right)}\sigma^D \\
&\Longleftrightarrow \omega^{se}\left(s^2 H(t) + \left(\sigma^{\phi}\right)^2\right) > \omega\left(\left(s^{se}\right)^2 H(t) + \left(\sigma^{\phi}\right)^2\right) \\
&\Longleftrightarrow \left(\omega^{se}s^2 - \omega\left(s^{se}\right)^2\right)H(t) + \left(\omega^{se} - \omega\right)\left(\sigma^{\phi}\right)^2 > 0 \\
&\Longleftrightarrow \left(\omega^{se}\left(\frac{\omega^i}{\omega^n}\right)^2 - \omega\left(\frac{\omega^{i,se}}{\omega^n}\right)^2\right)H(t) - \left(\omega^i\right)^2\left(\sigma^{\phi}\right)^2 > 0 \\
&\Longleftrightarrow \left(\frac{\omega^i}{\omega^n}\right)^2\left(\omega^{se} - \omega\left(1-\omega^i\right)^2\right)H(t) - \left(\omega^i\right)^2\left(\sigma^{\phi}\right)^2 > 0.
\end{aligned}$$

Eq. (83) in Lemma 37 shows $\omega^{se} - \omega\left(1-\omega^i\right)^2 = \omega^i\left(\omega^i(1-\omega) + 2\omega^n\right)$. Substituting in the formula above and rearranging gives $\omega^i\left(\omega^i(1-\omega) + 2\omega^n\right)H(t) > \left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2 > 0$.

To show Property (iii), note that,

$$\begin{aligned}
\gamma^{se}(t) &> \gamma(t) \Longleftrightarrow -\omega^{se}\frac{\left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\mu^D(T-t) - \omega^n\omega^{i,se}H(t)\mu^{\phi}}{H(t)M^{se}(t)}\sigma^D \\
&> -\omega\frac{\left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\mu^D(T-t) - \omega^n\omega^i H(t)\mu^{\phi}}{H(t)M(t)}\sigma^D \\
&\Longleftrightarrow \omega^{se}\left(\left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\mu^D(T-t) - \omega^n\omega^{i,se}H(t)\mu^{\phi}\right)M(t) \\
&< \omega\left(\left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\mu^D(T-t) - \omega^n\omega^i H(t)\mu^{\phi}\right)M^{se}(t) \\
&\Longleftrightarrow \left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\mu^D(T-t)\left(\omega^{se}M(t) - \omega M^{se}(t)\right) \\
&< \omega^n H(t)\mu^{\phi}\left(\omega^{se}\omega^{i,se}M(t) - \omega\omega^i M^{se}(t)\right).
\end{aligned}$$

Substituting (84)-(87) from Lemma 37 shows $\gamma^{se}(t) > \gamma(t)$ if and only if,

$$\begin{aligned}
&\left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\mu^D(T-t)\left(\omega^i\right)^2\left(\omega^i\left(\omega^i\left(1-\omega^i-\omega^n\right) + 2\omega^n\right)H(t) - \left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\right) \\
<\ &\omega^n H(t)\mu^{\phi}\left(\omega^i\right)^2\left[\omega^{i,se}\omega^i\omega^n H(t) - \left(\omega^i + \omega - \left(\omega^i\right)^2\right)\left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\right] \\
\Longleftrightarrow\ &\left(\sigma^{\phi}\right)^2\mu^D(T-t)\left(\omega^i\left(\omega^i\left(1-\omega^i-\omega^n\right) + 2\omega^n\right)H(t) - \left(\omega^n\right)^2\left(\sigma^{\phi}\right)^2\right) \\
<\ &H(t)\mu^{\phi}\left[\omega^{i,se}\omega^i H(t) - \left(\omega^i + \omega - \left(\omega^i\right)^2\right)\omega^n\left(\sigma^{\phi}\right)^2\right] \\
\Longleftrightarrow\ &0 < H(t)^2\mu^{\phi}\omega^{i,se}\omega^i - H(t)\left(\sigma^{\phi}\right)^2\Phi(t) + \mu^D(T-t)\left(\omega^n\right)^2\left(\sigma^{\phi}\right)^4
\end{aligned}$$

where,

$$\Phi(t) = \mu^{\phi}\left(\omega^{i}+\omega-\left(\omega^{i}\right)^{2}\right)\omega^{n}+\mu^{D}(T-t)\,\omega^{i}\left(\omega^{i}\left(1-\omega^{i}-\omega^{n}\right)+2\omega^{n}\right)$$
$$= \mu^{\phi}\left(\omega^{se}+\omega^{i}\right)\omega^{n}+\mu^{D}(T-t)\,\omega^{i}\left(\omega^{i,se}-\omega^{i}\omega^{n}+2\omega^{n}\right).$$

Hence, $\gamma^{se}(t) > \gamma(t) \iff H(t) > h^{+}(t)$ where $h^{+}(t)$ is the positive root of the quadratic equation above. ■

The next auxiliary lemma provides useful relations. Proofs are in the Technical Appendix.

Lemma 37 *The following relations hold,*

$$M(t) - M^{se}(t) = \left(\omega^{i}\right)^{3}\left(2-\omega^{i}\right)H(t) \tag{81}$$

$$\omega^{i,se}M(t) - \omega^{i}M^{se}(t) = \left(\omega^{i}\right)^{2}\left(\omega^{i}\omega^{i,se}H(t) - (\omega^{n})^{2}\left(\sigma^{\phi}\right)^{2}\right) \tag{82}$$

$$\omega^{se} - \omega\left(1-\omega^{i}\right)^{2} = \omega^{i}\left(\omega^{i}(1-\omega)+2\omega^{n}\right) \tag{83}$$

$$\omega^{se}M(t) - \omega M^{se}(t) = \left(\omega^{i}\right)^{2}\left(\omega^{i}\left(\omega^{i}(1-\omega)+2\omega^{n}\right)H(t) - (\omega^{n})^{2}\left(\sigma^{\phi}\right)^{2}\right) \tag{84}$$

$$\omega^{i,se}\omega^{se}\left(\omega^{i}\right)^{2} - \omega^{i}\omega\left(\omega^{i,se}\right)^{2} = \left(\omega^{i}\right)^{2}\omega^{i,se}\omega^{i}\omega^{n} \tag{85}$$

$$\omega^{i,se}\omega^{se} - \omega^{i}\omega = -\left(\omega^{i}\right)^{2}\left(\omega^{i}+\omega-\left(\omega^{i}\right)^{2}\right) \tag{86}$$

$$\omega^{se}\omega^{i,se}M(t) - \omega\omega^{i}M^{se}(t) = \left(\omega^{i}\right)^{2}\omega^{n}\left(\omega^{i,se}\omega^{i}H(t) - \left(\omega^{i}+\omega-\left(\omega^{i}\right)^{2}\right)\omega^{n}\left(\sigma^{\phi}\right)^{2}\right). \tag{87}$$

Proof of Corollary 29. The monopolistic equilibrium stock price volatility is greater if and only if $A^{se}(t)\,\sigma^{D} > A(t)\,\sigma^{D}$, i.e., $A^{se}(t) > A(t)$. Simple transformations show,

$$A^{se}(t) > A(t) \iff \left(\frac{H(T)}{H(t)}\right)^{\omega^{se}}\left(\frac{M^{se}(T)}{M^{se}(t)}\right)^{1-\omega^{se}} > \left(\frac{H(T)}{H(t)}\right)^{\omega}\left(\frac{M(T)}{M(t)}\right)^{1-\omega}$$
$$\iff \left(\frac{H(T)}{H(t)}\right)^{\omega^{se}-\omega}\left(\frac{M^{se}(T)}{M^{se}(t)}\right)^{1-\omega^{se}} > \left(\frac{M(T)}{M(t)}\right)^{1-\omega}$$
$$\iff \left(\frac{H(T)}{H(t)}\right)^{-\left(\omega^{i}\right)^{2}}\left(\frac{M^{se}(T)}{M^{se}(t)}\right)^{1-\omega^{se}} > \left(\frac{M(T)}{M(t)}\right)^{1-\omega}$$
$$\iff \left(\frac{H(T)}{H(t)}\right)^{-\left(\omega^{i}\right)^{2}}\left(\frac{M^{se}(T)}{M^{se}(t)}\right)^{1-\omega-\left(\omega^{i}\right)^{2}} > \left(\frac{M(T)}{M(t)}\right)^{1-\omega}$$
$$\iff \left(\frac{M^{se}(T)/M^{se}(t)}{M(T)/M(t)}\right)^{1-\omega}\left(\frac{M^{se}(T)}{M^{se}(t)}\frac{H(T)}{H(t)}\right)^{-\left(\omega^{i}\right)^{2}} > 1.$$

The result follows because each term on the left hand side is greater than 1. Indeed,

$$\begin{aligned}\frac{M^{se}(T)/M^{se}(t)}{M(T)/M(t)} &> 1 \Longleftrightarrow \frac{H(T)+\frac{1}{(s^{se})^2}\left(\sigma^{\phi}\right)^2}{H(t)+\frac{1}{(s^{se})^2}\left(\sigma^{\phi}\right)^2} > \frac{H(T)+\frac{1}{s^2}\left(\sigma^{\phi}\right)^2}{H(t)+\frac{1}{s^2}\left(\sigma^{\phi}\right)^2} \\ &\Longleftrightarrow \left(H(T)+\frac{1}{(s^{se})^2}\left(\sigma^{\phi}\right)^2\right)\left(H(t)+\frac{1}{s^2}\left(\sigma^{\phi}\right)^2\right) \\ &> \left(H(t)+\frac{1}{(s^{se})^2}\left(\sigma^{\phi}\right)^2\right)\left(H(T)+\frac{1}{s^2}\left(\sigma^{\phi}\right)^2\right) \\ &\Longleftrightarrow \frac{\left(\sigma^{\phi}\right)^2 H(t)}{(s^{se})^2}+\frac{\left(\sigma^{\phi}\right)^2 H(T)}{s^2} > \frac{\left(\sigma^{\phi}\right)^2 H(T)}{(s^{se})^2}+\frac{\left(\sigma^{\phi}\right)^2 H(t)}{s^2} \\ &\Longleftrightarrow \frac{H(t)-H(T)}{(s^{se})^2} > \frac{H(t)-H(T)}{s^2} \Longleftrightarrow s^2 > (s^{se})^2\end{aligned}$$

$$\frac{M^{se}(T)}{M^{se}(t)}\frac{H(T)}{H(t)} = \frac{H(T)+\left(\frac{\sigma^{\phi}}{s^{se}}\right)^2}{H(t)+\left(\frac{\sigma^{\phi}}{s^{se}}\right)^2}\frac{H(T)}{H(t)} < 1, \qquad \left(\frac{M^{se}(T)}{M^{se}(t)}\frac{H(T)}{H(t)}\right)^{-(\omega^i)^2} > 1.$$

This completes the proof. ■

Proof of Proposition 30. In order to preclude arbitrage opportunities, admissible strategies must be left-continuous and equilibrium price processes right continuous with left limits. It follows that investors can not control the jumps at the dividend payment dates. Preferences can then be redefined over the controllable part of wealth. That is, $\mathcal{U}^j = E\left[X_T^{c,j} - \frac{1}{2\Gamma}\int_0^T d\left[X^{c,j}\right]_t\right]$, where $X_t^{c,j}$ is the continuous component of the wealth process for agent $j \in \{i,u,n\}$.

The aggregate demand function $N_t^a \equiv \omega_n^u N_t^u + \omega_n^i N_t^i + \omega_n^n N_t^n$ is,

$$N_t^a = \omega_n^u \Gamma \frac{\sigma_t^S \theta_t^m}{\left(\sigma_t^S\right)^2} + \omega_n^i \Gamma \frac{\sigma_t^S\left(\theta_t^m + \theta_t^{G|m}(G)\right)}{\left(\sigma_t^S\right)^2} + \omega_n^n \Gamma \frac{\sigma_t^S\left(\theta_t^m + \theta_t^{G|m}(\phi)\right)}{\left(\sigma_t^S\right)^2}$$

for $t \in [T_{n-1}, T_n)$ and $n = 1, ...N$. Proceeding as in the model with a single dividend payment date, shows that the equilibrium PIPR and WAPR have the same structural forms with the appropriate coefficient adjustments.

The remainder of the proof is by induction. The stock price in the last period is, for $t \in [T_{N-1}, T_N)$, as in the one-dividend-cycle model $S_t = A_N^N(t) D_t + B_N^N(t) Z_N + F_N^N(t)$ where $A_N^N(t) = A_N(t)$, $B_N^N(t) = B_N(t)$ and $F_N^N(t) = F_N(t)$. Given the equilibrium stock price in $[T_n, T_{n+1})$ and assuming deterministic stock volatility σ_t^S for $t \in [T_{n-1}, T_n)$ the equilibrium stock price for $t \in [T_{n-1}, T_n)$ is,

$$S_t = E\left[S_{T_n} + D_{T_n} - \int_t^{T_n} \mu_v^S dv \,\middle|\, \mathcal{F}_t^m\right] = E\left[S_{T_n} \middle| \mathcal{F}_t^{Z_n, D}\right] + A_n(t) D_t + B_n(t) Z_n + F_n(t)$$

where the coefficients,

$$A_n(t) = \frac{M_n(T_n)}{M_n(t)} + \int_t^{T_n} \sigma_s^S \beta_n(s) \frac{M_n(s)}{M_n(t)} ds$$

$B_n(t)$ and $F_n(t)$ are the same as the formulas in the one-cycle model, but evaluated at the equilibrium volatility σ_t^S in the multi-cycle model. This volatility, as shown next, is determined by a backward Volterra integral equation adjusted by the feedback effect of the expected future stock price $E\left[S_{T_n}|\,\mathcal{F}_t^{Z_n,D}\right]$. Straightforward computations show that,

$$E\left[S_{T_n}|\,\mathcal{F}_t^{Z_n,D}\right]=A_{n+1}^N\left(T_n\right)E\left[D_{T_n}|\,\mathcal{F}_t^{Z_n,D}\right]+B_{n+1}^N\left(T_n\right)E\left[Z_{n+1}|\,\mathcal{F}_t^{Z_n,D}\right]+F_{n+1}^N\left(T_n\right)$$

$$\begin{aligned}E\left[D_{T_n}|\,\mathcal{F}_t^{Z_n,D}\right] &= E\left[D_{T_n}|\,D_t\right]+\frac{COV\left[D_{T_n},Z_n|\,D_t\right]}{VAR\left[Z_n|\,D_t\right]}\left(Z_n-E\left[Z_n|\,D_t\right]\right)\\ &= D_t+\mu^D\left(T_n-t\right)+\lambda_n\left(t,T_n\right)\left(Z_n-\omega^i\left(D_t+\mu^D\left(T_n-t\right)\right)-\omega_n^n\mu_n^\phi\right)\\ &\equiv A_n^D\left(t\right)D_t+B_n^D\left(t\right)Z_n+F_n^D\left(t\right)\end{aligned}$$

$$\begin{aligned}E\left[G_{n+1}|\,\mathcal{F}_t^{Z_n,D}\right] &= E\left[D_{T_{n+1}}+\zeta_{n+1}|\,\mathcal{F}_t^{Z_n,D}\right]=\mu^D\left(T_{n+1}-T_n\right)+E\left[D_{T_n}|\,\mathcal{F}_t^{Z_n,D}\right]\\ &= \mu^D\left(T_{n+1}-T_n\right)+A_n^D\left(t\right)D_t+B_n^D\left(t\right)Z_n+F_n^D\left(t\right)\end{aligned}$$

$$\begin{aligned}E\left[Z_{n+1}|\,\mathcal{F}_t^{Z_n,D}\right] &= \omega_{n+1}^iE\left[G_{n+1}|\,\mathcal{F}_t^{Z_n,D}\right]+\omega_{n+1}^nE\left[\phi_{n+1}|\,\mathcal{F}_t^{Z_n,D}\right]\\ &= \omega_{n+1}^i\left(\mu^D\left(T_{n+1}-T_n\right)+A_n^D\left(t\right)D_t+B_n^D\left(t\right)Z_n+F_n^D\left(t\right)\right)+\omega_{n+1}^n\mu_{n+1}^\phi\end{aligned}$$

with $B_n^D\left(t\right)\equiv\lambda_n\left(t,T_n\right)$, $A_n^D\left(t\right)\equiv 1-\omega_n^iB_n^D\left(t\right)$, $F_n^D\left(t\right)\equiv A_n^D\left(t\right)\mu^D\left(T_n-t\right)-\omega_n^nB_n^D\left(t\right)\mu_n^\phi$. Substituting and collecting terms,

$$\begin{aligned}E\left[S_{T_n}|\,\mathcal{F}_t^{Z_n,D}\right] &= \left(A_{n+1}^N\left(T_n\right)+B_{n+1}^N\left(T_n\right)\omega_{n+1}^i\right)\left(A_n^D\left(t\right)D_t+B_n^D\left(t\right)Z_n+F_n^D\left(t\right)\right)\\ &\quad +B_{n+1}^N\left(T_n\right)\left(\omega_{n+1}^i\mu^D\left(T_{n+1}-T_n\right)+\omega_{n+1}^n\mu_{n+1}^\phi\right)+F_{n+1}^N\left(T_n\right)\\ &\equiv A_n^S\left(t\right)D_t+B_n^S\left(t\right)Z_n+F_n^S\left(t\right)\end{aligned}$$

with,

$$A_n^S\left(t\right)\equiv L_n^NA_n^D\left(t\right),\qquad B_n^S\left(t\right)\equiv L_n^NB_n^D\left(t\right),\qquad L_n^N\equiv A_{n+1}^N\left(T_n\right)+\omega_{n+1}^iB_{n+1}^N\left(T_n\right)$$

$$F_n^S\left(t\right)\equiv L_n^NF_n^D\left(t\right)+K_n^N,\qquad K_n^N\equiv B_{n+1}^N\left(T_n\right)\left(\omega_{n+1}^i\mu^D\left(T_{n+1}-T_n\right)+\omega_{n+1}^n\mu_{n+1}^\phi\right)+F_{n+1}^N\left(T_n\right).$$

The stock price becomes $S_t=A_n^N\left(t\right)D_t+B_n^N\left(t\right)Z_n+F_n^N\left(t\right)$ with $A_n^N\left(t\right)=A_n^S\left(t\right)+A_n\left(t\right)$, $B_n^N\left(t\right)=B_n^S\left(t\right)+B_n\left(t\right)$, $F_n^N\left(t\right)=F_n^S\left(t\right)+F_n\left(t\right)$. The stock price volatility is $\sigma_t^S=A_n^N\left(t\right)\sigma^D$. The coefficient $A_n\left(t\right)$ therefore solves the backward Volterra integral equation,

$$A_n\left(t\right)=\frac{M_n\left(T_n\right)}{M_n\left(t\right)}+\sigma^D\int_t^{T_n}A_n\left(s\right)\beta_n\left(s\right)\frac{M_n\left(s\right)}{M_n\left(t\right)}ds+\sigma^D\int_t^{T_n}A_n^S\left(s\right)\beta_n\left(s\right)\frac{M_n\left(s\right)}{M_n\left(t\right)}ds. \tag{88}$$

Lemma A-2 shows $A_n\left(t\right)=A_n^1\left(t,T_n\right)+\sigma^D\int_t^{T_n}A_n^1\left(t,s\right)A_n^S\left(s\right)\beta_n\left(s\right)ds$ with $A_n^1\left(t,s\right)$ as defined. Formulas for $B_n\left(t\right)$, $F_n\left(t\right)$ follow by substituting $\sigma_t^S=\sigma_n^S\left(t\right)\equiv A_n^N\left(t\right)\sigma^D$. ■

Lemma 38 *The solution of the backward Volterra equation (88) is,*

$$A_n(t) = A_n^1(t, T_n) + \sigma^D \int_t^{T_n} \beta_n(s) A_n^S(s) A_n^1(t,s)\, ds, \quad A_n^1(t,s) \equiv \left(\frac{H_n(s)}{H_n(t)}\right)^{\omega_n} \left(\frac{M_n(s)}{M_n(t)}\right)^{1-\omega_n}.$$

Proof of Lemma 38. Let $C_n(t) = A_n(t) M_n(t)$ and $C_n^S(t) = A_n^S(t) M_n(t)$. (88) becomes,

$$C_n(t) = M_n(T_n) + \sigma^D \int_t^{T_n} \beta_n(s) C_n(s)\, ds + \sigma^D \int_t^{T_n} \beta_n(s) C_n^S(s)\, ds$$

or, $dC_n(t) = -\sigma^D \beta_n(t) C_n(t)\, dt - \sigma^D \beta_n(t) C_n^S(t)\, dt$, with boundary condition $C_n(T_n) = M_n(T_n)$. The solution is,

$$C_n(T) = C_n(t) e^{-\sigma^D \int_t^{T_n} \beta_n(s) ds} - \sigma^D \int_t^{T_n} \beta_n(s) C_n^S(s) e^{-\sigma^D \int_s^{T_n} \beta_n(v) dv} ds.$$

Solving for $C_n(t)$ gives,

$$\begin{aligned} C_n(t) &= C_n(T_n) e^{\sigma^D \int_t^{T_n} \beta_n(s) ds} + \sigma^D \int_t^{T_n} \beta_n(s) C_n^S(s) e^{\sigma^D \int_t^{s} \beta_n(v) dv} ds \\ &= M_n(T_n) e^{\sigma^D \int_t^{T_n} \beta_n(s) ds} + \sigma^D \int_t^{T_n} \beta_n(s) C_n^S(s) e^{\sigma^D \int_t^{s} \beta_n(v) dv} ds \end{aligned}$$

where,

$$e^{\sigma^D \int_t^s \beta_n(s) ds} = \left(\frac{H_n(s)}{H_n(t)}\right)^{\omega_n} \left(\frac{M_n(s)}{M_n(t)}\right)^{-\omega_n}$$

$$A_n^1(t,s) \equiv \frac{M_n(s)}{M_n(t)} e^{\sigma^D \int_t^s \beta_n(s) ds} = \left(\frac{H_n(s)}{H_n(t)}\right)^{\omega_n} \left(\frac{M_n(s)}{M_n(t)}\right)^{1-\omega_n}.$$

Substituting and simplifying gives $A_n(t) = A_n^1(t, T_n) + \sigma^D \int_t^{T_n} \beta_n(s) A_n^S(s) A_n^1(t,s)\, ds$. ■

Proposition 39 *Consider the model without private information and with N dividend cycles. The competitive equilibrium exists. The equilibrium stock price is piecewise continuous and given by $S_t^{ni} = A_n^{N,ni}(t) D_t + F_n^{N,ni}(t)$ for $t \in [T_{n-1}, T_n)$ and $n = 1, \ldots, N$, where,*

$$\begin{bmatrix} A_n^{N,ni}(t) \\ F_n^{N,ni}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ F_n^{ni}(t) \end{bmatrix} + \begin{bmatrix} A_n^{S,ni}(t) \\ F_n^{S,ni}(t) \end{bmatrix} 1_{n \le N-1} \tag{89}$$

$$\begin{bmatrix} A_n^{S,ni}(t) \\ F_n^{S,ni}(t) \end{bmatrix} = L_n^{N,ni} \begin{bmatrix} 1 \\ F_n^{D,ni}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K_n^{N,ni} \end{bmatrix}, \quad n \le N-1 \tag{90}$$

$$L_n^{N,ni} \equiv A_{n+1}^{N,ni}(T_n), \qquad K_n^{N,ni} \equiv F_{n+1}^{N,ni}(T_n), \qquad F_n^{D,ni}(t) \equiv \mu_n^D (T_n - t) \tag{91}$$

$$F_n^{ni}(t) = \mu_n^D (T_n - t) - \frac{(\sigma_n^D)^2}{\Gamma} \int_t^{T_n} A_n^{N,ni}(v)^2\, dv. \tag{92}$$

The equilibrium stock price coefficients are, for $t \in [T_{n-1}, T_n)$, $\mu_t^{S,ni} = \frac{1}{\Gamma}\left(\sigma_t^{S,ni}\right)^2$ and $\sigma_t^{S,ni} = A_n^{N,ni}(t)\, \sigma_n^D$. Innovations in the public filtration $\mathcal{F}_{(\cdot)}^m = \mathcal{F}_{(\cdot)}^D$ are $dW_t^S = dW_t^D$.

Proof of Proposition 39. In the absence of private information, the aggregate demand function $N_t^a \equiv \omega_n^u N_t^u + \omega_n^i N_t^i + \omega_n^n N_t^n$ is,

$$N_t^{a,ni} = \omega_n^u \Gamma \frac{\theta_t^{m,ni}}{\sigma_t^{S,ni}} + \omega_n^i \Gamma \frac{\theta_t^{m,ni}}{\sigma_t^{S,ni}} + \omega_n^n \Gamma \frac{\theta_t^{m,ni}}{\sigma_t^{S,ni}} = \Gamma \frac{\theta_t^{m,ni}}{\sigma_t^{S,ni}}$$

for $t \in [T_{n-1}, T_n)$ and $n = 1, ...N$. It follows that $\theta_t^{m,ni} = \sigma_t^{S,ni}/\Gamma$.

The equilibrium stock price in the last cycle is $S_t^{ni} = D_t + F_N^{ni}(t)$ where $F_N^{ni}(t) = \mu_N^D (T_N - t)$. Given the stock price in $[T_n, T_{n+1})$ and assuming deterministic volatility for $t \in [T_{n-1}, T_n)$, the equilibrium stock price for $t \in [T_{n-1}, T_n)$ is,

$$S_t^{ni} = E\left[S_{T_n}^{ni} + D_{T_n} - \int_t^{T_n} \mu_v^{S,ni} dv \middle| \mathcal{F}_t^m \right] = E\left[S_{T_n}^{ni} \middle| \mathcal{F}_t^D \right] + D_t + F_n^{ni}(t)$$

where $F_n^{ni}(t) = \mu_n^D (T_n - t) - \Gamma^{-1} \int_t^{T_n} \left(\sigma_v^{S,ni}\right)^2 dv$. Straightforward computations give,

$$\begin{aligned}
E\left[S_{T_n}^{ni} \middle| \mathcal{F}_t^D \right] &= A_{n+1}^{N,ni}(T_n) E\left[D_{T_n} \middle| \mathcal{F}_t^{Z_n,D} \right] + F_{n+1}^{N,ni}(T_n) \\
&= A_{n+1}^{N,ni}(T_n) \left(D_t + \mu_n^D (T_n - t) \right) + F_{n+1}^{N,ni}(T_n) \\
&= A_{n+1}^{N,ni}(T_n) \left(D_t + F_n^{D,ni}(t) \right) + F_{n+1}^{N,ni}(T_n) \equiv A_n^{S,ni}(t) D_t + F_n^{S,ni}(t)
\end{aligned}$$

where $F_n^{D,ni}(t) = \mu_n^{D,ni}(T_n - t)$, $A_n^{S,ni}(t) = A_{n+1}^{N,ni}(T_n)$ and $F_n^{S,ni}(t) = A_{n+1}^{N,ni}(T_n) F_n^{D.ni}(t) + F_{n+1}^{N,ni}(T_n)$. Substituting in the previous expression gives $S_t^{ni} = \left(1 + A_n^{S,ni}(t)\right) D_t + F_n^{ni}(t) + F_n^{S,ni}(t) \equiv A_n^{N,ni}(t) D_t + F_n^{N,ni}(t)$ where $A_n^{N,ni}(t) = 1 + A_n^{S,ni}(t)$ and $F_n^{N,ni}(t) = F_n^{ni}(t) + F_n^{S,ni}(t)$. The stock price volatility $\sigma_t^{S,ni} = A_n^{N,ni}(t) \sigma^D$ is deterministic as conjectured. ■

Corollary 40 *L_n^N satisfies the recursion $L_n^N = \varrho_{0,n+1} + \varrho_{1,n+1} L_{n+1}^N$, for $n \leq N-1$, subject to $L_N^N = 0$, where,*

$$\varrho_{0,n+1} \equiv A_{n+1}^1 (T_n, T_{n+1}) + \omega_{n+1}^i B_{n+1}^1 (T_n, T_{n+1}) \tag{93}$$

$$B_{n+1}^1 (T_n, T_{n+1}) \equiv \lambda_{n+1}(T_n, T_{n+1}) + \sigma_{n+1}^D \int_{T_n}^{T_{n+1}} A_{n+1}^1 (v, T_{n+1}) \delta_{n+1}(T_n, v) dv \tag{94}$$

$$\varrho_{1,n+1} \equiv 1 + \sigma_{n+1}^D \int_{T_n}^{T_{n+1}} \left(A_{n+1}^D (s) A_{n+1}^1 (T_n, s) \beta_{n+1}(s) + \omega_{n+1}^i J_{n+1}(s) \right) ds \tag{95}$$

$$J_{n+1}(s) \equiv \delta_{n+1}(T_n, s) \left(A_{n+1}^D (s) + \sigma_{n+1}^D \int_s^{T_{n+1}} A_{n+1}^1 (s, v) A_{n+1}^D (v) \beta_{n+1}(v) dv \right) \tag{96}$$

and $\delta_{n+1}(T_n, v) \equiv \alpha_{n+1}(v) + \beta_{n+1}(v) \lambda_{n+1}(T_n, v)$. The solution is,

$$L_n^N = \varrho_{0,n+1} + \sum_{i=2}^{N-n} \left(\left(\prod_{j=1}^{i-1} \varrho_{1,n+j} \right) \varrho_{0,n+i} \right) 1_{n<N-1}.$$

In the model without private information $L_n^{N,ni} = 1 + L_{n+1}^{N,ni}$, for $n \leq N-1$, subject to $L_N^{N,ni} = 0$, leading to $L_n^{N,ni} = N - n$.

Proof of Corollary 40. Substituting the expressions for $A_{n+1}^{N}(T_n), B_{n+1}^{N}(T_n)$ in $L_n^N \equiv A_{n+1}^{N}(T_n) + \omega_{n+1}^{i} B_{n+1}^{N}(T_n)$, using $A_{n+1}^{D}(t) + \omega_{n+1}^{i} B_{n+1}^{D}(t) = 1$ and collecting terms leads to the recursive equation stated. For the model without private information $L_n^{N,ni} \equiv A_{n+1}^{N,ni}(T_n)$. Substituting the expression for $A_{n+1}^{N,ni}(T_n)$ in Proposition 39 gives the result stated. ■

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Figure 1: **Sensitivity of WAPR to private information:** The figure presents the dynamic behavior of $\alpha(t)$ for $t \in [0, T]$. Parameter values are $T = 1$, $\sigma^D = 0.1$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.32$. The weight of informed is $\omega^i = \frac{1}{2}$. The weight of mimicking noise traders varies between $\omega^n = 0.05$ (left panel), $\omega^n = 0.22$ (middle panel) and $\omega^n = 0.25$ right panel.

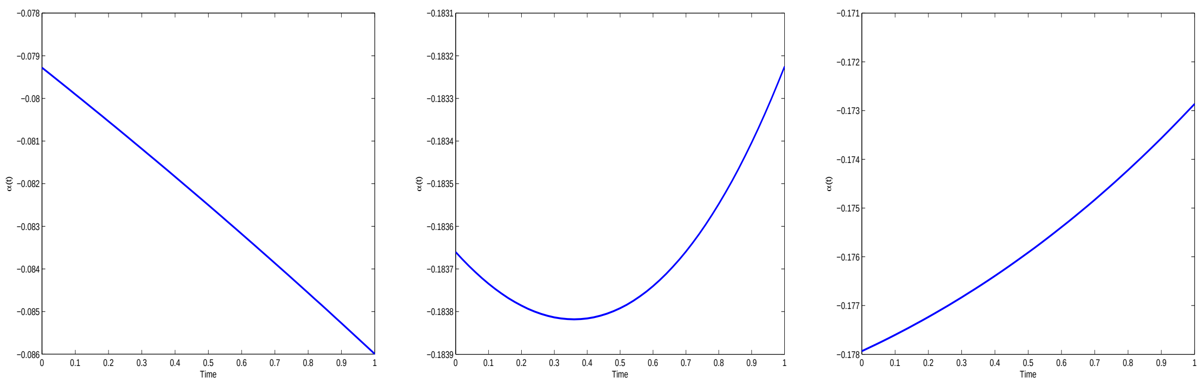

Figure 2: **Stock volatility and sensitivity of WAPR to fundamental information:** The figure presents the dynamic behavior of σ_t^S and $\beta(t)$ for $t \in [0, T]$. Parameter values are $T = 1$, $\sigma^D = 0.1$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.5$. The weight of informed and mimicking noise traders are $\omega^i = \omega^n = \frac{1}{3}$.

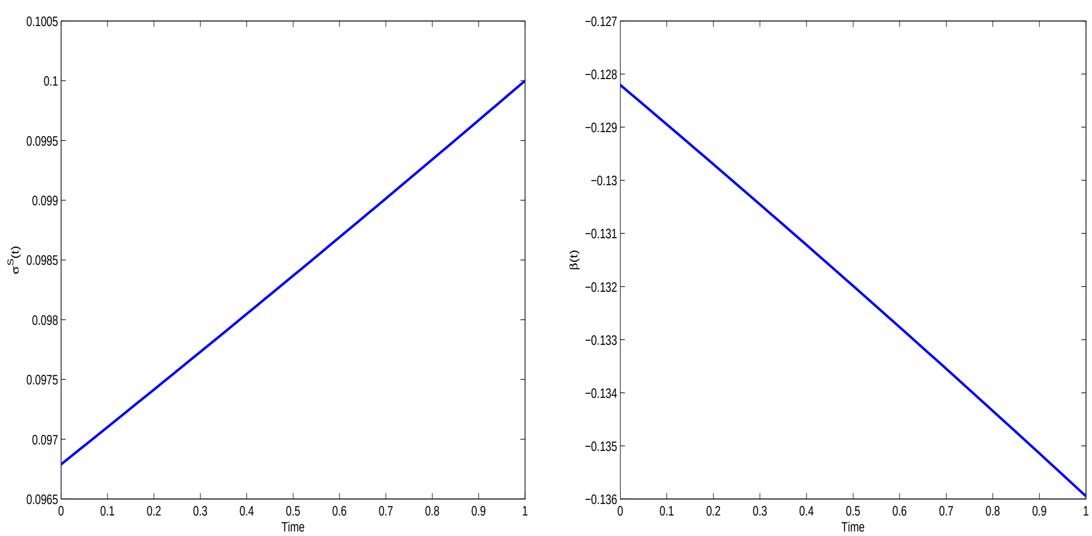

Figure 3: **Optimal portfolio holdings:** The figure shows the optimal portfolio holdings (z-axis) of the public (top panels) and the public (middle panels) in the competitive (left column) and the strategic (right column) equilibrium as a function of time t (x-axis) and dividends D_t (y-axis). The covariation of portfolio holdings with prices (y-axis) as a function of time t (x-axis) is shown in the bottom panels. Parameter values are $T = 1$, $\Gamma = 1/8$, $\sigma^D = 0.1$, $\mu^D = 0.05$, $D_0 = 1$, $\mu^\phi = E[G]$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.1$, $G = D_0 + \mu^D T$, $\phi = 0.9 \times G$. The weights of the informed and the mimicking noise traders are $\omega^i = \omega^n = \frac{1}{3}$.

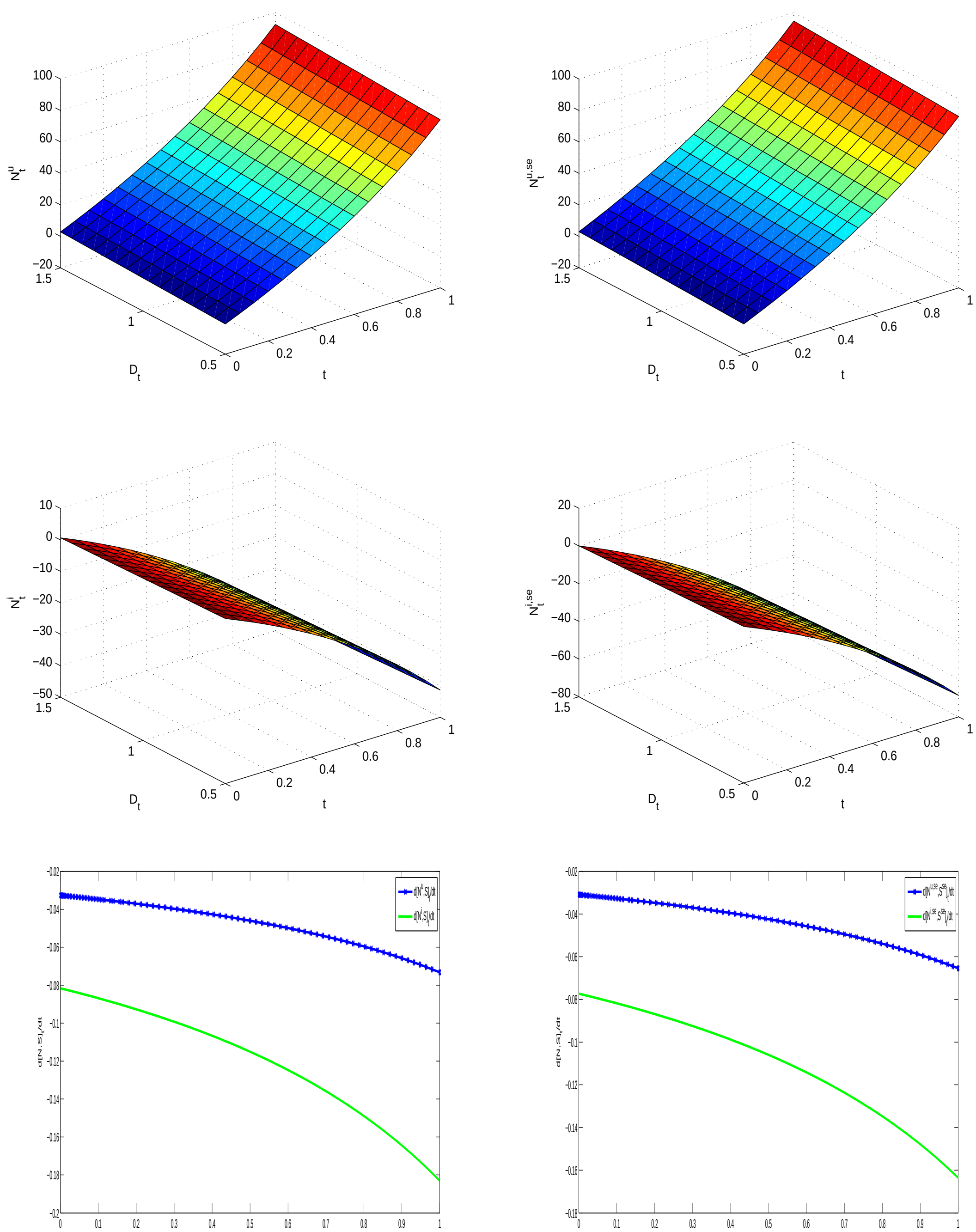

Figure 4: **Pareto ranking of competitive and strategic equilibria:** This figure presents the different components of the utility differential as a function of risk tolerance Γ. The top row presents results for the competitive equilibrium and the bottom row results for the strategic equilibrium. Parameter values are $T = 1$, $D_0 = 1$, $\omega^i = \omega^n = \frac{1}{3}$, $\sigma^D = 0.1$, $\mu^D = 0.05$, $\mu^\phi = E\,[G]$, $\sigma^\phi = STD\,[G]$. The precision of private information varies between $\sigma^\zeta = 5$ (left panels), $\sigma^\zeta = 0.5$, (middle panels) and $\sigma^\zeta = 0.05$ (right panels)

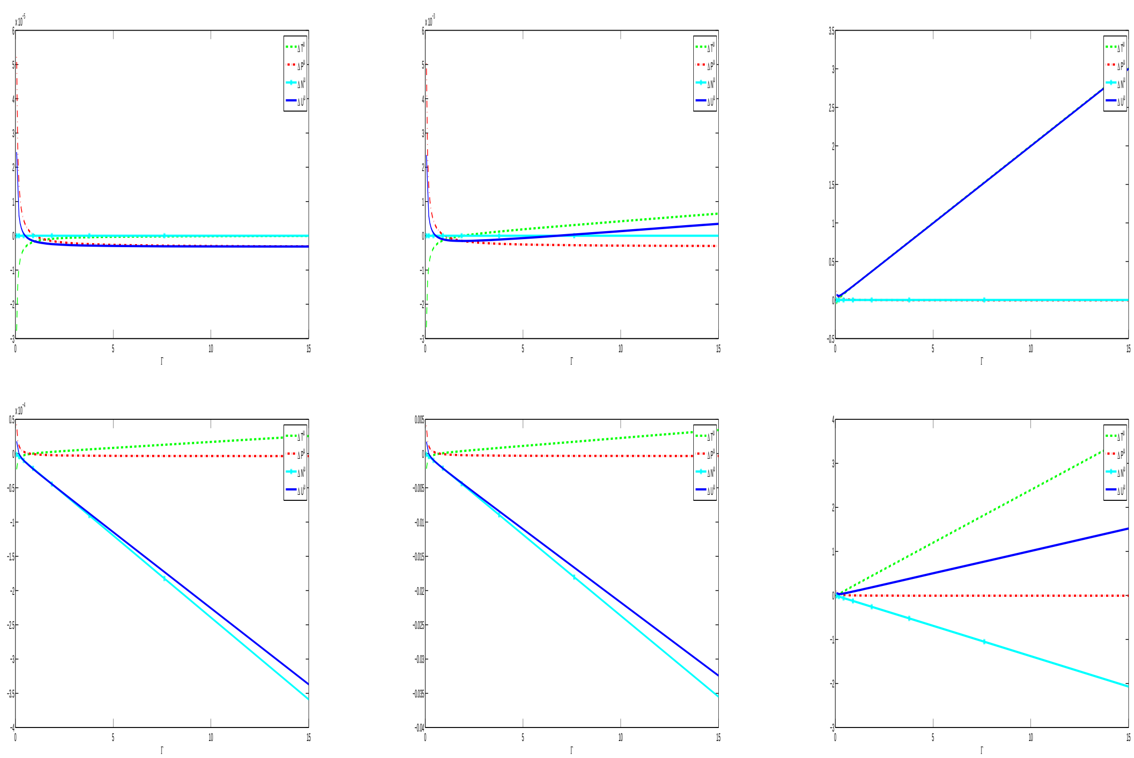

Figure 5: Sensitivity of WAPR to private information: The figure presents the dynamic behavior of $\alpha(t)$ for $t \in [0, T]$ in the competitive (top row) and strategic (bottom row) equilbria. Parameter values are $T = 1$, $\sigma^D = 0.1$, $\sigma^\phi = STD[G]$, $\sigma^\zeta = 0.32$. The weight of informed is $\omega^i = \frac{1}{2}$. The weight of mimicking noise traders varies between $\omega^n = 0.1$ (left panel), $\omega^n = 0.22$ (middle panel) and $\omega^n = 0.25$ (right panel).

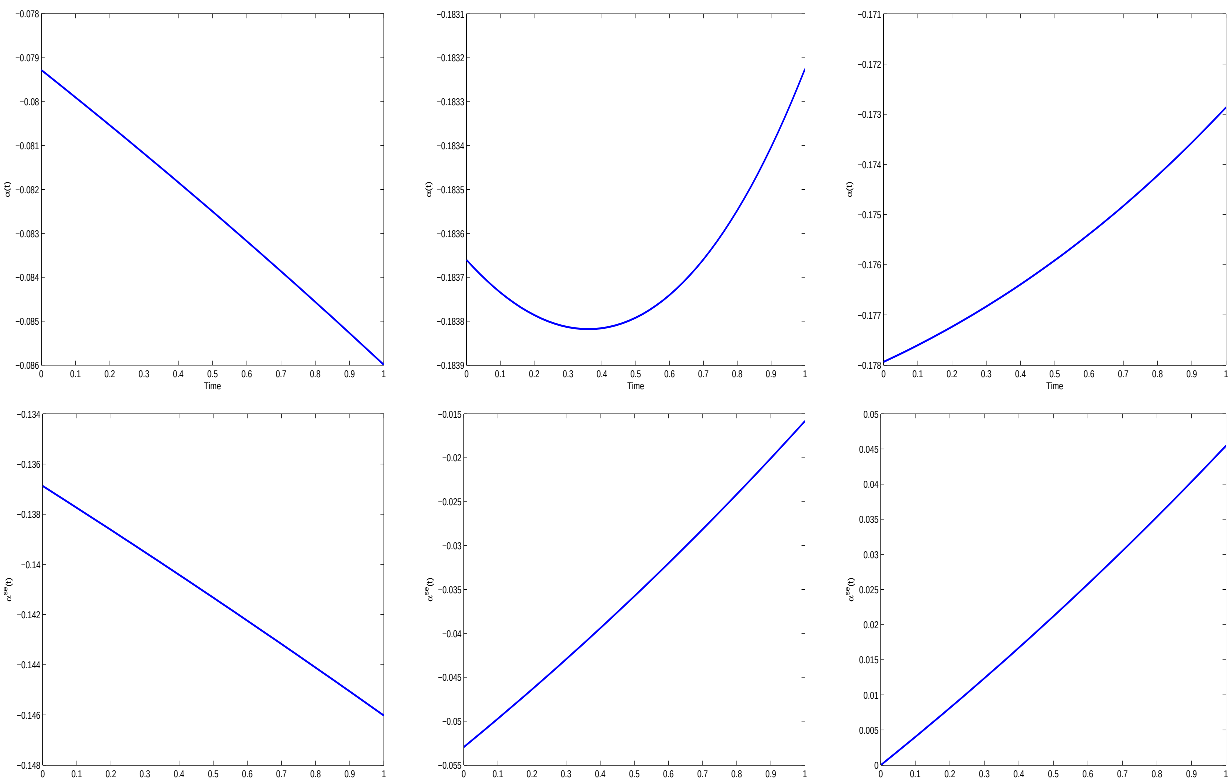

Figure 6: Equilibrium price coefficients over multiple dividend cycles: The figure presents the dynamic behavior of $A_n^N(t)$ (top row, left), $A_n^S(t)$ (top row, middle), $A_n(t)$ (top row, right), $B_n^N(t)$ (middle row, left), $B_n^S(t)$ (middle row, middle), $B_n(t)$ (middle row, right) and $F_n^N(t)$ (bottom row, left), $F_n^S(t)$ (bottom row, middle), $F_n(t)$ (bottom row, right) for $t \in [0, T_N]$ and $T_n = 0.25 \times n$ with $n = 1, \ldots, 40$. Parameter values are $D_0 = 1$, $\sigma_n^D = 0.1$, $\mu_n^D = 0.05$, $\sigma_n^\zeta = 0.05 \times T_n$, $\mu_n^\phi = E[G_n]$, $\sigma_n^\phi = STD[G_n]$. Population weights are $\omega_n^i = \omega_n^n = \frac{1}{3}$. The risk tolerance is $\Gamma = 1/2$.

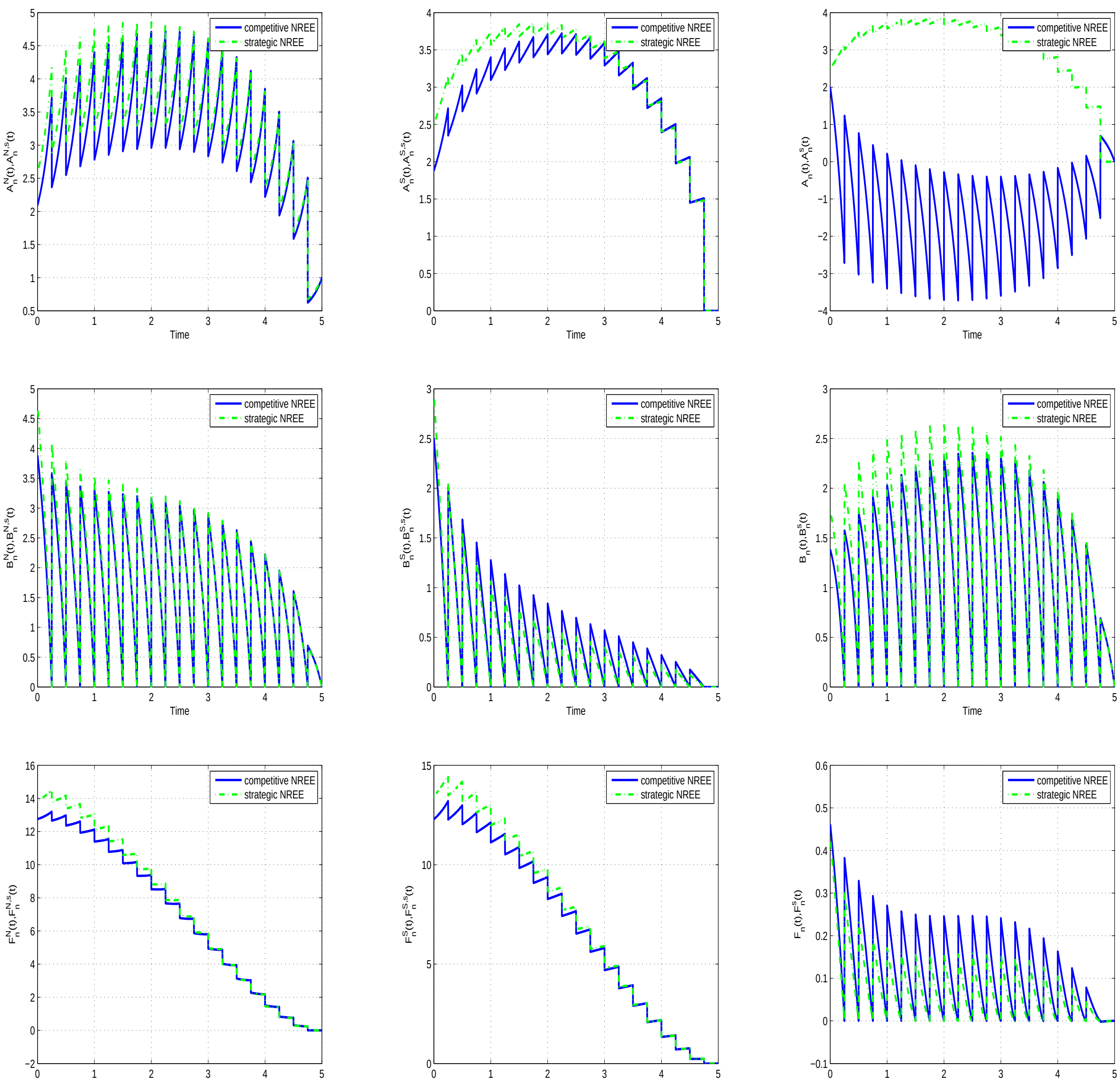

Figure 7: Volatility, stock premium, portfolio holdings and stock price over multiple dividend cycles: The figure presents the dynamic behavior of $\sigma_t^{S,N}$ (top row, left), $\mu_t^{S,N}$ (top row, middle) θ_t^m (top row, right), N_t^u (middle row, left), N_t^i (middle row, middle), N_t^n (middle row, right), S_t^N (bottom row, left), $\theta_t^{G|m}\left(Z_n;\omega_n^i,\omega_n^n\right)$ (bottom row, middle), D_t, G_n, ϕ_n, Z_n (bottom row, right) for $t \in [0, T_N]$ and $T_n = 0.25 \times n$ with $n = 1, \ldots, 20$ in the competitive and strategic equilibrium. Parameter values are $D_0 = 1$, $\sigma_n^D = 0.1$, $\mu_n^D = 0.05$, $\sigma_n^\zeta = 0.05 \times T_n$, $\mu_n^\phi = E\left[G_n\right]$, $\sigma_n^\phi = STD\left[G_n\right]$. Population weights are $\omega_n^i = \omega_n^n = \frac{1}{3}$. The risk tolerance is $\Gamma = 1/2$.

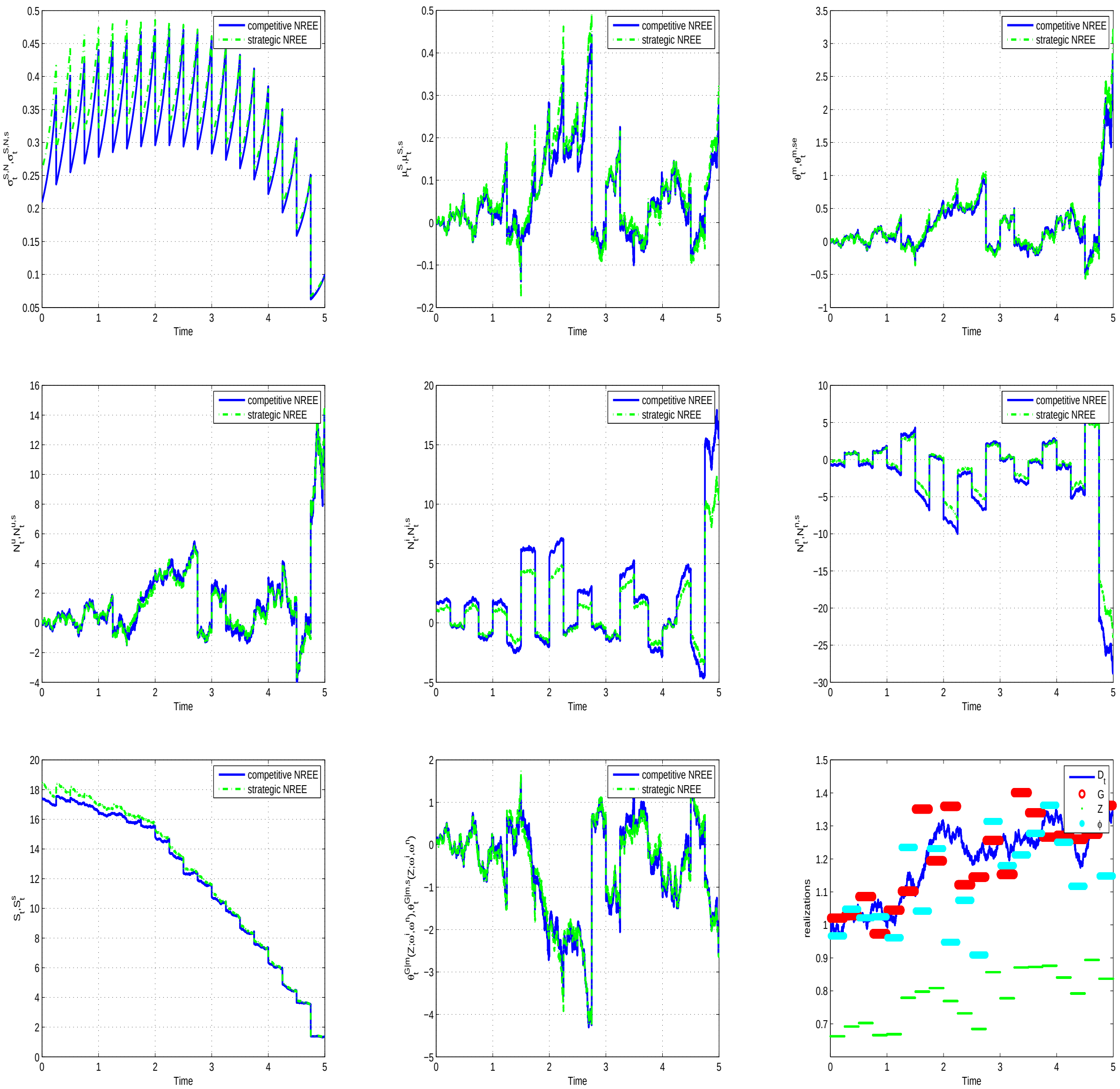

Figure 8: Variance stabilizing effects of information and Pareto ranking: The figure presents the dynamic behavior of the volatilities $\sigma_t^{S,N}$, $\sigma_t^{S,N,s}$ and $\sigma_t^{S,N,ni}$ for $t \in [0, T_N]$ and $T_n = 0.25 \times n$ with $n = 1, \ldots, 20$ in the competitive, strategic and no-information equilibrium for different levels of skill. The skill parameter values are $\sigma_n^{\zeta} = 0.05 \times T_n$ (left panel), $\sigma_n^{\zeta} = 0.15 \times T_n$ (middle panel), $\sigma_n^{\zeta} = 0.25 \times T_n$ (right panel). Ex ante utility gains for the corresponding levels of the skill parameter in the competitive (strategic) NREE are shown in the middle (bottom) row. Parameter values are $D_0 = 1$, $\sigma_n^D = 0.1$, $\mu_n^D = 0.05$, $\mu_n^{\phi} = E[G_n]$, $\sigma_n^{\phi} = STD[G_n]$. Population weights are $\omega_n^i = \omega_n^n = \frac{1}{3}$. The risk tolerance is $\Gamma = 1/2$.