

On $\beta = s/g$, Piketty's "Second Fundamental Law of Capitalism"¹

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$\beta = s/g$, is not helpfully called a "law of capitalism." Rather, $\beta_t = s_t/g_t$ is a stable dynamic equilibrium condition of capitalism.³

Where K_t , Y_t , s_t , and g_t , are respectively the aggregate stock of (nonhuman) capital (in a broad sense), the aggregate net flow of income (= product), the net savings rate, and the growth rate of income, all at (stock) or during (flow) time t :

$$\beta_t = K_t/Y_t$$
$$\frac{s_t}{g_t} = \frac{s_t Y_t / Y_t}{\Delta Y_t / Y_t} = \frac{s_t Y_t}{\Delta Y_t} = \frac{\Delta K_t}{\Delta Y_t}$$

$$\therefore \frac{K_t}{Y_t} = \frac{\Delta K_t}{\Delta Y_t}.$$

¹ Piketty (2014, p. 55, 166ff.).

² Thanks to insightful suggestions by Mehrene Larudee.

³ To my knowledge, this has not been previously explicitly noted and proved, at least not in Piketty's online "Technical Appendix." A more restricted version is noted in Stiglitz and Greenwald (2014), Theorem A.14 (p. 174) (See also *ibid.* footnote 43, p. 534-5.) Correction (13.12.14): That this is an equilibrium condition is noted though not proved by Branko Milanovic (Milanovic 2014, p. 523).

The existence of this condition does not depend on any particular theory of economic growth as has been suggested but not demonstrated by several critics, e.g., Krusell, et al. (2014) and Acemoglu, et al. (2014).

We might defensibly describe it as a "Tendential Law of Capitalism" which, e.g., the "Falling Rate of Profit" is not. (Thompson, (1995)). For better or worse, "The Fundamental Stable Equilibrium Condition of Capitalism" is not a catchy phrase.

I.e., in the only stable dynamic equilibrium state, the ratio of the stock of capital to the flow of income is equal to the ratio of the change in the stock of capital to the change in the flow of income.⁴

(A corollary: Where L_t is the flow of the labor input, $k_t = K_t/L_t$ and $y_t = Y_t/L_t$, and thus $\beta_t = k_t/y_t$, the equilibrium condition connects to (but does not depend on) Solow's theory of economic growth (taking account that Solow's Y is gross and Piketty's Y is net of depreciation and that $Y_{net} = Y_{gross} - \delta K$ where δ is the rate of depreciation.))

Convergence to the stable equilibrium:

Suppose:

$$\beta_{t-1} < \frac{s_{t-1}}{g_{t-1}}$$

Then:

$$\begin{aligned} \frac{K_{t-1}}{Y_{t-1}} &< \frac{\Delta K_{t-1}}{\Delta Y_{t-1}} \\ \frac{\Delta K_{t-1}}{K_{t-1}} &> \frac{\Delta Y_{t-1}}{Y_{t-1}} \\ 1 + \frac{\Delta K_{t-1}}{K_{t-1}} &> 1 + \frac{\Delta Y_{t-1}}{Y_{t-1}} \\ \frac{K_{t-1} + \Delta K_{t-1}}{K_{t-1}} &> \frac{Y_{t-1} + \Delta Y_{t-1}}{Y_{t-1}} \\ \frac{K_t}{K_{t-1}} &> \frac{Y_t}{Y_{t-1}} \\ \frac{K_t}{Y_t} &> \frac{K_{t-1}}{Y_{t-1}} \\ \therefore \beta_t &> \beta_{t-1} \end{aligned}$$

I.e., if $\beta_{t-1} < \frac{s_{t-1}}{g_{t-1}}$, then $\beta_t > \beta_{t-1}$.

Mutatis mutandis, if $\beta_{t-1} > \frac{s_{t-1}}{g_{t-1}}$, then $\beta_t < \beta_{t-1}$.

⁴ One could interpose some steps: Where S_t is aggregate saving and I_t is aggregate net investment, then $s_t Y_t = S_t$ and $S_t = I_t = \Delta K_t$.

$\beta = s/g$ is a profoundly *interesting* stable equilibrium condition.

Why? Because s and g are generally slow moving variables in the medium run and big movements generally have obvious exogenous causes, e.g., plagues, wars, revolutions, depressions, and, oppositely, recoveries from such. And β affects so much else, e.g., the distribution of income.

From non-proximate causes, s and g determine the equilibrium state. But if s/g is unchanging and $\beta_{t-1} \neq s/g$, then $\beta_t \neq s/g$ even though of course β_t is closer to s/g than β_{t-1} was.⁵ In this case the relation is asymptotic. But $\beta = s/g$ can be the case; the stable equilibrium condition can obtain in finite time. Here it must be noted that s and g and s/g will not in fact ever be perfectly constant from period to period even if for extended periods variations are quite small. $\beta = s/g$ is something like a “trembling hand” equilibrium. For example, a small change in s (even leaving g unchanged) $\beta_t = s/g$ can be reached in one period from $\beta_{t-1} = s/g$.⁶

⁵ Contrapositively, $\beta_t = s/g$ implies $\beta_{t-1} = s/g$ since

$$\beta_t = \frac{K_{t-1} + sY_{t-1}}{Y_{t-1} + gY_{t-1}} = \frac{K_{t-1} + \Delta K_{t-1}}{Y_{t-1} + \Delta Y_{t-1}} = \frac{K_t}{Y_t}.$$

⁶ Suppose:

$$\beta_{t-1} \neq \frac{s_{t-1}}{g_{t-1}}, \text{ e.g., that } \beta_{t-1} < \frac{s_{t-1}}{g_{t-1}} \text{ or } \beta_{t-1} > \frac{s_{t-1}}{g_{t-1}}.$$

In the next period:

$$\frac{K_t}{Y_t} = \frac{K_{t-1} + \Delta K_{t-1}}{Y_{t-1} + \Delta Y_{t-1}} = \frac{K_{t-1} + s_{t-1}Y_{t-1}}{(1 + g_{t-1})Y_{t-1}}.$$

Then there are values of s_t and g_t such that

$$\beta_t = \frac{s_t}{g_t}.$$

I.e., a stable equilibrium is reached in the next period. Those values are such that

$$s_t = \frac{g_t K_t}{Y_t}.$$

E.g., suppose (Cf., Piketty, p. 166, example):

$$K_{t-1} = 5.9, Y_{t-1} = 1/1.02, s_{t-1} = 10.2\%, \text{ and } g_{t-1} = 2\%,$$

A digression:

Consider briefly Piketty's "First Fundamental Law of Capitalism," $\alpha_t = r_t \beta_t$.

Here r_t is the rate of return on K_t , that is, the ratio of the flow of profits, Π_t , to the stock K_t , i.e., $r_t = \Pi_t/K_t$, and thus α_t is the share of profits, the capital share, in income Y_t , i.e., $\alpha_t = \Pi_t/Y_t$.

This r_t might or might not be construed as the marginal product of capital, $\frac{\partial Y_t}{\partial K_t}$, but it *is not* the reciprocal of $\frac{\Delta K_t}{\Delta Y_t} = \frac{s_t}{g_t}$ in the stable

equilibrium condition. For if $\beta_t = \frac{s_t}{g_t}$ and it were the case that $r_t = \frac{\Delta Y_t}{\Delta K_t}$, then $\alpha_t = 1$, i.e., profits equal all income, $\Pi_t = Y_t$, i.e., only capital would receive a return.⁷ I.e., $1/\beta_t$ is not the marginal product of capital;

$$\frac{\Delta Y_t}{\Delta K_t} \neq \frac{\partial Y_t}{\partial K_t}.$$

and thus that $\beta_{t-1} = 6.018$ and $s_{t-1}/g_{t-1} = 5.1$,

and thus that $\beta_{t-1} < \frac{s_{t-1}}{g_{t-1}}$.

Then $K_t = K_{t-1} + \Delta K_{t-1} = K_{t-1} + s_{t-1} Y_{t-1} = 5.9 + (0.102)(1/1.02) = 6$,

and $Y_t = Y_{t-1} + \Delta Y_{t-1} = Y_{t-1} + g_{t-1} Y_{t-1} = (1/1.02) + (0.02)(1/1.02) = 1$,

and thus $\beta_t = \frac{K_t}{Y_t} = 6$.

Then, again, with still $g_t = 0.02$, but now $s_t = 0.12$,

and $\frac{s_t}{g_t} = \frac{\Delta K_t}{\Delta Y_t} = 6$.

Now $\beta_t = \frac{s_t}{g_t}$ is a stable equilibrium.

I.e., raising the savings rate from 10.2% to 12% reaches a stable equilibrium in one period.

⁷ In such a (preventable!) case, if there were a labor input, it would be slave labor. Human capital need not be self-owned.

Thus $\beta = K/Y$ rising does not imply r falling due to diminishing returns to capital. L/Y , the labor/output ratio could be rising just as quickly, e.g., with constant returns to scale. Both are still consistent with $\Pi = rK$, capital income, rising faster than labor income, if r rises faster than the average return to labor, w , and thus with the capital share, $\alpha = \Pi/Y$, rising, or conversely.

Relating r and g :

Fundamentally interesting about the relationship between r , the rate of return on capital, and g , the rate of growth of the economy, is the entailed relationship between Π , the aggregate flow of profit and S , the aggregate flow of net saving, equivalently of Π to ΔK , the net change in the aggregate stock of capital.

For $r = \Pi/K$ and, at the stable equilibrium, $g = S/K$.⁸ And thus, at equilibrium, if $r > g$, $r = g$, or $r < g$, then, respectively, $\Pi > S = \Delta K$, $\Pi = S = \Delta K$, or $\Pi < S = \Delta K$.

Here it is helpful to make use of some additional standard notions and notation:

A Capitalist Economy:

A standard representation (not (yet) taking human capital into account):

Aggregate net income is either labor income, W_t , or nonlabor income, Π_t :

$$Y_t = W_t + \Pi_t$$

And is the sum of individual incomes:

$$Y_{t,i} = W_{t,i} + \Pi_{t,i}$$

Where:

$$W_{t,i} = w_{t,i}L_{t,i}; \quad \Pi_{t,i} = r_{t,i}K_{t,i}$$

And thus:

⁸ Since $\beta = K/Y = s/g$ and thus $g = sY/K = S/K$.

$$Y_{t,i} = w_{t,i}L_{t,i} + r_{t,i}K_{t,i}$$

In the aggregate:

$$W_t = \sum_{i=1}^n W_{t,i} ; \Pi_t = \sum_{i=1}^n \Pi_{t,i} ; L_t = \sum_{i=1}^n L_{t,i} ; K_t = \sum_{i=1}^n K_{t,i} ; w_t = \frac{W_t}{L_t} ; r_t = \frac{\Pi_t}{K_t}$$

$$Y_t = W_t + \Pi_t = w_t L_t + r_t K_t = \sum_{i=1}^n Y_{t,i} = \sum_{i=1}^n (w_{t,i} L_{t,i} + r_{t,i} K_{t,i})$$

Aggregate outgo is either consumption, C_t , or saving, $S_t = I_t = \Delta K_t$:

And is the sum of individual outgoes:

$$Y_{t,i} = C_{t,i} + S_{t,i}$$

In the aggregate:

$$C_t = \sum_{i=1}^n C_{t,i} ; S_t = \sum_{i=1}^n S_{t,i}$$

$$Y_t = C_t + S_t = \sum_{i=1}^n Y_{t,i} = \sum_{i=1}^n (C_{t,i} + S_{t,i})$$

Further relating r and g :

If $r_t > g_t$, then $\Pi_t > \Delta K_t$; the aggregate flow of profit exceeds the change in aggregate capital, i.e., $\Pi_t - S_t > 0$. Thus, since $Y_t = W_t + \Pi_t = C_t + S_t$, then $C_t > W_t$. In the aggregate there is some consumption from non-labor, i.e., capital, income.

If $r_t < g_t$, then $\Pi_t < \Delta K_t$; the aggregate flow of profit is less than the change in aggregate capital, i.e., $\Pi_t - S_t < 0$. Thus $C_t < W_t$. In the aggregate there is some investment from non-capital, i.e., labor, income.

Returning to the case in which $r_t = g_t$, even though (assuming the equilibrium condition obtains) in the aggregate $\Pi_t = \Delta K_t = S_t = I_t$, there is no reason to think that for any particular individual i ,

$\Pi_{t,i} = \Delta K_{t,i} = S_{t,i} = I_{t,i}$; i.e., it can be the case that $\Pi_{t,i} > \Delta K_{t,i} = S_{t,i} = I_{t,i}$ or that $\Pi_{t,i} < \Delta K_{t,i} = S_{t,i} = I_{t,i}$.

Of course, if every individual invests all of (and only of) their profit income, i.e., if $(\forall i)(\Pi_{t,i} = \Delta K_{t,i} = S_{t,i} = I_{t,i})$, then the distribution of shares of capital remains unchanged. In period t , every individual i owns $K_{t,i}$ (or, if $K_{t,i} < 0$, owes) and has the share of capital $K_{t,i}/K_t$. In period $t+1$, every individual i owns (or owes) $K_{t+1,i} = (K_{t,i} + \Delta K_{t,i})$ and has the share of capital $K_{t+1,i}/K_{t+1} = (K_{t,i} + \Delta K_{t,i})/(K_t + \Delta K_t)$. Every individual capital has changed in exactly the same proportion, $\Delta K_t/K_t$. But there is no reason to think that this generalization would hold. Instead, it can be the case that $(\exists i)(\Pi_{t,i} \neq \Delta K_{t,i} = S_{t,i} = I_{t,i})$.

More generally, if $\rho_t r_t = g_t$ for some scalar ρ_t , if every individual invests share ρ_t of (and only of) their profit income, i.e., if $(\forall i)(\rho_t \Pi_{t,i} = \Delta K_{t,i} = S_{t,i} = I_{t,i})$, then the distribution of capital again remains unchanged. Again, every individual capital has changed in exactly the same proportion, $\Delta K_t/K_t$. But, again, there is no reason to think that this generalization would hold. Instead, it can be the case that $(\exists i)(\rho_t \Pi_{t,i} \neq \Delta K_{t,i} = S_{t,i} = I_{t,i})$.

Further, if $\rho_t r_t = g_t$ for some scalar ρ_t : If, for an individual i , $S_{t,i} > \rho_t \Pi_{t,i}$, then the share of individual i of the whole capital stock, $K_t + \Delta K_t = K_{t+1}$, rises; i.e., $\frac{K_{t+1,i}}{K_{t+1}} > \frac{K_{t,i}}{K_t}$. If, for an individual i , $S_{t,i} < \rho_t \Pi_{t,i}$, then the share of

individual i of the whole capital stock, $K_t + \Delta K_t = K_{t+1}$, falls; i.e., $\frac{K_{t+1,i}}{K_{t+1}} < \frac{K_{t,i}}{K_t}$.

That individual saving is a constant function of individual profit, i.e., of individual capital income, is an instructive but quite counterfactual assumption. A more realistic hypothesis is that individual saving is a constant function of individual total income, $Y_{t,i} = W_{t,i} + \Pi_{t,i}$, i.e., that for some scalar γ_t , $(\forall i)(\gamma_t Y_{t,i} = \Delta K_{t,i} = S_{t,i} = I_{t,i})$. Then, if the equilibrium condition obtains and $\rho_t r_t = g_t$, then $\gamma_t Y_{t,i} = \rho_t \Pi_{t,i}$, i.e., $\gamma_t = \rho_t \Pi_{t,i}/Y_{t,i}$. A weaker requirement worth exploring would be that $(\forall i)(f_t(Y_{t,i}) = \Delta K_{t,i} = S_{t,i} = I_{t,i})$ where $f_t' > 0$.

In general:

If $\Delta K_{t,i} < 0$, then i becomes absolutely materially poorer;

if $\Delta K_{t,i} > 0$, then i becomes absolutely materially richer;

if $\Delta K_{t,i} < \Pi_{t,i}$, then i becomes relatively materially poorer;

if $\Delta K_{t,i} > \Pi_{t,i}$, then i becomes relatively materially richer.

A Capitalist Economy taking human capital into account.

A yawning gap in Piketty's analysis is its failure to integrate the determination of labor income (determined by labor inputs qualified by human capital) with the determination of non-labor income (determined by non-human capital inputs).

In outline it is clear how this can be done:

$H_{t,i}$ = investment in human capital by i

$h_{t,i}(H_{t,i})$ = earning capacity of human capital of i

$\sum_{i=1}^n h_{t,i}(H_{t,i})$ = aggregate earning capacity of human capital

Then:

$$Y_{t,i} = r_t \left(h_{t,i}(H_{t,i}) \right) \lambda_{t,i} + r_t K_{t,i} \kappa_{t,i} = r_t \left(\left(h_{t,i}(H_{t,i}) \right) \lambda_{t,i} + K_{t,i} \kappa_{t,i} \right)$$

$$\lambda_{t,i} = \frac{\text{individual labor in use } (L_{t,i})}{\text{individual available labor time } (L_{t,i}^{\max})} = \text{individual labor utilization ratio}$$

$$\kappa_{t,i} = \frac{\text{individual capital in use } (K_{t,i}^{\text{use}})}{\text{individual capital owned } (K_{t,i})} = \text{individual capital utilization ratio}$$

$w_{t,i} = r_t \left(h_{t,i}(H_{t,i}) \right)$, i.e., the rate of return on the use of human capital

$W_{t,i} = r_t \left(h_{t,i}(H_{t,i}) \right) \lambda_{t,i}$, labor income

Assuming there are diminishing returns to (individual) human capital (though not necessarily to individual nonhuman capital)⁹, then an individual labor-income-maximizing equilibrium condition is

$h_{t,i}(H_{t,i}) = H_{t,i}$. The individual choice variables are $L_{t,i}$, i.e. $\lambda_{t,i}$, and

$S_{t,i} = \Delta H_{t,i} + \Delta K_{t,i}$. Individual income is

$$Y_{t,i} = W_{t,i} + \Pi_{t,i} = r_t \left((h_{t,i}(H_{t,i})) \lambda_{t,i} + K_{t,i} K_{t,i} \right).$$

With this formalization one can specify Lorenz curves for:

$$(1 - \alpha_{t,i}) = W_{t,i}/Y_t \quad (\text{distribution of labor income (flow)})$$

$$\alpha_{t,i} = K_{t,i}/Y_t \quad (\text{distribution of capital income (flow)})$$

$$Y_{t,i}/Y_t \quad (\text{distribution of all income (flow)})$$

$$K_{t,i}/K_t \quad (\text{distribution of capital (stock)})^{10}$$

⁹ E.g., $h_{t,i}(H_{t,i}) = \Omega_{t,i} \sqrt{H_{t,i}}$, and thus $(\Omega_{t,i} > 0) \rightarrow (h'_{t,i}(H_{t,i}) > 0) \wedge (h''_{t,i}(H_{t,i}) < 0)$.

¹⁰ Consider first the sets $\Theta(\cdot)$ where

$$\Theta(1 - \alpha_{t,i}) = \left\{ \langle x_t, y_t \rangle : (\exists n) \left((n \leq N) \wedge (x_t = n/N) \right) \wedge (\exists i) \left(y_t = (1 - \alpha_{t,i}) \right) \right\}$$

$$\Theta(\alpha_{t,i}) = \left\{ \langle x_t, y_t \rangle : (\exists n) \left((n \leq N) \wedge (x_t = n/N) \right) \wedge (\exists i) \left(y_t = \alpha_{t,i} \right) \right\}$$

$$\Theta(Y_{t,i}/Y_t) = \left\{ \langle x_t, y_t \rangle : (\exists n) \left((n \leq N) \wedge (x_t = n/N) \right) \wedge (\exists i) \left(y_t = (Y_{t,i}/Y_t) \right) \right\}$$

$$\Theta(K_{t,i}/K_t) = \left\{ \langle x_t, y_t \rangle : (\exists n) \left((n \leq N) \wedge (x_t = n/N) \right) \wedge (\exists i) \left(y_t = (K_{t,i}/K_t) \right) \right\}$$

In a (one unit by one unit) plot of a Θ -set, each n/N column on the abscissa contains the same set of points representing the magnitude on the ordinate of the shares of each individual i .

The Lorenz Curves are then (neglecting possible ties) respectively the (one by one unit) plots of Λ -sets, e.g., for $\Lambda(\alpha_t)$ and correspondingly (and saving ink) for $\Lambda(1 - \alpha_{t,i})$, $\Lambda(Y_{t,i}/Y_t)$, and $\Lambda(K_{t,i}/K_t)$:

Further investigations

This framework can be elaborated to investigate theoretically an enormous variety of questions about the development of the distribution of wealth and income in capitalist economies and indeed the evolution or demise of capitalist systems.

Bibliography

Acemoglu, Daron, and James A. Robinson, 2014, "The Rise and Decline of General Laws of Capitalism," Massachusetts Institute of Technology, Department of Economics, Working Paper 14-18.

Krusell, Per, and Anthony A. Smith, Jr., 2014, "Is Piketty's 'Second Law of Capitalism' Fundamental," NBER ms.

Milanovic, Branko, 2014, "The Return of 'Patrimonial Capitalism': A Review of Thomas Piketty's *Capitalism in the Twenty-First Century*," *Journal of Economic Literature*, LII: 2, pp. 519-534.

Piketty, Thomas, 2014, *Capitalism in the Twenty-First Century*, Belknap Press.

Stiglitz, Joseph, and Bruce Greenwald, 2014, *Creating a Learning Society: A New Approach to Growth, Development, and Social Progress*, Columbia University Press.

Thompson, Frank, 1995, "Technical Change, Accumulation and the Rate of Profit," *Review of Radical Political Economics*, 27:1, pp. 97-126.

$$\langle x_{t,n/N}, y_{t,n/N} \rangle \in \Lambda(\alpha_{t,i}) \Leftrightarrow \left(\left((0 \leq n \leq N) \rightarrow (x_{t,n/N} = n/N) \right) \wedge \left((\forall \alpha_{t,i})(\forall \alpha_{t,j}) \left(\left((\forall s) \left((0 \leq s < n/N) \rightarrow (\alpha_{t,i}, \alpha_{t,j} > (y_s - y_{s-1})) \right) \right) \rightarrow \left((\alpha_{t,i} \leq \alpha_{t,j}) \rightarrow (y_{t,n/N} = \alpha_{t,i}) \right) \right) \right) \right)$$