Quadratic Voting *

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Abstract

While the one-person-one-vote rule often leads to the tyranny of the majority, alternatives proposed by economists have been complex and fragile. By contrast, we argue that a simple mechanism, Quadratic Voting (QV), is robustly very efficient. Voters making a binary decision purchase votes from a clearinghouse paying the square of the number of votes purchased. If individuals take the chance of a marginal vote being pivotal as given, like a market price, QV is the unique pricing rule that is always efficient. In an independent private values environment, any type-symmetric Bayes-Nash equilibrium converges towards this efficient limiting outcome as the population grows large, with inefficiency decaying as $1/N$. We use approximate calculations, which match our theorems in this case, to illustrate the robustness of QV, in contrast to existing mechanisms. We discuss applications in both (near-term) commercial and (long-term) social contexts.

Keywords: social choice, public goods, large markets, costly voting, vote trading

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> (T)he "one man one vote" rule gives everyone minimum share in public decision-making, but it also sets...a maximum...it does not permit the citizens to register the widely different intensities with which they hold their respective political convictions and opinions.

– Albert O. Hirschman, *Shifting Involvements: Private Interest and Public Action*

> (T)he will of those whose qualifications, when both sides are added up, are the greatest, should prevail.

– Aristotle, *The Politics*, Book VI, Part III

1 Introduction

Prohibitions on gay marriage seem destined to be remembered as classic examples of the "tyranny of the majority" that has plagued democracy since the ancient world. While in many countries a(n increasingly narrow) majority of voters oppose the practice, the value it brings to those directly affected seems likely to be an order of magnitude larger than the costs accruing to those opposed, as we discuss in greater detail in Section 2. However, one-person-one-vote (1p1v) offers no opportunity to express intensity of preference, allowing such inefficient policies to persist. While most developed countries have institutions, such as independent judiciaries and log-rolling, that help protect minorities, these are often slow, insufficient and plagued with their own inefficiencies. To address these limitations, in this paper we analyze a simple and robust mechanism, *Quadratic Voting* (QV), that we believe may offer a practical paradigm for collective decision-making.[1]

The basic problem is that 1p1v rations rather than prices votes, resulting in externalities across individuals. This contrast with the market mechanisms for allocating private goods where individuals pay the opportunity cost of their purchases, leading to social efficiency by the classic arguments of Smith (1776). We therefore, in Section 3, consider a simple class of *costly voting* rules under which individuals can purchase any continuous number of votes they wish using a quasi-linear numeraire. To study such rules, we propose a "price-taking" model where individuals take as given the *vote-price of influence*, the number of votes it takes to have a unit of influence on the outcome. This *price* for short plays the same role in coordinating behavior that prices do in a standard market for private goods. Optimization given price-taking, together with a fixed total supply of influence, inspired by general statistical limits on influence as derived by

[1] To our knowledge this mechanism was first proposed, in its present form, by Weyl in an earlier version of this paper (circulated in February 2012) as "Quadratic Vote Buying".

Al-Najjar and Smorodinsky (2000), and an essentially arbitrary rule for returning funds thus raised, define a *price-taking equilibrium*.

We show that for any convex vote costs there is a unique equilibrium. Limiting cases yield familiar predictions: in (nearly) linear vote buying equilibrium approaches the dictatorship of the single individual with the most intense preference typically derived from linear vote buying models and as the cost becomes extremely convex 1p1v results. We use this concept in Section 4 to extend to discrete decisions Hylland and Zeckhauser (1980)'s argument that efficiency occurs in equilibrium for all value configurations if and only the pricing rule is quadratic.[2] This uniqueness contrasts with the complete information, game theoretic environment studied by Groves and Ledyard (1977a) where many rules are efficient (Maskin, 1999).

Individuals in this model have an assumed-linear value of acquiring "influence", a concept without clear micro-foundation in a non-stochastic price theoretic environment. In Section 5 we therefore study a canonical, quasi-linear independent private values model with a small aggregate noise that smooths payoffs in the limit as the population size grows large.

We prove that in any type-symmetric, monotone Bayes-Nash equilibrium, at least one of which exists, any social waste associated with equilibrium is eliminated as the population grows large. Except in the non-generic case when the mean of the value distribution $\mu = 0$, equilibrium takes a surprising form, where a vanishingly small tail of "extremists", from the side of the distribution opposite to its mean, purchase enough votes to win the election with high probability. Their existence, despite occurring only with probability $1/N$ when the value distribution has bounded support, is sufficient to provide other individuals the incentive to buy sufficient votes to deter extremists from being more active.[3] The constant factor on this decay is on the order of magnitude of unity in realistic cases so that in calibrations to the gay marriage example highlighted above we find inefficiency is a small fraction of a percent, a result confirmed by preliminary numerical simulations.

Previous efficient mechanisms proposed by economists, such as the Vickrey (1961)-Clarke (1971)-Groves (1973) (VCG) mechanism are very fragile to small changes in the environment such as allowing for collusion or aggregate uncertainty. As we discuss in detail in Section 7, this has led most market designers, including the VCG mechanism's originators (Vickrey, 1961; Groves and Ledyard, 1977b), to reject these mechanisms as impracticable. We thus dedicate Section 6 to studying the robustness of QV to a variety of changes in the environment using approximations and numerical methods that match our proven results in the cases where we

[2]Also, our quasi-linearity assumption allows us to extend their result to a single decision.

[3]While our theorem applies only to the case of bounded value distributions, approximate calculations which match these results in the cases where we were able to prove our results, extend straightforwardly to the case of unbounded and, especially interesting, fat-tailed value distributions. As we discuss in Subsection 5.7, the fatter are the tails, the slower is the decay of inefficiency, though in all cases where the mean of the distribution exists inefficiency does decay in the population size.

have proved them but can be extended more broadly with relative analytic brevity. Some of these variations actually improves QV's performance, while others, such as aggregate uncertainty and collusion harm it or even allow some limiting inefficiency. However, QV is nearly always close to perfectly efficient and much more robustly so than 1p1v, and those features that do create limiting inefficiency do so in ways similar to their effect on market mechanisms like the double auction. This is in sharp contrast to the mechanisms discussed above, which, among other issues, admit first-best (for the colluders) collusion in (at least approximate) equilibrium for any two individuals.

These analyses, and the success of initial laboratory (Goeree and Zhang, 2013) and field (Cárdenas et al., 2014) experiments with QV, persuaded us that QV may be relevant as a paradigm for improving practical decision-making, though we do not advocate its immediate and mechanical application on a broad scale. Weyl has therefore been pursuing, in collaboration with Eric Posner and other co-authors, several applications of QV, from the near-term commercial to the long-term social, that we briefly describe in Section 8. We conclude in Section 9 with a discussion of directions for future research, both that Weyl is conducting with various co-authors and that others might consider.

Except where explicitly noted otherwise, all extended calculations, proofs and computational techniques are grouped into appendices following the main text. Readers who are interested primarily in the core economic intuitions behind and applications of QV may wish to skip Sections 5-6, which are quite technical. On the other hand, readers primarily interested in the formal game theoretic results of the paper may wish to focus on these sections and in particular skip Sections 2-4.

2 Motivation

According to the Census, in 2010 lesbian, gay, bisexual and transgender (LGBT) voters constituted approximately 4% of the population of California and voters in same-sex couple households constitute approximately .7% of California's population. Given that, according to a survey by The Wedding Report, the average wedding alone costs more than $25,000 and LGBT couples are on average wealthier than non-LGBT couples[4], it seems reasonable to suppose that the benefit of marriage to the same-sex household couples is at least $100,000 per voter and given both the option value and dignity concerns that the option to marry is worth at least $20,000 to other LGBT voters.

Assuming that LGBT voters voted on California's 2008 Proposition 8, which banned gay marriage, at the same rates as other voters in the population and that all LGBT voters opposed the measure, the measure's passage by 52% to 48% implies that, among the 96% of non-LGBT

[4]See Experian Marketing's 2013 Lesbian, Gay, Bisexual, Transgender Demographic Report.

voters, the measure was supported 52% to 44%. Assuming that, on average, these not-directly-affected voters had on average the same willingness to pay (for ideological or ethical reasons) to see the initiative go their way and that this was no greater than $5000 on average (which seems quite high) the full-population average willingness-to-pay resulting from non-LGBT voters is $960 in favor of Proposition 8.

Thus, unless this calculation is significantly off, Proposition 8 seems a clear example of Pareto-inefficient tyranny of the majority. If a proposition involving appropriate transfers could have been arranged, it likely would have received overwhelming support. However, arranging such transfers to achieve Pareto-improvements is typically infeasible in large-scale political contexts both because of the incentives they create for rent-seeking (Coate and Morris, 1995) (viz. passing oppressive measures just to be paid off) and because of incomplete information (Mailath and Postelwaite, 1990) (viz. individuals claiming to oppose gay marriage just to receive a payment). Nonetheless, Proposition 8 seems likely to have not just been unjust in the sense many of have claimed but also inefficient in the standard utilitarian sense and a system that would consistently systematically avoid such tyranny of the majority would likely be Pareto-improving, or nearly so, by a substantial amount. A leading reason is that Proposition 8 is merely one salient example of problem of tyranny of the majority about which political theorists have been concerned at least since classical times, and individuals who are in the oppressive majority on one issue may well be in an oppressed minority on another issue. Other examples arise in nearly every walk of life, from the trivial to the epochal:

- A promising recruitment candidate with a relatively narrow focus is rejected because they cannot muster a majority in a diverse economics department.
- Latin American polities elect redistributive populist governments that wreck their economies and Middle Eastern polities elect divisive sectarian governments that lead to coups and civil wars.

The best-functioning organizations typically have mechanisms in place designed to make such inefficient outcomes less likely, such as log-rolling, favor-trading, lobbying, absolute protections of minority rights, etc. These checks and balances, however, are both often insufficient and carry with them inefficiencies in the form of governmental paralysis and corruption that are all too familiar (Posner and Weyl, Forthcoming). A practical formal mechanism that can, as the market economy does for private goods, facilitate efficient trade on collective decisions, is therefore badly needed.

3 Price-Taking Equilibrium of Costly Voting

As we discuss in Section 7, the mechanisms economists have previously proposed to address these issues strike most as complex and fitted tightly to the formal modeling environments that motivated them. Furthermore they have proved fragile in other analyses and have thus been widely dismissed as impractical. Like over-fitted statistical models, they appear to perform poorly out of sample.

We therefore take an alternative approach inspired by the literature on over-fitting (Vapnik and Chervonenkis, 1971): we consider a simple class of mechanisms that are as analogous as possible to the linear-pricing market mechanism studied by Smith (1776) and study them using a price-taking approximation. This allows us to uniquely identify a simple mechanism that we hope, as a result, will be more robust than those previously considered by economists. While placing such arbitrary restraints on the class of mechanisms considered may seem undesirable from the perspective of abstract logic, statistical learning theory (Blumer et al., 1987) suggests it increases the reliability of extrapolation based upon such an analysis. Similarly while our price-taking concept is somewhat ad-hoc, Section 5 shows that it is the limit of any equilibrium of a canonical game theoretic micro-foundation and Section 6 indicates that it provides broader intuitions beyond that canonical game theoretic context.

3.1 Model

Consider a finite collection N of individuals trying to make a binary collective decision about whether to maintain a shared status quo or to adopt an alternative A to this status quo. Each individual $i = 1, \ldots, N$ is characterized by a *value* $u_i \in \mathbb{R}$ describing her willingness to pay, out of a quasi-linear numeraire, to see the alternative adopted over the status quo; negative values represent the willingness to pay to maintain the status quo.

We study a class of *costly voting mechanisms*. Whether the alternative is implemented is determined by a vote in which each individual selects a scalar $v_i \in \mathbb{R}$ and the alternative is implemented if and only if $\sum_i v_i \geq 0$. Each individual pays a cost $c(v_i)$ where c is differentiable, convex, even and strictly monotone increasing in $|v_i|$ and receives a refund $r_i(v_{-i})$ such that $\sum_i c(v_i) = \sum_i r_i(v_{-i})$, where v_{-i} is the vector of votes by other individuals. We do not specify precisely which refund rule is used as it is irrelevant to the analysis that follows in this and the next two sections (given that it in no way depends on the individual's own choice), but a simple one obeying this budget balance condition is that each individual receives $\frac{\sum_{j \neq i} v_j}{N-1}$. Thus $t_i(v_i) = c(v_i) - r_i(v_{-i})$.

3.2 Definition of equilibrium

We begin by defining our equilibrium concept formally and then motivate it.

Definition 1. A *collective decision problem* is a three-tuple $\{N, S, u\}$ of a *number of individuals* $N \in \mathbb{Z}_{++}$, a *supply of influence* $S \in \mathbb{R}_{++}$ and an N-dimensional *value vector* $u \in \mathbb{R}^N$.

Definition 2. A *price-taking equilibrium* of a collective decision problem $\{N, S, u\}$ under costly voting rule c is an influence vector $I^\star \in \mathbb{R}^N$ and a *price* $p^\star \in \mathbb{R}_{++}$ such that

1. *Price-taking*: for each $i \in 1, \ldots, N$, $I_i^\star$ maximizes $2u_i I_i - c\left(pI_i\right)$ over all choices of $I_i \in \mathbb{R}$.
2. *Market clearing*: $\sum_{i=1}^N \left|I_i^\star\right| = S$.

The *equilibrium votes* corresponding to an equilibrium $\{I^\star, p^\star\}$ is the vector $v^\star$ whose ith entry is $v_i^\star = p^\star I_i^\star$.

Under our solution concept, individuals choose an amount of influence to exert over the decision, taking as given the linear price of this influence in terms of votes. The market clears if the total absolute value of influence acquired equals a pre-specified supply of influence S. The most natural micro-foundation for this concept is the chance that individuals perceive of their changing the outcome in their preferred direction. Section 5 develops this interpretation rigorously (but narrowly).

The chance of any vote influencing the outcome of the aggregate decision is likely to be small in large elections for reasons familiar to economists. As Mailath and Postelwaite (1990) show, only in very special, complete information environments is it possible to make a large number of individuals pivotal for a single binary decision each with a large probability. In fact, Al-Najjar and Smorodinsky (2000) prove strong upper bounds on the total influence exerted on average on the outcome. Extensive empirical evidence confirms these predictions (Mulligan and Hunter, 2003; Gelman et al., 2010). While the precise amount of influence "available" depends on the particular form and parameters of the information environment, the basic principles of it being in limited supply and thus it being small for almost all individuals is robust across all such environments (Gelman et al., 2002). This motivates our assumption that there is a limited total supply of influence.

Clearly this influence arises (only) from the possibility that the decision may be tied. Thus the price of influence in units of votes is the inverse of the chance (density with which) a tie occurs. We refer to this price as the *vote-price of influence* or simply *price* for short. Mueller (1973, 1977) and Laine (1977) argue that, in a somewhat different context, this price is insensitive to number of votes an individual purchases. This seems a reasonable extension of the previous intuition limiting the total size of influence, because it appears impossible for an individual with very little influence over the final decision to significantly impact the chance a tie occurs. But by Al-Najjar and Smorodinsky's arguments, it is impossible for a large number of individuals to have significant influence. This is the intuition behind our price-taking assumption.

3.3 Existence and uniqueness

As a prelude to our main analysis, we now consider two results, one technical and one substantive, that illustrate attractive properties of price-taking equilibrium.

Lemma 1. Given our assumptions, particularly the convexity of c and quasi-linearity, for any collective decision problem $\{N, S, u\}$ and any vote cost $c(\cdot)$ there exists a unique price-taking equilibrium.

Convex c ensures that the $I_i^\star$ satisfying price-taking is single-valued (and thus convex). Raising p lowers the left-hand side of the market-clearing condition, by standard comparative statics arguments (Milgrom and Shannon, 1994) and mechanically raises its right-hand side. Thus there may be at most one market-clearing price and each price corresponds to a unique influence vector. Finally, the influence vector that satisfies price-taking must approach the 0 vector as $p \to \infty$ and the absolute value of all its entries must approach infinity as $p \to 0$ by differentiability so there must be an equilibrium. Thus, at least at a technical level in this environment, price-taking equilibrium is a tractable solution concept.[5]

3.4 Limit cases as validation

Does the concept yield substantively reasonable conclusions? Consider the special case of convex power costs $c(v) = k|v|^x$, where $x > 1$. Then price taking requires that

$$\frac{u_i}{p} = \frac{kx}{2}\text{sign}\left(v_i^\star\right)\left|v_i^\star\right|^{x-1} \iff v_i^\star = \left(\frac{2}{kxp}\right)^{\frac{1}{x-1}} \text{sign}(u_i)\left|u_i\right|^{\frac{1}{x-1}},$$

where $v_i^\star \equiv \frac{I_i^\star}{p}$. Thus, in this class of mechanisms, the decision always favors whichever side has a greater value of $\sum_i |u_i|^{\frac{1}{x-1}}$. Two extreme cases yield particularly simple and familiar results.

For any fixed vector u and for all i, $\lim_{x\to\infty} |u_i|^{\frac{1}{x-1}} \to 1$. Thus, as the cost of voting becomes arbitrarily convex the decision is determined by which side has more individuals, that is 1p1v majority rule. A case yielding a less obvious conclusion is the limit as $x \to 1$ of linear voting

[5] Note the role that each of convexity and quasi-linearity play here. Absent convexity, equilibrium could easily fail to exist because it might be that individuals would, at each value of p, choose either a very large number of votes or a very small number of votes, potentially violating market clearing. Absent quasi-linearity, income effects of changes in p could result in multiple equilibria just as in classical general equilibrium theory.

costs. Let $i^\star$ be the index of the (generically unique) individual with the largest $|u_i|$. Note that for any $j \neq i^\star$

$$\lim_{x\to 1} \frac{|u_{i^\star}|^{\frac{1}{x-1}}}{|u_j|^{\frac{1}{x-1}}} = \infty \implies \lim_{x\to 1} \frac{|u_{i^\star}|^{\frac{1}{x-1}}}{\sum_{j\neq i^\star} |u_j|^{\frac{1}{x-1}}} \to \infty.$$

Thus as $x \to 1$ the outcome is generically the dictatorship of the single individual with the most intense preference.

This result is predicted by several recent studies of equilibria in linear vote buying models with new solution concepts or set-ups that resolve classic problems nonexistence in linear vote-buying models, arising from lack of convexity.[6] Casella et al. (2012) propose a notion of *ex-ante* equilibrium under which they, and Casella and Turban (2014), show that in every case they can study the single individual with the most intense preference wins with high probability, regardless of all other parameters. Dekel et al. (2008, 2009) and Dekel and Wolinsky (2012) find similar and in many cases identical results in a range of game theoretic vote buying models relating to public, legislative and corporate voting.

4 Efficiency under Price-Taking

Given that our solution concept is tractable and appears to pick out reasonable outcomes in at least in extreme cases, we now characterize the class of voting costs yielding efficient outcomes. However, we first define a notion of robust efficiency.

Definition 3. A voting rule $c(\cdot)$ is *robustly efficient* if, for all collective decision problems $\{N, S, u\}$, in the unique equilibrium $\text{sign}\left(\sum_i v_i^\star\right) = \text{sign}\left(\sum_i u_i\right)$.

4.1 The uniquely robust efficiency of Quadratic Voting

Theorem 1. $c(\cdot)$ is robustly efficient if and only if $c(v) = kv^2$ for some $k > 0$.

The proof formalizes the intuition that quadratic functions are the only ones with linear derivatives and thus the only ones where individuals equating their marginal utility to the *marginal cost* of a vote will buy votes in proportion to their utility. As Smith (1776) observed about the linear pricing of private goods, quadratic pricing leads a voter who intends only her own gain to be led by an invisible hand to promote an end that is no part of her intention.

[6]It is also consistent with a long tradition of informal argument about the impact of (linear) vote buying and lobbying on political outcomes (Olson, 1965) and formal results on the private provision of public goods with linear technology (Bergstrom et al., 1986).

4.2 Relationship to the literature

Our result is closely related to three from the literature. Groves and Ledyard (1977a) show that quadratic pricing can be used to achieve optimality in the provision of continuous public goods under complete information. However, under complete information, many other pricing schemes (Greenberg et al., 1977), many of them (Maskin, 1999) far more fragile than the quadratic mechanism, achieve optimality. This led much of the literature to consider such schemes generally unattractive (Bailey, 1994).

A closer result, therefore, is that in unpublished and publicly unavailable work by Hylland and Zeckhauser (1980).[7] Hylland and Zeckhauser show, in a Walrasian model analogous to ours where individuals take the price of influence as constant, that quadratic pricing of continuous public goods using artificial currency is the unique pricing rule that achieves an analog to the First and Second Fundamental Welfare Theorems. We extend their analysis to the discrete decision case through our notion of price-taking equilibrium for discrete decisions and consider a case with only a single choice through a partial-equilibrium quasi-linear utility model. They also do not consider the convergence of non-cooperative, game theoretic equilibrium of their mechanism to efficiency in large populations as we shortly take up.[8]

Finally, Goeree and Zhang (2013), in work that was circulated after the first draft of this paper. consider the large population limit of the expected externality mechanism of Arrow (1979) and d'Aspremont and Gérard-Varet (1979) for a binary, quasilinear collective decision like ours. In the case when the mean of the distribution of values is 0, this limit is payments that are approximately quadratic, so that a quadratic pricing mechanism (with a particular value of k) gives approximate incentives for truthful value revelation. Little of our game theoretic analysis below is concerned with the mean 0 independent private values case in which their mechanism is defined and we discuss the relationship between the mechanisms in detail in Subsection 7.3 below. However, it was also this connection to the expected externality mechanism that led us to consider the quadratic form.

Furthermore their logic connects our result to an observation of Vickrey (1961) highlighted by Tideman (1983). While Vickrey's studies how taxes might be used to offset the incentives of an oligopolist to manipulate market prices. Unlike the competitive profits that an oligopolist makes, which are proportional to her quantity, this incentive is typically proportional to the *square* of her quantity, as both the amount by which price moves and the number of units impacted by the price change are proportional to the oligopolist's production. Thus the necessary offsetting "counter-speculation" tax is proportional to the square firm's quantity. In the competitive limit,

[7]This was recently revived by Benjamin et al. (2013) and Chung and Duggan (2014) who, like us until after the first version of this paper was published, were unaware of Hylland and Zeckhauser's previous work.

[8]See our conclusion for a discussion of Weyl's on-going work with Hylland and Zeckhauser to update and publish their original work.

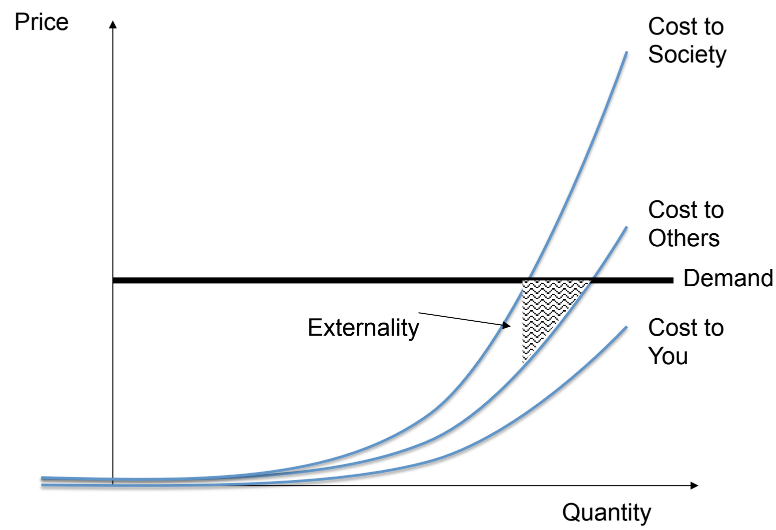

Figure 1: The quadratic growth of the deadweight loss triangle in an individual's influence as an analogy for the optimality of quadratic pricing of votes.

where each firm is a small part of the market, the quadratic term vanishes, leading to competitive behavior (Satterthwaite and Williams, 1989). On the other hand, in the limit of public goods, where individuals are only concerned with the aggregate outcome rather their own production, the linear term vanishes and the quadratic term becomes the leading term for small individuals.

This quadratic approximation argument is illustrated in Figure 1, where we consider the classic problem of choosing the level of consumption of a good causing a negative externality. Each individual reports her schedule of harms and the optimal level of the externality is determined by equating demand to the vertical sum of the harm schedules. The Vickrey payment is the externality on other individuals of a given individual's report, which is the area between private demand and the cost curve for all other individuals between the quantity that would prevail absent an individual's report and the quantity that prevails with this report. Note that this is a deadweight loss triangle and, as such, grows quadratically in the change in quantity induced by the individual's report as shown in the figure.

5 Convergence from Independent Private Values

While the analysis above provides a useful baseline, it is incompletely satisfying in the same way that general equilibrium theory is, as it assumes individuals take as given a price that their actions clearly impact. It is therefore useful to investigate whether there is a coherent non-cooperative game-theoretic model in which individual equilibrium behavior converges in large populations to that suggested by the price-taking model. In the private goods context, this micro-foundation was provided by Cournot (1838) and, more recently for a more plausible description of market functioning, by Satterthwaite and Williams (1989), Rustichini et al. (1994) and Cripps and Swinkels (2006).

Beyond the results that we can formally prove we have several results that we have derived from detailed calculations but we have not established rigorously thus far. If we believe we see clearly how to prove these results but have simply not written the proof down we refer to the results as "claims"; if we believe they are true but do not clearly see how to prove them formally we refer to them as "conjectures".

5.1 Model

We consider an environment of symmetric, independent private values, analogous to the most canonical models of double auctions (Satterthwaite and Williams, 1989). Some of these assumptions are relaxed in Section 6.

There are N voters $i = 1, \ldots, N$. Each voter is characterized by a value, u_i; these are drawn independently and identically from a smooth, atomless distribution $F = F_U$ supported by a finite interval $[\underline{u}, \overline{u}]$, with associated density $f = f_U$. We denote by μ, σ^2 and μ_3 respectively the mean, variance and raw third moment of u under F, and we assume that f is bounded away from 0 on $[\underline{u}, \overline{u}]$.[9] Each individual knows her own value, but knows nothing about those of the other agents except that they were obtained by random sampling from F_U.

Each individual buys v_i votes, where $v_i \in \mathbb{R}$, and earns utility

$$u_i \Psi(V) - v_i^2 + \frac{1}{N-1} \sum_{j \neq i} v_j^2, \tag{1}$$

where $V \equiv \sum_{i=1}^{N} v_i$. The payoff function $\Psi : \mathbb{R} \to [-1, 1]$ is C^∞ and odd, and its derivative $\psi = \Psi'$ is twice an even probability density with compact support $[-\delta, \delta]$ that is strictly positive on $(-\delta, \delta)$.[10] In addition, we shall assume throughout that Ψ satisfies the following *steepness hypothesis*.

Assumption 1. There exist a unique pair (α, w) of real numbers such that $-\delta \leq w < \delta < \alpha$ and

$$\begin{aligned} (1 - \Psi(w)) \, |\underline{u}| &= (\alpha - w)^2 \quad \text{and} \\ (1 - \Psi(w')) \, |\underline{u}| &\leq (\alpha - w')^2 \quad \text{for all } w' \neq w, \end{aligned} \tag{2}$$

and that

$$2 + \psi'(w) \, |\underline{u}| > 0. \tag{3}$$

[9] In Subsection 5.7 we discuss the case of distributions with unbounded support.

[10] Thus, $\Psi = 1$ on $[\delta, \infty)$ and $\Psi = -1$ on $(-\infty, \delta]$, and Ψ is strictly increasing on $[-\delta, \delta]$.

The hypotheses (2) and (3) are discussed further in Appendix B. Roughly, they guarantee that the payoff function Ψ is sufficiently "steep" that an agent with value $\underline{u}$ would find it worthwhile to "buy" the election if she knew that the sum of the other agents' votes were sufficiently close to δ. For sufficiently small δ, hypotheses (2) and (3) hold generically.

The function Ψ can be interpreted as representing some exogenous uncertainty around close elections, arising from judicially supervised recount procedures such as those during the 2000 United States Presidential election.[11] The hypotheses on Ψ are also useful for technical purposes: we have been unable to establish our results for natural payoff functions with discontinuities, such as $\Psi = \mathbf{1}_{(0,\infty)} - \mathbf{1}_{(-\infty,0]}$, although we conjecture that they hold. However, our results hold for arbitrarily small δ.

The last term in Expression 1 represents a balancing transfer in which each voter receives $\frac{1}{N-1}$ of the revenues paid in by all other voters and none of the revenue collected directly from him; however, none of the results below depend on this particular redistributive scheme.[12]

Remark 1. QV is budget balanced as

$$\sum_i \left(v_i^2 - \frac{1}{N-1} \sum_{j \neq i} v_j^2 \right) = \sum_i v_i^2 - \frac{N-1}{N-1} \sum_j v_j^2 = 0.$$

Voters are expected wealth maximizers and thus voter i chooses her v_i to maximize

$$\mathbb{E}_i \left[u_i \Psi \left(V_{-i} + v_i \right) \right] - v_i^2,$$

where $V_{-i} \equiv \sum_{j \neq i} v_i$.

We define the (expected) *welfare* achieved as $W \equiv \frac{\mathbb{E}[U\Psi(V)]}{2\mathbb{E}[|U|]} + \frac{1}{2}$. Note that welfare is always between 0 and 1, assuming $U \equiv \sum_i u_i$ is not (identically) 0; 0 arises from (always) making the wrong decision and 1 arises from (always) making the right decision. Expected inefficiency $EI \equiv 1 - W$.

Much of what follows concerns limiting behavior and as such we make heavy use of standard Bachmann-Landau notation: for functions h and g of N, $h \in O(g)$ denotes that h is eventually (in N) no greater than αg for some constant α, $h \in o(g)$ denotes that h is eventually strictly less than αg for any α, $h \in \Omega(g)$ if $g \in O(h)$ and $h \in \Theta(g)$ denotes that $h \in O(g)$ and $h \in \Omega(h)$.

5.2 Existence of equilibria

Lemma 2. For any $N > 1$ there exists a type-symmetric Bayes-Nash Equilibrium v that is monotone increasing.

[11] Ψ may also be interpreted as representing the possibility of only-partial victories in sufficiently tight elections.

[12] Any rule in which all revenues are returned and each voter receives the same share of the revenues she herself pays suffices to establish essentially all results that follow.

This result follows directly from Reny (2011)'s Theorem 4.5 for symmetric games.[13] Now consider the optimal behavior of an individual in such an equilibrium, given that the other individuals use the Bayes-Nash strategy $v(\cdot)$.[14] The expected utility of an individual with value u who buys votes v is $\mathbb{E}\left[u\Psi\left(v+V_{N-1}\right)\right]-v^2$, where V_{N-1} is the sum of $N-1$ independent draws of $v(u)$, where u has distribution F. We denote the distribution of V_{N-1} by G and (if applicable) its density by g. By smoothness of Ψ, maximization of expected utility implies the necessary condition

$$u\mathbb{E}\left[\psi\left(v+V_{N-1}\right)\right]=2v \implies v=\frac{1}{p(v)}u,$$

where $p(v) \equiv \frac{2}{\mathbb{E}[\psi(v+V_{N-1})]}$ is the price perceived by an individual. The following subsections will be devoted to establishing a sense in which p is approximately constant for individuals when N is large and therefore establishing that our results from the previous section hold approximately for large N.

5.3 Efficiency

The following theorem is our main rigorous result about the efficiency of quadratic voting. It will follow from the explicit characterizations of Bayes-Nash equilibria discussed in the following sections.

Theorem 2. For a given sampling distribution F_U and payoff function Ψ satisfying the hypotheses specified above, there exist constants $\alpha_N > 0$ satisfying $\lim_{N\to\infty}\alpha_N = 0$ such that for any type-symmetric Bayes-Nash equilibrium, EI is bounded above by α_N.

We conjecture that $\alpha_N = w/(N-1)$ for some constant w, and we give explicit expressions for w in terms of F in the next two subsections. These depend on whether $\mu = 0$ or $\mu \neq 0$.

5.4 Characterization of equilibrium in the zero mean case

The structure of a type-symmetric Bayes-Nash equilibrium differs radically depending on whether $\mu = 0$ or $\mu \neq 0$. Although non-generic, the case $\mu = 0$ corresponds most closely to the simplest intuition for why p is approximately constant in the limit. Furthermore, it may arise frequently in the equilibrium of a broader political game where candidates or candidate initiatives converge toward efficiency (Ledyard, 1984).

[13] All of Reny's conditions can easily be checked, so we highlight only the less obvious ones. Continuity of payoffs in actions follows from the smoothed payoffs imposed through Ψ. Type-conditional utility is only bounded from above, not below, but boundedness from below can easily be restored by simply deleting for each value type u votes of magnitude greater $\sqrt{|u|}$. The existence of a monotone best-response follows from the clear super-modularity of payoffs.

[14] We conjecture that these results are true of asymmetric equilibria as well, given that the "smallness" of each agent rules out significant asymmetries. However, we have not proved this.

Theorem 3. For any sampling distribution $F = F_U$ with mean $\mu = 0$ that satisfies the hypotheses of section 5.1 there exist constants $\epsilon_N \to 0$ such that in any type-symmetric Bayes-Nash equilibrium, $v(u)$ is C^∞ on $[\underline{u}, \overline{u}]$ and satisfies the following approximate proportionality rule:

$$\left| \frac{v(u)}{u} - \frac{1}{2p_N} \right| \leq \frac{\epsilon_N}{p_N} \quad \text{where} \quad p_N = \frac{\sqrt{\sigma}\sqrt[4]{\pi(N-1)}}{\sqrt[4]{2}}. \tag{4}$$

Furthermore, there exist constants $\alpha_N, \beta_N \to 0$ such that in any equilibrium the vote total $V = V_N$ and expected inefficiency satisfy

$$|EV| \leq \alpha_N \sqrt{\text{var}(V)} \quad \text{and} \quad EI < \beta_N. \tag{5}$$

Thus, in any equilibrium agents buy votes approximately in proportion to their values u_i, which corresponds to their behavior under price-taking, as described in the previous section. The Central Limit Theorem suggests that the distribution G of the vote total is approximately normal with standard deviation roughly proportional to $\sqrt{N}/p_N$. For large N this is of the order $\sqrt[4]{N}$, and so the density g (which exists by the smoothness assertion of the theorem) should be relatively flat. Thus, all individuals perceive approximately equal densities of pivotality.

We conjecture, based on heuristic arguments in Subappendix D.1, that the inefficiency of QV decays at a rate $1/N$ and with constant $\frac{\mu_3^2}{16\sigma^6}$. To see the size of this constant, consider the following calibrated example, which we draw from the case of gay marriage discussed in Section 2.

Suppose population groups have the same average willingness-to-pay on a gay marriage referendum as we described in Section 2 and that in each group there is a uniform distribution of willingness to pay between \$0 and twice the average willingness to pay. However, suppose the proportions supporting and opposing gay marriage among heterosexual individuals are such that aggregate willingness to pay has mean 0 (approximately 64% of heterosexual individuals oppose the practices). Then the value of the constant is approximately 4.5. Thus in a community of 101 individuals, inefficiency is 4.5%, all resulting from gay marriage being too frequently defeated as it would be with near-certainty in democracy. In a city of $100,001$ it is a negligible .0045%.

5.5 Characterization of equilibrium in the non-zero mean case

When μ is not zero but the nature of equilibrium is quite different: for sufficiently large N, any type-symmetric Bayes-Nash equilibrium has a massive jump discontinuity in the extreme tail of the sampling distribution.

Theorem 4. Assume that the sampling distribution $F = F_U$ has mean $\mu > 0$ and satisfies the hypotheses of section 5.1,and that the payoff function satisfies Assumption 1. Then for any

$\epsilon > 0$, if N is sufficiently large then for any Bayes-Nash equilibrium $v(u)$,

(i) $v(u)$ has a single discontinuity at u_*, where $|u_* + |\underline{u}| - \gamma n^{-2}| < \epsilon n^{-2}$;

(ii) $|v(u) + \sqrt{|\underline{u}|}| < \epsilon$ for $u \in [\underline{u}, u_*)$; and

(iii) $P\{|V - \alpha| > \epsilon\} < \epsilon$, where $\alpha = \gamma\psi(w) f_U(\underline{u})$.

Here $\gamma > 0$ is a constant that depends only on the distribution F_U and $n \equiv N - 1$.

Thus, an agent with value u will buy approximately $CN^{-1}u$ votes, where C is a constant $C > 0$ depending on the sampling distribution F_U, *unless* u is in the extreme lower tail of F_U, in which case the agent will buy approximately $\alpha - w \approx -\sqrt{|\underline{u}|}$ votes, enough to single-handedly win the election. Agents of the first type will be called *moderates*, and agents of the second kind *extremists* for short. Because the tail region in which extremists reside has F_U-probability on the order N^{-2}, the sample of agents will contain an extremist with probability only on the order N^{-1}, and will contain two or more extremists with probability on the order N^{-2}. Given that the sample contains no extremists, the conditional probability that $|V - \alpha| > \epsilon$ is $O(e^{-\varrho n})$ for some $\varrho > 0$, by standard large deviations estimates, and so the event that $V < 0$ essentially coincides with the event that the sample contains an extremist. Thus, we have the following corollary.

Corollary 2. Under the hypotheses of Theorem 4, expected inefficiency is of order $O(N^{-1})$.

In fact, heuristic calculations (discussed below) suggest that the expected inefficiency is bounded by $|\underline{u}|/(\mu(N-1))$. Again we can calibrate these constants to our running example of gay marriage. Consider the figures from Section 2, but with the assumption of the last subsection of uniform distributions of willingness to pay within group. Then $|\underline{u}| = \$10,000$ and $\mu = \$960$. This gives a constant of approximately 10 and thus ballpark similar quantitative results on inefficiency to those from the previous subsection (a bit more than twice as large).[15]

Why does equilibrium take this somewhat counter-intuitive form in this case? For an agent i with value u_i in the "bulk" of the sampling distribution F_U, there is very little information about the vote total V in the agent's value u_i, and so for most such agents the pivotal constant $E\psi(V_{-i} + v(u_i))$ will be approximately $E\psi(V)$. Thus, in the bulk of the distribution the function $v(u)$ will be approximately linear in u. Therefore, by the law of large numbers, the vote total will, with high probability, be near $E\psi(V)\mu$. Since $\mu > 0$, agents with negative values will, with high probability, be on the losing side of the election. However, if $E\psi(V)$ were too small – small enough that $E\psi(V)\mu$ is near δ – then agents with values near $\underline{u}$ would find it worthwhile to steal the election; thus, $E\psi(V)$ cannot be too small, else extremists would occur with sufficiently high probability to significantly erode the expected utility of agents with positive values. On the other hand, $E\psi(V)$ cannot be so large that it would no longer be worthwhile for any agent to try

[15] Here again all this inefficiency arises by inefficiency favoring the majority. However, in this case this is not a general property and is instead an artifact of assuming the minority is right. If the majority were right then the inefficiency would arise from a chance of going against the majority and thus 1p1v would converge much faster (exponentially in that case) towards efficiency than does QV.

to buy the election, because then, for an agent with positive value, the probability of losing the election would be so small (in fact, exponentially small in N) that she would have incentive to "defect".

Elaboration of this argument leads to our conjectured value of the expected inefficiency. Monotonicity of Bayes-Nash equilibria implies that if a voter with some value u is an extremist, then so is any other voter with a value lower than u; thus, the extremist regime must be an interval $[\underline{u}, u_*]$. For large N, the probability that the sample contains an extremist must be vanishingly small, because otherwise moderates with positive values u_i, faced with the strong possibility that an extremist will steal the election, would have an incentive to buy a small number of additional votes to thwart the extremist. Hence, the extremist interval $[\underline{u}, u_*]$ must be vanishingly small as $N \to \infty$, and so $v(u_*-)$ must be approximately $-\sqrt{|\underline{u}|}$. In order that a voter with value $u_* \approx \underline{u}$ be indifferent to the extremist and moderate strategies, the total cost of the votes needed to successfully steal an election must be approximately $2\,|\underline{u}|$; consequently, the average number of votes per person among non-extremists must be approximately $\sqrt{2\,|\underline{u}|}/(N-1)$. If moderates act essentially as price-takers (later we will prove rigorously that they must) then the perceived price among moderates must be approximately $\frac{\mu(N-1)}{2\sqrt{2|\underline{u}|}}$. Let q be the probability an extremist exists in the full population; then the price perceived by an extremist is $q\frac{\mu(N-1)}{2\sqrt{2|\underline{u}|}}$, because an extremist knows that she exists while moderates believe this only occurs with probability q. Therefore, the necessary condition $2v(\underline{u}) = E\psi(V + v(\underline{u}))\underline{u}$ and the requirement that $v(u_*-) \approx -\sqrt{|\underline{u}|}$ yield the identity

$$|\underline{u}| \approx 2\sqrt{2\,|\underline{u}|}q\frac{\mu(N-1)}{2\sqrt{2\,|\underline{u}|}} \implies q \approx \frac{|\underline{u}|}{\mu(N-1)}.$$

5.6 Proof sketch

The proofs of Theorems 3 and 4, which are given in full in Appendix B, involve intricate technical arguments requiring somewhat arcane results from the classical limit theory of random sums. Most of the difficulties arise from the fact that *a priori* we know almost nothing about the distribution G of the vote total, and so information about the vote function $v(u)$ must be obtained by bootstrapping of successively more precise characterizations of this function. Following is a brief discussion of some of the main points in the argument. The necessary condition $2v(u) = E\psi(V + v(u))u$ plays a central role throughout.

1. *Strict monotonicity*: First we establish (Lemma 5) the relatively straightforward result that any type-symmetric equilibrium strategies must be *strictly* monotone. This then implies that any equilibrium strategy is almost continuous except at perhaps countably many points, which in turn implies that any type-symmetric Bayes-Nash equilibrium almost surely coincides with a pure-strategy Bayes-Nash equilibrium (Lemma 4).

2. *Weak consensus*: We then consider voters in the "bulk" of the value distribution. We show (Lemma 6) that for any such voter the conditional distribution (given the voter's value u) of the vote total nearly coincides with the unconditional distribution, and use this to deduce from the necessary condition $2v(u) = E\psi(V+v(u))u$ that the ratio $v(u)/u$ is bounded above and below by positive constants independent of N and of the particular equilibrium.

3. *Concentration*: Because $2v(u) = E\psi(V + v(u))u$, a voter with value u will only purchase a large number $|v(u)|$ of votes if the distribution of V is highly concentrated in an interval of width 2δ. Using the weak consensus estimates on $v(u)/u$, we use concentration inequalities for sums of i.i.d. random variables to show (Lemma 7) that when N is large either $|v(u)|$ is bounded above by $C/\sqrt{N}$, for some C, or no voter (including extremists) will buy more than a vanishingly small number of votes. We then deduce from this that the F_U-probability of the extremist regions cannot be larger than $N^{-3/2}$ for large N.

4. *Discontinuity size and smoothness*: Next we show (Lemma 11), using the necessary condition and another concentration argument, that the size of a discontinuity in the equilibrium vote function $v(u)$ be bounded below. In view of the concentration results discussed above, this implies that discontinuities can only occur at values u within distance $O(N^{-3/2})$ of one of the endpoints $\underline{u}, \overline{u}$. By differentiating in the necessary condition (using the smoothness of $\psi = \Psi'$) we then obtain a first-order differential equation for $v(u)$ that has no singularities in the region where v is continuous; this implies that v is smooth up to within distance $O(N^{-3/2})$ of one of the endpoints $\underline{u}, \overline{u}$.

5. *Approximate proportionality*: The various results on the size of the vote function $v(u)$ in the bulk of the distribution, the size and location of discontinuities, and concentration of the vote total are then brought to bear on the necessary condition $2v(u) = E\psi(V + v(u))u$ to prove the *approximate proportionality* rule for Bayes-Nash equilibria (Lemma 6): in the bulk of the value distribution (up to within distance $O(N^{-3/2})$ of one of the endpoints), the ratio $v(u)/u$ must be almost constant, with relative error converging to zero uniformly as $N \to \infty$. Furthermore, if v has no discontinuities then the approximate proportionality rule extends all the way to the endpoints $\underline{u}, \overline{u}$. Given the approximate proportionality of the vote function $v(u)$, concentration inequalities and uniform versions of the Central Limit Theorem (the Berry-Esseen Theorem) can then be deduced for the vote total.

6. *Non-zero mean case*: The analysis of the case $\mu \neq 0$ now proceeds largely along the lines of the heuristic arguments given earlier. First, we rule out discontinuities near the upper endpoint $\overline{u}$ by arguing that, unless the sample contains an extremist with value near $\underline{u}$, an event of vanishingly small probability, the conditional probability that the sum of the moderate votes will exceed δ is exponentially close to 1, so voters with values near $\overline{u}$ need

not buy more than $O(N^{-1})$ votes to ensure that their side wins the election. Second, we show that a discontinuity near the lower bound of the value distribution must exist, and must occur at a point u_* within distance $O(N^{-2})$ of $\underline{u}$. This follows by the rough argument outlined earlier. The fact that $v(u)$ can have no small discontinuities (Lemma 11) ensures that there will only be one discontinuity near $\underline{u}$. Finally, we use the indifference condition for an extremist at the discontinuity point to tie down the number of votes that extremists buy.

7. *Zero-mean case*: This case is technically more difficult than the non-zero-mean case, and requires more delicate tools, notably the *Edgeworth expansion* for the density of a sum of i.i.d. random variables; this is why our result on the rate in this case is still conjectural. The difficulty arises because the approximate proportionality rule as formulated in Proposition 6 is too weak to rule out the possibility that the sum of errors in the approximate proportionality rule might be of larger order of magnitude than the standard deviation of the vote total. Thus, a sharper analysis is needed; this is based on a Taylor expansion $2v(u) = E\psi(V)u + E\psi'(V + \tilde{v}(u))v(u)u$ of the necessary condition. The Edgeworth approximation is used to obtain sharp approximations to the expectations $E\psi(V)$ and $E\psi'(V + \tilde{v}(u))$. This analysis is complicated by the fact that the Edgeworth approximations involve the mean EV of the vote total, which is, *a priori*, unknown; thus, a bootstrapping argument is needed to deduce that EV is of smaller order of magnitude than the standard deviation of V. Given this, it then follows that the lead term in the Edgeworth series (the usual normal approximation) dominates, and the results claimed in Theorem 3 then follow.

5.7 Approximate calculations for unbounded distributions

The rigorous arguments sketched in the previous subsection applies only to the case of bounded distributions. However, we strongly believe they can be extended more broadly, as it is relatively straightforward to extend our approximate calculations to these cases. To illustrate this, we now extend these calculations to the most interesting case not covered above, and probably the most realistic case overall, namely when the (unbounded) value distribution has Pareto tails and μ (which exists as the Pareto tails have coefficient of at least unity) is greater than 0.

We now assume that the distribution of values is smooth and has support on the full real line. We further assume that μ exists and is positive and that f has a Pareto lower tail in the sense that $\lim_{u \to -\infty} F(u) = \frac{k}{(-u)^\alpha}$ for some constants $k > 0, \alpha > 1$; whether or not the distribution has a Pareto upper tail is irrelevant to our analysis so we take no stand on this.

Claim 1. Suppose that F has $\mu > 0$ and a Pareto lower tail with tail coefficient $\alpha > 1$, then there exists an $\underline{N}$ such that for $N > \underline{N}$ in any type-symmetric Bayes-Nash equilibrium

$$v(u) = \begin{cases} \frac{1}{2\tilde{p}_N}\left[1 + O\left(\frac{1}{N}\right)\right] u & \text{if } u \geq -\left[\frac{(N-1)\mu}{2\tilde{p}_N}\right]^2 [1 + o(1)] \\ -\frac{(N-1)\mu}{2\tilde{p}_N}\left[1 + O\left(\frac{1}{\sqrt{N}}\right)\right] & \text{if } u < -\left(\frac{(N-1)\mu}{2\tilde{p}_N}\right)^2 [1 + o(1)] \end{cases}$$

where $\tilde{p}_N \equiv {}^1\!/_2 \left(\frac{1}{k} + \frac{1}{k\alpha}\right)^{\frac{1}{2(\alpha+1)}} (N-1)^{\frac{\alpha}{\alpha+1}} \mu^{\frac{1+2\alpha}{2(\alpha+1)}}$. This implies that EI is bounded above by $\frac{k^{\frac{1}{1+\alpha}}\left(\frac{1}{\mu}+\frac{1}{\mu\alpha}\right)^{\frac{\alpha}{1+\alpha}}}{(N-1)^{\frac{\alpha-1}{\alpha+1}}}$.

Note this rate converges toward that in Corollary 2 as $\alpha \to \infty$ (that is, as the tail of the Pareto distribution becomes increasingly thin). However, for smaller α (fatter tails), convergence is slower. For example, when $\alpha = 3$, a realistic estimate of the Pareto tail of the United States income distribution (Saez, 2001), inefficiency only decays at a ${}^1\!/\!\sqrt{N}$ rate. Intuitively, the fatter tails are the larger moderate votes must grow to deter extremists, but this larger number of votes is only justified if there is a larger chance of an extremist existing. Despite this slower decay, however, in calibrated examples inefficiency is still very small.

To see this let us return to our gay marriage example, using the numbers from Subsection 5.5. However, instead of assuming a uniform distribution of willingness to pay among gay marriage opponents, let us suppose that they have a distribution of willingness to pay that is a scaling down of (to have a mean of \$5000) of a double Pareto lognormal approximation to the United States income distribution. In particular we scale down the calibration of Fabinger and Weyl (2014), which has an (upper) Pareto tail coefficient of 3.

This gives $k \approx 5.7 \cdot 10^{10}$ and thus

$$k^{\frac{1}{1+\alpha}}\left(\frac{1}{\mu} + \frac{1}{\mu\alpha}\right)^{\frac{\alpha}{1+\alpha}} \approx \sqrt[4]{5.7 \cdot 10^{10}}\left(\frac{4}{3 \cdot 960}\right)^{\frac{3}{4}} \approx 3.5.$$

This leads to significantly more inefficiency than in the bounded distribution cases. With 101 individuals, there is still 35% inefficiency and it takes a small town of 10,001 to reach single digit percentage points. However, it is still the case that in any reasonable-sized polity in which a question such as gay marriage would be decided on, inefficiency is very small. With 1,000,001 individuals, inefficiency is down to a third of a percent and with the population of California in 2008 when Proposition 8 passed (just over 36 million) it would be $.058\%$. This compares, of course, to near complete inefficiency of 1p1v in this example.

5.8 Numerical results

We now consider computational solutions for equilibrium with moderately large populations to examine both the quantitative reliability of our asymptotic results and the validity of our partly

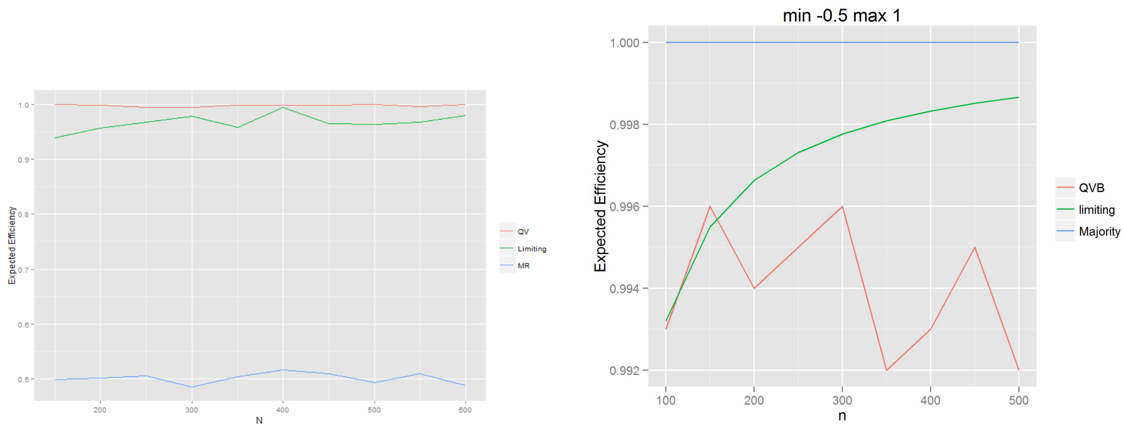

Figure 2: Expected efficiency as a function of the number of voters in our numerical analysis in extreme cases. The left shows values drawn from a correlated product of a double Pareto lognormal distribution and a normal distribution. The right shows results from a uniform distribution on the interval $[-.5, 1]$.

conjectural results.[16] We describe our numerical methods in detail in Subappendix E.1; here we only provide a brief summary. We used a grid-based, iterative approach beginning at an initialization corresponding to the approximate equilibrium suggested by our limiting results. We allowed for a single discontinuity in the vote function at each round, calculating this point using a binary search routine, and on either side of this discontinuity considered a grid of values on which we computed optimal votes. We used local linear interpolation between these grid points to generate a vote function given our continuum of valuations. We then represented the distribution of the sum of votes as a mixture of normal distributions, with the parameters of the normal set to match a Monte Carlo simulation of the moments of the distribution of the sum of $N-1$ votes among moderates. This normal was then "mixed" by the possibility of a single extremist exist, continuously spread out across the possible votes purchased by extremists. The resulting distribution of $N-1$ votes was then used to compute best responses and the discontinuity point. Movements of all quantities were limited in relative terms to ensure convergence by avoiding over-shooting.

While we have been unable to prove the convergence of this scheme, fluctuations generally dampened; we are still working to achieve high precision, but we only report results that had been converging for several thousand iterations. There also appears to be an error in our current computation of welfare that systematically overstates inefficiency (it counts only correct decisions, not the magnitudes, dampening QV's tendency to get things right when the value is largest and only get it wrong when it is small). We will shortly correct these errors.

[16] These numerical techniques can also be used to examine the behavior of QV in environments where our bounds are entirely uninformative (when N is very small), a direction we take in Subsection 6.5 below.

We did this for a wide range of distributions of values and many examples are presented in Appendix E. Here we discuss just the two examples at the extreme range of what we found. The results for these are shown in Figure 2. The left panel shows an example where values are the product of a double Pareto lognormal (dPln) component calibrated to the United States income distribution by Fabinger and Weyl (2014) and a standard normal component. These were linked by a standard Gaussian copula and correlated $.72$, which results in a positive mean and 0 median. The right panel shows a uniform distribution on the interval $[-.5, 1]$. For each case we calculate the expected efficiency of QV, 1p1v and the our limiting prediction of QV's efficiency. Even the latter has some variability across samples for the dPln cases as the parameters of the resulting distribution had to be calculated numerically. Results were calculated for N between 100 and 500 (left) or 600 (right).

Results on the left represent the "best case" performance of QV relative to our limiting results. QV typically does much better than predicted by limiting analysis which, as we discussed in the previous subsection, predicts significant inefficiency even in fairly large populations with a large tail. Inefficiency is consistently below 1% despite the limiting prediction being several percent. On the other hand 1p1v has an inefficiency close to 50% for all values of N. The results on the right represent the "worst case" performance of QV. In this case the median is on the same side as the mean so for this fairly large sample 1p1v is perfectly efficient. Furthermore QV appears not to be converging very rapidly towards perfect efficiency over this range, though its inefficiency is always less than 1%. Furthermore, over this range our limiting analysis is a weak predictor. This was also the case where the convergence of our numerical methods was noisiest, so we trust our results least. More broadly, in almost all cases, the limiting results gave good or overly conservative guidance about the results we obtained.

6 Robustness

While QV is unique in the class of mechanisms we discussed in Section 3 above, there are many other efficient mechanisms, as we discuss in the next section, that do not fall into this class. The fundamental virtue that therefore recommends QV to us is not its efficiency in the narrow settings we focus on above but rather the robustness of its efficiency to a wide range of changes in the economic environment. We have analyzed the mechanism in a range of environments with varying degrees of formalism to understand this robustness. In this section we report the key conclusions of our analysis in what we consider to be the five most important robustness checks. Essentially all formalism is left for Appendix F.[17]

[17]We have not rigorously proved almost any of the results in this section, given the evident difficulty of such proofs and the space requirements for reporting such analysis. Most results are thus claims or conjectures. Within the claims, results related to the $\mu = 0$ are closer to being conjectures than the rest of the result, as they depend (but

6.1 Collusion and other manipulations

A severe limitations of existing efficient mechanisms is sensitivity to collusion. An important question therefore is whether QV is more robust to such coordination.

First note that an efficacious collusive group maximizing the joint payoffs of its members will ask all members to purchase the same number of votes.[18] Second note is that a collusive group of size M will buy M times as many votes per unit of aggregate utility as its individual members would buy if they had such a utility. To see this suppose that every individual in the group had the same utility. Each would create positive externalities on the others for each vote she purchased of the same magnitude of the value she obtains from a vote. Collusion internalizes these externalities, magnifying the optimal amount the group would vote. Only this second effect impacts efficiency; the first only reduces revenues.

This sort of collusion is most harmful when it involves a large number with large values pointing in the same direction. The worst case scenario is thus that the M individuals involved in collusion are the M most extreme individuals whose values point in the same direction, on either side of the distribution when $\mu = 0$ and in the negative direction when $\mu > 0$.[19] Perhaps this case is also close to the one that would be most plausible in practice. It also, if undetected, clearly can wreak havoc for QV even with a very small number of members when $\mu > 0$ and the distribution has bounded support, because the chance that two individuals together have values in total beyond the extremist threshold is typically quite high even when the chance that any individual is beyond this threshold is small. Thus, at least in this case, QV might seem as sensitive to collusion as other efficient mechanisms.

However, we claim and illustrate in Subappendix F.1 that several forces limit the possibility of collusion even in this worst-case scenario. When $\mu = 0$, unilateral incentives for deviating from a collusive group are the strongest limit; they grow dramatically to the extent that the collusive group becomes large and ambitious because this requires larger departures from the unilaterally optimal actions and payoffs are roughly quadratic in large populations. While such incentives do not limit the possibility of collusion as strongly when $\mu \neq 0$ and the collusive group engages in a conspiracy to act as extremists, reactions by other players to the possibility of an extremist conspiracy, which raises their chances of being pivotal, do. In particular, if the rest of the population believes there is any non-vanishing chance of such a conspiracy, the group of conspirators (conditional on their existing) must have size on order of $N^{2/5}$ (when $\alpha = 3$). This

should be fairly straightforward to derive) on our conjectural characterization of the decay rates of EI in the $\mu = 0$ case stated in the previous section.

[18]The reason is that the utility of the group depends only on the aggregate number of votes it buys and the aggregate payments it makes. Conditional on the first, the second is minimized when all individuals split evenly the aggregate votes because the quadratic function is convex.

[19]In a previous analysis we considered also an average case with randomly drawn individuals. This case seems fairly unrealistic, however, and our results are strictly better in this case so we did not consider it to be worth including.

requires roughly a thousand conspirators to succeed in a state like California if the population is aware of the possibility that such a conspiracy may form.

On the one hand, this resembles the forces of unilateral deviation and encouragement of entry that limit collusion under the standard market mechanism (Stigler, 1964). On the other hand, it contrasts sharply with VCG and EE, where collusive agreements are (at least approximate) equilibria and give no significant incentives either for other players to change their strategies nor for agents to unilaterally deviate from the collusive agreement. A similar analysis shows that an individual would have to fraudulently pass herself off as a large number of individuals as N grows large to significantly decrease the efficiency of QV, unlike in other efficient mechanism (see Subsection 7.2-**??** below).

6.2 Aggregate uncertainty

Our baseline model assumes that the distribution of values is commonly known and thus that, except in the knife-edge $\mu = 0$ case, there is no uncertainty about the optimal action. This both seems unrealistic and makes the informational problem somewhat trivial in large populations.[20] It thus seems natural to consider how QV performs when the distribution of valuations is uncertain among voters. Unlike with collusion, this case has a different mathematical structure than our baseline analysis. However, by analogy to similar models of costly 1p1v we are able to make a detailed set of conjectures in some special cases.

In particular we study a case when values are drawn iid conditional on a scalar, γ, which orders the value distributions according to strict first-order stochastic dominance. We focus on the case when there is some value of γ in the support of the distribution over it such that $\mathbb{E}\left[u|\gamma\right] < 0$ and another such that $\mathbb{E}\left[u|\gamma\right] > 0$. Analyses of costly 1p1v (Krishna and Morgan, 2012; Myatt, 2012) in this case suggest that equilibrium will have the structure that there exists a unique threshold $\gamma^{\star}$ such that for γ above this threshold the alternative is implemented with probability near 1 in large populations and for γ below this threshold the status quo is maintained with high probability in large populations. All pivotal events thus occur when γ is very close to $\gamma^{\star}$ and welfare is by how close $\mathbb{E}\left[u|\gamma^{\star}\right]$ is to 0.

This characterization yields simple equations for analyzing the outcome of any particular parametric set-up. While we have not yet derived any general principles from these equations, one broad intuition emerges from the examples we have studied, what Myatt (2012) labels the "Bayesian Underdog effect". If the unconditional value of $\mathbb{E}\left[u\right] > 0$, it will often be the case

[20] However, as McLean and Postelwaite (Forthcoming) argue, it may be the existence of an efficient mechanism given aggregate certainty that provides correct incentives for individuals to reveal their information to the group and thus create this aggregate certainty. Thus aggregate certainty may be the appropriate framework for analysis of a robust mechanism like QV even if it would admit other, non-robust mechanisms as described in Subsection 7.4 below.

that those with $u < 0$ will expect a tie to be more likely than do those with $u > 0$. Intuitively, in the 2012 election Romney supporters believed the election was more likely to be tied than did Obama supporters as each updated prevailing polls with the additional information of their own personal views. This led Obama supporters to believe a bit more in a large blow-out and Romney supports to believe a bit more in a tie.

Given that raising the chance of a tie lowers the price, this would lead supporters of the underdog to vote more than those of the favorite. This in turn biases the vote in favor of the underdog, leading to some inefficiency. In every example we have studied where $\mathbb{E}[u] \neq 0$, this effect emerges and leads to some limiting inefficiency in QV. However, this inefficiency is never very large because the very logic that brings it about prevents it from being: if the favorite becomes the underdog because of large inefficiency the bias would reverse.

We quantify this intuition with specific examples and find that, over all of our examples, the largest inefficiency of QV is 5%, while 1p1v typically has quite high inefficiency, approaching 1 in some cases and usually above 10%. However, when the conditional distribution of u given γ is symmetry, 1p1v is always fully efficient but QV is not; when the distributions of γ and u given γ are both normal, for example, QV reaches 2.2% inefficiency in the worst case. In a calibrated gay marriage example, QV has 4% inefficiency and 1p1v 47%. In all cases as the aggregate uncertainty vanishes full efficiency is achieved, an interesting general result to shoot for in the future.

6.3 Voter behavior

If voting has any cost, standard instrumental models predict very few individuals should vote (Downs, 1957), contrary to empirical observations. In this subsection we consider how QV would perform if individuals behaved according to models capable of explaining observed turnout. Many such models, including as voting to tell others that one has voted (DellaVigna et al., 2014), would imply that behavior conditional on voting will follow standard rational choice.[21] Thus it may be that our preceding analysis is an accurate prediction of voting behavior in QV, given that we assumed universal turnout.

However, we consider the two models discussed in a survey on the paradox of voting by Blais (2000) that we could clearly determine would lead to different behavior in QV even conditional on voting. In the first model individuals overestimate the chance of their being pivotal. In the second, individuals gain some direct, "expressive" utility for each of their votes in addition to a chance of changing the outcome.

In these models the extent to which voters over-estimate their chance of being pivotal and

[21] In fact, Kawai and Watanabe (2013) and Spenkuch (2014) find empirically that, conditional on voting at all, voters *do* behave quite strategically.

the proportion between their instrumental value and their expressive value are (noisily) independent of their values. The only systematic deviation usually discussed in the literature is that those with small chances of being pivotal may over-estimate their chances of being pivotal by more than those who are more likely to be pivotal. To the extent this is true, these behaviors only increase the effective extent of proportionality between votes and values, at least on average.

They may cause some inefficiency in the $\mu = 0$ case by adding noise that is not of vanishing importance in this knife-edge case. However, as long as the noise is relatively small compared to the value heterogeneity, it does not substantially lessen efficiency. Furthermore, QV will typically outperform 1p1v, which entirely neglects any cardinal signals from values. In our calibrated mean 0 examples, noise in voting would have to have a standard deviation of many times its mean before 1p1v would out-perform QV. This is similar to the intuition that, in a market economy, misoptimization must be quite large relative to preference heterogeneity for rationing to be more efficiency to market allocation. Furthermore when $\mu \neq 0$ these motives will typically increase the rate of convergence towards efficiency, sometimes even making it exponential, by eliminating the need for extremists to prop up beliefs about a tie having some reasonable probability. The limited laboratory (Goeree and Zhang, 2013) and field (Cárdenas et al., 2014) experimentation on QV thus far indicate these forces play an important role in the efficiency of QV in practice. Thus we do not think that non-instrumental or irrational voter behavior is a significant threat to, and may even enhance, QV's efficiency.

6.4 Common values

In our analysis above we assumed that elections served to aggregate preferences. Another role of voting to aggregate information, even when preferences are known to be aligned across individuals. There are two distinct types of such common values settings. The first is perfect common values. Because interests are aligned, voting is then essential just a form of communication as McLennan (1998) shows. There QV, like any mechanism that allows the reporting of a continuous variable, should allow more information aggregation than will 1p1v. While there are some subtle issues about the nature of equilibrium given that any scaling down of reports yields the same decision rule without added noise, we are hopeful we can show that in a fairly broad range of cases where individuals receive a continuous signal, QV will lead to the best possible decision in equilibrium with high probability, while 1p1v will throw information away.

The second type of common values setting proposed by Feddersen and Pesendorfer (1996, 1997) features individuals who differ both in their information about a common value component and their preferences over a private value component. Feddersen and Pesendorfer (1997) show that 1p1v aggregates information in large populations if there are arbitrarily unbiased, but still well-informed, individuals: these individuals who vote on the basis of information rather than preference. We believe QV can achieve information aggregation in a wider range of cases, and more quickly in the realistic case where those that are nearly indifferent on preferences also have worse information, because information is continuously incorporated into the quantity of votes purchased by all individuals rather than just the direction of the vote of those who are nearly indifferent. We presently have one example illustrating this and are working to generalize that example.

6.5 Small populations

The efficiency bounds we provide for finite populations are very weak for small N. To investigate the small N performance of QV (relative to 1p1v), we extended our numerical approach from Subsection 5.8 above to the case of small populations by replacing our normal approximation with non-parametric log-spline interpolations. We considered normal and Pearson Type I distributions. We considered very small populations, with N ranging from 2 to 10.

We summarize our qualitative findings here and discuss the quantitative results in Subappendix E.2. The worst relative performs of QV comes in cases with $N = 10$ when the value distributions have small variance, a mean and median of these same sign, and N on the order of 10 as the "majority is right" in these cases and 1p1v appears to converge to large population behavior more quickly. The worst case we found was one in which QV had an EI of 15% compared to 7.5% for 1p1v. Deviations in any directions from these conditions led QV to outperform 1p1v. For very small N, such as $2 - 4$ QV did significantly better than (half to one-third the EI of) 1p1v by incorporating cardinal information.

Regardless of N, whenever the distribution had mean 0 or a variance much larger than its mean, QV did significantly better, usually an order of magnitude smaller than 1p1v. Goeree and Zhang (2013) find related results in a laboratory experiment using a variant on QV discussed further in Subsection 7.3 below. QV outperformed 1p1v most starkly when the mean and the median of the distribution had opposite signs, for obvious reasons.

The above results are similar to the results we obtained under aggregate uncertainty in Subsection 6.2 above. QV appears much more robustly efficient than majority rule and outperforms it in most cases. However, in cases where there is a strong prior reason to believe that the majority is right, majority rule appears to perform better in both contexts.

7 Relationship to Other Mechanisms

These conclusions about the robustness of QV appear to be confirmed by the first laboratory (Goeree and Zhang, 2013) and field (Cárdenas et al., 2014) experiments on QV. We now compare these conclusions to the properties of other mechanisms economists have proposed; these comparisons are summarized in Table 1.

7.1 One-person-one-vote

Because most of this paper compares QV to majority rule 1p1v, we only cover here a few points not covered above. 1p1v is somewhat more robust to collusion than is QV along some dimensions (a colluding group of extremists may accomplish less unless it is even larger than needed under QV), but along others it is less robust (incentives for unilateral deviation are smaller and reactions by those outside the colluding group are non-existent).

Ledyard and Palfrey (1994) and Ledyard and Palfrey (2002) show that in large populations if the distribution of valuations is known then by choosing the threshold for voting equal to the quantile corresponding to its mean, the limit-efficiency of voting may be restored. This mechanism suffers from the same limitations of the mechanisms, discussed in Subsections 7.3 and 7.4 below, that require the planner to know the distribution of valuations. For any fixed super- or sub-majority rule, voting typically performs worse than it would under a simple majority rule; see Posner and Weyl (Forthcoming) for a detailed discussion.

7.2 The Vickrey-Clarke-Groves mechanism

The most canonical mechanism that has been suggested by economists as an alternative to 1p1v is the Vickrey (1961)-Clarke (1971)-Groves (1973) (VCG) mechanism. In its application to binary collective decisions (Tideman and Tullock, 1976; Green et al., 1976), individuals report their cardinal value for the alternative and the decision is chosen based on the sign of the sum of the reports. Any individual who is pivotal in the sense that, had she reported 0, the decision would have gone the other way pays the amount by which all those other than her preferred the decision she opposed. In addition to sharing with QV its detail-freeness, this system is appealing because if every individual plays the weakly-dominant strategy of reporting her valuation truthfully then, in an extremely broad range of circumstances, VCG implements the efficient outcome. VCG is fully efficient (in this equilibrium) in a sense that is very robust to the information structure and the number of participating individuals, unlike QV.

Despite this, somewhat narrow sense of robustness, the VCG mechanism has almost never been used for collective choices. The reason that VCG is "lovely, but lonely" (Ausubel and Milgrom, 2005) is that a number of other severe failures of robustness make it "not practical" (Rothkopf, 2007). These flaws were recognized by the originators of the mechanism from the start (Vickrey, 1961; Groves and Ledyard, 1977b), though their severity and implications for implementing VCG were not well-understood until more recently (Tideman and Tullock, 1977) when the first laboratory experiments based on the simplest Tideman and Tullock environments gave disastrous results (Attiyeh et al., 2000).

	Detail-dependence	Standard efficiency	Aggregate uncertainty	Strategic complexity	Collusion and de-mergers	Budget	Other problems
QV	Detail-free	Limit-efficient	Limit-efficient with large polls or if $\mathbb{E}[u] = 0$; inefficiency small otherwise	Non-direct	Moderately robust	Balanced	
1p1v	Detail-free	Limit-efficient when mean and median have same definite sign, typically inefficient	Typically limit-inefficient	Strategy-proof	Moderately robust	Transfer-free	Inefficient with common values
VCG	Detail-free	Efficient	Efficient	Strategy-proof	Extremely sensitive	Surplus	Bankruptcy, uncertain payments
Expected Externality	Detail-based, requires aggregate certainty	Efficient	Ill-defined	Direct but Bayesian	Highly sensitive	Balanced	Ill-defined with common values

Table 1: Comparison of binary collective choice mechanisms

Perhaps the most severe defect of the VCG is that under complete information, in addition to its efficient equilibria, VCG has a very large number of other equilibria, including, for any two individuals, equilibria where they attain their desired outcome and make no payments. In particular, any two individuals may announce sufficiently large values in the same direction so that neither is individually pivotal. Anticipating this, other individuals can do no better than to report 0 (or her true value). Similarly, any individual who can pretend to be two individuals can "break" the mechanism. Even with incomplete information, if value reports are unbounded, for any two individuals this strategy can be implemented in an ϵ equilibrium for any $\epsilon > 0$ as if each conspirator reports a sufficiently extreme value the chances of the other conspirator being pivotal become negligible.[22]

Among the other practical problems with VCG are that any revenue raised must be destroyed to avoid creating perverse incentives, which may be hard for the government to commit to; absent such a commitment, the scheme falls apart.[23] Even when such commitments are possible, the revenue that must be destroyed can often be greater than the improvement in efficiency over even simple mechanisms like 1p1v (Groves and Ledyard, 1977b; Attiyeh et al., 2000; Drexl and Kleiner, 2013). VCG requires much larger liquidity among participants than does QV; individuals must have in cash the full magnitude of their value and place this into escrow when submitting their report. Unless this "bankruptcy" problem is addressed, individuals have an incentive to exaggerate their report and fall back on judgement-proofness if called upon to pay (which occurs with very small probability in a large population). Under QV payments are limited with probability 1 to a very small portion of underlying values and are certain conditional on the report made.[24]

7.3 Expected Externality mechanism

The next-most canonical mechanism economists have suggested is the Expected Externality (EE) mechanism of Arrow (1979) and d'Aspremont and Gérard-Varet (1979), which was first applied to binary collective decisions by Goeree and Li (2008). This is similar to VCG, except that individuals pay the planner's ex-ante expectation of their VCG payments rather than their actual payments. Because individuals can thus not affect others' payments, the revenue raised may

[22] While limiting the report space can help reduce this risk, it either must sacrifice some efficiency to do so (especially in the fat tail case) by reducing its sensitivity to large values or remain open to collusion by slightly larger groups. Furthermore such ad-hoc fixes make the mechanism ever-more complex and eliminate most of its small-sample advantages over QV.

[23] Some have suggested schemes to get around this problem in large populations. See, for example, the work of (Bailey, 1997). However these schemes also eliminate the perfect small population efficiency of VCG. Perhaps more importantly, these variants take an already complex and fragile system and make it more complex and more fragile along other dimensions.

[24] It seems likely, though we have not tried to show it, that this would lead to strict dominance of QV when agents are risk-averse.

be refunded much as under QV, though, like QV, the mechanism is Bayesian rather than having a dominant-strategy equilibrium. QV was partly inspired by this mechanism as, in the case when $\mu = 0$, these EE payments are approximately quadratic in large populations. The intuition is closely related to our discussion of Tideman (1983)'s interpretation of Vickrey (1961) in Subsection 4.2.

However, this is at most a starting point for QV, as when $\mu \neq 0$ EE payments are nothing like quadratic and in the richer information environments we consider they are not even well-defined. However, Goeree and Zhang (2013) take this logic more literally and propose a mechanism, limited to the case when $\mu = 0$, in which individuals pay the exact quadratic approximation to their EE payments. This quadratic schedule has a coefficient on it that depends on the number of individuals and the standard deviation of their value and thus, like EE, depends on the planner knowing the distribution of valuations. Thus both the EE and Goeree and Zhang mechanisms are not detail-free and only apply under aggregate certainty (and the later also requires $\mu = 0$). Additionally, when $\mu \neq 0$, the EE mechanism suffers from essentially the same collusion problem as VCG in large populations, though this is less well-known.[25]

7.4 Other mechanisms proposed by economists

Many other, even more fragile, mechanisms have been proposed by economists and we briefly discuss a few:

1. *Implement the alternative if and only if $\mu > 0$*: In addition to requiring the planner to know the distribution of valuations, this suggestion places great and easily-abused power in the hands of the authority charged with determining the sign of μ (Maskin, 1999).

2. *Maskin (1999)'s mechanism*: In this mechanism, all individuals are asked to report μ and the alternative is implemented if and only if all report $\mu > 0$. If all do not agree, all agents are "killed" (pay a very large and inefficient penalty). In addition to requiring aggregate certainty, this mechanism is likely to be difficult to commit to and has a very large multiplicity of inefficient equilibria.[26]

3. *Crémer and McLean (1988)-McAfee and Reny (1992)-style mechanisms:* Roughly, individuals are asked (via their report of their type) to guess other individuals' report of their type and

[25]Suppose two individuals report $-\frac{2\mu}{3}$. Each will make vanishingly small EE payments as the probability of either of these reports being pivotal is exponentially small in N by standard large deviation theory results. However, together they will ensure the outcome goes in the inefficient direction.

[26]Other mechanisms requiring individuals to have complete information have been proposed (Hurwicz, 1977; Walker, 1981), but are widely viewed as highly fragile (Bailey, 1994). Another mechanism proposed by Thompson (1966) rely very sensitively on the absence of heterogeneity in risk attitudes and beliefs and on detailed knowledge by the planner.

are given large rewards for guessing correctly. Like the Maskin mechanism, this mechanism has a large multiplicity of equilibria (McLean and Postelwaite, Forthcoming), depends both on the mechanism designer having a very precise knowledge of the distribution of types and on individuals having preferences that are "appropriately" correlated with their beliefs about other individuals' types (Heifetz and Neeman, 2006).

All of these mechanisms are also quite complicated to explain and strain credibility along a variety of other practical dimensions. While there is not the space here to discuss all in detail, a large literature has established their impracticability.[27] An important reason, we believe, for their limitations, and for those of the VCG and EE mechanisms, is the "over-fitting" we discussed in Section 3.

7.5 Mechanisms used in practice

Existing informal institutions may be more efficient than formal 1p1v is. The fact that voting may be costly is one such institution (Ledyard, 1984; Myerson, 2000; Krishna and Morgan, Forthcoming). Ledyard considers a model where individuals have non-pecuniary costs of voting. If these costs are all strictly positive and follow a non-atomic distribution independent of voters' values, then the "representative voter" with a given value effectively faces a quadratic cost as a function of the fraction of cost types she represents in the limit as the population size grows large and thus both the density of pivotality and thus turn out grow very small.[28] Thus non-pecuniary costs of voting may approximate QV and thus efficiency.

Ledyard argues that this is unlikely to provide a good approximation to reality as it requires turnout to approach zero in large populations, which is rarely observed in practice. If some voters have zero or negative non-pecuniary costs of voting or overestimate their chance of being pivotal to such an extent that turnout remains large in large elections, as much empirical research on voting suggests (Blais, 2000), the result clearly fails, though voluntary voting can somewhat mitigate some of the inefficiency of compulsory voting (Borgers, 2004). On the other hand, we showed in Subsection 6.3 that QV remains as efficient or even may be more efficient than if voters are "standard". Thus costly voting does not seem a promising approximation to QV in practice.[29]

More comprehensive solutions to the inefficiency of 1p1v are genuinely costly activities undertaken to influence collective decisions, such as log-rolling on committees and legislative bodies and lobbying, campaigning or out-right illegal vote-buying in elections. At least in some

[27] See Tideman (2006) for a detailed and excellent survey from which much of the discussion here is derived.

[28] The argument is same argument underlying the analogy to deadweight loss triangle in Subsection 4.2.

[29] At a more technical level, note that costly voting is is a wasteful means of vote pricing compared with QV, because the costs are not refunded back.

contexts, costs may be somewhat convex. In votes on committees it is usually easy for any individual to influence one vote of a colleague, harder to obtain the second, even harder to sway the third and so forth.[30] Exactly how close various contexts come to approximating QV is an interesting empirical question, which we discuss further in Section 9.

8 Applications

What seems clear, however, is that none of the institutions discussed in the previous subsection approximate QV precisely. Similarly we believe that, to whatever extent some informal practical institutions resemble QV, its formalization is has the potential to significantly improve welfare.[31]

Therefore, in this section we discuss several contexts where we believe it could improve the allocation of public goods compared with existing mechanisms. We begin with the most modest applications that are likely to be feasible over the shortest time horizons and gradually build to those that, if implemented, could have the greatest impact on social welfare. Weyl and Eric Posner (see below) have founded, with Kevin Slavin, a start-up venture, Collective Decision Engines, that is designing software to ease implementation in these applications.

8.1 Private sector

The easiest, near-term application is likely that to the decision-making of committees that repeatedly interact such as recruitment committees in academic departments or honors-granting committees. Such groups would likely use artificial currency in lieu of actual money, which would allow Pareto improvements over majority rule internal to the decision process but would not permit global Kaldor-Hicks efficiency (Budish, 2011). We have had detailed discussions with a number of such committees and expect implementations in this setting to begin in several locations by year's end. Somewhat more ambitious are applications to surveying consumers for market research. This is the initial focus of Collective Decision Engines and initial engagements are underway.[32]

More ambitious is the use of QV for the governance of public corporations, as advocated by Posner and Weyl (2014). Posner and Weyl propose an alternative implementation, *Square Root Voting* (*SRV*), in which investors obtain votes in proxy battles equal to the square root of the number of shares they own. Such an implementation is different more in design than in type

[30]Similar effects may occur in campaigning, where it is typically much easier for individuals to influence their close friends or at least those they are acquainted with than to influence those further away from them.

[31]The situation might be compared to the institutions of informal trading barter and reciprocity that are thought to have preceded more formalized exchange economies and to persist in less developed societies (Sahlins, 1972). Most believe that the emergence of formalized market economies allowed exchange to occur more efficiently, between larger numbers of individuals, over a broader temporal and spatial extent and at lower cost (Greif, 1994).

[32]Given the proprietary nature of this work, we do not discuss it further here.

from existing minority shareholder protections, such as poison pills, and thus has attracted some attention on Wall Street. Similar discussions are underway about the use of QV for corporate restructuring (Posner and Weyl, 2013).

8.2 Public sector

As important as such private applications are, the most important public goods are provided by various levels of the public sector. Probably the most plausible applications in the short term are ones where money already plays an important role, such as in the assembly of complementary goods that would otherwise be subject to holdout by coercive means (Kominers and Weyl, 2012b). Kominers and Weyl (2012a) advocate allowing any potential purchaser of a large number of complementary goods (such as land, spectrum, patents, etc.) to make an offer for the package of goods and having the current owners decide by QV whether to accept the offer, thus avoiding holdout. They show this procedure has many advantages in terms of ex-post efficiency, property right protection and fairness over traditional "eminent domain" procedures.

The next most modest such applications would be decisions about local public goods and amenities. State, local or national referenda, as we used to motivate the paper, are another promising, though much more ambitious, application. At least as important but perhaps somewhat more plausible in the medium term is the use of QV to formalize log-rolling in representative bodies. Rather than inefficient expenditures on pork-barrel projects being used to buy votes close to linearly, representatives could directly use funds from their district to (at quadratic cost) purchase influence on national legislation. Weyl, in collaboration with Hylland and Zeckhauser, plans to show that if representatives act to maximize the interests of those in their district, as they should if elected by QV, then the resulting decisions in a QV-governed representative body will be efficient as well.

Likely the most important application of such a representative property of QV would be governance of international organizations, such as the European Union, the United Nations, the International Monetary Fund and the World Trade Organization. Such organizations have long been plagued by an inability to make decisions that respect national sovereignty when a particular nation's vital interests are at stake while simultaneously allowing efficient decisions on issues that primarily impact great powers (Posner and Sykes, 2014). This has led to great disappointment about the capacity for large-scale international cooperation (Mearsheimer, 1994–1995). QV could offer a method for overcoming these stumbling blocks.

Of course prejudices in democratic culture and theory against the use of money will make such implementations an uphill battle and many difficult philosophical issues are involved. Posner and Weyl (Forthcoming) discuss these issues in detail and argue that they may not impede the adoption of QV in the longer-term for several reasons. First, the public may really oppose linear vote-buying (for its dictatorial equilibrium) rather than QV. Second, existing institutions allow domination of politics by the wealthy (**?**) far beyond what QV would enable (given that dollars only buy their square-root in influence) and QV might actually dampen the ability to "buy votes" through advertising by making voting on whim costly. Third, theory only predicts that the wealthy would have greater influence over social, not economic, decisions under QV and thus may actually benefit the poor economically.[33] Finally, if such concerns remained strong, versions of QV using artificial currency spread over many decisions could be developed, though they would offer smaller efficiency gains.

9 Conclusion

Economists have typically been skeptical of the possibility of public decisions being taken as efficiently as private goods are allocated, as reflected in the formal results of Arrow (1951), Gibbard (1973) and Satterthwaite (1975) and in informal attitudes in work such as Friedman (1962). In this paper we have argued that this attitude may be an artifact of particular institutions. Public goods do not pose a fundamentally harder mechanism design problem than that posed by private goods. In fact, the mechanism we propose has a number of formal symmetries with market mechanisms for the allocation of private goods, such as the double auction.[34]

This symmetry suggests both applied and theoretical directions for future research. On the applied side, experimental work on QV would be highly complementary with the practical implementations of QV we discussed in Section 8, helping to both shape practical application and providing a more realistic setting for testing than might be available in lab experiments such as those considered by Goeree and Zhang (2013). The first such experiments we are aware of have been conducted by Cárdenas et al. (2014) and by Collective Decision Engines. This parallels the way in which mechanisms for the allocation of private goods have co-evolved with laboratory and field testing of those mechanisms (Roth, 2002; Milgrom, 2004).

Another standard analytic approach in the allocation of private goods is to study how well markets conform to the idealized conditions under which the welfare theorems apply; this is much of the focus of the field of industrial organization. A natural analog is to consider empiri-

[33]Another potential issue is "repugnance", but we see no reason in the long-term that all public goods transactions should be repugnant, while so few private goods transactions are.

[34]It is a Bayesian, separable, budget balanced that is simple to explain rather than a VCG mechanism which is dominant strategy, generally not budget balanced and opaque to most participants. It is perfectly efficient only with large numbers of voters, but out-performs natural alternatives when there are a small number of voters. Collusion is possible and potentially profitable, but requires a significant fraction of the market to participate in order to be viable, generates unilateral incentives for deviation and will be exploited to the harm of the colluders by other market participants if discovered. Equilibrium is summarized by a small-dimensional price-like object (the density of pivotality and the threshold for extreme behavior) and it is the constancy of these across voters that induces efficiency.In fact the symmetries between competitive markets and QV extend further when one considers issues, such as incentives for information acquisition, that are beyond the scope of our analysis here.

cally how well practical institutions, such as lobbying, log-rolling and other informal institutions discussed in Subsection 7.5 correspond to an efficient quadratic rule mapping costly expenditures to influence on the decision and thus how efficient the allocation of public goods is in practice in various settings.

On the theoretical side, a more formal statement of the sense in which the Arrow, Gibbard and Satterthwaite results may be misleading about the distinctions between private and public goods would be useful. In particular, building off of Hylland and Zeckhauser (1980), Weyl plans a collaboration with Hylland and Zeckhauser to show that the welfare theorems apply to an economy where public goods are allocated by QV and that if strategy-proofness is relaxed in precisely the same way that makes the double auction to be "approximately strategy-proof" in large markets the welfare theorems can be approximated using QV in finite populations. They also plan to explore variations on QV that expand the range of cases where it can be applied.

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Appendix

A Price-Taking

Proof of Lemma 1. Let the $I(p;u)$ be the unique vector of price-taking influences voters acquire when the price is p. By convexity and differentiability of c this is defined for any value of p uniquely by the first-order condition

$$I_i(p;u) = \frac{\gamma\left(\frac{u_i}{p}\right)}{p} \tag{6}$$

in the notation of the proof of Theorem 1. γ is strictly monotone increasing by strict convexity of c, has the same sign as I_i by the fact that c is even and increasing in the absolute value of its argument and is continuous by differentiability of c. $|I_i(p;u)|$, and thus $|I(p;u)|$, is thus strictly monotone decreasing and continuous in p. Thus $D(p) \equiv \sum_{i=1}^{N} |I_i(p;u)|$ is strictly monotone decreasing in p and continuous. By definition, any equilibrium must have $D(p) = S$; thus there may be at most one equilibrium.

Furthermore note that as $p \to \infty$, $I_i(p;u) \to 0$ for all i because $\gamma\left(\frac{u_i}{p}\right)$ is strictly declining in p. By the same logic, $I_i(p;u) \to \pm\infty$ as $p \to 0$. Thus $\lim_{p\to 0} D(p) = \infty$ and $\lim_{p\to\infty} D(p) = 0$. Thus by continuity and the intermediate value theorem there is some p for which $D(p) = S$. This p and $I(p;u)$ constitute an equilibrium as Equation 6 fully characterizes the price-taking condition by our arguments above. □

Proof of Theorem 1. First consider the "if" direction. By price-taking and our analysis in Subsection 3.4, at any price-taking equilibrium we must have

$$v_i^\star = \frac{1}{kp}\text{sign}\left(u_i\right)|u_i| = \frac{u_i}{kp}.$$

Thus

$$\text{sign}\left(\sum_i v_i^\star\right) = \text{sign}\left(\sum_i \frac{u_i}{kp}\right) = \text{sign}\left(\sum_i u_i\right),$$

because $k, p > 0$.

For the "only if" direction, by differentiability the general necessary first-order condition for price-taking is that

$$2u_i = pc'\left(pI_i\right).$$

By convexity and differentiability, c' is invertible and its inverse is continuous; denote this inverse $\gamma(\cdot)$. Thus at an equilibrium with price p,

$$v_i^\star = \gamma\left(\frac{2u_i}{p}\right). \tag{7}$$

By the proof of Lemma 1 (and by the argument in the text above), for any number of individuals N and value vector u, adjusting S can lead to any desired value of p. Thus we can, without loss of generality with respect to considering robust efficiency, assume that $p = 2$ as long as we do not adjust S in the rest of the proof. Thus Equation 7 becomes $v_i^\star = \gamma(u_i)$. The only homogeneous of degree one functions of a single variable are linear, so either γ is linear or it is not homogeneous of degree one. In the first case, inversion and integration yields that c takes the desired form, given that γ is also even by the evenness of c. In the second case, there must exist some values $u' > 0, \kappa > 1$ (again by evenness) such that $\gamma(\kappa u') \neq \kappa\gamma(u')$. Suppose, without loss of generality, that $\gamma(\kappa u') > \kappa\gamma(u')$ and let $\Delta \equiv \frac{\gamma(\kappa u')}{\kappa\gamma(u')} - 1$. Let $N^\star$ be the least integer strictly greater than $\frac{2\kappa(1+\Delta)}{\Delta}$ and let $N^{\star\star}$ be the greatest integer strictly less than $\frac{N^\star}{k}$.

Consider a collective decision problem where $N^{\star\star}$ individuals have value $-\kappa u'$ and $N^\star$ individuals have value u' and there are no other individuals in the economy. Then note that

$$\sum_i u_i = N^\star u' - N^{\star\star}\kappa u' > N^\star u' - \frac{N^\star}{\kappa}\kappa u' = 0$$

so the sign of $\sum_i u_i > 0$. However,

$$\sum_i v_i^\star = N^\star\gamma(u') - N^{\star\star}\gamma(\kappa u') = \gamma(u')\left[N^\star - N^{\star\star}\kappa(1+\Delta)\right] <$$

$$\gamma(u')\left[N^\star - (N^\star - \kappa)(1+\Delta)\right] = \kappa\gamma(u')\left[1 + \Delta - \Delta\frac{N^\star}{\kappa}\right] < \kappa\gamma(u')\left[1 + \Delta - 2(1+\Delta)\right] < 0,$$

using the fact that $k, \gamma > 0$ by the monotonicity of γ. Thus unless $\gamma(\cdot)$ is homogeneous of degree 1, $c(\cdot)$ cannot be robustly efficient. □

B Nash Equilibria for Compactly Supported Utility Densities

B.1 Assumptions and Terminology

In this appendix we shall analyze the multi-player game described in section 5 under several different sets of hypotheses on the utility distribution $F = F_U$. In all of the scenarios to be considered here F_U is a probability distribution on $\mathbb{R}$ with mean $\mu_U \geq 0$, variance $\sigma_U^2 > 0$, and with density $f_U(u)$ supported by the interval $[\underline{u}, \overline{u}]$. We shall assume that $f_U(u)$ is C^∞ on the interval $[\underline{u}, \overline{u}]$, and also (to avoid having to discuss a multitude of different cases) that $f_U(\underline{u}) > 0$ and $f_U(\overline{u}) > 0$.

A *pure strategy* is a Borel measurable function $v : [\underline{u}, \overline{u}] \to [-\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}}]$; when a pure strategy v is adopted, each agent buys $v(u)$ votes, where u is the agent's utility. A *mixed strategy* is a Borel measurable[35] function $\pi_V : [\underline{u}, \overline{u}] \to \Pi$, where Π is the collection of Borel probability measures on $[-\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}}]$; when a mixed strategy π_V is adopted, each agent i will buy a random number V_i of votes, where $V_1, V_2, \ldots$ are conditionally independent given the utilities $U_1, U_2, \ldots$ and V_i has conditional distribution $\pi_V(U_i)$. Clearly, the set of mixed strategies contains the pure strategies.

[35]The space of Borel probability measures on $[-\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}}]$ is given the topology of weak convergence; Borel measurability of a function with range Π is relative to the Borel field induced by this topology. Proposition 4 below implies that in the quadratic voting game only pure strategies are relevant, so measurability issues will play no role in this paper.

A *best response* for an agent with utility u to a strategy (either pure or mixed) is a value v such that

$$E\Psi(v+S_n)u - v^2 = \sup_{\tilde{v}} E\Psi(\tilde{v}+S_n)u - \tilde{v}^2, \tag{8}$$

where S_n is the sum of the votes of the other n agents when these agents all play the specified strategy and E denotes expectation. (Thus, under E, the random variables V_i of the n other voters are distributed in accordance with the strategy and the sampling rule for utility values U_i described above.) Since Ψ is continuous and bounded, the equation (8) and the dominated convergence theorem imply that for each u the set of best responses is closed, and hence has well-defined maximal and minimal elements $v_+(u), v_-(u)$.

A mixed strategy π_V is a *Nash equilibrium* if for every $u \in [\underline{u}, \overline{u}]$ the measure $\pi_V(u)$ is supported by the set of best responses to π_V for an agent with utility u. (Thus, the term *Nash equilibrium* is identical to what is called a "type-symmetric Bayes-Nash equilibrium" in the text.

Assumption 3. *(Steepness)* There exists $w \in [-\delta, \delta]$ such that

$$(1 - \Psi(w))\,|\underline{u}| > (\delta - w)^2. \tag{9}$$

Furthermore, there exist a unique pair (α, w) such that $\alpha > \delta$ and

$$\begin{aligned} &(1 - \Psi(w))\,|\underline{u}| = (\alpha - w)^2 \quad \text{and} \\ &(1 - \Psi(w'))\,|\underline{u}| \leq (\alpha - w')^2 \quad \text{for all } w' \neq w. \end{aligned} \tag{10}$$

For this pair,

$$2 + \psi'(w)\,|\underline{u}| > 0. \tag{11}$$

We shall only consider payoff functions Ψ for which Assumption 3 holds. Observe that for given values of $\underline{u}, \overline{u}$ the condition 9 will hold for any Ψ for which the support $[-\delta, \delta]$ of $\psi = \Psi'$ is sufficiently small; since the payoff function Ψ is an approximation to the Heaviside function, this is not an unrealistic restriction. For some functions Ψ satisfying the steepness condition (9) there might be several pairs (α, w) for which inequalities (10) hold, but for small perturbations of such Ψ the extremal pair (α, w) will be unique, and the second derivative condition (11) will hold. Thus, although Assumption 3 need not be valid for every payoff function Ψ, at least it holds *generically*.

B.2 Main results

Nash equilibria need not be unique, as we will show, nor are they necessarily in pure strategies. However, every Nash equilibrium is "essentially" a pure strategy, as the following theorem implies.

Proposition 4. If a mixed strategy π_V is a Nash equilibrium, then the set of utility values $u \in [\underline{u}, \overline{u}]$ for which there is more than one best response (and hence the set of values u such that $\pi_V(u)$ is not supported by just a single point $v(u)$) is at most countable.

Since by hypothesis the values U_i are sampled from a distribution F that is absolutely continuous with respect to Lebesgue measure, it follows that for every Nash equilibrium there is an equivalent pure-strategy Nash equilibrium $v(u)$.

Proposition 5. Every pure-strategy Nash equilibrium $v(u)$ is a strictly increasing function of $u \in [\underline{u}, \overline{u}]$, and therefore is continuous except at possibly countably many points. If π_V is a mixed-strategy Nash equilibrium and $v(u)$ the equivalent pure-strategy Nash equilibrium, then the utility values $u \in [\underline{u}, \overline{u}]$ for which there is more than one best response are precisely the points at which $v(u)$ is discontinuous.

Propositions 4 and 5 both follow by routine monotonicity arguments, using the fact that the action space $[-\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}}]$ and the type space $[\underline{u}, \overline{u}]$ are totally ordered. The details are laid out in section B.3. Our main results concern the nature of the pure-strategy Nash equilibria. First, we will prove that in all cases Nash equilibria are nearly linear functions, except in the extreme tails of the utility distribution.

Proposition 6. There exists $\beta > 0$ such that for all sufficiently large n, every Nash equilibrium $v(u)$ is C^∞ in the interval $[\underline{u} + \beta n^{-3/2}, \overline{u} - \beta n^{-3/2}]$. Moreover, for any $\epsilon > 0$ there exist constants $n_\epsilon < \infty$ and $C < \infty$ such that if $n \geq n_\epsilon$ then for any Nash equilibrium $v(u)$ and for all $u \in \left[\underline{u} + Cn^{-3/2}, \overline{u} - Cn^{-3/2}\right]$,

$$(1 - \epsilon)E\psi(S_n)|u| \leq |2v(u)| \leq (1 + \epsilon)E\psi(S_n)|u|. \tag{12}$$

Furthermore, for all sufficiently large n any Nash equilibrium $v(u)$ with no discontinuities must satisfy (12) for all $u \in [\underline{u}, \overline{u}]$.

The smoothness assertion will be proved in sections B.4.c–B.4.d, and the approximate proportionality relation (12) in section B.4.e.

Proposition 6 implies that, except for agents with extreme utilities, the number of votes $v(u)$ that will be purchased will be roughly proportional to the agent's utility u, with proportionality constant $E\psi(S_n)$. The size of this proportionality constant will vary from order $1/n$ to $1/n^{1/4}$, depending on the properties of the utility distribution and the payoff function Ψ. There are two main cases, depending on whether the mean μ of the utility distribution is zero or non-zero; the case where the mean is non-zero splits into two sub-cases, depending on the relative "steepness" of the payoff function. Recall that Ψ is strictly increasing on the interval $[-\delta, \delta]$ and flat on both $[\delta, \infty)$ and $(-\infty, -\delta]$.

Proposition 7. When $\mu > 0$, there exists $\gamma > 0$ such that for any $\epsilon > 0$, all sufficiently large values of the sample size n, and any Nash equilibrium $v(u)$,

(i) $v(u)$ has a single discontinuity at u_*, where $|u_* + |\underline{u}| - \gamma n^{-2}| < \epsilon n^{-2}$;
(ii) $|v(u) - \alpha + w| < \epsilon$ for $u \in [\underline{u}, u_*)$;
(iii) $\alpha = \gamma\psi(w) f_U(\underline{u})$; and
(iv) $P\{|S_n - \alpha| > \epsilon\} < \epsilon$.

Under the hypotheses of Proposition 7 Nash equilibria are not unique, for a trivial reason: any Nash equilibrium v has a discontinuity u_*, and both the left limit $v(u_*-)$ and the right limit $v(u_*+)$ are best responses at u_*, so a Nash equilibrium can be obtained by setting $v(u_*)$ to either (or to any convex combination, for a mixed-strategy Nash equilibrium).

What is perhaps most interesting about Proposition 7 is that the structure of a Nash equilibrium is determined by the possible presence in the sample of an *extremist* (an agent with utility $u < \underline{u} + \gamma n^{-2}$) who will singlehandedly buy the election. Such an agent will occur only with probability $\approx n^{-1} f_U\,(\underline{u})\,\gamma$, and so will affect the expected payoff to a non-extremist by an amount of order only $1/n$, but this is enough to force non-extremists to purchase enough votes so that the aggregate vote of the non-extremists in the sample will be near the point α where only agents in the $1/n^2$ tail would find it beneficial to use the extremist strategy.

The case $\mu = 0$, where the utility distribution is (at least in a crude sense) balanced is more technically challenging, and here we have less complete results. First, there are no extremists and no discontinuities.

Proposition 8. If $\mu = 0$, then for all sufficiently large values of the sample size n no Nash equilibrium $v(u)$ has a discontinuity in $[\underline{u}, \overline{u}]$. Moreover, for any $\epsilon > 0$, if n is sufficiently large then every Nash equilibrium $v(u)$ satisfies

$$\|v\|_\infty \leq \epsilon. \tag{13}$$

Under additional hypotheses, the proportionality constant in (12) is of order $n^{-1/4}$, as the next theorem shows.

Proposition 9. Assume that the function $\psi = \Psi'$ is even. If $\mu = 0$ then for any $\epsilon > 0$, if n is sufficiently large then every Nash equilibrium satisfies

$$|\sqrt{\pi n/2}(E\psi(S_n))^2\sigma_U - 2| < \epsilon, \tag{14}$$

and therefore, by the proportionality rule (12), for all $u \neq 0$,

$$|\sqrt{2\sigma_U}\sqrt[4]{\pi n/2}v(u)/u - \sqrt{2}| < \epsilon \tag{15}$$

Furthermore, there are constants $\alpha_n > 0$ satisfying $\lim \alpha_n = 0$ such that for all sufficiently large n and every Nash equilibrium,

$$|ES_n| \leq \alpha_n \sqrt{\mathrm{var} S_n}; \tag{16}$$

consequently, there are constants $\beta_n > 0$ satisfying $\lim \beta_n = 0$ such that the expected inefficiency of any Nash equilibrium is bounded by β_n.

The proofs of Propositions 8 and 9 will be given in section B.6.

B.3 Necessary Conditions for Nash Equilibrium

Let π_V be a mixed-strategy Nash equilibrium, and let S_n be the sum of the votes of n agents with utilities U_i gotten by random sampling from F_u, all acting in accordance with the strategy π_V. For an agent with utility u, a best response v must satisfy Equation (8), and so in particular for every $\Delta > 0$, if $u > 0$ then

$$\begin{aligned} &E\{\Psi(S_n + v + \Delta) - \Psi(S_n + v)\}\, u \leq 2\Delta v + \Delta^2 \quad \text{and} \\ &E\{\Psi(S_n + v - \Delta) - \Psi(S_n + v)\}\, u \leq -2\Delta v + \Delta^2 \end{aligned} \tag{17}$$

Similarly, if $u < 0$ and $\Delta > 0$ then

$$\begin{aligned} &E\{\Psi(S_n + v - \Delta) - \Psi(S_n + v)\}\, u \leq -2\Delta v + \Delta^2 \quad \text{and} \\ &E\{\Psi(S_n + v + \Delta) - \Psi(S_n + v)\}\, u \leq 2\Delta v + \Delta^2 \end{aligned} \tag{18}$$

Since Ψ is C^∞ and its derivative ψ has compact support, differentiation under the expectation if permissible. Thus, we have the following necessary condition.

Lemma 3. If π_V is a mixed-strategy Nash equilibrium then for every u a best response v must satisfy

$$E\psi(S_n + v)u = 2v. \tag{19}$$

Consequently, every pure-strategy Nash equilibrium $v(u)$ must satisfy the functional equation

$$E\psi(S_n + v(u))u = 2v(u). \tag{20}$$

□

Lemma 4. Let π_V be a mixed-strategy Nash equilibrium, and let $v, \tilde{v}$ be best responses for agents with utilities $u, \tilde{u}$, respectively. If $u = 0$ then $v = 0$, and if $u < \tilde{u}$, then $v \leq \tilde{v}$. Consequently, any pure-strategy Nash equilibrium $v(u)$ is a nondecreasing function of u, and therefore has at most countably many discontinuities and is differentiable almost everywhere.

Proof. It is obvious that the only best response for an agent with $u = 0$ is $v = 0$, and the monotonicity of the payoff function Ψ implies that a best response v for an agent with utility u must be of the same sign as u. If $v, \tilde{v}$ are best responses for agents with utilities $0 \leq u < \tilde{u}$, then by definition

$$\begin{aligned} E\Psi(\tilde{v} + S_n)\tilde{u} - \tilde{v}^2 &\geq E\Psi(v + S_n)\tilde{u} - v^2 \quad \text{and} \\ E\Psi(v + S_n)u - v^2 &\geq E\Psi(\tilde{v} + S_n)u - \tilde{v}^2, \end{aligned}$$

and so, after re-arrangement of terms,

$$\begin{aligned} (E\Psi(\tilde{v} + S_n) - E\Psi(v + S_n))\tilde{u} &\geq \tilde{v}^2 - v^2 \quad \text{and} \\ (E\Psi(\tilde{v} + S_n) - E\Psi(v + S_n))u &\leq \tilde{v}^2 - v^2. \end{aligned}$$

Hence,

$$(E\Psi(\tilde{v} + S_n) - E\Psi(v + S_n))(\tilde{u} - u) \geq 0.$$

The monotonicity of Ψ implies that if $0 \leq \tilde{v} < v$ then $E\Psi(\tilde{v} + S_n) \leq E\Psi(v + S_n)$, and so it follows that the two expectations must be equal, since $\tilde{u} - u > 0$. But if the two expectations were equal then v could not possibly be a best response at u, because an agent with utility u could obtain the same expected payoff $E\Psi(v + S_n)u$ at a lower vote cost by purchasing $\tilde{v}$ votes. This proves that if $0 \leq u < \tilde{u}$ then best responses $v, \tilde{v}$ for agents with utilities $u, \tilde{u}$ must satisfy $0 \leq v \leq \tilde{v}$. A similar argument shows that if $u < \tilde{u} \leq 0$ then best responses $v, \tilde{v}$ for agents with utilities $u, \tilde{u}$ must satisfy $v \leq \tilde{v} \leq 0$.

□

Proof of Proposition 4. For each u denote by $v_-(u)$ and $v_+(u)$ the minimal and maximal best responses at u. Proposition 4 implies that if $u < \tilde{u}$ then $v_+(u) \leq v_-(\tilde{u})$. Consequently, for any $\epsilon > 0$ the set of utilities values u at which $v_+(u) - v_-(u) \geq \epsilon$ must be finite, because otherwise $v_+(u) \to \infty$ as $u \to \overline{u}$, which is impossible since best responses must take values between $-\sqrt{2\,|\underline{u}|}$ and $\sqrt{2\overline{u}}$. □

Remark 10. It follows from Proposition 4 that a mixed-strategy Nash equilibrium is essentially equivalent to a pure-strategy Nash equilibrium, specifically, the pure strategy $v(u) = v_-(u)$. This is because by hypothesis the sampling distribution F_U is absolutely continuous with respect to

Lebesgue measure, and so with probability one the sample $U_1, U_2, \dots, U_{n+1}$ will contain only values u at which $v_-(u) = v_+(u)$. This proves the second assertion of Proposition 5.

Henceforth, we shall consider only pure-strategy Nash equilibria; whenever we refer to a *Nash equilibrium* we will mean a pure-strategy Nash equilibrium.

Lemma 5. If $v(u)$ is a Nash equilibrium, then $v(u) \neq 0$ for all $u \neq 0$.

Proof. If $v(u) = 0$ for some $u > 0$ then by Proposition 4 $v(u') = 0$ for all $u' \in (0, u)$. Since the utility density $f_U(u)$ is strictly positive on $[\underline{u}, \overline{u}]$, it follows that there is positive probability p that every agent in the sample casts vote $V_i = 0$. But then an agent with utility u could improve her expectation by buying $\varepsilon > 0$ votes, where $\varepsilon \ll u\psi(0)p$, because then the expected utility gain would be at least

$$u\Psi(\varepsilon)p \sim u\psi(0)p\varepsilon$$

at a cost of ε^2. Since by hypothesis $\psi(0) > 0$, the expected utility gain would overwhelm the increased vote cost for small $\varepsilon > 0$. □

Corollary 11. Any Nash equilibrium $v(u)$ is *strictly* monotone on $[\underline{u}, \overline{u}]$.

Proof. Propositions 3 and 5 imply that $E\psi(S_n + v(u)) > 0$ for every $u \neq 0$. Now differentiation of the necessary condition (20) gives

$$E\psi(S_n + v(u)) = (2 - E\psi'(S_n + v(u)))v'(u)$$

at every u where $v(u)$ is differentiable. Since such points are dense in $[\underline{u}, \overline{u}]$, and since ψ and ψ' are C^∞ functions with compact support, it follows that $v'(u) \neq 0$ on a dense set. But $v'(u) \geq 0$ at every point where the derivative exists, so it follows that $v'(u) > 0$ almost everywhere, and this implies that v is strictly monotone. □

B.4 Continuity and Smoothness Properties of Nash Equilibria

B.4.a Weak consensus bounds

According to Proposition the necessary condition 3, in any Nash equilibrium the number of votes $v(u)$ purchased by an agent with utility u must satisfy the necessary condition (20). It is natural to expect that when the sample size $n + 1$ is large the effect of adding a single vote v to the aggregate total S_n should be small, and so the function $v(u)$ should satisfy the approximate proportionality rule

$$2v(u) \approx E\psi(S_n)u.$$

As we will show later, this naive approximation can fail badly for utility values u in the extreme tails of the distribution F_U, and even in the bulk of the distribution the relative error in the approximation can be significant. Nevertheless, the idea of approximate population consensus on the expectations $E\psi(v(u) + S_n)$ can be used to obtain weak bounds that we will find useful. The following lemma states, roughly, that if it is optimal for some agent in the bulk of the population to buy a moderately large number of votes, then *most* agents will be forced to buy a moderately large number of votes.

Lemma 6. For every $\epsilon > 0$ there exist constants $\alpha, \beta > 0$ such that for all sufficiently large n and any Nash equilibrium $v(u)$

$$\frac{v(u)}{u} \geq \alpha \max(-v(\underline{u} + \epsilon), v(\overline{u} - \epsilon)) - e^{-\beta n} \quad \text{for all } \; |u| > 2\epsilon. \tag{21}$$

Proof. It suffices to establish the lower bound $\alpha v(\overline{u} - \epsilon) - e^{-\beta n}$, as the other half of (21) can be proved in virtually the same way. The main idea is that, for an agent with utility u not in the tails of the distribution F_U, the joint distribution of the sample $U_1, U_2, \dots, U_{n+1}$ conditional on the agent's value u is not appreciably different than the *unconditional* distribution; that is, the agent gets very little information from knowing her own utility value u.

Set $u_\epsilon = \overline{u} - \epsilon$ and $p_\epsilon = 1 - F_U(u_\epsilon)$ where F_U is the cumulative distribution function of the utility distribution. Let $N = N_\epsilon$ be the number of points in the sample $U_1, U_2, \dots, U_n$ that fall in the interval $[\overline{u} - \epsilon, \overline{u}]$, and let $U = U_{n+1}$ be independent of $U_1, U_2, \dots, U_n$. Then

$$\begin{aligned} E(\psi(v(U) + S_n) \,|\, U < u_\epsilon) &= \sum_{m \geq 0} \binom{n}{m} p_\epsilon^m (1 - p_\epsilon)^{n-m} E(\psi(v(U) + S_n) \,|\, U < u_\epsilon, N = m), \\ E(\psi(v(U) + S_n) \,|\, U \geq u_\epsilon) &= \sum_{m \geq 0} \binom{n}{m} p_\epsilon^m (1 - p_\epsilon)^{n-m} E(\psi(v(U) + S_n) \,|\, U \geq u_\epsilon, N = m) \end{aligned}$$

Now conditional on $N = m$, the sample $U_1, U_2, \dots, U_n$ is obtained by choosing m points at random according to the conditional distribution of U given $U \geq u_\epsilon$ and $n - m$ according to the conditional distribution of U given $U < u_\epsilon$. Consequently, for each $m \geq 0$,

$$E(\psi(v(U) + S_n) \,|\, U \geq u_\epsilon, N = m) = E(\psi(v(U) + S_n) \,|\, U < u_\epsilon, N = m + 1).$$

Furthermore, for any small $\epsilon' > 0$ and for m in the range $[np_\epsilon - n\epsilon', np_\epsilon + n\epsilon']$, the ratio

$$\binom{n}{m} p_\epsilon^m (1 - p_\epsilon)^{n-m} \Big/ \binom{n}{m+1} p_\epsilon^{m+1} (1 - p_\epsilon)^{n-m-1}$$

is between $1/2$ and 2. Since the binomial-(n, p_ϵ) distribution puts only an exponentially small (in n) mass outside the interval $[np_\epsilon - n\epsilon', np_\epsilon + n\epsilon']$, it follows that for some constants α', β' depending on ϵ but not n,

$$E(\psi(v(U) + S_n) \,|\, U < u_\epsilon) \geq \alpha' E(\psi(v(U) + S_n) \,|\, U \geq u_\epsilon) - e^{-\beta' n} \tag{22}$$

for all sufficiently large n.

A similar argument proves that for suitable constants $\alpha'', \beta'' > 0$, and for any interval $J \subset [\underline{u}, \overline{u}]$ of length ϵ not overlapping $[\overline{u} - \epsilon, \overline{u}]$,

$$E(\psi(v(U) + S_n) \,|\, U \in J) \geq \alpha'' E(\psi(v(U) + S_n) \,|\, U \geq u_\epsilon) - e^{-\beta'' n}. \tag{23}$$

To see this, let N be the number of points in the sample $U_1, U_2, \dots, U_n$ that fall in the interval $[\overline{u} - \epsilon, \overline{u}]$, and let N' be the number of points in the sample that fall in J. Decompose the conditional expectations $E(\psi(v(U) + S_n) \,|\, U \in J)$ and $E(\psi(v(U) + S_n) \,|\, U > u_\epsilon)$ according to the values of N

and N', and use the identity

$$E(\psi(v(U) + S_n) \,|\, U \in J, N = m+1, N' = m') = E(\psi(v(U) + S_n) \,|\, U > u_\epsilon, N = m, N' = m'+1).$$

As in the proof of (22), the ratio

$$\frac{P\{N = m+1, N' = m'\}}{P\{N = m, N' = m'+1\}}$$

is near one for all pairs (m, m') except those in the tails of the joint distribution, and the tails are exponentially small, by standard estimates for the multinomial distribution.

Now recall that any Nash equilibrium $v(u)$ is monotone, and satisfies the necessary condition $2v(u) = E\psi(v(u) + S_n)u$. Since any $u \in [\epsilon, u_\epsilon - \epsilon]$ is the right endpoint of an interval $J = [u - \epsilon, u]$ of length ϵ that does not intersect $[u_\epsilon, \overline{u}]$, it follows that for any such u,

$$\begin{aligned}
E\psi(v(u) + S_n) &\geq E(\psi(v(U) + S_n) \,|\, U \in [u - \epsilon, u]) \\
&\geq \alpha'' E(\psi(v(U) + S_n) \,|\, U \geq u_\epsilon) - e^{-\beta'' n} \\
&\geq \alpha'' E\psi(v(u_\epsilon + S_n)) - e^{-\beta'' n} \\
&= \alpha'' v(u_\epsilon)/\overline{u} - e^{-\beta'' n}
\end{aligned}$$

and similarly for any $u \in [\underline{u} + 2\epsilon, -\epsilon]$. The assertion (21) now follows from another application of the necessary condition (20). □

B.4.b Concentration and size constraints

Since the vote total S_n is the sum of independent, identically distributed random variables $v(U_i)$ (albeit with unknown distribution), its distribution is subject to concentration restrictions, such as those imposed by the following lemma.

Lemma 7. For any $\epsilon > 0$ there exists a constant $\gamma = \gamma(\epsilon) < \infty$ such that for all sufficiently large values of n and any Nash equilibrium $v(u)$, if

$$\max(v(\overline{u} - \epsilon), -v(\underline{u} + \epsilon) \geq \gamma/\sqrt{n}, \tag{24}$$

then

$$P\{|S_n + v| \leq \delta\} < \epsilon \quad \text{for all } v \in \mathbb{R} \tag{25}$$

and therefore

$$\frac{|2v(u)|}{|u|} \leq \epsilon \|\psi\|_\infty \quad \text{for all } u \in [\underline{u}, \overline{u}]\,. \tag{26}$$

We will deduce Lemma 7 from the following general fact about sums of independent, identically distributed random variables.

Lemma 8. Fix $\delta > 0$. For any $\epsilon > 0$ and any $C < \infty$ there exists $C' = C'(\epsilon, C) > 0$ and $n' = n'(\epsilon, C) < \infty$ such that the following statement is true: if $n \geq n'$ and $Y_1, Y_2, \ldots, Y_n$ are independent random variables such that

$$E|Y_1 - EY_1|^3 \leq C\text{var}(Y_1)^{3/2} \quad \text{and} \quad \text{var}(Y_1) \geq C'/n \tag{27}$$

then for every interval $J \subset \mathbb{R}$ of length δ or greater, the sum $S_n = \sum_{i=1}^n Y_i$ satisfies

$$P\{S_n \in J\} \leq \epsilon |J|/\delta. \tag{28}$$

The proof of the proposition, a routine exercise in the use of Fourier methods, is relegated to the appendix.

Proof of Lemma 7. Inequality (26) follows from (25), by the necessary condition (20) for Nash equilibria. Hence, it suffices to show that (24) implies (25).

Lemma 6 implies that there are constants $\alpha, \beta > 0$ such that for every $u \in [\underline{u}, \overline{u}] \setminus [-2\epsilon, 2\epsilon]$ the ratio $v(u)/u$ is at least $\alpha v(u_\epsilon) - e^{-\beta n}$, where $u_\epsilon = \overline{u} - \epsilon$. Since the utility density f_U is bounded below, it follows that for suitable constants $0 < C < \infty$ and $p > 0$, for every sufficiently large n and every Nash equilibrium $v(u)$ there is an interval $[u', u''] \subset [\overline{u}/2, \overline{u})$ of probability p such that $u' < u_\epsilon < u''$ and

$$v(u_\epsilon) \leq C v(u'') \leq C^2 v(u'). \tag{29}$$

Similarly, there exists an interval $[u_*, u_{**}] \subset [\underline{u}, 0]$ of probability p such that

$$|v(u_*)| \leq C |v(u_{**})|. \tag{30}$$

Let N be the number of points U_i in the sample $U_1, U_2, \ldots, U_n$ that fall in $[u_*, u_{**}] \cup [u', u'']$, and let S_n^* be the sum of the votes $v(U_i)$ for those agents whose utility values fall in this range. Observe that N has the binomial-$(n, 2p)$ distribution, and that conditional on the event $N = m$ and $S_n - S_n^* = w$, the random variable S_n^* is the sum of m independent random variables Y_i whose variance is at least $v(u')^2/4$ and whose third moment obeys the restriction (27) (this follows from the inequalities (29)–(30)). Consequently, by Lemma 8, if $v(u_\epsilon)\sqrt{n}$ is sufficiently large then the conditional probability, given $N = n \geq np$ and $S_n - S_n^* = w$, that S_n^* lies in any interval of length δ is bounded above by $\epsilon/2$. Since $P\{N \leq np\}$ is, for large n, much less than $\epsilon/2$, the inequality (25) follows.

□

Lemma 7 implies that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium $v(u)$ the absolute value $|v(u)|$ can assume large values only at utility values u within distance ϵ of one of the endpoints $\underline{u}, \overline{u}$. The following proposition improves this to the extreme tails of the distribution.

Lemma 9. For any $0 < C < \infty$ there exists $C' > 0$ such that for all sufficiently large n and any Nash equilibrium $v(u)$ satisfies the inequality

$$|v(u)| \leq C \quad \text{for all} \quad u \in \left[\underline{u} + C' n^{-3/2}, \overline{u} - C' n^{-3/2}\right]. \tag{31}$$

Proof. Fix $C > 0$, and suppose that $2v(u_*) \geq C$ for some $u_* > 0$. Since any Nash equilibrium v is monotone, we must have $2v(u) \geq C$ for all $u \geq u_*$, and by the necessary condition (20) it follows that

$$E\psi(v(u) + S_n)u \geq C \quad \Longrightarrow \quad E\psi(v(u) + S_n) \geq C/\overline{u} \quad \forall\, u \geq u_*. \tag{32}$$

Consequently, the distribution of S_n is concentrated: since the function ψ has support $[-\delta, \delta]$, the probability that $S_n + v(u) \in [-\delta, \delta]$ must be at least $C/\overline{u}\|\psi'\|_\infty$. Thus, Lemma 7 implies that for any $\epsilon > 0$ there exists $\gamma_\epsilon > 0$ (depending on both ϵ and C, but not on n) such that

$$\max\left(-v\left(\underline{u}+\epsilon\right), v(\overline{u}-\epsilon)\right) \leq \gamma_\epsilon/\sqrt{n} \tag{33}$$

In particular, for all sufficiently large n,

$$\begin{aligned} v(\overline{u}/2) \leq \frac{\gamma_{\overline{u}/2}}{\sqrt{n}} \quad &\Longrightarrow \quad E\psi(v(\overline{u}/2)+S_n) \leq \frac{2\gamma_{\overline{u}/2}}{\overline{u}\sqrt{n}} \\ &\Longrightarrow \quad E\psi(S_n) \leq \frac{2\gamma_{\overline{u}/2}}{\overline{u}\sqrt{n}} + \|\psi'\|_\infty v(\overline{u}/2) \\ &\Longrightarrow \quad E\psi(S_n) \leq \frac{C_{\overline{u}/2}}{\sqrt{n}} \end{aligned} \tag{34}$$

for a constant $C_{\overline{u}/2} < \infty$ that may depend on $\overline{u}/2$ and C but not on either n or the particular Nash equilibrium.

Fix C' large, and suppose that $2v(u_*) \geq C$ for $u_* = \overline{u} - C'n^{-3/2}$. Let N_* be the number of points U_i in the sample $U_1, U_2, \dots, U_n$ that fall in the interval $[u_*, \overline{u}]$; by our assumptions concerning the sampling procedure, the random variable N_* has the binomial distribution with mean

$$EN_* = n\int_{u_*}^{\overline{u}} f_U(u)\,du = C'C_f n^{-1/2}$$

where C_f is the mean value of f_U on the interval $[u_*, \overline{u}]$ (which for large n will be close to $f_U(\overline{u}) > 0$). Since EN_* is vanishingly small for large n, the assumption $v(u_*) \geq C$ implies that

$$E\psi(v(u)+S_n)\mathbf{1}\{N_*=0\} \geq C/2\overline{u} \quad \text{for all } u \geq u_*. \tag{35}$$

This expectation can be decomposed by partitioning the probability space into the event $G = \{U_n \in [\underline{u}+\epsilon, \overline{u}-\epsilon]\}$ and its complement. On the event G, the contribution of $v(U_n)$ to the vote total S_n is at most $\gamma_\epsilon/\sqrt{n}$ in absolute value, by (33). On the complementary event G^c the integrand is bounded above by $\|\psi\|_\infty$. Therefore,

$$\begin{aligned} E\psi(v(u)+S_n)\mathbf{1}\{N_*=0\} &\leq P(G^c)\|\psi\|_\infty + E\psi(v(u)+S_n)\mathbf{1}\{N_*=0\}\mathbf{1}_G \\ &\leq P(G^c)\|\psi\|_\infty + E\psi(v(u)+S_{n-1})\mathbf{1}\{N_*=0\} + \|\psi'\|_\infty(\gamma_\epsilon/\sqrt{n}) \\ &\leq \epsilon' + E\psi(v(u)+S_{n-1})\mathbf{1}\{N_*=0\} \end{aligned}$$

where $\epsilon' > 0$ can be made arbitrarily small by choosing $\epsilon > 0$ small and n large. This together with inequality (35) implies that for large n,

$$E\psi(v(u)+S_{n-1})\mathbf{1}\{N_*=0\} \geq C/4\overline{u} \quad \text{for all } u \geq u_*. \tag{36}$$

Now consider the conditional distribution of S_n given that $N_* = 1$: this can be simulated by generating S_{n-1} from the conditional distribution of S_{n-1} given that $N_* = 0$ and then adding an independent $v(U)$ where $U = U_n$ is drawn from the conditional distribution of U given that $U \geq u_*$. Consequently, by inequality (36),

$$E(\psi(S_n)\,|\,N_*=1) = E(\psi(S_{n-1}+v(U))\,|\,N_*=0) \geq C/4\overline{u}.$$

But this implies that

$$E(\psi(S_n)) \geq (C/2\overline{u})P\{N_* \geq 1\} \approx CC'C_f/(2\overline{u}\sqrt{n}).$$

For large C' this is incompatible with inequality (34) when n is sufficiently large. $\square$

B.4.c Discontinuities

Since any Nash equilibrium $v(u)$ is monotone in the utility u, it can have at most countably many discontinuities. Moreover, since any Nash equilibrium is bounded in absolute value by $\sqrt{2\max(|\underline{u}|, \overline{u})}$ (as no agent will pay more for votes than she could gain in expected utility) the sum of the jumps is bounded by $\sqrt{2\max(|\underline{u}|, \overline{u})}$. We will now show that there is a *lower* bound on the size of $|v|$ at a discontinuity.

Lemma 10. Let $v(u)$ be a Nash equilibrium. If v is discontinuous at $u \in (\underline{u}, \overline{u})$ then

$$E\psi'(\tilde{v} + S_n)u = 2 \tag{37}$$

for some $\tilde{v} \in [v_-, v_+]$, where v_- and v_+ are the left and right limits of $v(u')$ as $u' \to u$.

Proof. The necessary Condition (20) holds at all u' in a neighborhood of u, so by monotonicity of v and continuity of ψ, the Equation (20) must hold when $v(u)$ is replaced by either of $v_\pm$, that is,

$$\begin{aligned} 2v_+ &= E\psi(v_+ + S_n)u \quad \text{and} \\ 2v_- &= E\psi(v_- + S_n)u. \end{aligned}$$

Subtracting one equation from the other and using the differentiability of ψ we obtain

$$2v_+ - 2v_- = uE\int_{v_-}^{v_+} \psi'(t + S_n)\,dt = u\int_{v_-}^{v_+} E\psi'(t + S_n)\,dt.$$

The result then follows from the mean value theorem of calculus. $\square$

Lemma 11. There is a constant $\Delta > 0$ such that for all sufficiently large n, at any point u of discontinuity of a Nash equilibrium,

$$\begin{aligned} v(u_+) &\geq \Delta \quad \text{if } u \geq 0, \quad \text{and} \\ v(u_-) &\leq -\Delta \quad \text{if } u \leq 0. \end{aligned} \tag{38}$$

Consequently, there is a constant $\beta < \infty$ not depending on the sample size n such that for all sufficiently large n no Nash equilibrium $v(u)$ has a discontinuity at a point u at distance greater than $\beta n^{-3/2}$ from one of the endpoints $\underline{u}, \overline{u}$.

Proof. Since the function ψ has support contained in the interval $[-\delta, \delta]$, equation (37) implies that v can have a discontinuity only if the distribution of S_n is highly concentrated: specifically,

$$P\{S_n + \tilde{v} \in [-\delta, \delta]\} \geq \frac{2}{\|\psi'\| \max(|\underline{u}|, \overline{u})}. \tag{39}$$

In fact, since ψ' vanishes at the endpoints of $[-\delta, \delta]$, there exists $0 < \delta' < \delta$ such that

$$P\{S_n + \tilde{v} \in [-\delta', \delta']\} \geq \frac{1}{\|\psi'\| \max(|\underline{u}|, \overline{u})}. \tag{40}$$

Lemma 7 asserts that strong concentration of the distribution of S_n can occur only if $|v(u)|$ is vanishingly small in the interior of the interval $[\underline{u}, \overline{u}]$. In particular, if $\epsilon < (\|\psi'\| \max(|\underline{u}|, \overline{u}))^{-1}$ and n is sufficiently large then $|v(u)| < \gamma_\epsilon/\sqrt{n}$ for all $u \in [\underline{+}\epsilon, \overline{u} - \epsilon]$. But $v(u)$ must satisfy the necessary Condition (20) at all such u, so

$$E\psi(v(u) + S_n)|u| \leq 2\gamma_\epsilon/\sqrt{n}$$

for all $u \in [\underline{u} + \epsilon, \overline{u} - \epsilon]$. Since the function ψ is positive and bounded away from 0 in any interval $[-\delta'', \delta'']$ where $0 < \delta'' < \delta$, it follows from (40) that for sufficiently large n,

$$|\tilde{v}| \geq (\delta - \delta')/3 := \Delta.$$

Thus, by the monotonicity of Nash equilibria, at every point u of discontinuity we must have (38). Lemma 9 now implies that any such discontinuities can occur only within a distance $\beta n^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. $\square$

B.4.d Smoothness

Since Nash equilibria are monotone, by Lemma 4, they are necessarily differentiable almost everywhere. We will show that in fact differentiability must hold at *every* u, except near the endpoints $\underline{u}, \overline{u}$.

Lemma 12. If $v(u)$ is a Nash equilibrium then at every u where v is differentiable,

$$E\psi(S_n + v(u)) + E\psi'(S_n + v(u))uv'(u) = 2v'(u). \tag{41}$$

Proof. This is a routine consequence of the necessary condition (20) and the smoothness of the function ψ. $\square$

Equation (41) can be rewritten as a first-order differential equation:

$$v'(u) = \frac{E\psi(S_n + v(u))}{2 - E\psi'(S_n + v(u))u}. \tag{42}$$

This differential equation becomes singular at any point where the denominator approaches 0, but is regular in any interval where $E\psi'(S_n + v(u))u \leq 1$. The following lemma implies that this will be the case on any interval where $|v(u)|$ remains sufficiently small.

Lemma 13. For any $\alpha > 0$ there exists a constant $\beta = \beta_\alpha > 0$ such that for any strategy $v(u)$, any $\tilde{v} \in \mathbb{R}$, any $u \in [\underline{u}, \overline{u}]$, and all n,

$$\begin{aligned} E|\psi'(\tilde{v} + S_n)u| \geq \alpha \quad &\Longrightarrow \quad E\psi(\tilde{v} + S_n)|u| \geq \beta \text{ and} \\ E|\psi''(\tilde{v} + S_n)u| \geq \alpha \quad &\Longrightarrow \quad E\psi(\tilde{v} + S_n)|u| \geq \beta \end{aligned} \tag{43}$$

Proof. Recall that $\psi/2$ is a C^∞ probability density with support $[-\delta, \delta]$ and such that ψ is *strictly* positive in the open interval $(-\delta, \delta)$. Consequently, on any interval $J \subset (-\delta, \delta)$ where $|\psi'|$ (or $|\psi''|$) is bounded below by a positive number, so is ψ.

Fix $\epsilon > 0$ so small that $\epsilon \max(\underline{u}, \overline{u}) < \alpha/2$. In order that $E|\psi'(\tilde{v} + S_n)u| \geq \alpha$, it must be the case that the event $\{|\psi'(\tilde{v} + S_n)| \geq \epsilon\}$ contributes at least $\alpha/2$ to the expectation; hence,

$$P\{|\psi'(\tilde{v} + S_n)| \geq \epsilon\} \geq \frac{\alpha}{2\|\psi'\|_\infty \max(\underline{u}, \overline{u})}.$$

But on this event the random variable $\psi(\tilde{v} + S_n)$ is bounded below by a positive number $\eta = \eta_\epsilon$, so it follows that

$$E\psi(\tilde{v} + S_n)|u| \geq \frac{\eta\alpha}{2\|\psi'\|_\infty \max(\underline{u}, \overline{u})}.$$

A similar argument proves the corresponding result for ψ''. $\square$

Lemma 14. There exist constants $C, \alpha > 0$ such that for all sufficiently large n, any Nash equilibrium $v(u)$ is continuously differentiable on any interval where $|v(u)| \leq C$ (and therefore, by Proposition 9, on $\left(\underline{u} + C'n^{-3/2}, \overline{u} - C'n^{-3/2}\right)$), and the derivative satisfies

$$\alpha \leq \frac{v'(u)}{E\psi(v(u) + S_n)} \leq \alpha^{-1}. \tag{44}$$

Proof. The function $v(u)$ is differentiable almost everywhere, by Lemma 4, and at every point u where $v(u)$ is differentiable the differential equation (42) holds. By Lemma 11, the sizes of discontinuities are bounded below, and so if $C > 0$ is sufficiently small then a Nash equilibrium $v(u)$ can have no discontinuities on any interval where $|v(u)| \leq C$. Furthermore, if $C > 0$ is sufficiently small then by Lemma 13 and the necessary Condition (20), we must have $E\psi'(v(u) + S_n) \leq 1$ on any interval where $|v(u)| \leq C$. Since the functions $v \mapsto E\psi(S_n + v)$ and $v \mapsto E\psi'(S_n + v)$ are continuous (by dominated convergence), it now follows from Equation (42) that if $C > 0$ is sufficiently small then on any interval where $|v(u)| \leq C$ the function $v'(u)$ extends to a continuous function. Finally, since the denominator in equation (42) is at least 1 and no larger than $2 + \|\psi'\|_\infty$, the inequalities (44) follow. $\square$

Similar arguments show that Nash equilibria have derivatives of higher orders provided the sample size is sufficiently large. The proof of Lemma 9 in section B.6 will require information about the second derivative $v''(u)$. This can be obtained by differentiating under the expectations in (42):

$$v''(u) = \frac{E\psi'(v(u) + S_n)v'(u)}{2 - E\psi'(S_n + v(u))u} + \frac{E\psi(v(u) + S_n)(E\psi''(v(u) + S_n)v'(u)u + E\psi'(v(u) + S_n)}{(2 - E\psi'(S_n + v(u)u))^2}. \tag{45}$$

A repetition of the proof of Lemma 14 now shows that for suitable constants $C, \beta > 0$ and all sufficiently large n, any Nash equilibrium $v(u)$ is twice continuously differentiable on any interval where $|v(u)| \leq C$ and satisfies the inequalities

$$\beta \leq \frac{v''(u)}{E\psi(v(u) + S_n)} \leq \beta^{-1}. \tag{46}$$

B.4.e Approximate proportionality

The information that we now have about the form of Nash equilibria can be used to sharpen the heuristic argument given in paragraph B.4.a to support the "approximate proportionality rule". Recall that in a Nash equilibrium the number of votes $v(u)$ purchased by an agent with utility u must satisfy the equation $2v(u) = E\psi(v(u) + S_n)u$. We have shown in Proposition 9 that for any Nash equilibrium, $v(u)$ must be small except in the extreme tails of the distribution (in particular, for all u at distance much more than $n^{-3/2}$ from both endpoints $\underline{u}, \overline{u}$). Since ψ is uniformly continuous, it follows that the expectation $E\psi(v(u) + S_n)$ cannot differ by very much from $E\psi(S_n)$.

Unfortunately, this argument only shows that the approximation $2v(u) \approx E\psi(S_n)u$ is valid up to an error of size $\epsilon_n|u|$ where $\epsilon_n \to 0$ as $n \to \infty$. However, as $n \to \infty$ the expectation $E\psi(S_n) \to 0$, and so the error in the approximation above might be considerably larger than the approximation itself. Proposition 6 makes the stronger assertion that when n is large the relative error in the approximate proportionality rule is small.

Proof of Proposition 6. Since ψ has compact support, it and all of its derivatives are uniformly continuous and uniformly bounded, and so the function $v \mapsto E\psi(v + S_n)$ is differentiable with derivative $E\psi'(v+S_n)$. Consequently, by Taylor's theorem, for every u there exists $\tilde{v}(u)$ intermediate between 0 and $v(u)$ such that

$$2v(u) = E\psi(v(u) + S_n)u = E\psi(S_n)u + E\psi'(\tilde{v}(u) + S_n)v(u)u. \tag{47}$$

We will argue that for all $C > 0$ sufficiently small, if $|v(u)| \leq C$ then the expectation $E\psi'(\tilde{v}(u) + S_n)$ remains below ϵ in absolute value, provided n is sufficiently large. Proposition 9 will then imply that there exists $C' < \infty$ such that (12) holds for all $u \in (\underline{u}, \overline{u})$ at distance greater than $C'n^{-3/2}$ from the endpoints $\underline{u}, \overline{u}$.

If $|2v(u)| \leq C$ then $|E\psi(v(u) + S_n)| \leq C/\max(|\underline{u}|, \overline{u})$, by the necessary condition (20). By Lemma 11, if $C < \Delta$, where Δ is the discontinuity threshold, then $v(u)$ is continuous on any interval $[0, u_C]$ where $|v(u)| \leq C$, and so for each u in this interval there is a $u' \in [0, u]$ such that $\tilde{v}(u) = v(u')$. Consequently, $|E\psi(\tilde{v}(u) + S_n)| \leq C/\max(|\underline{u}|, \overline{u})$. But Lemma 13 implies that for any $\epsilon > 0$, if $C > 0$ is sufficiently small then for all n and any Nash equilibrium $v(u)$,

$$|E\psi'(\tilde{v}(u) + S_n)| < \epsilon$$

on any interval $[0, u_C]$ where $|v(u)| \leq C$. Thus, the error in the approximation (47) will be small when n is large and $|v(u)| < C$, for $u > 0$. A similar argument applies for $u \leq 0$.

Finally, suppose that $v(u)$ is a Nash equilibrium with no discontinuities. By Lemma 9, for any $C > 0$ there exists $C' < \infty$ such that $|v(u)| \leq C/2$ except at arguments u within distance $C'/n^{3/2}$ of one of the endpoints. Moreover, Lemma 14 implies that if C is sufficiently small then on any interval where $|v(u)| \leq C$ the function v is differentiable, with derivative $v'(u) \leq C''$ for some constant $C'' < \infty$ not depending on n or on the particular Nash equilibrium. It then follows that

$$v(\overline{u}) \leq C/2 + C'C''n^{-3/2} \leq C$$

provided n is large. Since $C > 0$ can be chosen arbitrarily small, it follows that $v(u)$ must satisfy the proportionality relations (12) on $[0, \overline{u}]$. A similar argument applies to the interval $[\underline{u}, 0]$. $\square$

B.4.f Consequences of Proposition 6

Proposition 6 puts strong constraints on the distribution of the vote total S_n in a Nash equilibrium. According to this proposition, the approximate proportionality rule (12) holds for all $u \in [\underline{u}, \overline{u}]$ except those values u within distance $Cn^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. Call such values *extremists*, and denote by G the event that the sample $U_1, U_2, \ldots, U_n$ contains no extremists. By Proposition 6, on the event G the approximate proportionality rule (12) will apply for each agent; furthermore, for Nash equilibria with no discontinuities, (12) holds for all $u \in [\underline{u}, \overline{u}]$. Thus, *conditional* on the event G (or, for continuous Nash equilibria, *unconditionally*) the random variables $v(U_i)$ are (at least for sufficiently large n) bounded above and below by $E\psi(S_n)\overline{u}$ and $E\psi(S_n)\underline{u}$, and so Hoeffding's inequality Hoeffding (1963) applies.

Corollary 12. Let G be the event that the sample U_i contains no extremists. Then for all sufficiently large n and any Nash equilibrium $v(u)$,

$$P(|S_n - ES_n| \geq tE\psi(S_n) \,|\, G) \leq \exp\{-2t^2/n \max\left(|\underline{u}|^2, \overline{u}^2\right)\}; \tag{48}$$

and for any Nash equilibrium with no discontinuities,

$$P(|S_n - ES_n| \geq tE\psi(S_n)) \leq \exp\{-2t^2/n \max\left(|\underline{u}|^2, \overline{u}^2\right)\}. \tag{49}$$

Proposition 6 also implies uniformity in the normal approximation to the distribution of S_n, because the proportionality rule (12) guarantees that the ratio of the third moment to the $3/2$ power of the variance of $v(U_i)$ is uniformly bounded. Hence, by the Berry-Esseen theorem, we have the following corollary.

Corollary 13. There exists $\kappa < \infty$ such that for all sufficiently large n and any Nash equilibrium $v(u)$, the vote total S_n satisfies

$$\sup |P((S_n - ES_n) \leq t\sqrt{\mathrm{var}(S_n)} \,|\, G) - \Phi(t)| \leq \kappa n^{-1/2}; \tag{50}$$

and for any Nash equilibrium with no discontinuities,

$$\sup |P((S_n - ES_n) \leq t\sqrt{\mathrm{var}(S_n)}) - \Phi(t)| \leq \kappa n^{-1/2}. \tag{51}$$

Here Φ denotes the standard normal cumulative distribution function.

B.5 Unbalanced populations: The case $\mu > 0$

B.5.a Concentration of the vote total

Lemma 15. If $\mu > 0$ then for all large n no Nash equilibrium $v(u)$ has a discontinuity at a non-negative value of u. Moreover, if $\mu > 0$ then for any $\epsilon > 0$, if n is sufficiently large then in any Nash equilibrium the vote total S_n must satisfy
(i) $ES_n \in [\delta - \epsilon, \delta + \epsilon + \sqrt{2\,|\underline{u}|}]$ and
(ii) $P\{|S_n - ES_n| > \epsilon\} < \epsilon$.
Furthermore, there is a constant $\gamma > 0$ such that for any $\epsilon > 0$, if n is sufficiently large and $v(u)$ is a Nash equilibrium with no discontinuities, then
(iii) $P\{|S_n - ES_n| > \epsilon\} < e^{-\gamma n}$.

Proof. By Lemma 11, a Nash equilibrium $v(u)$ can have no discontinuities at distance greater than $Cn^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. Agents with such utilities are designated *extremists*; the event G that the sample $U_1, U_2, \dots, U_n$ contains no extremists has probability $1 - O(n^{-1/2})$.

By Proposition 6, any Nash equilibrium $v(u)$ obeys the approximate proportionality Rule (12) except in the extremist regime. The contribution of extremists to ES_n is vanishingly small for large n, since $P(G^c) = O(n^{-1/2})$ and $|v| \leq \max(\sqrt{2\,|\underline{u}|}, \sqrt{2\overline{u}})$. Consequently, (12) implies that for any $\epsilon > 0$, if n is large then

$$E\psi(S_n)\mu_U(1-\epsilon) \leq ES_n/n \leq E\psi(S_n)\mu_U(1+\epsilon). \tag{52}$$

Since $\mu_U > 0$, this implies that $ES_n \geq 0$ for all sufficiently large n.

Suppose now that $ES_n < \delta - 2\epsilon'$ for some small $\epsilon' > 0$. If $\epsilon > 0$ is sufficiently small relative to ϵ' then (52) implies that $nE\psi(S_n)\mu_U \leq \delta - \epsilon'/2$. But then Hoeffding's inequality (48) (for this a weaker Chebyshev bound would suffice), together with the fact that $P(G^c) \leq Kn^{-1/2}$, implies that

$$P\{S_n \in [-\delta/2, \delta - \epsilon'/4]\} \geq 1 - \epsilon$$

for large n. This is impossible, though, because we would then have

$$E\psi(S_n) \geq (1-\epsilon) \min_{v \in [-\delta/2, \delta-\epsilon'/4]} \psi(v),$$

and since ψ is bounded away from 0 on any compact sub-interval of $(-\delta, \delta)$ this contradicts the fact that $nE\psi(S_n) < \delta - \epsilon'/2$. This proves that for all large n and all Nash equilibria, $ES_n \geq \delta - 2\epsilon'$.

Next suppose that $ES_n > \delta + \sqrt{2\,|\underline{u}|} + 2\epsilon'$, where $\epsilon' > 0$. The proportionality rule (12) (applied with some $\epsilon > 0$ small relative to ϵ') then implies that $nE\psi(S_n) > \delta + \sqrt{2\,|\underline{u}|} + \epsilon'$. Hence, by the Hoeffding inequality (48), there exists $\gamma = \gamma(\epsilon') > 0$ such that

$$P(S_n \leq \delta + \sqrt{2\,|\underline{u}|} \,|\, G) \leq e^{-\gamma n},$$

because on the event $S_n \leq \delta + \sqrt{2\,|\underline{u}|}$ the sum S_n must deviate from its expectation by more than $nE\psi(S_n)\epsilon'$. Hence, for all $v \in [-\sqrt{2\,|\underline{u}|}, 0] \leq 0$,

$$E\psi(v + S_n) \leq e^{-\gamma n}\|\psi\|_\infty + P(G^c)\|\psi\|_\infty.$$

Thus, $|v\,(\underline{u})\,|$ must be vanishingly small, and so by Lemma 11 there can be no discontinuities in $[\underline{u}, 0]$. But this implies that the proportionality rule (12) holds for all $u \in \left[\underline{u}, \overline{u} - Cn^{-3/2}\right]$, and so another application of Hoeffding's inequality (coupled with the observation that $v(u)/u \geq (1-\epsilon)E\psi(S_n)$ holds for all $u \in [\underline{u}, \overline{u}]$ if v has no discontinuities at negative values of u) implies that

$$P(S_n \leq \delta + \sqrt{2\,|\underline{u}|}) \leq e^{-\gamma n} \quad \Longrightarrow \quad E\psi(S_n) \leq e^{-\gamma n}\|\psi\|_\infty,$$

which is a contradiction. This proves assertion (i).

Since ES_n is now bounded away from 0 and ∞, it follows as before that $nE\psi(S_n)$ is bounded away from 0 and ∞, and so the proportionality rule (12) implies that the conditional variance of S_n given the event G is $O(n^{-1})$. The assertion (ii) therefore follows from Chebyshev's inequality and the bound $P(G^c) = O(n^{-1/2})$. Given (i) and (ii), we can now conclude that there can be no discontinuities at nonnegative values of u, because in view of Proposition 11, the monotonicity of Nash equilibria, and the necessary condition (20), this would entail that

$$E\psi(v(\overline{u}) + S_n)\overline{u} \geq 2\Delta,$$

which is incompatible with (i) and (ii).

Finally, if v is a Nash equilibrium with no discontinuities then Corollary 12 implies the exponential bound (iii). □

B.5.b Proof of Proposition 7

Lemma 15 implies that for large n the distribution of S_n must be highly concentrated near ES_n in any Nash equilibrium, and for any $\epsilon > 0$ there exists $\gamma > 0$ such that for any Nash equilibrium with no discontinuities,

$$P\{|S_n - ES_n| \geq \epsilon\} \leq e^{-\gamma n}.$$

Hence, if $ES_n > \delta + \epsilon$ then $E\psi(S_n) < e^{-\gamma n}$. But Proposition 6 asserts that if a Nash equilibrium $v(u)$ has no discontinuities then the proportionality rule (12) holds for all $u \in [\underline{u}, \overline{u}]$, and so

$$ES_n \leq (1+\epsilon')nE\psi(S_n) \leq (1+\epsilon')ne^{-\gamma n},$$

contradicting the fact that $ES_n \geq \delta - \epsilon$. This proves that for large n, any Nash equilibrium $v(u)$ with no discontinuities must satisfy $ES_n < \delta + \epsilon$.

Suppose that $ES_n < \alpha - 2\epsilon$ for some $\epsilon > 0$. If $\epsilon > 0$ is sufficiently small, then for some $\epsilon' > 0$ depending on ϵ,

$$(1 - \Psi(w+\epsilon)))(|\underline{u}| - \epsilon') > (\alpha - \epsilon - w)^2.$$

Consequently, if $ES_n \leq \alpha - 2\epsilon$, then an agent with utility $u \in [\underline{u}, \underline{u} + \epsilon']$ purchasing $\alpha + \epsilon - w$ votes would have expected payoff at least

$$-\Psi(w+\epsilon))\,(|\underline{u}| - \epsilon')\,P\{S_n \leq \alpha - \epsilon\} - (\alpha - \epsilon - w)^2.$$

This strictly dominates the expected payoff $\approx \underline{u}P\{S_n \geq \alpha - 3\epsilon\}$ for buying votes in accordance with the approximate proportionality rule (12). But any Nash equilibrium must satisfy the rule (12) except in the extremist regime, so we have a contradiction. This proves that for all sufficiently large n, in any Nash equilibrium we must have $ES_n > \alpha - 2\epsilon$. It follows that for all sufficiently large n, every Nash equilibrium has a discontinuity. The discontinuity must be located within distance $Cn^{-3/2}$ of the endpoint $\underline{u}$, by Lemma 11.

Now suppose that $ES_n > \alpha + 3\epsilon$. Then, by Hoeffding's inequality, $P(S_n \leq \alpha + 2\epsilon \,|\, G)$ is exponentially small for large n. Furthermore, since (α, w) is the unique pair satisfying (10),

$$(1 - \Psi(w' + \epsilon)))\,|\underline{u}| + 2\,|\underline{u}|\,e^{-\gamma n} < (\alpha + 2\epsilon - w')^2 \quad \text{for all } w' \in [-\delta, \delta]$$

and so it would be suboptimal for an agent with utility value $\underline{u}$ to buy more than $\alpha + 2\epsilon - \delta$ votes. Clearly it would also be suboptimal to buy more than Δ but no more than $\alpha + 2\epsilon - \delta$ votes, where Δ is the discontinuity threshold (cf. Lemma 11), because this would leave the expected utility payoff below $\underline{u}(1 - e^{-\gamma n})$. Consequently, if $ES_n > \alpha + 3\epsilon$ then for large n no Nash equilibrium would have a discontinuity; since we have shown that for large n every Nash equilibrium has a discontinuity it follows that ES_n cannot exceed $\alpha + 3\epsilon$ for large n. We have therefore proved that for any $\epsilon > 0$, if n is sufficiently large then (a) every Nash equilibrium has a discontinuity in the extremist regime near $\underline{u}$; and (b) $|ES_n - \alpha| < \epsilon$. Assertion (iv) of the theorem follows, by Proposition 15.

Let $v(u)$ be a Nash equilibrium, and let u_* be the rightmost point u_* of discontinuity of v. Consider the strategy $v(u)$ for an agent with utility value $u < u_*$: since v is monotone, $v(u) \leq -\Delta$. Moreover, the expected payoff for an agent with utility u must exceed the expected payoff under the alternative strategy of buying no votes. The latter expectation is approximately $\underline{u}$, because S_n is highly concentrated near $ES_n > \alpha - \epsilon$ and so $E\Psi(S_n) \approx 1$. On the other hand, the expected payoff at u for an agent playing the Nash strategy v is approximately

$$\Psi(\alpha - v(u)) (\underline{u}) - v(u)^2.$$

Consequently, since (α, w) is the unique pair such that relations (10) hold, we must have

$$|v(u)| \approx \alpha - w.$$

This proves assertion (ii).

That v has only a single point of discontinuity u_* follows from the hypothesis (11). Recall (cf. Lemma 10) that if v is discontinuous at u then $E\psi'(\tilde{v} + S_n) = 2$ for some $\tilde{v}$ intermediate between the right and left limits $v(u+)$ and $v(u-)$. But any discontinuity u must occur within distance $\beta n^{-3/2}$ of $\underline{u}$, and if $u < u_*$ then $v(u) \approx -\alpha + w$. Hence, since the distribution of S_n is concentrated in a neighborhood of α,

$$E\psi'(v(u\pm) + S_n) \approx \psi'(w),$$

and so by (11), for $u \in [\underline{u}, u_*)$ there cannot be a value $\tilde{v} \in [v(u-), v(u+)]$ satisfying the necessary condition $E\psi'(\tilde{v} + S_n) = 2$ for a discontinuity.

Finally, since u_* must be within distance $Cn^{-3/2}$ of $\underline{u}$, the conditional probability that there are at least two extremists in the sample $U_1, U_2, \ldots, U_n$ given that there is at least one is of order $O(n^{-1/2})$. Consequently,

$$E\psi(S_n) = \psi(w) f_U(\underline{u}) (u_* + |\underline{u}|) + O(n^{-1/2}(u_* + |\underline{u}|)).$$

On the other hand, since $ES_n \approx \alpha$, the proportionality rule (12) implies that $nE\psi(S_n) \approx \alpha$. Therefore,

$$u_* + \underline{u} \sim \gamma n^{-2}$$

where γ is the unique solution of the equation $\alpha = \gamma\psi(w) f_U(\underline{u})$. This proves assertions (i) and (iii).

□

B.6 Balanced Populations: $\mu_U = 0$

B.6.a Continuity of Nash Equilibria

Proof of Proposition 8. The size of any discontinuity is bounded below by a positive constant Δ, by Proposition 11, so it suffices to prove the assertion (13). By Proposition 7, for any $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon)$ such that if n is sufficiently large then any Nash equilibrium $v(u)$ satisfying $\|v\|_\infty > \epsilon$ must also satisfy $|v(u)| \leq \gamma/\sqrt{n}$ for all u not within distance ϵ of one of the endpoints

$\underline{u}, \overline{u}$. Hence, the approximate proportionality relation (12) implies that

$$E\psi(S_n) \leq \frac{C}{\sqrt{n}} \tag{53}$$

for a suitable $C = C(\gamma)$. Since $v(u)/u$ is within a factor $(1+\epsilon)^{\pm 1}$ of $E\psi(S_n)$ for all u not within distance $C'_\epsilon n^{-3/2}$ of $\underline{u}$ or $\overline{u}$, it follows from Chebyshev's inequality that for any $\alpha > 0$ there exists $\beta = \beta(\alpha)$ such that

$$P\{|S_n - ES_n| \geq \beta\} \leq \alpha.$$

On the other hand, if $\|v\|_\infty \geq \epsilon$, then by the necessary condition (20), there is some u such that

$$P\{S_n + v(u) \in [-\delta, \delta]\} \geq \frac{\epsilon}{\|\psi\|_\infty \max\left(|\underline{u}|, \overline{u}\right)}.$$

Since S_n is concentrated around ES_n, it follows that ES_n must be at bounded distance from $v(u)$, and so the Berry–Esseen bound (50) implies that $P\{S_n \in [-\delta/2, \delta/2]\}$ is bounded below. But this in turn implies that $E\psi(S_n)$ is bounded below, which for large n is impossible in view of (53). Thus, if n is sufficiently large then no Nash equilibrium $v(u)$ can have $\|v\|_\infty \geq \epsilon$.

□

Since $\|v\|_\infty$ is small for any Nash equilibrium v, the distribution of the vote total S_n cannot be too highly concentrated. This in turn implies that the proportionality constant $E\psi(S_n)$ in (12) cannot be too small.

Lemma 16. For any $C < \infty$ there exists $n_C < \infty$ such that for all $n \geq n_C$ and every Nash equilibrium,

$$nE\psi(S_n) \geq C. \tag{54}$$

Proof. By the approximate proportionality rule (12) and the necessary condition (20), for any $\epsilon > 0$ and all sufficiently large n,

$$|ES_n| \leq n\epsilon E\psi(S_n)E|U|.$$

Thus, by Hoeffding's inequality (Corollary 12), if $nE\psi(S_n) < C$ then the distribution of S_n must be highly concentrated in a neighborhood of 0. But if this were so we would have, for all large n,

$$E\psi(S_n) \approx \psi(0) > 0,$$

which is a contradiction. □

B.6.b Edgeworth expansions

For the analysis of the case $\mu_U = 0$ refined estimates of the errors in the approximate proportionality rule (12) will be necessary. These we will derive from the Edgeworth expansion for the density of a sum of independent, identically distributed random variables (cf. Feller Feller (1971), Ch. XVI, sec. 2, Th. 2). The relevant summands here are the random variables $v(U_i)$, and since the function $v(u)$ depends on the particular Nash equilibrium (and hence also on n), it will be necessary to have a version of the Edgeworth expansion in which the error is precisely quantified. The following variant of Feller's Theorem 2 (which can be proved in the same manner as in Feller (1971)) will suffice for our purposes.

Proposition 14. Let $Y_1, Y_2, \ldots, Y_n$ be independent, identically distributed random variables with mean $EY_1 = 0$, variance $EY_1^2 = 1$, and finite $2r$th moment $E|Y_1|^{2r} = \mu_{2r} \leq m_{2r}$. Assume that the distribution of Y_1 has a density $f_1(y)$ whose Fourier transform $\hat{f}_1$ satisfies $|\hat{f}_1(\theta)| \leq g(\theta)$, where g is a C^{2r} function such that $g \in L^\nu$ for some $\nu \geq 1$ and such that for every $\epsilon > 0$,

$$\sup_{|\theta| \geq \epsilon} g(\theta) < 1. \tag{55}$$

Then there is a sequence $\epsilon_n \to 0$ depending only on m_{2r} and on the function g such that the density $f_n(y)$ of $\sum_{i=1}^n Y_i/\sqrt{n}$ satisfies

$$\left| f_n(x) - \frac{e^{-x^2/2}}{\sqrt{2\pi n}} \left(1 + \sum_{k=3}^{2r} n^{-(k-2)/2} P_k(x) \right) \right| \leq \frac{\epsilon_n}{n^{-r+1}} \tag{56}$$

for all $x \in \mathbb{R}$, where $P_k(x) = C_k H_k(x)$ is a multiple of the kth Hermite polynomial $H_k(x)$, and C_k is a continuous function of the moments $\mu_3, \mu_4, \ldots, \mu_k$ of Y_1.

The following lemma ensures that in any Nash equilibrium the sums $S_n = \sum_{i=1}^n v(U_i)$, after suitable renormalization, meet the requirements of Proposition 14.

Lemma 17. There exist constants $0 < \sigma_1 < \sigma_2 < m_{2r} < \infty$ and a function $g(\theta)$ satisfying the hypotheses of Proposition 14 (with $r = 4$) such that for all sufficiently large n and any Nash equilibrium $v(u)$ the following statement holds. If $w(u) = 2v(u)/E\psi(S_n)$ then

(a) $\sigma_1^2 < \mathrm{var}(w(U_i)) < \sigma_2^2$;

(b) $E|w(U_i) - Ew(U_i)|^{2r} \leq m_{2r}$; and

(c) the random variables $w(U_i)$ have density $f_W(w)$ whose Fourier transform is bounded in absolute value by g.

Proof. These statements are consequences of the proportionality relations (12) and the smoothness of Nash equilibria. By Proposition 8, Nash equilibria are continuous on $[\underline{u}, \overline{u}]$ and for large n satisfy $\|v\|_\infty < \epsilon$, where $\epsilon > 0$ is any small constant. Consequently, by Proposition 6, the proportionality relations (12) hold on the entire interval $[\underline{u}, \overline{u}]$. Since $EU_1 = 0$, it follows that for any $\epsilon > 0$, if n is sufficiently large then $|Ew(U_i)| < \epsilon$, and so assertions (a)–(b) follow routinely from (12).

The existence of the density $f_W(w)$ follows from the smoothness of Nash equilibria, which was established in section B.4.d. In particular, by Proposition 14, inequalities (46), and the proportionality relations (12), if the sample size n is sufficiently large and v is any continuous Nash equilibrium then v is twice continuously differentiable on $[\underline{u}, \overline{u}]$, and there are constants $\alpha, \beta > 0$ not depending on n or on the particular Nash equilibrium such that the derivatives satisfy

$$\alpha \leq \frac{v'(u)}{E\psi(S_n)} \leq \alpha^{-1} \quad \text{and} \quad \beta \leq \frac{v''(u)}{E\psi(S_n)} \leq \beta^{-1} \tag{57}$$

for all $u \in [\underline{u}, \overline{u}]$. Consequently, if U is a random variable with density $f_U(u)$ then the random variable $W := 2v(U)/E\psi(S_n)$ has density

$$f_W(w) = f_U(u) E\psi(S_n)/(2v'(u)) \quad \text{where } w = 2v(u)/E\psi(S_n) \tag{58}$$

Furthermore, the density $f_W(w)$ is continuously differentiable, and its derivative

$$f'_W(w) = \frac{f'_U(u)(E\psi(S_n))^2}{4v'(u)^2} - \frac{f_U(u)(E\psi(S_n))^2 v''(u)}{4v'(u)^3}$$

satisfies

$$|f'_W(w)| \leq \kappa \tag{59}$$

where $\kappa < \infty$ is a constant that does not depend on either n or on the choice of Nash equilibrium.

It remains to prove the existence of a dominating function $g(\theta)$ for the Fourier transform of f_W. This will be done in three pieces: (i) for values $|\theta| \leq \gamma$, where $\gamma > 0$ is a small fixed constant; (ii) for values $|\theta| \geq K$, where K is a large but fixed constant; and (iii) for $\gamma < |\theta| < K$. Region (i) is easily dealt with, in view of the bounds (a)–(b) on the second and third moments and the estimate $|Ew(U)| < \epsilon'$, as these together with Taylor's theorem imply that for all $|\theta| < 1$,

$$|\hat{f}_W(\theta) - (1 + i\theta Ew(U) - \theta^2 \mathrm{var}(w(U))/2| \leq m_3|\theta|^3.$$

Next consider region (ii), where $|\theta|$ is large. Integration by parts shows that

$$\hat{f}_W(\theta) = \int_{w\underline{u}}^{w(\overline{u})} f_W(w)e^{i\theta w}\,dw = -\int_{w\underline{u}}^{w(\overline{u})} \frac{e^{i\theta w}}{i\theta} f'_W(w)\,dw + \frac{e^{i\theta w}}{i\theta} f_W(w)\bigg|_{w\underline{u}}^{w(\overline{u})};$$

since $f_W(w)$ is uniformly bounded at $w\underline{u}$ and $w(\overline{u})$, by (57) and (58), and since $|f'_W(w)| \leq \kappa$, by (59), it follows that there is a constant $C < \infty$ such that for all sufficiently large n and all Nash equilibria,

$$|\hat{f}_W(\theta)| \leq C/|\theta| \quad \forall\, \theta \neq 0.$$

Thus, setting $g(\theta) = C/|\theta|$ for all $|\theta| \geq 2C$, we have a uniform bound for the Fourier transforms $\hat{f}_W(\theta)$ in the region (ii).

Finally, to bound $|\hat{f}_W(\theta)|$ in the region (iii) of intermediate $\theta-$values, we use the proportionality rule once again to deduce that $|w(u) - u| < \epsilon$. Therefore,

$$\begin{aligned}
\hat{f}_W(\theta) &= \int_{\underline{u}}^{\overline{u}} e^{i\theta w(u)} f_U(u)\,du \\
&= \int_{\underline{u}}^{\overline{u}} e^{i\theta u} f_U(u)\,du + \int_{\underline{u}}^{\overline{u}} (e^{i\theta w(u)} - e^{i\theta u}) f_U(u)\,du \\
&= \hat{f}_U(\theta) + R(\theta)
\end{aligned}$$

where $|R(\theta)| < \epsilon'$ uniformly for $|\theta| \leq C$ and $\epsilon' \to 0$ as $\epsilon \to 0$. Since $\hat{f}_U$ is the Fourier transform of an absolutely continuous probability density, its absolute value is bounded away from 1 on the complement of $[-\gamma, \gamma]$, for any $\gamma > 0$. Since $\epsilon > 0$ can be made arbitrarily small (cf. Proposition 6), it follows that there is a continuous, positive function $g(\theta)$ that is bounded away from 1 on $|\theta| \in [\gamma, C]$ such that $|\hat{f}_W)\theta| \leq g(\theta)$ for all $|\theta| \in [\gamma, C]$. The extension of g to the whole real line can now be done by smoothly interpolating at the boundaries of regions (i), (ii), and (iii). $\square$

B.6.c Proof of Proposition 9

Since the function ψ is smooth and has compact support, differentiation under the expectation in the necessary condition $2v(u) = E\psi(v(u) + S_n)u$ is permissible, and so for every $u \in [-\underline{u}, \overline{u}]$ there exists $\tilde{v}(u)$ intermediate between 0 and $v(u)$ such that

$$2v(u) = E\psi(S_n)u + E\psi'(S_n + \tilde{v}(u))v(u)u. \tag{60}$$

The proof of Proposition 9 will hinge on the use of the Edgeworth expansion (Proposition 14) to approximate each of the two expectations in (60) precisely.

As in Lemma 17, let $w(u) = 2v(u)/E\psi(S_n)$. We have already observed, in the proof of Lemma 17, that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium, $|Ew(U)| < \epsilon$. It therefore follows from the proportionality rule that

$$\left| \frac{4\,\mathrm{var}(v(U))}{(E\psi(S_n))^2\sigma_U^2} - 1 \right| \leq \epsilon \quad \text{and} \quad \left| \frac{E|v(u) - Ev(u)|^k}{(E\psi(S_n))^k E|U|^k} \right| < \epsilon \ \ \forall\, k \leq 8. \tag{61}$$

Moreover, Lemma 17 and Proposition 14 imply that the distribution of S_n has a density with an Edgeworth expansion, and so for any continuous function $\varphi : [-\delta, \delta] \to \mathbb{R}$,

$$E\varphi(S_n) = \int_{-\delta}^{\delta} \varphi(x) \frac{e^{-y^2/2}}{\sqrt{2\pi n}\sigma_V} \left(1 + \sum_{k=3}^{m} n^{-(k-2)/2} P_k(y) \right) dx + r_n(\varphi) \tag{62}$$

where

$$\begin{aligned} &\sigma_V^2 := \mathrm{var}(v(U)), \\ &y = y(x) = (x - ES_n)/\sqrt{\mathrm{var}(S_n)}, \end{aligned}$$

and $P_k(y) = C_k H_3(y)$ is a multiple of the kth Hermite polynomial. The constants C_k depend only on the first k moments of $w(U)$, and consequently are uniformly bounded by constants C_k' not depending on n or on the choice of Nash equilibrium. The error term $r_n(\varphi)$ satisfies

$$|r_n(\varphi)| \leq \frac{\epsilon_n}{n^{(m-2)/2}} \int_{-\delta}^{\delta} \frac{|\varphi(x)|}{\sqrt{2\pi \mathrm{var}(S_n)}}\, dx. \tag{63}$$

In the special case $\varphi = \psi$, (62) and the remainder estimate (63) (with $m = 4$) imply that

$$E\psi(S_n) \leq \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x)\, dx + o(n^{-1}\sigma_V^{-1}).$$

Since $4\,\sigma_V^2 \approx (E\psi(S_n))^2\sigma_U^2$ for large n, this implies that for a suitable constant $\kappa < \infty$,

$$E\psi(S_n) \leq \frac{\kappa}{\sqrt[4]{n}}. \tag{64}$$

Claim 2. There exist constants $\alpha_n \to \infty$ such that in every Nash equilibrium,

$$|ES_n| \le \alpha_n^{-1}\sqrt{\mathrm{var}(S_n)} \quad \text{and} \tag{65}$$

$$\mathrm{var}(S_n) \ge \alpha_n^2. \tag{66}$$

Proof of Proposition 9. Before we begin the proof of the claim, we indicate how it will imply Proposition 9. If (65) and (66) hold, then for every $x \in [-\delta, \delta]$,

$$|y(x)| \le (1+2\delta)/\alpha_n \to 0.$$

Consequently, the dominant term in the Edgeworth expansion (62) for $\varphi = \psi$ (with $m = 4$), is the first, and so for any $\epsilon > 0$, if n is sufficiently large,

$$E\psi(S_n) = \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x)\,dx(1 \pm \epsilon).$$

(Here the notation $(1 \pm \epsilon)$ means that the ratio of the two sides is bounded above and below by $(1 \pm \epsilon)$.) Since $4\,\sigma_V^2 \approx (E\psi(S_n))^2\sigma_U^2$ this will imply that

$$\sqrt{\pi n/2}\sigma_U(E\psi(S_n))^2 = \int_{-\delta}^{\delta} \psi(x)\,dx(1 \pm \epsilon) = 2 \pm 2\epsilon,$$

proving the assertion (14). □

Proof of Claim 2. First we deal with the remainder term $r_n(\varphi)$ in the Edgeworth expansion (62). By Lemma 16, the expectation $E\psi(S_n)$ is at least C/n for large n, and so by (61) the variance of S_n must be at least C'/n. Consequently, by (63), the remainder term $r_n(\varphi)$ in (62) satisfies

$$|r_n(\varphi)| \le C'' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-2)/2}\sqrt{\mathrm{var}(S_n)}} \le C''' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-3)/2}}.$$

Suitable choice of m will make this bound small compared to any desired monomial n^{-A}, and so we may ignore the remainder term in the arguments to follow.

Suppose that there were a constant $C < \infty$ such that along some sequence $n \to \infty$ there were Nash equilibria satisfying $\mathrm{var}(S_n) \le C$. By (61), this would force $C/n \le E\psi(S_n) \le C'/\sqrt{n}$. This in turn would force

$$C''\mathrm{var}(S_n)\log n \ge |ES_n|^2 \ge C'''\mathrm{var}(S_n)\log n, \tag{67}$$

because otherwise the dominant term in the Edgeworth series for $E\psi(S_n)$ would be either too large or too small asymptotically (along the sequence $n \to \infty$) to match the requirement that $C/n \le E\psi(S_n) \le C'/\sqrt{n}$. (Observe that since the ratio $|ES_n|^2/\mathrm{var}(S_n)$ is bounded above by $C''\log n$, the terms $e^{-y^2/2}P_k(y)$ in the integral (62) are of size at most $(\log n)^A$ for some A depending on m, and so the first term in the Edgeworth series is dominant.) We will show that (67) leads to a contradiction.

Suppose that $ES_n > 0$ (the case $ES_n < 0$ is similar). The Taylor expansion (60) for $v(u)$ and the hypothesis $EU = 0$ implies that

$$2Ev(U) = E\psi'(S_n + \tilde{v}(U))v(U)U. \tag{68}$$

The Edgeworth expansion (62) for $E\psi'(S_n+\tilde{v}(u))$ together with the independence of S_n and U and the inequalities (67), implies that for any $\epsilon>0$, if n is sufficiently large then

$$E\psi'(S_n+\tilde{v}(u)) \\ = \frac{1}{\sqrt{2\pi\mathrm{var}(S_n)}}\int_{-\delta}^{\delta}\psi'(x)\exp\{-(x+\tilde{v}(u)-ES_n)^2/2\mathrm{var}(S_n)\}\,dx(1\pm\epsilon). \quad (69)$$

Now since ψ and ψ' have support $[-\delta,\delta]$, integration by parts yields

$$\int_{-\delta}^{\delta}\psi'(x)\exp\{-(x+\tilde{v}(u)-ES_n)^2/2\mathrm{var}(S_n)\}\,dx \\ = \int_{-\delta}^{\delta}\psi(x)\exp\{-(x+\tilde{v}(u)-ES_n)^2/2\mathrm{var}(S_n)\}\frac{x+\tilde{v}(u)-ES_n}{\mathrm{var}(S_n)}\,dx, \quad (70)$$

and since $x+\tilde{v}(u)$ is of smaller order of magnitude than ES_n, it follows that for large n

$$E\psi'(S_n+\tilde{v}(u)) = -\frac{ES_n}{\mathrm{var}(S_n)}E\psi(S_n)(1\pm\epsilon). \quad (71)$$

But it now follows from the Taylor series for $2Ev(U_i)$ (by summing over i) that

$$2ES_n = -n\frac{ES_n}{\mathrm{var}(S_n)}E\psi(S_n)Ev(U)U(1\pm\epsilon), \quad (72)$$

which is a contradiction, because the right side is negative and the left side positive. This proves the assertion (66).

The proof of inequality (65) is similar. Suppose that for some $C>0$ there were Nash equilibria along a sequence $n\to\infty$ for which $ES_n\geq C\sqrt{\mathrm{var}(S_n)}$. In view of (66), this implies in particular that $ES_n\to\infty$, and also that $|y(x)|\geq C/2$ for all $x\in[-\delta,\delta]$. Thus, the Edgeworth approximation (69) remains valid, as does the integration by parts identity (70). Since $ES_n\to\infty$, the terms $x+\tilde{v}(u)$ are of smaller order of magnitude that ES_n, and so once again (71) and therefore (72) follow. This is, once again, a contradiction, because the right side of (72) is negative while the left side diverges to $+\infty$.

$\square$

C Proof of Lemma 8

Lemma 8. *Fix $\delta>0$. For any $\epsilon>0$ and any $C<\infty$ there exists $\beta=\beta(\epsilon,C)>0$ and $n'=n'(\epsilon,C)<\infty$ such that the following statement is true: if $n\geq n'$ and $Y_1,Y_2,\dots,Y_n$ are independent random variables such that*

$$E|Y_1-EY_1|^3\leq C\mathrm{var}(Y_1)^{3/2} \quad \textit{and} \quad \mathrm{var}(Y_1)\geq\beta/n \quad (73)$$

then for every interval $J\subset\mathbb{R}$ of length δ or greater, the sum $S_n=\sum_{i=1}^n Y_i$ satisfies

$$P\{S_n\in J\}\leq\epsilon|J|/\delta. \quad (74)$$

Proof. It suffices to prove this for intervals of length δ, because any interval of length $n\delta$ can be partitioned into n pairwise disjoint intervals each of length δ. Without loss of generality, $EY_1 = 0$ and $\delta = 1$ (if not, translate and re-scale). Let g be a nonnegative, even, C^∞ function with $\|g\|_\infty = 1$ that takes the value 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and is identically zero outside $[-1, 1]$. It is enough to show that for any $x \in \mathbb{R}$,

$$Eg(S_n + x) \leq \epsilon.$$

Since g is C^∞ and has compact support, its Fourier transform is real-valued and integrable, so the Fourier inversion theorem implies that

$$Eg(S_n + x) = \frac{1}{2\pi} \int \hat{g}(\theta)\varphi(-\theta)^n e^{-i\theta x}\, d\theta,$$

where $\varphi(\theta) = Ee^{i\theta Y_1}$ is the characteristic function of Y_1. Because $EY_1 = 0$, the derivative of the characteristic function at $\theta = 0$ is 0, and hence φ has Taylor expansion

$$|1 - \varphi(\theta) - \frac{1}{2} EY_1^2 \theta^2| \leq \frac{1}{6} E|Y_1|^3 |\theta|^3.$$

Consequently, if the hypotheses (27) hold then for any $\gamma > 0$, if n is sufficiently large,

$$|\varphi(\theta)^n| \leq e^{-\beta^2\theta^2/4}$$

for all $|\theta| \leq \gamma$. This implies (since $|\hat{g}| \leq 2$) that

$$Eg(S_n + x) \leq \frac{1}{\pi} \int_{|\theta|<\gamma} e^{-\beta^2\theta^2/4}\, d\theta + \frac{1}{2\pi} \int_{|\theta|\geq\gamma} |\hat{g}(\theta)|\, d\theta$$

Since $\hat{g}$ is integrable, the constant γ can be chosen so that the second integral is less that $\epsilon/2$, and if β is sufficiently large then the first integral will be bounded by $\epsilon/2$. □

D Approximate Convergence Calculations

D.1 $\mu = 0$

Our analytic results show that the equilibrium when $\mu = 0$ involves strategies that are approximately linear and thus the sum of votes is approximately normally distributed. To calculate the rates and constants of convergence for this approximation, we use a Taylor approximation to the equilibrium's non-linearities and use an Edgeworth expansion of the density of the sum of $N-1$ votes. We also ignore the role of the smoothing function Ψ; that is we focus on the limit as $\delta \to 0$.

Suppose there are N individuals and $\mu = 0$. Let μ_z for $z = 3, 4, \ldots$ be the zth raw moment of f. We approximate the equilibrium v_N with N individuals by a Taylor expansion about 0, where we know $v_N = 0$:

$$v_N(u) \approx \hat{v}_N(u) = a_N u + b_N u^2 + c_N u^3.$$

Furthermore we assume that $\frac{b_N\sqrt{N-1}}{a_N}, \frac{c_N\sqrt{N-1}}{a_N} \to 0$ by approximate proportionality in the limit.

Let G be the cumulative distribution function and g the probability density function of the sum of any $N-1$ votes of individuals whose values are drawn according to f and who use strategy $\hat{v}_N$. This distribution is approximately normal with mean

$$\hat{M} = (N-1)\left[b_N\sigma^2 + c_N\mu_3\right]$$

and standard deviation

$$\hat{S} = a_N\sqrt{N-1}\sigma + o(1),$$

where the $o(1)$ terms arises from the higher order terms in $\hat{v}_N$ and vanish for large N by approximate proportionality; it can easily be verified that the varnish quickly enough to make no contribution to the leading terms in the approximate calculations below. We thus from hereon neglect these terms.

The second-order Edgeworth expansion of g about its mean is thus

$$\hat{g}(-v) = \frac{e^{-\frac{(N-1)^2\left[b_N\sigma^2+c_N\mu_3\right]^2}{2a_N^2(N-1)\sigma^2}}}{a_N\sigma\sqrt{2\pi(N-1)}}\left[1 - \frac{(N-1)\left[b_N\sigma^2+c_N\mu_3\right]}{a_N^2(N-1)\sigma^2}v - \left(\frac{1}{a_N^2(N-1)\sigma^2} - \frac{(N-1)^2\left[b_N\sigma^2+c_N\mu_3\right]^2}{a_N^4(N-1)^2\sigma^4}\right)v^2\right] \approx$$

$$\frac{1}{a_N\sigma\sqrt{2\pi(N-1)}}\left[1 - \frac{b_N\sigma^2+c_N\mu_3}{a_N^2\sigma^2}v - \frac{1}{a_N^2(N-1)\sigma^2}v^2\right],$$

where the approximation drops the vanishing terms that shown to make no contribution to the leading limiting terms. Note that this approximation depends on the assumption that $\mu_3 \neq 0$ and we will turn to the case when $\mu_3 = 0$ below.

Now we can substitute $\hat{v}_N(u)$ for v and match terms to solve from a_N, b_N and c_N. All but the linear terms in the expansion vanish rapidly and thus make no contribution to the limiting rates or constants,[36] so we obtain

$$g\left(-\hat{v}_N(u)\right) \approx \frac{1}{a_N\sigma\sqrt{2\pi(N-1)}}\left[1 - \frac{b_N\sigma^2+c_N\mu_3}{a_N\sigma^2}u - \frac{1}{(N-1)\sigma^2}u^2\right].$$

But $g = \frac{1}{p}$ so that, by the first-order condition for maximization, we must have

$$\hat{v}_N = \frac{1}{2a_N\sigma\sqrt{2\pi(N-1)}}\left[1 - \frac{b_N\sigma^2+c_N\mu_3}{a_N\sigma^2}u - \frac{1}{(N-1)\sigma^2}u^2\right] \implies$$

by matching coefficients

$$a_N = \frac{1}{2a_N\sigma\sqrt{2\pi(N-1)}} \implies a_N = \frac{1}{\sqrt{2\sigma}\sqrt[4]{2\pi(N-1)}} = -c_N(N-1)\sigma^2 \implies c_N = -\frac{1}{(N-1)^{\frac{5}{4}}\sigma^{\frac{5}{2}}\sqrt{2}\sqrt[4]{2\pi}},$$

and

$$b_N = -\frac{b_N\sigma^2+c_N\mu_3}{\sigma^2} \implies b_N = -\frac{c_N\mu_3}{2} = \frac{\mu_3}{(N-1)^{\frac{5}{4}}\sigma^{\frac{5}{2}}2^{\frac{7}{4}}\sqrt[4]{\pi}}.$$

[36] In fact, it can be shown that more accurate approximations make (higher order) contributions to *greater* efficiency, so including them is more favorable to our analysis.

Note that this is clearly consistent with the initial supposition that $\lim_{N\to\infty} \frac{b_N\sqrt{N-1}}{a_N} = 0$ and similarly for c_N.

Welfare of the first best is just $\mathbb{E}[|U|] \approx \sqrt{\frac{2N}{\pi}}\sigma$ by the formula of the mean deviance of the normal distribution. Thus EI is

$$\frac{\mathbb{E}\left[\left(2 \cdot 1_{V>0} - 1\right) U\right]}{2\sqrt{\frac{2N}{\pi}}\sigma}. \tag{75}$$

Note that $V > 0$ is the same as $\frac{V}{a_N} > 0$ as a_N is a positive constant.

Let $U_z \equiv \sum_{i=1}^{N} u_i^z$ for $z = 2, 3, \ldots$. $\frac{V}{a_N} = U + \frac{b_N}{a_N}U_2 + \frac{c_N}{a_N}U_3$. By the Central Limit Theorem, ignoring terms that vanish more rapidly than the leading terms for large N,

$$\frac{1}{N}\begin{pmatrix} U \\ \frac{V}{a_N} \end{pmatrix} \sim \mathcal{N}\left(\begin{array}{c} 0 \\ -\frac{\mu_3}{2\sigma^2(N-1)} \end{array}, \frac{1}{N}\begin{bmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{bmatrix}\right).$$

Inefficiency can arise only when U has the opposite sign to $U + \frac{b_N}{a_N}U_2 + \frac{c_N}{a_N}U_3$ and its magnitude is the integrated absolute value of U over the events when this occurs. Note that, by asymptotic perfect correlation of $U + \frac{b_N}{a_N}U_2 + \frac{c_N}{a_N}U_3$ and U, this sign switching can occur only when U has the same sign as μ_3. Without loss of generality, let us assume μ_3 is positive. Then inefficiency results approximately whenever $U < \frac{\mu_3 N}{2(N-1)}$. We can thus calculate inefficiency as

$$\int_0^{\frac{N\mu_3}{2\sigma^2(N-1)}} \frac{Ue^{-\frac{U^2}{2N\sigma^2}}}{\sqrt{2\pi N}\sigma} dU \leq \int_0^{\frac{N\mu_3}{2\sigma^2(N-1)}} \frac{U}{\sqrt{2\pi N}\sigma} dU = \frac{N^2\mu_3^2}{8\sigma^4(N-1)^2\sqrt{2\pi N}\sigma}.$$

Thus EI is bounded above by

$$\frac{N\mu_3^2}{16\sigma^6(N-1)^2}$$

as reported in the text.

When $\mu_3 = 0$ this analysis breaks down as the approximation to the mean loses its leading term. By the symmetry of the normal distribution, the fourth moment, like the second moment, only makes a complementary contribution and one that vanishes relative to that of the second moment. Following analogous calculations to those above but allowing a fifth-order rather than third-order approximation yields that EI is bounded above by

$$\frac{9N\mu_5^2}{16\sigma^{10}(N-1)^3},$$

which clearly has a faster rate then when $\mu_3 = 0$. Calculating the value of these constants is tricky in our calibrated examples because there is no simple way to make both the mean and the third moment vanish without dramatically changing the set-up. The simplest fix we could find was adjusting both the balance in the population among non-gays and their average valuations until $\mu = \mu_3 = 0$. This requires that about 51% of non-gays support oppose gay marriage but that they are on average willing to pay $73k to see it defeated...a pretty unrealistic example, but one where the calculation is at least possible to conduct.

In this case, the constant is quite large, approximately 4800. But the rate is so much faster that even in a population of 101 individuals this yields 4.8% inefficiency, coincidentally almost precisely the same as in the $\mu_3 \neq 0$ case considered in the text. Clearly the decay rate is much more rapid from there on out, with inefficiency dwindling to about a few hundredths of a percent in a population of only a thousand. Thus it seems likely that not only is the formal rate much faster in this case, but this case leads to much lower inefficiency in practical examples.

Another interesting thing to note is that this inefficiency is in the opposite direction to that created by μ_3 as long as the two moments have the same sign. This is because the normal distribution, while concave near its peak, becomes convex towards its tails. Given the similar magnitude of these opposite distortion for small N, it may be that our analysis significantly overstates inefficiency in many cases; this may account for a significant part of the better performance of QV in small- and medium-sized populations relative to the limiting predictions discussed in Appendix E below.

D.2 Fat tail $\mu \neq 0$ constant calculations

To include a $u^\star$ cut-off, the perceived price among moderates must be $\frac{(N-1)\mu}{2\sqrt{|u^\star|}}$. Thus the inverse of the price perceived by the average extremist must be $\frac{2\sqrt{|u^\star|}}{q(N-1)\mu}$. Because extremists buy approximately the mean number of votes up to a converging-to-unity multiplicative constant, extremist with value u approximately has first-order condition

$$|u| \approx 2\sqrt{|u^\star|}qp(u) \implies p(u) = \frac{|u|}{2q\sqrt{|u^\star|}} \implies \frac{1}{p(u)} \approx \frac{2q\sqrt{|u^\star|}}{|u|}.$$

Averaging over the Pareto tail yields that, on average across all extremists,

$$\frac{1}{p} \approx \frac{2\alpha q\sqrt{|u^\star|}}{(\alpha+1)\,|u^\star|}.$$

Thus the price perceived by moderates is approximately $\frac{(\alpha+1)|u^\star|}{2\alpha q\sqrt{|u^\star|}}$. Thus, employing our calculation from above, the "supply" of extremists consistent with the cut-off $u^\star$ is

$$\frac{(\alpha+1)\,|u^\star|}{2\alpha q\sqrt{|u^\star|}} \approx \frac{(N-1)\mu}{2\sqrt{|u^\star|}} \implies q \approx \frac{(\alpha+1)\,|u^\star|}{\alpha\mu(N-1)}. \tag{76}$$

By the Pareto tail assumption and the Bonferroni approximation, the demand for being an extremist beyond $u^\star$ is $k(N-1)\left(|u^\star|\right)^{-\alpha}$. Combining demand with supply we thus obtain

$$k(N-1)\left(|u^\star|\right)^{-\alpha} \approx \frac{(\alpha+1)\,|u^\star|}{\alpha\mu(N-1)} \implies |u^\star| = \left[\frac{k(N-1)^2\alpha\mu}{\alpha+1}\right]^{\frac{1}{\alpha+1}} \implies$$

$$q \approx k(N-1)\left[\frac{\alpha+1}{k(N-1)^2\alpha\mu}\right]^{\frac{\alpha}{\alpha+1}} = k^{\frac{1}{1+\alpha}}\left(\frac{1}{\mu}+\frac{1}{\mu\alpha}\right)^{\frac{\alpha}{\alpha+1}}(N-1)^{-\frac{\alpha-1}{\alpha+1}}.$$

Note that this calculation of q overestimates limiting inefficiency for several reasons:

1. Moderate strategies are not perfectly proportional and because they are entirely driven in the limit by the event an extremist exists, they deviate from proportionality by moderates

opposing the extremists having a lower price than those supporting the extremists. This leads to a (slowly) vanishing bias that works against the extremists and reduces q somewhat.

2. Bonferroni's approximation overestimates the chance of an extremist.

3. An extremist does not always win, even when she exists (though she does always win asymptotically).

4. In the event where an extremist exists and manages to win, the expected value of U is somewhat below its mean value, so this event causes somewhat less than damage than a completely loss of efficiency in these cases.

Thus our calculation of q upper bounds the inefficiency for large N.

E Numerical Methods and Results

We now describe our methods and some of our numerical results for first medium-large and then for small values of N. Many additional results were generated and not reported here.

E.1 Medium and large populations

We employ a grid-based iterative procedure that repeatedly computes the optimal vote levels $v_t(u)$ for each utility level u *given* the vote levels v_{t-1} last period, and continues until the between-period change in vote levels is below some threshold $\epsilon_V > 0$. We assume that, since the population size N is large enough so that the sum of $N-1$ other votes is approximately normally distributed.

E.1.a Iterative procedure

At any period $t+1$ in our program, we have a grid of utility values, a matrix of sample utility values, a discontinuity point u_t and a grid of values that represents the voting function v_t. We compute

$$\mu_v := \mathbb{E}[v_t|u > u_t], \ \sigma_v := \sqrt{\mathbb{V}[v_t|u > u_t]}$$

using a Monte-Carlo simulation over the matrix of sample utilities. Define the normal density $g_e := \phi_{\mu_v,\sigma_v}$ with mean μ_v and standard deviation σ_v, and define the "smeared" density

$$g_m(v) := \frac{\int_{-\infty}^{u_t} f(u) g_e(v - v(u))\, du}{F(u_t)}.$$

The mixture

$$g(v) := (1 - (N-1)F(u_t))g_e(v) + (N-1)F(u_t)g_m(v)$$

is then the density of pivotality, accounting for the possibility that there is an extremist. Note that g_e determines the optimal vote level (given v) for extremists (since the contribution from the g_m near $-\mu_v$ is negligible).

The two vote levels v_a, v_b and utility levels u_a, u_b at which the rays v/u_a and v/u_b are tangent to $g_e(-v)/2$ are given by[37]

$$v_a = \frac{-\mu_v - \sqrt{\mu_v^2 - 4\sigma_v^2}}{2},\ v_b = \frac{-\mu_v + \sqrt{\mu_v^2 - 4\sigma_v^2}}{2}$$

and

$$u_a = \frac{1}{g_e'(v_a)},\ u_b = \frac{1}{g_e'(v_b)}.$$

The mixture g is sufficiently close to the normal density g_e near v_a and v_b so that (u_b, u_a) will contain the (unique) discontinuity point. Note that $u_a > u_b$ and $v_a < v_b$, and that (u_b, u_a) is the interval of utility levels u whose rays v/u intersect $g_e(-v)/2$ three times. We may now define the moderate voting function $v_m\colon [u_b, \infty) \to \mathbb{R}$ and the extremist voting function $v_e\colon (-\infty, u_a] \to \mathbb{R}$ by the relations

$$\frac{g(-v_m(u))}{2} = \frac{v_m(u)}{u},\ \frac{g_e(-v_e(u))}{2} = \frac{v_e(u)}{u},$$

where we define v_e as the minimum and v_m as the maximum if there are multiple such vote levels for a given u.

We can now write the extremist welfare as

$$W_e(u) := u(1 - G_e(-v_e(u))) - v_e(u)^2$$

and the moderate welfare as

$$W_m(u) := u(1 - G_m(-v_m(u))) - v_m(u)^2.$$

The new discontinuity point u_{t+1} is the unique point in (u_b, u_a) for which $W_e(u_{t+1}) = W_m(u_{t+1})$. We use a binary-search algorithm[38] to find this point in logarithmic time and define the new voting function v_{t+1} so that $v_{t+1} \equiv v_e$ on $(-\infty, u_{t+1}]$ and $v_{t+1} \equiv v_m$ on (u_{t+1}, ∞). Figure 3 displays this situation graphically: the yellow dotted line is the ray v/u_a, the green dotted line is the ray v/u_b, the black curve is $g_e(-v)/2$, the purple curve is $g_m(-v)/2$ and the red line is the ray v/u_t.

E.1.b Initialization

We initialize the voting function for a distribution with CDF F and mean μ to have a discontinuity at the unique utility level u_d for which

$$\frac{|u_d|}{F(u_d)} = \mu(N-1)^2.$$

To this end, we begin the procedure by finding a real number $a > 0$ such that the linear voting function $v(u) = au$ restricted to (u_d, ∞) with a discontinuity initially at u_d induces a discontinuity at u_d. For any $a > 0$ we define the voting function $v(u) = au$ and apply the iterative procedure

[37] This follows since the tangency condition $g_e'(v)v = g_e(-v)/2$ simplifies to the quadratic equation $v^2 + \mu_v v + \sigma_v^2 = 0$.

[38] Let $\epsilon = u_a - u_b$, and let $u_1 = \frac{\epsilon}{2}$. If $u_t < u_b$ or $W_m(u_t) < W_e(u_t)$ then we update $u_{t+1} = u_t + \frac{\epsilon}{2^{t+1}}$, and if not then we update $u_{t+1} = u_t - \frac{\epsilon}{2^{t+1}}$.

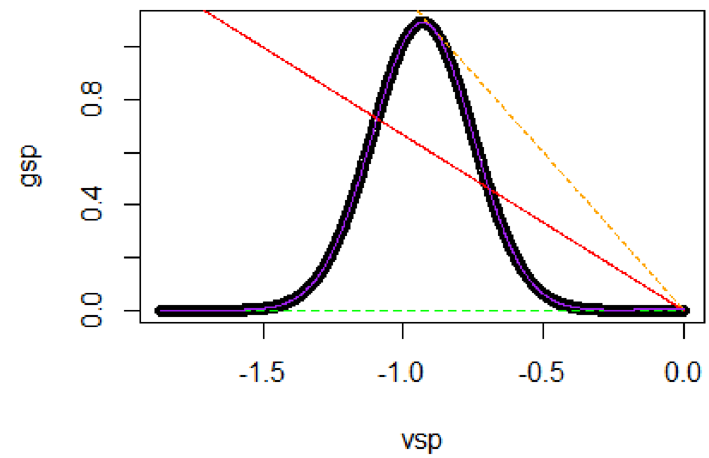

Figure 3: Our procedure for locating the discontinuity point that marks the beginning of extremist behavior.

in Section 1 to find the discontinuity point $u(a)$ induced by this voting function, relative to our matrix of sample utility levels. Let W_m and W_e denote the moderate and extremist welfare functions induced by $v \equiv au$. Assuming that the a value that induces this discontinuity is in $[0, 1]$, we initialize with $a_1 = 0.5$. Note that $W_e(u_d) > W_m(u_d)$ iff $u(a) < u_d$ and $W_e(u_d) < W_m(u_d)$ iff $u(a) > u_d$, so we update $a_{t+1} = a_t + \frac{1}{2^{t+1}}$ in the first case and $a_{t+1} = a_t - \frac{1}{2^{t+1}}$ in the latter case. We repeat this procedure until we obtain a value $a_d > 0$ such that $u(a)$ is sufficiently close to u_d, and we store the grid of extreme vote levels v_e from the step $a = a_d$ to construct the initial voting function v_0 so that $v_0 \equiv v_e$ on $(-\infty, u_d]$ and $v_0 \equiv au$ on (u_d, ∞). To ensure the stability of our procedure we limit the between-period movement of the new discontinuity point u_{t+1} so that $|F(u_{t+1}) - F(u_t)| \leq kF(u_t)$, and we similarly bound the amount that the moderate votes can change between period by k. In our current implementation we used the value $k = 0.1$.

E.1.c Computing welfare

The matrix $\mathbf{sample_u}$ has S_{size} rows and N columns, where N is the number of voters and $S_{\text{size}} = 1000$ in our implementation. Each row thus corresponds to a collection of N voters with utilities drawn from the same distribution. Let u_{ij} denote the utility of the j-th individual in the i-th row (i.e. the (i, j)-th entry of $\mathbf{sample_u}$). After we obtain the equilibrium voting function v^* from the above procedure we define the relative efficiency of QV as the fraction of rows j of $\mathbf{sample_u}$ in which $\sum_{i=1}^{N} v^*(u_{ij})$ the same sign as $\sum_{i=1}^{N} u_{ij}$. The relative efficiency of majority rule is defined as the fraction of rows j in which the median utility (of row j) has the same sign as $\sum_{i=1}^{N} u_{ij}$.

E.1.d Double-Pareto log-normal

We calibrated a double-Pareto log-normal distribution with upper tail index $\alpha = 3$, lower tail index $\beta = 1.43, \mu = 10.9$ and $\sigma = 0.45$ to fit the US distribution of income; the distribution has

PDF

$$f_W(x;\alpha,\beta,\mu,\sigma)=\frac{\alpha+\beta}{x(\alpha+\beta)}\phi\left(\frac{\log(x)-\mu}{\sigma}\right)\left(R\left(\alpha\sigma-\frac{\log(x)-\mu}{\sigma}\right)+R\left(\beta\sigma+\frac{\log(x)-\mu}{\sigma}\right)\right),$$

where $R(x)=\frac{1-\Phi(x)}{\phi(x)}$. We then generate a sample of normally distributed idiosyncratic values $v\sim N(0,1)$ and double-Pareto log-normally distributed income levels $w\sim \text{dPln}(\alpha,\beta,\mu,\sigma)$ with correlation $\rho=0.72$ and define each voter's utility as $u=vw$ (the correlation must be positive so that $\mathbb{E}[u]>0$). The CDF of $u=vw$ can then be expressed analytically as

$$F(u)=\int_{-\infty}^{\infty}\Phi\left(\frac{\frac{u}{F_W^{-1}(\Phi(x))}-\rho x}{1-\rho^2}\right)\phi(x)\,dx,$$

where F_W is the CDF of the double-Pareto log-normal distribution (i.e. the integral of f_W).

E.1.e Bounded Distributions

For bounded distributions, we define the smeared density by

$$g_m(v)=g_e(v-v_{min})$$

where v_{min} is the vote level for an individual with the minimum utility level.

E.1.f Beta Distribution

When the parameters α and β are integer-valued, we use the closed-form expression for the beta distribution function

$$F(x;\alpha,\beta,a,c)=\sum_{i=\alpha}^{\alpha+\beta-1}\binom{\alpha+\beta-1}{i}\Big(\frac{x-a}{c-a}\Big)^i\Big(1-\frac{x-a}{c-a}\Big)^{\alpha+\beta-1-i}$$

where a is the minimum of the distribution and c is the maximum. In the case where α and β are not integer-valued, we approximate the CDF evaluated at the discontinuous utility level u_d by

$$F(u_d;\alpha,\beta,a,c)\approx\frac{1}{B(\alpha,\beta)}\int_a^{u_d}\big(\frac{y-a}{c-a}\big)^{\alpha-1}dy=\frac{1}{B(\alpha,\beta)}\frac{c-a}{\alpha}\big(\frac{u_d-a}{c-a}\big)^{\alpha}$$

In our simulations with non-integer α and β u_d is within 10^{-11} of the minimum of the distribution, so this is approximation is very close to the true value. Since u_d is so close to the minimum in these cases, we used the R package Rmpfr to increase the floating point precision for u_d to 80 bits.

E.1.g Gay marriage example

Let $X_1,...,X_4$ be the utility distributions in each subpopulation and $\alpha_1,...,\alpha_4$ be the corresponding proportions for each subpopulation. Refer to table 2. Let Y_1 be the distribution of votes for

Subpopulation	Proportion	Min	Max
X_1	.52	-10000	0
X_2	.44	0	10000
X_3	.033	0	40000
X_4	.007	0	200000

Table 2: Gay Marriage Example with Uniform Subpopulations.

people with utility drawn from X_1 conditioned on them being moderates, and let M be the proportion of people drawn from X_1 who behave as moderates. Similarly, let Y_2, Y_3, and Y_4 be the distributions of votes for people drawn from X_2, X_3, and X_4. Let Y be the distribution of votes for moderates across the entire population. We calculate the expectation and variance of Y as follows:

$$E[Y] = \frac{M\alpha_1 E[Y_1] + \alpha_2 E[Y_2] + \alpha_3 E[Y_3] + \alpha_4 E[Y_4]}{M\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}$$

$$E[Y^2] = \frac{M\alpha_1 E[Y_1^2] + \alpha_2 E[Y_2^2] + \alpha_3 E[Y_3^2] + \alpha_4 E[Y_4^2]}{M\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}$$

$$Var(Y) = E[Y^2] - E[Y]^2$$

Then the expectation and variance for the sum of votes G are

$$E[G] = (N-1)E[Y], Var(G) = (N-1)Var(Y)$$

E.1.h Computing Welfare in Cases with a Discontinuity

Let p be the probability that a voter is an extremist and q be the probability that a voter swings the election given they are an extremist.Then

$$p = \int_{-\infty}^{u_d} G(-v(u)) f(u|u < u_d)\, du,$$

$$q = F(u_d)$$

and the expected inefficiency

$$EI = pq = \int_{-\infty}^{u_d} G(-v(u)) f(u)\, du$$

We estimate this integral using a Riemann sum approximation.

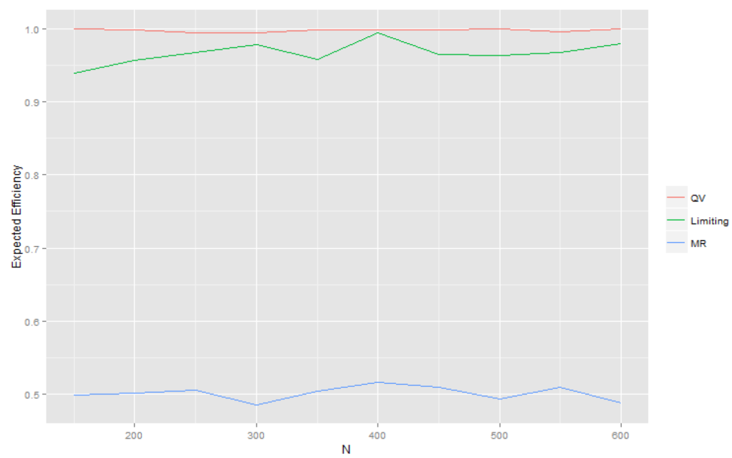

Figure 4: Double-Pareto log-normal results with a correlation $\rho = 0.71$.

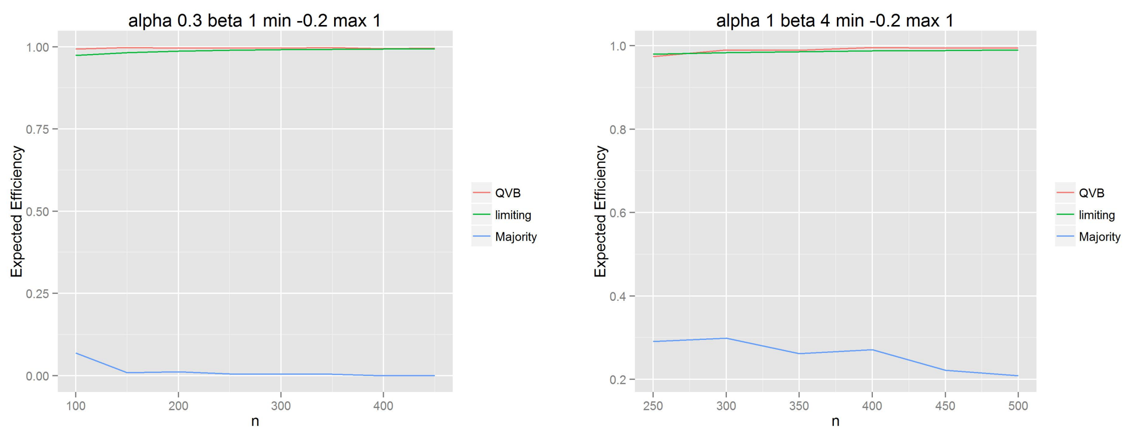

Figure 5: Pearson Type I distribution results

E.1.i Difficulties

For values of N close to the minimum value for which there is a discontinuity in a given bounded utility distribution, the program may find after some iterations that it is not optimal for any individual to act as an extremist. In such cases we continue adjusting the discontinuity towards the minimum of the distribution on each iteration while setting the extremists' vote levels to their vote levels from the previous iteration. However, the program continues to find that there should not be a discontinuity in the voting function on all successive iterations.

E.1.j Welfare results

The double-Pareto log-normal results below were obtained with a correlation $\rho = 0.71$.

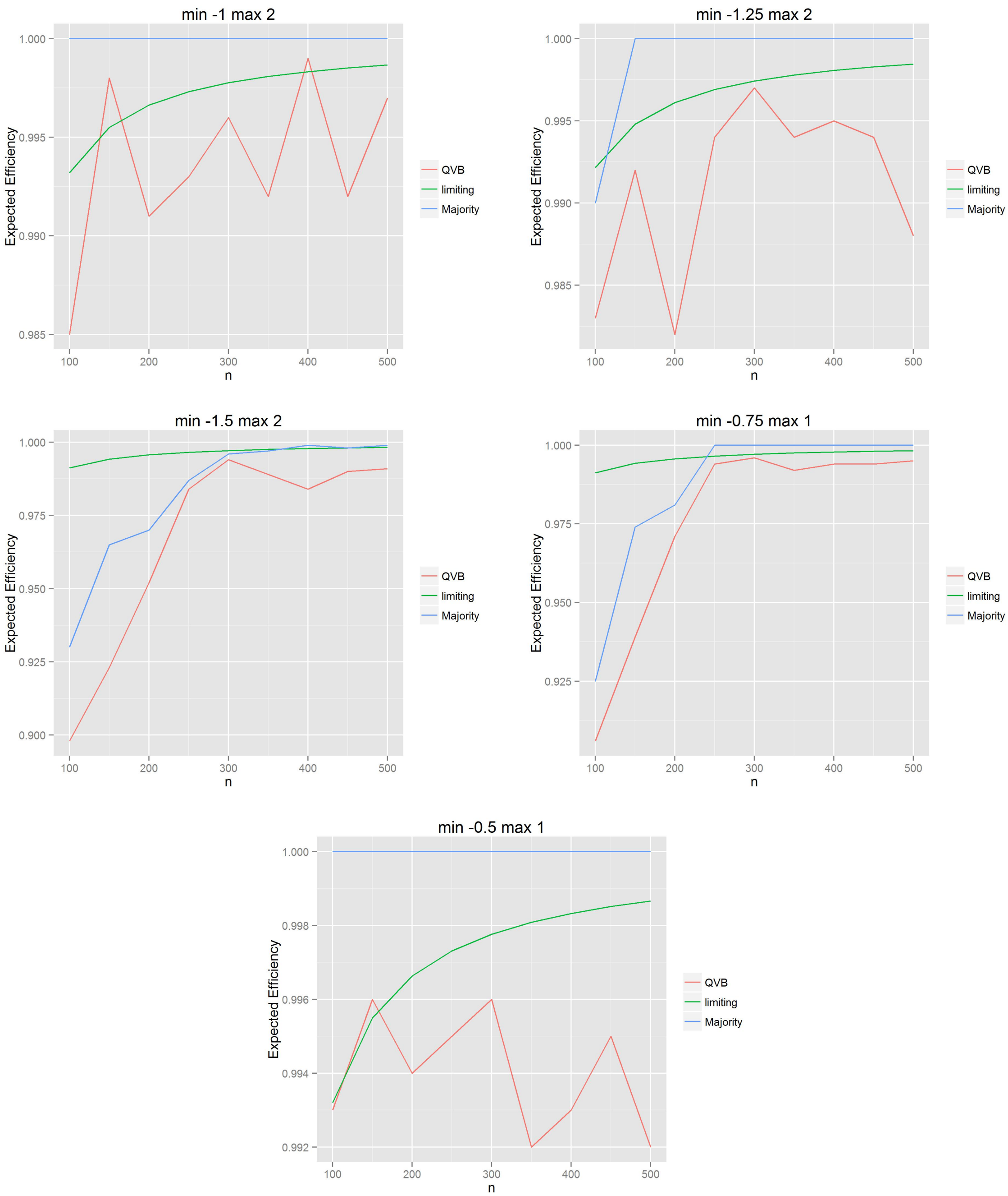

Figure 6: Uniform distribution results

E.2 Small populations

To calculate approximate equilibria for small populations we used standard computational game theory techniques for solving for equilibria.We began by initializing voting functions for each individual separately to $v_{i0}(u) = \frac{u}{2}$, so as to allow for the potential identification of asymmetric equilibria.[39] We then entered a loop to calculate equilibrium values of v_{i0} for each individual until it "converged" in the sense that the "update error" ϵ_t defined in the loop below being less than $.005$ or until $8N$ loop iterations had passed, in which case it was determined that the loop was not converging. These and all other numbers below were obtained by trial-and-error to involve the minimum computation time necessary for consistent and reliable results. The loop in period t ran the following steps:

1. If t $\mod$ N=0 check to see if all individuals have converged or if $t = 8N$. If either has occurred, terminate. If not, continue the loop.

2. For individual $i = t \mod N + 1$, draw $500,000$ random values for each of the other $N - 1$ individuals. Use each set of draws to calculate the sum of all other votes using v_{it-1}. Tabulate a histogram of these values on 5000-point, evenly-spaced grid from the lowest to the highest observed value of the sum of other votes. To this pure empirical PDF, fit a 13th degree polynomial approximation for smoothing that minimizes mean-squared error to the pure empirical PDF.

3. Divide the support of u, or in the case of the normal distribution the mean of u plus and minus 5.8 standard deviations, into 5000 evenly-spaced grid points. For each grid point, numerically solve for the number of votes maximizing expected utility using Newton's Method on the first-order condition $v_i(u) = \frac{p(-v)}{2}u$, with an approximation accurate two significant digits. Approximate this function using a piecewise cubic spline interpolation (except in the case when $N = 2$ and the distribution is uniform, in which case a 10th degree polynomial offers a better approximation consistently).

4. Calculate the Euclidean norm between this function and v_{it-1}. If this is less than $.005$ label individual i as "converged"; if it is greater than $.005$ label individual i as "not converged".

5. Store this as v_{it} and for all $j \neq t \mod N$replace set $v_{jt} \equiv v_{jt-1}$.

6. Loop.

Before moving forward, the distribution of the sum of votes was inspected visually to ensure that there was only one solution to the first-order conditions and that it corresponded to a maximum in the final results. Then, using the output values of v_{iT}, where T was the final period of the loop, we calculated Expected Inefficiency (EI) as follows

1. Draw 500,000 random values of each individual. Let u_i^j but the$j$th utility values for individual i. Let $U_j \equiv \sum_{i=1}^N u_i^j$, $V_j \equiv \sum_{i=1}^N v_{iT}\left(u_i^j\right)$ and let $M_j \equiv 2\sum_{i=1}^N 1_{u_i^j \geq 0} - N$.

[39] Other initialization values were tried, including asymmetric ones, but very similar results were typically obtained though often after more interactions.

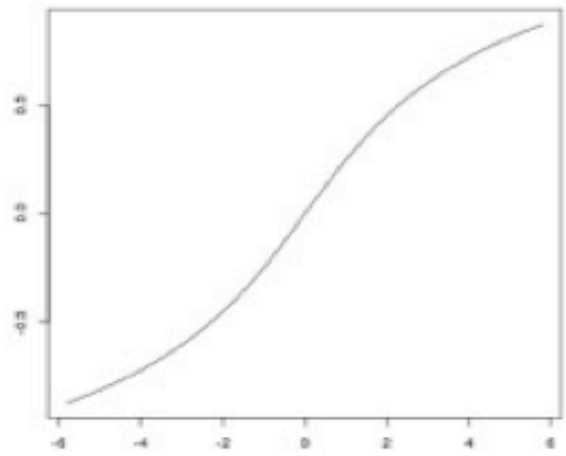

Figure 7: Approximate equilibrium voting function for $N = 10$ given values are drawn from a standard normal distribution.

2. EI of QV is

$$\frac{1}{2} + \frac{1}{500,000} \sum_{j=1}^{500,000} \frac{U_j 1_{V_j \geq 0} - U_j 1_{V_j < 0}}{2\left(U_j 1_{U_j \geq 0} - U_j 1_{U_j < 0}\right)}$$

and of majority rule[40] is

$$\frac{1}{2} + \frac{1}{500,000} \sum_{j=1}^{500,000} \frac{U_j \left(1_{M_j > 0} + \frac{1_{M_j = 0}}{2}\right) - U_j \left(1_{M_j < 0} + \frac{1_{M_j = 0}}{2}\right)}{2\left(U_j 1_{U_j \geq 0} - U_j 1_{U_j < 0}\right)}.$$

We used these methods for N ranging from 2 to 50. The distributions considered were the normal distribution, with parameters μ and σ^2, and the Pearson Type I distribution with parameters $\underline{u}$ and $\overline{u}$ for the bounds of the support, α and β for the shape and position within these bounds. We tried many values for these parameters and N ranging from 2 to 10. In this appendix we only show results that directly support claims in the text.

Figure 7 shows the converged voting rule when $N = 10$ and values are drawn from a standard normal distribution. The shape is close to linear but has a gentle version of the S-shape that would be predicted by the characterization in Subsection 5.4: individuals with large values in either direction buy fewer votes per unit of value because they are less likely to be pivotal with a marginal vote.

Figure 8 shows our first set of results, for the uniform distribution varying over different ranges of the bounds and different values of N. Expected efficiency, rather than expected inefficiency, is graphed. For N below 4 and/or an upper bound on the distribution less than 1.33 QV always outperforms majority rule. However as the mean and the median shift up and N becomes large, 1p1v outperforms QV, though never by a large amount.

Figure 9 shows results for normal distributions with varying μ and σ^2 (as well as N) in the left panel and varying μ holding σ^2 fixed at 1 in the right panel. In the left panel, all cases with $\mu = 0$

[40]Note that ties are non-generic under QV and thus we simply assume they are broken in favor of the alternative, but under majority rule they matter and thus are broken by a coin flip.

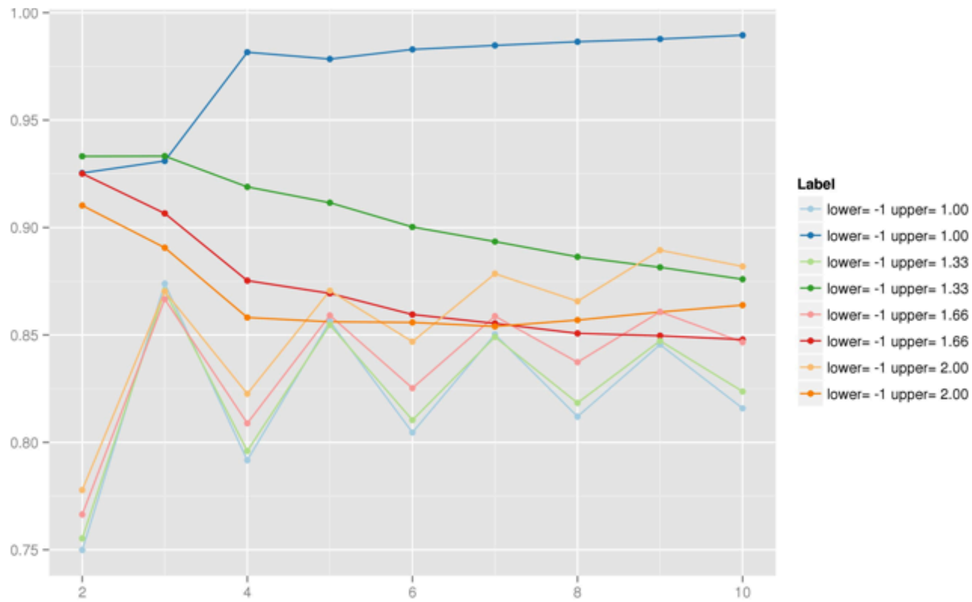

Figure 8: Expected efficiency of QV and 1p1v under a uniform value distribution for the various values of N (horizontal axis) and different upper and lower bounds for the distribution (the colors, correspond to the legend at right). The dark plots are QV, the light plots 1p1v.

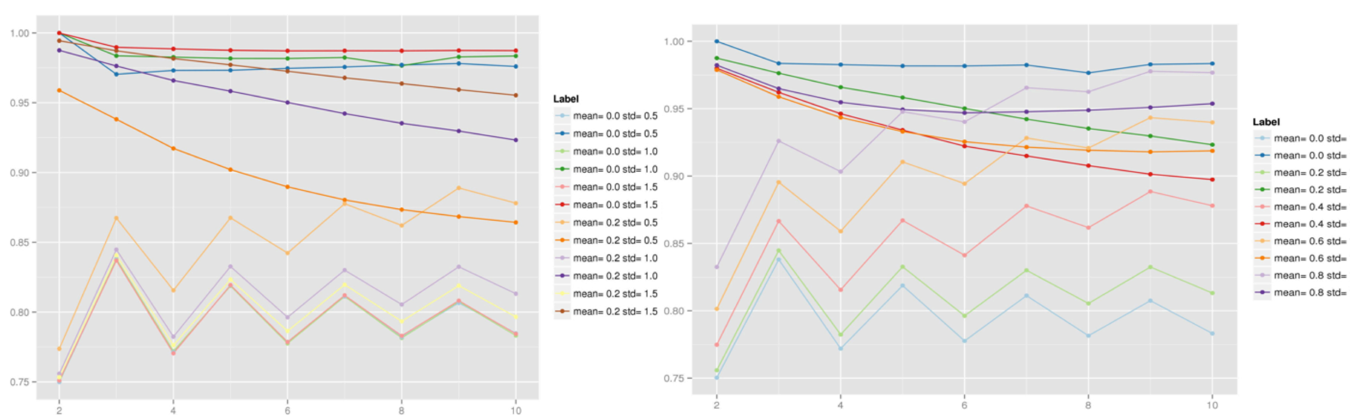

Figure 9: Expected efficiency of QV and 1p1v under a normal value distribution for the various values of N (horizontal axis) and means and variances (the colors, correspond to the legend at right). The dark plots are QV, the light plots 1p1v.

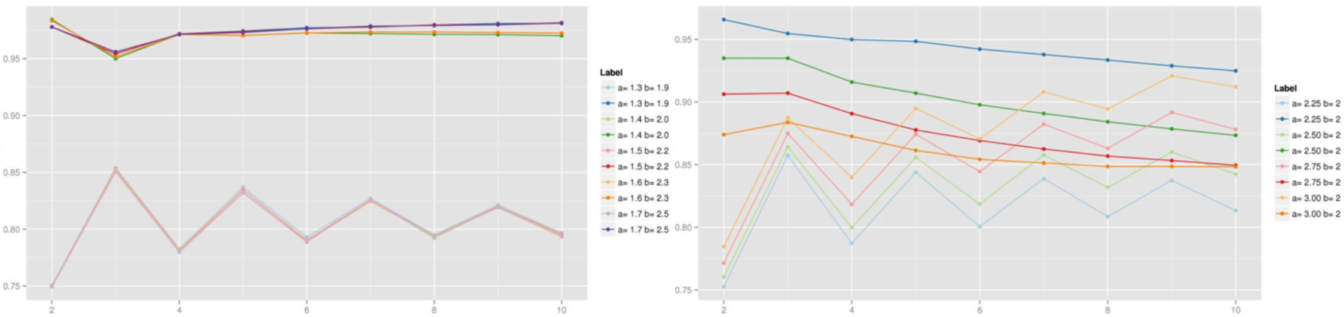

Figure 10: Expected efficiency of QV and 1p1v under a Pearson Type I. The left panel shows cases with, a negative median but positive mean, where QV consistently outperforms 1p1v. These have $\underline{u} = -1$ and $\overline{u} = 1.5$, with the values of α and β varying by colors as indicated at the legend at right. The right panel shows cases, with a positive mean and median, where QV sometimes underperforms 1p1v. There $\underline{u} = -\overline{u} = -1$.

have QV dominating. Even when the $\mu = .2$ QV dominates except in the case when $N = 10$ and $\sigma = .5$; for larger σ or smaller N, QV again dominates. The right panel varies the mean over a wider range and exhibits a wider range of behaviors as a consequence. For $N = 10$, where the gap between QV and 1p1v is largest, QV's performance is non-monotone in the mean, falling and then rising, while 1p1v monotonically improves as the mean grows larger, leading 1p1v to outperform QV by 4-5 percentage points in some cases.

Finally, Figure 10 highlights our most striking results, which occur under various version of the Pearson Type I distribution. In the left panel is pictured a case where the support of the distribution is right-skewed but $\beta > \alpha$ so that the mean of the distribution is positive while its median is negative. The different colors represent different values of α and β, which increase in tandem across curves thereby holding the mean approximately constant while reducing the variance. In all cases in this class, QV dramatically outperforms 1p1v, achieving near-perfect efficiency, though this is most extreme when the variance is smallest. In the right panel is pictured a case when the support of the distribution is symmetric: $[-1, 1]$. β is fixed at 2 and $\alpha > 2$ varies. As it increases (moving towards brighter/lighter colors, the mean increases and the variance declines, making the setting less favorable to QV. When α is only 2.25 QV still significantly dominates majority rule for all values of N. However as α rises, especially for large N, QV's performs declines (though in a concave fashion; if α increase further beyond this point it begins to improve again) while 1p1v improves. Once $\alpha = 3$ QV only outperforms 1p1v for small even numbers of N where tie breaking is important.

F Robustness

F.1 Collusion

For brevity we only discuss the calculations of rates in this appendix. The more concrete quantities reported in the text use calculations of constants that we omit here.

Let u_M be the sum of the values of all the individuals in the collusive conspiracy. First we consider the approximate distribution of u_M in the worst case. Without loss of generality, let

$\mu \geq 0$ and when $\mu = 0$ that $\alpha_- \leq \alpha_+$; as in the text we suppress the index on α. In both cases efficiency is most distorted when all collusive members are drawn as the most extreme members of the lower tail of the distribution, that is the M most negative order statistics. This is true because this maximizes the chance of inefficient extremist behavior when $\mu > 0$ and it creates the largest deviation from proportionality when $\mu = 0$.

For large N and very negative $u^\star$, the probability that the lowest order statistic is less than $u^\star$ is less than $Nk\left(u^\star\right)^{-\alpha}$, for some constants $\alpha > 1, k > 0$ using the Pareto tail approximation. By standard results from extreme value theory, the jth-to-last order statistic is approximately $\frac{1}{j^{\frac{1}{\alpha}}}$ times the last order statistic. Thus, using the continuous approximation, the sum of the last M order statistics is approximately

$$\int_0^M \frac{1}{x^{\frac{1}{\alpha}}} dx = \frac{\alpha}{\alpha - 1} M^{\frac{\alpha-1}{\alpha}}$$

times the first-order statistic. Thus the probability that u_M is larger in absolute value than u is approximately

$$Nk\left[\frac{u(\alpha-1)}{\alpha M^{\frac{\alpha-1}{\alpha}}}\right]^{-\alpha} = Nk\left[\frac{u\left(\alpha-1\right)}{\alpha}\right]^{-\alpha} M^{\alpha-1}.$$

That is, a collusive group raises the probability of extreme values by a factor of order approximately $M^{\alpha-1}$.

On the other hand the typical value of u_m is $\frac{\alpha}{\alpha-1} M^{\frac{\alpha-1}{\alpha}}$ times the last order statistic. This last order statistic is in $\Theta\left(N^{\frac{1}{\alpha}}\right)$. Thus the typical size of u_M is in $\Theta\left(M^{\frac{\alpha-1}{\alpha}} N^{\frac{1}{\alpha}}\right)$.

There are (at least) three basic challenges that limit the efficacy of such a collusive group. First, the group may need to be relatively large to have a significant impact as its optimal vote purchases are only magnified by a factor of the group size. Second, individual members may face unilateral incentives for deviation. Third, to the extent it is anticipated, even probabilistically, collusion will meet with offsetting reactions by other non-colluding agents that may undermine the efficacy of the collusion and even make it counter-productive. We now informally discuss and state propositions formally establishing the limits each of these forces places on the possibility of collusion.

The case in which collusion poses the greatest threat to the efficiency of QV is when $\mu > 0$ and the distribution of values is bounded. Then if the collusive group is entirely unanticipated by the rest of the population even a very small, extreme collusive group can wreck the efficiency of QV. The reason is that while the chance of any single individual wishing to act as an extremist is small, the chance of two individuals together wishing to act as an extremist (given the equilibrium voting behavior) is $\Theta(1)$. While it may be rare to have a single individual extremely close to $\underline{u}$, is very common to have two individuals the sum of whose values are larger than $\frac{\underline{u}}{2}$ in absolute value, which is all that is needed for it to be in their collective interest to buy the approximately requisite $\frac{\sqrt{|\underline{u}|}}{2}$ votes each. This problem is slightly mitigated when tails are fat, as the single most extreme individual accounts for a greater fraction of the top two most extreme individuals' valuations in this case.

To see this note that, by our reasoning in the text, the cost to an extremist coalition of size M of behaving as extremists is only $1/M$ of that of any single individual acting as an extremist. Thus if $u^\star$ is the threshold utility for a single individual to act as an extremist the threshold value for a collusive group to do so is ${u^\star}/{M}$. And the extremist threshold is in $\Theta\left(N^{\frac{2}{1+\alpha}}\right)$ from our analysis in the Subappendix D.2.

Thus the order of the probability of the collusive group finding an extremist strategy attractive is

$$N^{-\frac{2\alpha}{1+\alpha}} N M^{\alpha} M^{\alpha-1} = N^{-\frac{\alpha-1}{1+\alpha}} M^{2\alpha-1}.$$

Thus, depending on the size of M relative to N, the rate of decay of inefficiency changes from $\Theta\left(N^{-\frac{\alpha-1}{\alpha+1}}\right)$ to $\Theta\left(N^{-\frac{\alpha-1}{\alpha+1}} M^{2\alpha-1}\right)$. Inefficiency still dies in the limit as long as M is in $O\left(N^{\frac{\alpha-1}{(\alpha+1)(2\alpha-1)}}\right)$.

On the other hand, when $\mu = 0$, the outcome is somewhat less sensitive to the votes of a few, even quite extreme, individuals buying a large number of votes, as the equilibrium aggregate number of votes is quite large. To see this note that when $\mu = 0$, u_M is of order $M^{\frac{\alpha-1}{\alpha}} N^{\frac{1}{\alpha}}$. By the logic of Subappendix D.1, the inefficiency this causes is proportional to the square of the distortion in the sum of the votes normalized by ${1}/{a_N}$, which is $M-1$ times the normalized value of the collusive group as they magnify their votes by a factor of M, and inversely related to $\sqrt{N}$. $a_N = \Theta\left({1}/{\sqrt[4]{N}}\right)$, so the squared shift in the mean of votes is $\Theta\left(M^{\frac{2\alpha-1}{\alpha}} N^{\frac{4+\alpha}{4\alpha}}\right)$; dividing by $\sqrt{N}$ gives $\Theta\left(M^{\frac{2\alpha-1}{\alpha}} N^{\frac{4-\alpha}{4\alpha}}\right)$. Squaring this distortion and normalizing it relative to aggregate efficiency implies that the EI created by worst case collusion is $O\left(M^{\frac{2(2\alpha-1)}{\alpha}} N^{-\frac{\alpha-2}{\alpha}}\right)$. This dies with N as long as $M = O\left(N^{\frac{\alpha-2}{2(2\alpha-1)}}\right)$, as reported in the text. These results are summarized in the following claim.

Claim 3. If there is a single perfect collusive group of size M and other individuals play as in equilibrium where they believe there is no possibility of collusion taking place, $EI \to 0$ as N grows large as long as

1. $\mu = 0$, colluders are drawn as in the worst case and $M \in O\left(N^{\frac{\alpha-2}{2(2\alpha-1)}}\right)$ so that in the bounded support case $M \in O\left(\sqrt[4]{N}\right)$ or
2. $\mu \neq 0$, colluders are drawn as in the worst case and $M \in O\left(N^{\frac{\alpha-1}{(\alpha+1)(2\alpha+1)}}\right)$ so that in the bounded support case even a fixed number of colluders may stop EI converging to 0.

Here α denotes the smaller of the two α values in 1) and that from the negative tail in 2).

This result offers only a small benefit over VCG and the Expected Externality mechanism in realistic cases. For example, when $\alpha = 3$, the size of the collusive group grows only as $N^{1/10}$ when $\mu = 0$ and only as $N^{1/14}$ when $\mu \neq 0$. In both cases calculating and calibrating constants suggests realistic collusion even in the full California population would only require $3-7$ individuals, which is likely quite feasible without detection. While this is a bit better than the 2 needed for collusion against VCG and EE, it is hardly better.

However, recall that Theorem 2 shows that QV's efficiency occurs at *all* type-symmetric thus its efficiency is coalition-proof, again unlike VCG.[41] This suggests that equilibrium considerations may further limit collusion. The most natural such consideration is incentives for unilateral deviation, which are usually thought to be the primary deterrent to collusion in standard

[41]Note, however, that this is only true of "type-symmetric" collusive strategies where it is individuals of particular types, rather than individuals of particular names, that collude.

markets when there is not a very large number of competitors (Stigler, 1964). These can be quite powerful in the case when $\mu = 0$.

Note that in the case when $\mu = 0$ the payoff of all individuals is approximately quadratic and thus any deviation from unilaterally optimal actions leads to a) an incentive to deviate, b) a marginal incentive to deviate proportional to the deviation from the unilaterally optimal action and c) an aggregate incentive to deviation that is proportional to the square of the deviation from the unilaterally optimal action. Furthermore, to the extent that the payoff is not perfectly quadratic, large deviations of the votes in any direction diminish the chance of a tie and make it optimal for each individual to purchase fewer votes, further depressing the incentives of the collusive group to buy so many votes.

However, in the case when $\mu \neq 0$ they accomplish little. To see this note that when $\mu \neq 0$, members of an "extremist conspiracy" who band together to behave like extremists have strong strategic complementarity: once one member of the group has purchased, say, half the necessary number of votes, it becomes much cheaper for the other member of the conspiracy to buy the other half and thus swing the decision. Thus while the possibility of unilateral deviation makes collusion a bit harder (requiring both colluding individuals' values, rather than their average value, to exceed a threshold) it does not dramatically change the picture.

More formally, for an extremist conspiracy to maintain each individuals' unilateral incentive to participate, it must be that each individual purchase votes no greater than the square root of the magnitude of her value. This implies that the aggregate vote purchased by an extremist coalition cannot exceed the sum of the squares of values of its members. By our logic above, this approximately equals

$$\int_0^M x^{-\frac{1}{2\alpha}} dx = \frac{2\alpha}{2\alpha - 1} M^{\frac{2\alpha-1}{2\alpha}}$$

times the value of the square root of the last-order statistic. The total votes bought by moderates is $\Theta\left(N^{\frac{1}{\alpha}}\right)$ as it is on the order of square root the extremist threshold. Dividing this by $\frac{2\alpha}{2\alpha-1} M^{\frac{2\alpha-1}{2\alpha}}$, we have that, for the extremist conspiracy to succeed, the square root of the first-order statistic must be in $\Theta\left(N^{\frac{1}{\alpha}} M^{-\frac{2\alpha-1}{2\alpha}}\right)$. This implies that that first-order statistic itself must be in $\Theta\left(N^{\frac{2}{\alpha}} M^{-\frac{2\alpha-1}{\alpha}}\right)$. The probability of the first-order statistic being that large is, by our logic above

$$\Theta\left(N N^{-2} M^{2\alpha-1}\right) = \Theta\left(\frac{M^{2\alpha-1}}{N}\right),$$

which clearly dies with N if $M = O\left(N^{1/2\alpha-1}\right)$. These results are summarized in the following claim.

Claim 4. Suppose there is a single perfect collusive group of size M and other individuals play as in equilibrium where they believe there is no possibility of collusion taking place. Then if $\mu = 0$ there are always unilateral deviation incentives for any collusive behavior for large N and the size of marginal deviation incentives is at least proportional to the deviation from unilateral behavior. If $\mu \neq 0$ a collusive agreement non-vanishing inefficiency with no unilateral deviation incentives is possible only when $M \in \Omega\left(N^{\frac{1}{2\alpha-1}}\right)$ in the worst case.

Thus when $\mu = 0$, the unilateral incentives for deviation from a collusive group are likely to prevent collusion from significantly denting large-population efficiency. This would be especially true if collusion was made illegal, as vote-buying and collusion in markets are illegal in most developed societies, and was rigorously policed with whistle-blowing incentives that further increase the incentive for unilateral deviations. When $\mu \neq 0$, such incentives do limit the ability to engage in collusion when $\alpha = 3$ and the population is very large; in the calibrated example, a collusive group would have to have nearly 100 individuals. This might well be large enough that the collusive group would find it difficult to conceal their activity from authorities, especially given it would be fairly easy to identify the individuals likely to be the most extreme opponents of the alternative, as they would all be quite wealthy and opinionated. This is a sharp contrast to VCG and EE, where collusion is exactly or approximately individually incentive compatible, requiring only that two individuals coordinate their behavior, as we discuss in the next section. However, QV performs no better in this regard when the value distribution is bounded.

Therefore, the strongest check on extremist collusion may arise not from the explicit enforcement activities of the authorities, but from the implicit enforcement by the public. If the public realizes that an extremist collusive group is operating with significant probability, this will induce them to believe that there is a significant chance of a tie and thus dramatically raise the number of votes they buy, with which the extremist coalition must then compete. This is analogous to the observation that in standard markets collusive arrangements attract entry that undermines them (Stigler, 1964).

It is again instructive to consider this logic in the familiar case of bounded support. Suppose that individuals believe that the chance of an extremist coalition forming and succeeding is r. Then the ratio of votes purchased by an average moderate to those purchased by the average member of the extremist coalition will be, by the logic of Subsection 5.5, approximately ${}^{r\mu}/_{M|\underline{u}|}$ because the extremists know they exist when they do and multiply their votes by their coalition size to internalize the externalities to other members of the coalition. For the extremist coalition to win when it does exist, on the other hand, this requires that the total votes purchased by the extremist coalition (more than) balance those purchased by others on average so that

$$r\mu N \leq M^2 \left|\underline{u}\right| \implies r \leq \frac{M^2 \left|\underline{u}\right|}{\mu N},$$

so that M must be $\Omega\left(\sqrt{N}\right)$ in order for the rational-expectations chance of the extremist coalition not to vanish as N grows large. Even if the moderates significantly underestimate the chance of such a coalition, as long as they do so only by a constant factor, this will impact only the constant and not the rate at which extremist coalitions must vanish.

The following claim extends this to the Pareto tail case, but does not consider $\mu = 0$ because unilateral deviations are the binding constraint there. A collusive group of size M that is rationally expected to exist with (non-vanishing) probability r will, by the logic in the text, buy votes relative to those of population of moderates in

$$\Theta\left(\frac{M M^{\frac{\alpha-1}{\alpha}} N^{\frac{1}{\alpha}}}{r\mu N}\right).$$

But we know that these magnitudes must approximately match or the extremist group to be both effective and not wasteful. Thus, for r not to vanish, we must have

$$M^{\frac{2\alpha-1}{\alpha}} \in \Omega\left(N^{\frac{\alpha-1}{\alpha}}\right) \implies M \in \Omega\left(N^{\frac{\alpha-1}{2\alpha-1}}\right).$$

Claim 5. Suppose that $\mu > 0$ and a single perfectly collusive group of size M drawn from the most extreme individuals with negative utility emerges with common knowledge probability r. Then r must vanish in N as long as $M = \in O\left(N^{\frac{\alpha-1}{2\alpha-1}}\right)$.

The constants in this result remain essentially the same (except for a small factor equal to approximately 1.4 when $\alpha = 3$) as in the baseline analysis of Subsection 5.7. Thus if we consider the case of Proposition 8 again with $\alpha = 3$ and a collusive group of the 100 strongest gay marriage opponents, inefficiency would be approximately .5%, significantly higher than .058%, but still quite small. Roughly a thousand colluders would be needed to significantly dent efficiency. Thus if participants have rational expectations about collusion and collusive groups are not so large as to be easily detected, it seems unlikely they will significantly decrease efficiency compared to the non-cooperative baseline, in contrast to VCG and EE, where knowledge of collusion occurring does not change optimal behavior.

A similar analysis may be applied to a single individual who fraudulently "de-mergers", representing herself as more than a single individual. Such de-merger attacks are known to be highly effective against VCG as discussed in the next section.

De-mergers are simpler to analyze as only the value of the most extreme individual is relevant. Repeating our analyses from above, the threshold for an extremist with L identities who is not anticipated is $^1/_L$ of that for an extremist with a single identity. Thus the threshold is in $\Theta\left(N^{2/1+\alpha}L^{-1}\right)$. Such an individual exists with probability in

$$\Theta\left(N \cdot N^{-\frac{2\alpha}{1+\alpha}} L^{\alpha}\right) = \Theta\left(N^{-\frac{\alpha-1}{\alpha+1}} L^{\alpha}\right),$$

which dwindles with N as long as $L \in O\left(N^{\alpha-1/\alpha(\alpha+1)}\right)$.

For the case of $\mu = 0$ the typical size of the votes purchased by the fraudulent extremist is $LN^{\frac{1}{\alpha}}$, which causes inefficiency on the order of $L^2 N^{\frac{4-\alpha}{2\alpha}}$ or, normalized by total welfare, $L^2 N^{-\frac{\alpha-2}{\alpha}}$. Clearly this dies with N as long as $L \in O\left(N^{-\frac{\alpha-2}{2\alpha}}\right)$.

Finally, a fraudulent extremist rationally expected to exist with non-vanishing probability r will buy votes relative to those of moderate in

$$\Theta\left(\frac{LN^{\frac{1}{\alpha}}}{rN}\right),$$

so that if $L \in O\left(N^{\frac{\alpha-1}{\alpha}}\right)$, r must dwindle to 0 with N. These results are summarized in the following claim.

Claim 6. If a single individual can fraudulently misrepresent herself as L individuals, in any equilibrium $EI \to 0$ as N grows large as long as

1. $\mu = 0$, the fraudulent individual is the most extreme individual in the population, all other individuals behave as in the non-cooperative equilibrium and $L \in O\left(N^{\frac{\alpha-2}{2\alpha}}\right)$;

2. $\mu \neq 0$, the fraudulent individual is the most extreme individual in the population, other individuals behave as in the equilibrium without fraud and $L \in O\left(N^{\frac{\alpha-1}{\alpha(\alpha+1)}}\right)$ or

3. $\mu \neq 0$, the fraudulent individual is the most extreme individual in the population, all other individuals are aware of the fraudulent behavior, an equilibrium is played given this common knowledge of fraud and $L \in O\left(N^{\frac{\alpha-1}{\alpha}}\right)$.

Here α denotes the smaller of the two α values in 1) and that from the side opposite to μ in sign in 2) and 3).

Thus the number of identities that a perpetrator of a fraud would have have to take on to significantly impact efficiency is much larger even than the size of an effective collusive group. Such large-scale fraud is likely to be detected and thus is unlikely to be a serious threat. Thus QV appears to be roughly as robust as standard market institutions like the double auction are to collusion, though certainly not more so.

F.2 Aggregate Uncertainty

Consider the simplest possible case of aggregate uncertainty, when there is an unknown scalar parameter $\gamma \in \left(\underline{\gamma}, \overline{\gamma}\right) \subseteq \mathbb{R}$ that determines the density of valuations, $f(u|\gamma)$, has a prior density distribution g and and is affiliated with u, that is it orders f by first-order stochastic dominance (Milgrom, 1981; Milgrom and Weber, 1982). We maintain all of our assumptions on the distribution from above also assume that g is non-atomic and that our assumptions apply to the unconditional distribution of u. Assume that $\exists \gamma_-, \gamma_+ \in \left(\underline{\gamma}, \overline{\gamma}\right) : \mathbb{E}[u|\gamma_+] > 0 > \mathbb{E}[u|\gamma_-]$. The following is an intuitive conjecture about the structure of equilibrium in a large population: there exists a threshold $\gamma^\star$ such that if $\gamma > \gamma^\star$ then the alternative is chosen with probability near 1 and if $\gamma < \gamma^\star$ then the alternative is chosen with probability near 0.

Conjecture 1. Under the assumptions of this section, there exists a unique $\gamma^\star \in \left(\underline{\gamma}, \overline{\gamma}\right)$ such that in any equilibrium as $N \to \infty$, $\mathbb{P}(V > 0|\gamma) \to 1$ if $\gamma > \gamma^\star$ and $\mathbb{P}(V > 0|\gamma) \to 0$ if $\gamma < \gamma^\star$.

This conjecture greatly simplifies the analysis of equilibrium for several reasons. First, note that there is also a unique $\gamma_0 : \mathbb{E}[u|\gamma_+] > 0 > \mathbb{E}[u|\gamma_-]$ whenever $\gamma_+ > 0 > \gamma_-$. As a result, perfect limiting efficiency is achieved if and only if $\gamma_0 = \gamma^\star$. Second, by the analysis of Good and Mayer (1975) and Chamberlain and Rothschild (1981), for large N all ties occur when γ is very close to $\gamma^\star$. This leads to a very simple description of equilibrium behavior.

Corollary 15. At any equilibrium $v_i(u) = \left[\frac{g(\gamma^\star|u)+o(1/\sqrt{N})}{N}\right] u$.

This characterization states that, in large populations individuals buy votes in proportion to the chance they perceive of $\gamma^\star$ realizing. This in turn leads to a simple integral equation for $\gamma^\star$:

$$\mathbb{E}\left[g\left(\gamma^\star|u\right)u|\gamma^\star\right] = 0. \tag{77}$$

We have not been able to derive from this fully general results about efficiency. However, we have studied several examples that admit an analytic solution of Equation 77; others can easily be studied by solving Equation 77 computationally. A common thread running throughout these analyses is the "Bayesian Underdog effect" (BUE) identified by Myatt (2012) in the context of

1p1v when it is costly to turn out. Suppose, without loss of generality, that $\mathbb{E}[u] > 0$ so that the alternative is the ex-ante "favorite" in welfare terms and that the status quo is the ex-ante "underdog". If efficiency were to result, that is if $\gamma^\star = \gamma_0$, then individuals with $u < 0$ would tend to put a higher probability on $\gamma^\star$ intuitively because their own utility is a poll of one person indicating a lower value of γ. Because the alternative is the favorite lowering γ increases the chance of a tie: Republicans believed that in 2012 a close election was more likely than did Democrats. This BUE thus raises the votes of the ex-ante underdog and thus $\gamma^\star$, leading to inefficiency because there are some values of $\gamma \in (\gamma_0, \gamma^\star)$ when the favorite should win but the alternative does. We have not been able to identify general conditions under which this logic is valid as it is based on a frequentist intuition, while it is the Bayesian probability of $\gamma^\star$ that is relevant. However, it plays an important role in all of the examples we have explored.

By Bayes's rule

$$\mathbb{E}\left[g\left(\gamma^\star|u\right)u|\gamma^\star\right] = \int_u u\frac{f\left(u|\gamma^\star\right)g\left(\gamma^\star\right)}{f(u)}f\left(u|\gamma^\star\right)du = g\left(\gamma^\star\right)\int_u u\frac{f^2\left(u|\gamma^\star\right)}{f(u)}du.$$

Thus any $\gamma^\star$ solving $\int_u u\frac{f^2(u|\gamma^\star)}{f(u)}du = 0$ also solves $\mathbb{E}\left[g\left(\gamma^\star|u\right)u|\gamma^\star\right] = 0$ and thus is an equilibrium value of $\gamma^\star$.

In the following examples, we compute efficiency and inefficiency as in Section 5, but with an additional average taken over all possible realizations of γ according to the measure over γ.

Example 1. Suppose that u is equal to γ plus normally distributed noise with standard deviation σ_1^2 and that γ is normally distributed with mean μ and variance σ_2^2.

We assume, throughout and without loss of generality given the symmetry of the normal distribution, that $\mu > 0$. The marginal distribution of u is $\mathcal{N}\left(\mu, \sigma_1^2 + \sigma_2^2\right)$ while the γ-conditional distribution is $\mathcal{N}\left(\gamma, \sigma_1^2\right)$ by standard properties of the normal distribution. We use this to solve out for $\gamma^\star$:

$$\int_u u\frac{f^2\left(u|\gamma^\star\right)}{f(u)}du = 0 \iff \int_u ue^{\frac{(u-\mu)^2}{2\left(\sigma_1^2+\sigma_2^2\right)}-\frac{(u-\gamma^\star)^2}{\sigma_1^2}} = 0 \iff \frac{(u-\mu)^2}{2\left(\sigma_1^2+\sigma_2^2\right)} - \frac{(u-\gamma^\star)^2}{\sigma_1^2} = au^2 + b$$

for some constants a and b independent of u as this is the only quadratic form symmetric about 0, and symmetry about 0 is clearly necessary to yield a 0 expectation given the normal form of the density.

$$\frac{(u-\mu)^2}{2\left(\sigma_1^2+\sigma_2^2\right)} - \frac{(u-\gamma^\star)^2}{\sigma_1^2} = \frac{\sigma_1^2(u-\mu)^2 - 2\left(\sigma_1^2+\sigma_2^2\right)(u-\gamma^\star)^2}{2\left(\sigma_1^2+\sigma_2^2\right)\sigma_1^2} = au^2 + b - 2\frac{\sigma_1^2\mu - 2\left(\sigma_1^2+\sigma_2^2\right)\gamma^\star}{2\left(\sigma_1^2+\sigma_2^2\right)\sigma_1^2}u.$$

Thus $\gamma^\star$ solves

$$2\frac{\sigma_1^2\mu - 2\left(\sigma_1^2+\sigma_2^2\right)\gamma^\star}{2\left(\sigma_1^2+\sigma_2^2\right)\sigma_1^2} = 0 \iff \gamma^\star = \frac{\sigma_1^2}{2\left(\sigma_1^2+\sigma_2^2\right)}\mu.$$

In a large population, the first-best welfare is proportional to $\mathbb{E}\left[|\gamma|\right] = \sigma_2\sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma_2^2}} + \mu\left[1 - 2\Phi\left(-\frac{\mu}{\sigma_2}\right)\right]$. Welfare loss relative to this occurs in a large population when $\gamma \in (0, \gamma^\star)$ and, in these cases, is proportional to $|\gamma|$. This loss equals

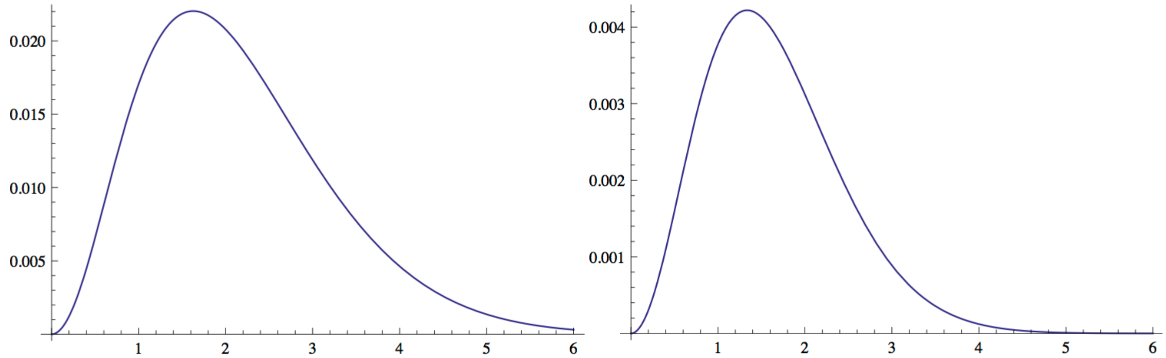

Figure 11: EI for the joint normal example when $\sigma_1 \to \infty$ (left) and $\sigma_1 = \sigma_2$ (right) as a function of $\frac{\mu}{\sigma_2}$.

$$\int_0^{\frac{\sigma_1^2}{2\left(\sigma_1^2+\sigma_2^2\right)}\mu} \frac{\gamma e^{-\frac{(\gamma-\mu)^2}{2n\sigma_2^2}}}{\sigma_2\sqrt{2\pi}} d\gamma$$

which is clearly monotonically increasing in $\frac{\sigma_1^2}{2\left(\sigma_1^2+\sigma_2^2\right)}\mu$, which in turn monotonically increases in σ_1^2. We can further compute analytically using Mathematica that

$$\int_0^{\frac{\sigma_1^2}{2\left(\sigma_1^2+\sigma_2^2\right)}\mu} \frac{\gamma e^{-\frac{(\gamma-\mu)^2}{2\sigma_2}}}{\sigma_2\sqrt{2\pi}} d\gamma = \mu\left[\Phi\left(\frac{\mu}{\sigma_2}\right) - \Phi\left(\frac{\mu\left(\sigma_1^2+2\sigma_2^2\right)}{2\sigma_2\left(\sigma_1^2+\sigma_2^2\right)}\right)\right] - \frac{\sigma_2\left(e^{-\frac{\mu^2\left(\sigma_1^2+2\sigma_2^2\right)^2}{8\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)^2}} - e^{-\frac{\mu^2}{2\sigma_2^2}}\right)}{\sqrt{2\pi}}.$$

Thus EI is

$$\frac{x\left[\Phi(x) - \Phi\left(\frac{x\left(\sigma_1^2+2\sigma_2^2\right)}{2\left(\sigma_1^2+\sigma_2^2\right)}\right)\right] - \frac{e^{-\frac{x^2\left(\sigma_1^2+2\sigma_2^2\right)^2}{8\left(\sigma_1^2+\sigma_2^2\right)^2}} - e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}}{2\left(\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} + x\left[1-2\Phi(-x)\right]\right)}$$

where $x \equiv \frac{\mu}{\sigma_2}$. In the limit as $\sigma_1 \to \infty$ this becomes

$$\frac{x\left[\Phi(x) - \Phi\left(\frac{x}{2}\right)\right] - \frac{e^{-\frac{x^2}{8}} - e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}}{2\left(\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} + x\left[1-2\Phi(-x)\right]\right)}$$

and when $\sigma_1 = \sigma_2$

$$\frac{x\left[\Phi(x) - \Phi\left(\frac{3x}{4}\right)\right] - \frac{e^{-\frac{9x^2}{32}} - e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}}{2\left(\sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} + x\left[1-2\Phi(-x)\right]\right)}.$$

Figure 11 shows the EI expression in both of these cases. Note that 1p1v is limit-efficient by symmetry (1p1v always chooses the sign of the median, which is the same as the sign of the mean) and QV is not. $\frac{\sigma_1^2}{2\left(\sigma_1^2+\sigma_2^2\right)}\mu = \gamma^\star > \gamma_0 = 0$. For large N and a fixed σ_2^2 maximal limit-EI occurs as $\sigma_1^2 \to \infty$; globally maximal limit-EI occurs when $\frac{\mu}{\sigma_2^2} \approx \pm 1.6$ and equals approximately 2.2%. Typically it is much less; for example if $\sigma_1^2 \to \infty$ but $\frac{\mu}{\sigma_2}$ is less than 75 or greater than 3 inefficiency is below 1% and if $\sigma_1^2 = \sigma_2^2$ then inefficiency is always below .5%.

Because the normal distribution is symmetric, standard voting, which always selects the preference of the median voter, achieves perfect efficiency in this example, while QV is not perfectly efficient. However, even in the worst case, QV still achieves more than 97% efficiency; usually it does much better. A natural intuition is that as the noise of individual values, σ_1^2, grows large and thus value heterogeneity becomes important relative to the aggregate uncertainty, QV should become perfectly efficient. This turns out to be wrong: the greater σ_1^2 the lower efficiency, presumably because the combination of extreme values and clearly separated likelihood ratio leads to greater over-weighting of the underdogs.

We now consider an example based on a set-up proposed by Krishna and Morgan (2012) to study costly voting.

Example 2. Suppose that γ is the fraction of individuals who have positive value, but that the distribution of the magnitude of value conditional on its sign is fixed and commonly known. Let μ_+, μ_- be respectively the mean magnitude of values for those with positive and negative values respectively.

The average value conditional on γ is $\gamma\mu_+ - (1-\gamma)\mu_-$ so that $\gamma_0 = \frac{\mu_-}{\mu_- + \mu_+}$. $f(u|\gamma) = \gamma$ for $u > 0$ and $f(u|\gamma) = 1-\gamma$ for $u < 0$. As a result, $f(u) = \mathbb{E}[\gamma]$ for $u > 0$ and $1 - \mathbb{E}[u]$ for $u < 0$. Thus $\gamma^\star$ solves

$$\mu_+ \frac{\gamma^2}{\mathbb{E}[\gamma]} - \mu_- \frac{(1-\gamma)^2}{1-\mathbb{E}(\gamma)} = 0 \implies \gamma^2 k = (1-\gamma)^2,$$

where $k \equiv \frac{\mu_+(1-\mathbb{E}[\gamma])}{\mu_-\mathbb{E}[\gamma]}$. Solving this quadratic equation yields

$$\gamma = \frac{-1 \pm \sqrt{k}}{k-1}.$$

The solution must be in the interval $[0, 1]$, which the negative solution never is and the positive solution always is. Thus

$$\gamma^\star = \frac{\sqrt{k}-1}{k-1} = \frac{1}{\sqrt{k}+1}.$$

Efficiency results if and only if $\gamma_0 = \gamma^\star$, that is if

$$\frac{1}{1+\sqrt{k}} = \frac{\mu_-}{\mu_- + \mu_+} \iff \sqrt{k} = \frac{\mu_+}{\mu_-} \iff \frac{\mu_+^2}{\mu_-^2} = \frac{\mu_+\left(1-E[\gamma]\right)}{\mu_- E[\gamma]} \iff \mu_+ E[\gamma] = \mu_-\left(1-E[\gamma]\right),$$

hat is the election is an expected welfare tie ex-ante.

$$\frac{\mu_+}{\mu_-\sqrt{k}} = \sqrt{\frac{\mu_+ E[\gamma]}{\mu_-\left(1-E[\gamma]\right)}}$$

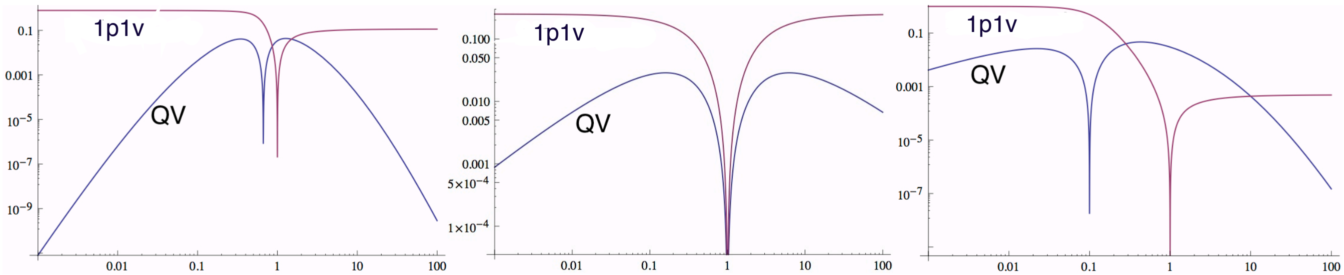

Figure 12: EI of QV compared to 1p1v in the Krishna and Morgan (2012) example with γ distributed uniform (left), $\beta(15, 10)$ (center) and $\beta(10, 1)$ (right). Both axes are on a log-scale, though labeled linearly; the x-axis, measures $r = \frac{\mu_+}{\mu_-}$.

so that $\frac{\mu_+}{\mu_-} > (<)\sqrt{k} \iff \mu_+ E[\gamma] > (<)\mu_- (1 - E[\gamma])$. Thus

$$\gamma_0 < (>)\gamma^\star \iff \mu_+ E[\gamma] > (<)\mu_- (1 - E[\gamma]),$$

the BUE.

Under 1p1v the threshold in γ for implementing the alternative with high probability is 1/2. For each regime we compute EI as

$$1 - \frac{\int_0^{\gamma_t} [\mu_- (1-\gamma) - \mu_+ \gamma] h(\gamma) d\gamma + \int_{\gamma_t}^1 [\mu_+ \gamma - \mu_- (1-\gamma)] h(\gamma) d\gamma}{\int_0^{\gamma_0} [\mu_- (1-\gamma) - \mu_+ \gamma] h(\gamma) d\gamma + \int_{\gamma_0}^1 [\mu_+ \gamma - \mu_- (1-\gamma)] h(\gamma) d\gamma},$$

where γ_t is the appropriate threshold value f γ. Using this method and explicit integration on Mathematica, we computed the relative (to the first best) efficiency of QV and 1p1v assuming g follows a Beta distribution. Note that, if one divides the numerator and denominator by μ_-,

$$\frac{\int_0^{\gamma_t} [\mu_- (1-\gamma) - \mu_+ \gamma] h(\gamma) d\gamma + \int_{\gamma_t}^1 [\mu_+ \gamma - \mu_- (1-\gamma)] h(\gamma) d\gamma}{\int_0^{\gamma_0} [\mu_- (1-\gamma) - \mu_+ \gamma] h(\gamma) d\gamma + \int_{\gamma_0}^1 [\mu_+ \gamma - \mu_- (1-\gamma)] h(\gamma) d\gamma} =$$

$$\frac{\int_0^{\gamma_t} \left[(1-\gamma) - \frac{\mu_+}{\mu_-}\gamma\right] h(\gamma) d\gamma + \int_{\gamma_t}^1 \left[\frac{\mu_+}{\mu_-}\gamma - (1-\gamma)\right] h(\gamma) d\gamma}{\int_0^{\gamma_0} \left[(1-\gamma) - \frac{\mu_+}{\mu_-}\gamma\right] h(\gamma) d\gamma + \int_{\gamma_0}^1 \left[\frac{\mu_+}{\mu_-}\gamma - (1-\gamma)\right] h(\gamma) d\gamma},$$

So EI depends only on the ratio $r \equiv \frac{\mu_+}{\mu_-}$ and on the parameters of the Beta distribution, not on both μ_+ and μ_- independently

Figure 12 shows three examples that are representative of the more than 100 cases we experimented with. Whenever $\alpha = \beta$ (the distribution of γ is symmetric), QV always dominates 1p1v as it does in the left panel shown, which is $\alpha = \beta = 1$, the uniform distribution. 1p1v obviously performs best when r, shown on the horizontal axis, is near to unity. When α is larger than β, 1p1v may out-perform QV near $r = 1$. This is shown in the center and right panels where $(\alpha, \beta) = (15, 10)$ and $(10, 1)$ respectively. The larger α is relative to β, the larger the region over which voting outperforms QV. However, it is precisely in these cases where, if r is very small, voting is most dramatically inefficient. Intuitively majority may outperform QV by blindly favoring the majority which is almost always in favor of taking the action for $\alpha \gg \beta$, while QV may be a bit too conservative in favoring the action because of the BUE. However this blind

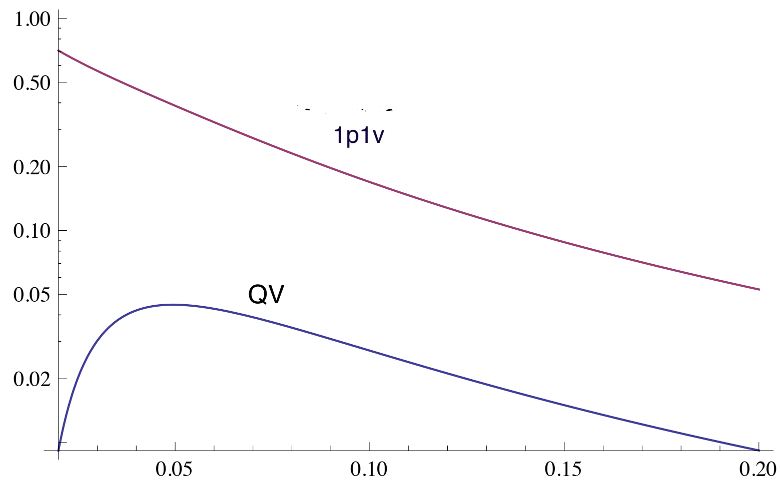

Figure 13: EI as a function of the standard deviation of the total vote share in favor of Proposition 8. The vertical axis is on a logarithmic scale with linear labels.

favoritism towards the majority view can be highly destructive under 1p1v, but not under QV, when the minority has an intense preference. In fact, while voting becomes highly inefficient when the minority preference becomes very intense, QV actually becomes closer to the first best. In all cases (shown here and that we have sampled) QV's efficiency is above 95\% and usually it is well above this.

In summary EI is never greater than 5% for QV and it can be arbitrarily large for majority-rules voting. In the special case when γ has a uniform distribution, QV dominates voting, which may have EI as high as 25% while for QV it is never greater than 3%. For "most" parameter ranges QV appears to outperform majority-rules voting, often quite significantly.

Finally we consider an example similar to the previous one but calibrated to the evaluation of Proposition 8 in California discussed in Section 2 above.

Example 3. Suppose that 4% of the electorate is commonly known to oppose the alternative and is willing to pay on average \$34k to see it defeated. Suppose that the other 96% of the electorate is willing to pay on average \$5k to either support or oppose the alternative with the intensity of their values being independent of γ. γ is the fraction of the 96% that support the alternative and is assumed to have a Beta distribution with parameters set so that on the mean fraction of the population in favor of the alternative is 52%. The average value from implementing the alternative is therefore $\$4800(2\gamma-1)-1360$ and $\gamma_0 = .64$.

Individuals in the 4% receive no signal about γ and thus $f(u|\gamma)$ for this group is simply $f(u)$. For proponents of the alternative among the 96%, $f(u|\gamma)$ is, by the logic of the previous proof, $.96f(u)\gamma$ and for opponents among the 96% $.96f(u)(1-\gamma)$; $f(u)$ is formed by taking expectations over γ as in the previous proof. Using the same techniques derivations as there we can solve for $\gamma^\star$.

To calibrate, we assume that a Beta distribution of γ and that

$$.96\frac{\alpha}{\alpha+\beta} = .96\mathbb{E}[\gamma] = .52.$$

Solving this out implies that $\alpha = 1.18\beta$. The variance of γ is given by the standard formula for the variance of a Beta distributed variable:

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1.18\beta^2}{2.18^2\beta^2(2.18\beta+1)} = \frac{.25}{2.18\beta+1}.$$

Thus the standard deviation of the total fraction of the population supporting the alternative is $\frac{.96 \cdot .5}{\sqrt{2.18\beta+1}} = \frac{.48}{\sqrt{2.18\beta+1}}$ and thus if the standard deviation of the vote share for the alternative under standard voting is σ then $\beta = \frac{.45\left(.23-\sigma^2\right)}{\sigma^2}$. Figure 13 shows the dominance of QV.

Then QV is thus always superior to 1p1v and the gap is larger the smaller is the standard deviation of the vote share. When the standard deviation of the population share supporting the alternative is 5 percentage points (well above the margin of error in most individual polls), QV has 4% EI and 1p1v 47% EI. Even when the standard deviation is 20 percentage points QV achieves 1.2% EI while 1p1v has 7%.

F.3 Voter Behavior

First, suppose that individuals misestimate $g(-v)$ as $e\left(g(-v)\right)\epsilon$ where e is smooth and $e, e_1, e_2(0) > 0 > e_{11}, e_{12}$ and, but that individuals never take a strictly dominated actions and any individual with a large impact (who subjectively rationally changes the chances of the outcome by 50% or more) has no mispercpetions. ϵ is a random variable drawn from $(\underline{\epsilon}, \overline{\epsilon}) \subseteq \mathbb{R}$ according to a smooth distribution h that is independent of u and embodies the extent to which individuals over-estimate the chance of being pivotal. ϵ is assumed to have mean 1 and denote its standard deviation by σ_ϵ.

Our assumptions on e ensure that over-estimation is greater the small is the chance of being pivotal; individuals may even underestimate the chance when it is very large.[42] This is consistent with experimental evidence reported by Blais. Our assumption of independence of ϵ from u ensures that no type of individual systematically over-estimates more than others *conditional on the true chance of their being pivotal with a marginal vote*. However, individuals with endogenously different chances of being pivotal (because of the number of votes their value induces them to buy) may over-estimate to differing extents. We label this model as the "misperception model".

Second, suppose that, in addition to the instrumental utility each individual earns, each also receives a benefit $\epsilon\left(x + \frac{\xi}{\sum_i |v_i|}\right)uv$ where $x, \xi \geq 0$. ϵ is again an independent-of-u random variable with the same properties as in the misperception model. x represents a per-unit-of-value expressive utility for each vote she purchases and ξ is a per-unit-of-value expressive utility she earns from the *fraction of total votes cast* that she represents. These two possibilities correspond to two different interpretations of expressive utility in the literature. The first corresponds to traditional expressivist accounts, such as that of Fiorina (1976), where expression creates a personal psychological benefit for the voter. The second corresponds to a more semi-instrumental motive, suggested by Myerson (2000), where voters vote to influence perceptions of political

[42] An alternative model that we have also considered by do not report here for brevity is one where individuals overestimate the chance of their being pivotal unless they pay a cost to obtain a better estimate. In this case QV behaves more like 1p1v, thus losing some of its efficiency benefits over 1p1v. However, it may perform better for finite populations as this is the case that most effectively deters extremists and it always continues to outperform 1p1v, at least if the costs of acquiring information about p are excluded. If these are included, 1p1v may perform better.

support, assuming only aggregate vote shares are reported by the media. We label this model as the "expressivist model".

In all cases we evaluate welfare based on the simple instrumentalist welfare we apply in the rest of the paper, as we do not have a coherent approach to incorporating the value individuals gain from expressing themselves or acting on their misperceptions. Furthermore we typically think the latter will be quantitatively small relative to the former. We also assume that the distribution of ϵ either has thin tails or has a tail index weakly greater than those of the value distribution (has thinner tails). As in our baseline analysis, but even more strongly, the conclusions we reach differ dramatically based on whether we consider the case of $\mu = 0$ or $\mu \neq 0$, though interestingly they do not differ dramatically between the misperception and expressivist models.

First consider the case of $\mu = 0$. First consider either the expressivist model when $x \neq 0$ or the misperception model. In the first case, the expressivist motive must be dominant in large populations as the instrumental motive dies as the population grows large, while the expressivist motive does not. Thus in both cases the variation in votes, orthogonal to u, created by ϵ must be the leading source of inefficiency as it does not die with population size and, given the smoothness of e, our arguments for other sources of inefficiency vanishing can easily be replicated here.

Because ϵ is independent of u, the variance of ϵu is $\sigma^2 \left(1 + \sigma_\epsilon^2\right)$ and the covariance of u and ϵu is σ^2. Thus by the Central Limit Theorem (CLT) logic, letting $\hat{V}$ be an appropriately normalization of $\sum_i v_i$, in equilibrium approximately for large N

$$\frac{1}{N}\begin{pmatrix} U \\ \hat{V} \end{pmatrix} \sim \mathcal{N}\left(\begin{matrix} 0 \\ 0 \end{matrix}, \frac{1}{N}\begin{bmatrix} \sigma^2 & \sigma^2 \\ \sigma^2 & \sigma^2\left(1+\sigma_\epsilon^2\right) \end{bmatrix}\right).$$

Thus $\hat{V}$ is just U plus mean zero noise of magnitude $\sigma^2\sigma_\epsilon^2/N$. Efficiency is the expected total value of U in events where the vote goes the right way, or in this case approximately

$$2\sqrt{N}\int_0^\infty \frac{xe^{-\frac{x^2}{2\sigma^2}}\Phi\left(\frac{x}{\sigma\sigma_\epsilon}\right)}{\sqrt{2\pi}\sigma}dx = \frac{\left(1+\frac{1}{\sqrt{1+\sigma_\epsilon^2}}\right)\sigma}{\sqrt{2\pi}}.$$

Normalizing this by $\sqrt{2/\pi}\sigma$, the total possible efficiency, yields

$$\frac{1}{2}+\frac{1}{2\sqrt{1+\sigma_\epsilon^2}} \implies EI \approx 1-\left(\frac{1}{2}+\frac{1}{2\sqrt{1+\sigma_\epsilon^2}}\right) = \frac{1}{2}-\frac{1}{2\sqrt{1+\sigma_\epsilon^2}}.$$

In the expressivist model when $x = 0$ suppose that everyone behaves as in the limiting equilibrium where the expressivist motive is entirely ignored. Then aggregate votes are on the order of $N^{\frac{3}{4}}$ by our argument in Subappendix D.1 above. Thus the size of the expressivist motive is in $\Theta\left(1/N^{3/4}\right)$. On the other hand by our calculations in Subappendix D.1 the instrumental voting motive is in $\Theta\left(1/\sqrt[4]{N}\right)$, while the inefficiency arises from deviations from proportionality that are in $\Theta\left(1/N^{5/4}\right)$. Thus the expressivist motive is larger than the sources of inefficiency in this case, dying only at a relative rate $1/\sqrt{N}$, but still dies relative to the vote total. In this case the same logic as in the two cases considered above yields the same constant on the large population inefficiency, but now because this motives dies at a relative rate $1/\sqrt{N}$, EI is bounded above by

$$\frac{1}{2}-\frac{1}{2\sqrt{1+\frac{\sigma_\epsilon^2}{\sqrt{N}}}}\approx\frac{\sigma_\epsilon^2}{4\sqrt{N}},$$

where the approximation is based on the Taylor expansion for large N. Furthermore this function is always concave, so this Taylor expansion overstates the limiting inefficiency, implying this is an upper-bound for large N. These results are summarized in the following claim.

Claim 7. If F has $\mu = 0$ then in any type-symmetric (across the joint type (u, ϵ)) Bayes-Nash equilibrium, then

1. In the expressivist model if $x = 0$ then there exists an $\underline{N}$ such that $EI \leq \frac{\sigma_\epsilon^2}{4\sqrt{N}}$.
2. In any other case, EI converges as N grows large to $\frac{1}{2} - \frac{1}{2\sqrt{1+\sigma_\epsilon^2}}$.

When the expressivist motive dies off with the total number of votes, limiting efficiency still obtains, but more slowly than before; however as we discuss presently a value of σ_ϵ much above unity is implausible so in practice inefficiency in reasonably large populations will be quite small. If its value is unity, for example, then even in a town of 10,000 it would be a quarter of a percent.

Even in the case when there is limiting inefficiency, QV will often out-perform 1p1v. Goeree and Zhang (2013) show that when $\mu = 0$ the limiting EI of majority rule is

$$\frac{1}{2}-\frac{\mathbb{E}\left[|u|\right]}{2\sigma}.$$

This equals about 10% for the normal distribution and about 24% for our mean-zero gay marriage example with uniform distributions. In order to match these levels of inefficiency the noise from ϵ have to be quite large: σ_ϵ would have to equal 57% in the case of the normal distribution and 270% in the uniform distribution gay marriage example. In cases with fat-tailed distributions of values the inefficiency of 1p1v is far greater, as it is more important to incorporate the highly heterogeneous intensities of the population into the decision: it is about 43%, requiring an insanely large 5000% value of σ_ϵ to match it, in our mean-zero gay marriage example if the conditional-on-group value distributions follow a double Pareto log-normal distribution as in Subsection 5.7. Thus while QV is not perfectly efficient in this case, noise has be extremely large (often much larger than the signal coming from values) in realistic cases for 1p1v to out-perform QV.

Note, however, that this is not always the case; for example, suppose that noise took on a value of 0 with very high probability and some extremely large value with very low probability to maintain the mean of 1. Then QV could be worse than 1p1v, especially if intensities are relatively homogeneous, as QV would effectively count only a very small fraction of all valid votes. Intuitively, 1p1v throws away all cardinal signal. Unless this signal is of little value because individuals have homogeneous intensities and/or the noise in individuals' estimations is so large that these signals are overwhelmed, QV will perform better even though it responds to spurious noise that 1p1v neglects.

Now we turn to the $\mu \neq 0$ case and as usual focus on $\mu > 0$ WLOG. In the expressivist model, clearly unless $x = 0$ the expressive motive will become dominant in large populations as the number of votes each individual purchases dies off with N, while the expressive motive does not. Even when $x = 0$ note that if individuals purchase votes primarily from the expressive motive, they will buy votes of magnitude approximately (up to unimportant fluctuations in ϵ given that the mean is not 0) ξ/Vu and thus $V \approx \frac{N\xi\mu}{V}$ by the law of large numbers. Thus V is in $\Theta\left(\sqrt{N}\right)$. Thus only individuals with utility on the order of N will be willing to act as extremists. Thus, by the logic of Subappendix D.1 the chance of an extremist existing must die at rate $N^{-(\alpha-1)}$. Furthermore the price of influence $\hat{p}$ perceived by an extremist must satisfy her first-order condition for her value, which will be in $\Theta(N)$ and the votes she buys which are $\Theta\left(\sqrt{N}\right)$

$$\Theta\left(\frac{N}{\hat{p}}\right) = \Theta\left(\sqrt{N}\right) \implies \Theta\left(\hat{p}\right) = \Theta\left(\sqrt{N}\right).$$

Thus the price perceived by moderates is in $\Theta\left(N^{\alpha-.5}\right)$. This implies that the instrumental motive for buying votes must die at least a $1/N^{.5+\alpha-1}$, that is more quickly than does the expressive motive. So the expressive motive's dominance is confirmed. A reverse calculation, based on the assumption of dominance of the instrumental motive, leads to the same conclusion.

Thus in either case of the expressivist model the expressive motive is dominant for large N. Thus, up to the error arising from ϵ, U and V will be proportional to each other in large populations. For ϵ to make a difference, given that $\mu > 0$, it must create a large deviation or be so close to zero for a significant individual that it cancels out a large deviation driven by that individual's value as in every other event U and V are both positive and efficiency results. However, by our assumption that ϵ is either thin-tailed or has a tail index weakly greater than that of u, we have that the probability of such a large deviation in either U or V is in $O\left(N^{-(\alpha-1)}\right)$, because the tail index of $u\epsilon$ is the minimum of the tail index of u and ϵ given their independence. Thus a superset of the events causing inefficiency has probability in $O\left(N^{\alpha-1}\right)$ and thus the probability of inefficiency must also be in $O\left(N^{\alpha-1}\right)$.

In the misperception model clearly if overestimation of the chance of being pivotal is severe enough for very small values of g the same logic as in the expressivist model will hold. Suppose, instead, that as $g \to 0$, $e(g) \in \Theta\left(\sqrt[\beta]{g}\right)$ for some $g > 1$ so that the over-estimation is greater than a constant fraction, but not extremely rapid. Under this assumption, we can trace through the logic of Subappendix D.2 to derive limiting inefficiency. Following the logic there, the price perceived by a rational moderate must now be not in $\Theta\left(N/\sqrt{|u^\star|}\right)$ but instead in $\Theta\left(N^\beta/(|u^\star|)^{\beta/2}\right)$, where $u^\star$ is the threshold for being such an extremist, as this will be seen as $\Theta\left(N/\sqrt{|u^\star|}\right)$ by a misperceiving moderate. Thus the (assumed-rational given that she has significant impact, though this assumption can be dispensed) extremists must perceive a price in $\Theta\left(qN^\beta/(|u^\star|)^{\beta/2}\right)$, where q is the probability an extremist (actually) exists. By extremist rationality, her first-order condition and our arguments about integrating over all ponytail extremists through a power law gives that

$$\Theta\left(\sqrt{|u^\star|}\right) = \Theta\left(\hat{p}\right),$$

where $\hat{p}$ is the price perceived by the average extremist. Combining this with our observation above gives that

$$\Theta\left(\sqrt{|u^\star|}\right) = \Theta\left(\frac{qN^\beta}{(|u^\star|)^{\frac{\beta}{2}}}\right) \implies \Theta\left(q\right) = \Theta\left(\frac{(|u^\star|)^{\frac{1+\beta}{2}}}{N^\beta}\right).$$

We can then set this modified "supply" equation equal to the standard "demand" equation, given this involves no misperceptions.

$$\Theta\left(N\left(|u^\star|\right)^{-\alpha}\right)=\Theta\left(\frac{\left(|u^\star|\right)^{\frac{1+\beta}{2}}}{N^\beta}\right)\implies\Theta\left(N^{1+\beta}\right)=\Theta\left(\left(|u^\star|\right)^{\frac{1+\beta+2\alpha}{2}}\right)\implies\Theta\left(|u^\star|\right)=\Theta\left(N^{\frac{2+2\beta}{1+\beta+2\alpha}}\right).$$

Again applying the demand equation

$$\Theta(q)=\Theta\left(NN^{\frac{-2\alpha(1+\beta)}{1+\beta+2\alpha}}\right)=\Theta\left(N^{-\frac{2\alpha\beta-1-\beta}{1+\beta+2\alpha}}\right).$$

Whether the inefficiency created by extremism or that created by the large deviations relevant in the expressivist model clearly turns on whether the extremist event predicted by this analysis is asymptotically larger or smaller than the large deviations event. This yields the following claim.

Claim 8. If F has $\mu \neq 0$ then in any type-symmetric Bayes-Nash equilibrium, then

1. In the expressivist model, so long as either $\max\{x,\xi\}>0$, there $EI \in O\left(N^{\alpha-1}\right)$.

2. In the misperception model, if for sufficiently small p, $e(g)\in\Omega\left(\sqrt[\beta]{g}\right)$, then there exists a $\underline{N}$ such that for $N \geq \underline{N}$, $EI \in O\left(1/N^\kappa\right)$, where $\kappa=\max\left\{\alpha-1,\frac{2\alpha\beta-1-\beta}{1+\beta+2\alpha}\right\}$.

In the expressivist model or the misperception model for strong enough misperception (β large enough) the rate here is much faster than those reported in Section 5; even in our most fat-tailed case ($\alpha=3$), EI dies off with the square of N. How large does β have to be for these faster rates to take hold? Consider our running example of $\alpha=3$. Then β must equal 5 (the decay of beliefs in pivotality must be no faster than the fifth root of the decay of the true chance of being pivotal) before the new rates dominate. When β is greater than 5, the leading source of inefficiency is insufficient rather than excessive behavior by extreme players. When β is smaller but not as dramatically. Consider $\beta=3$, roughly the value necessary according to the data of Gelman et al. (2010) to justify voting in a presidential election worth \$5000 to a citizen who must spend \$10 to vote, perhaps therefore a reasonable ball-park for real-world marginal voters. Then inefficiency decays with $N^{-7/5}$, much faster than the $1/\sqrt{N}$ rate predicted by the fully rational model in this case.

On the other hand in the bounded or thin-tailed distribution case, when $\alpha \to \infty$, excessive extremism is always the leading source of harm, the decay of EI is exponential in the expressivist model and occurs at rate $1/N^\beta$ in the misperception model; if in addition the perceived chance of being pivotal stays significant even when g is truly exponentially small, arguably a finding consistent with Blais (2000)'s results, then the decay of EI is exponential in the misperception model as well.

To summarize:

1. In the Myerson-style expressivist model, the long-term-oriented voter behavior never causes limiting inefficiency, though it does slow its decay when $\mu=0$, and generically dramatically increases the rate of decay of EI.

2. In the Fiorina-style expressivist model, the non-instrumental voter behavior causes some limiting inefficiency but probably does not make QV less efficient than 1p1v in many realistic cases, even constrained to the non-generic $\mu=0$ class, and generically dramatically increases the rate of decay of EI.

3. In the Blais-styel misperception model, irrational voter behavior causes some limiting inefficiency but probably does not make QV less efficient than 1p1v in many realistic cases, even constrained to the non-generic $\mu = 0$ class, and generically significantly decreases the rate of decay of EI, so long as the overestimation of the chance of being pivotal affects the rate and not just constant of the decay of g near 0.

Of course if ϵ, rather than being independent of u, were correlated with it this would likely cause limiting inefficiency for reasons similar to those in the aggregate uncertainty case we considered in Subsection 6.2 and Subappendix F.2, though this might be worse than there as there would be little rationality to keep such divergent estimates in check. We do not consider this a particularly realistic case, however, and thus do not discuss it further here.

F.4 Common Values

We begin by considering the simplest information aggregation setting, that of pure common interest considered by de Condorcet. McLennan (1998) shows that the optimal information aggregation procedure using the actions available to agents is always an equilibrium. By essentially the same argument this is also true of QV. However, because QV allows expression of cardinal values, it allows the expression of strictly more information and thus, for generic information structures, achieves more efficient information aggregation in some equilibrium than voting does in any equilibrium. The one complicating factor is that under QV individuals must pay for their votes and thus interests are not perfectly aligned.

Without the smoothing through uncertainty (Ψ), this could lead to problems of existence, because individual could want to shirk in incorporating their information, given that an arbitrarily small down-scaling of all individuals' reports would be sufficient to implement any given efficient equilibrium. We have to investigate further these issues. With Ψ added for smoothness existence and approximate efficiency should not be a problem, but perfect efficiency may be in a small population because of the aggregate uncertainty created by the smoothing. In either case we believe that QV will significantly outperform 1p1v in almost any case where individuals receive signals that are richer than binary.

As Feddersen and Pesendorfer (1997) argue, most collective decisions involve a mixture of conflicting preferences and dispersed information. They show that in large, majority-rule elections this mixture does not prevent information aggregation because a large number of individuals who constitute a small fraction of the population and are close to being indifferent conditional on information leading to an expected tie vote on the basis of their information. The fact that all information aggregation occurs through the votes of a narrow segment of the population, however, does put important limits on information aggregation. If, for example, all individuals have some minimum intensity of preferences, information does not aggregate and if those who are nearly indifferent also have very poor information, information aggregates very slowly. Under QV, by contrast, information aggregates by small adjustments to *all individuals' vote quantities* rather than large adjustments to a small fraction of the population's votes. We believe this leads to information aggregation in settings where it does not under 1p1v and faster aggregation even when it does, but very slowly, under 1p1v.

We plan to investigate this with examples. Calculations justifying the following result will be added shortly

Example 4. Suppose that each individual's value $v_i = \mu + \epsilon_i$ where μ is a common value component and ϵ_i is the individual's idiosyncratic preference. μ and ϵ_i are drawn identically and independently (in the latter case across individuals) from a distribution that equals -1 with $.5$ probability and 1 with $.5$ probability. Individuals receive signals s_i that are drawn independently and identically conditional on μ, taking on the same value as μ with probability $p \in (.5, 1)$ and $-\mu$ with probability $1-p$.

Suppose that (something close to) full information aggregation occurs in 1p1v. Then the decision in a large population must have the same sign as μ: if μ is negative the alternative is not adopted and if μ is positive the alternative is adopted. Thus the only event where an individual may be pivotal is when μ is very close to 0. Whenever this is the case, however, every individual will want to vote purely based on her preference as this makes a bigger difference to the individual than does μ. Thus full information aggregation under voting is impossible. The reason is that there are not any individuals with preferences sufficiently close to 0 so that they will still want to vote based on their information even if aggregation approximately occurs.

Now consider QV. We conjecture an approximate equilibrium where every individual votes $v_i = as_i + k\epsilon_i$. We now verify if an equilibrium of this sort is possible. If this holds, then $V_{-i} = a(N-1)\mu + a\sum_{j\neq i} s_i - \mu + k\sum_{j\neq i}\epsilon_i$. For large N this sum is distributed, conditional on μ, as a normal with mean $a(N-1)\mu$ and variance $a^2\sigma_s^2(N-1) + k^2(N-1)$. Thus $\frac{V_{-i}}{a(N-1)}$ is distributed, conditional on μ, normally with mean μ and variance $\frac{\sigma_s^2 + \frac{1}{\alpha^2}}{N-1}$ where $\alpha \equiv \frac{a}{k}$. Thus V_{-i} is a signal with precision $\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}}$ of μ. Thus i best estimate, by the standard normal updating formula, of μ is

$$\frac{\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}}}{\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}} + \frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_s^2}}V_{-i} + \frac{\frac{1}{\sigma_s^2}}{\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}} + \frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_s^2}}s_i + \frac{\frac{1}{\sigma_\mu^2}}{\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}} + \frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_s^2}}0.$$

Note that this is affine in s_i and V_{-i} so that individual i's best estimate of u_i for the purposes of choosing v_i, given that V_{-i} is already incorporated into the sum of votes, is just a constant plus

$$\frac{\frac{1}{\sigma_s^2}}{\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}} + \frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_s^2}}s_i + \epsilon_i.$$

Crucially this is true regardless of the value of V_{-i}, including conditioning on the individual being pivotal. Thus the ratio of the coefficient on s_i to that on ϵ_i, which must equal α in a large market equilibrium given that individuals buy votes proportional to their utility, is $\frac{\frac{1}{\sigma_s^2}}{\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}} + \frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_s^2}}$. This lets us determine the value of α according to the following equation:

$$\alpha = \frac{\frac{1}{\sigma_s^2}}{\frac{N-1}{\sigma_s^2+\frac{1}{\alpha^2}} + \frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_s^2}}.$$

As N grows large, the denominator must shrink, leading α to shrink, unless α shrinks. Thus we conclude that α must approach zero for large N. This allows us to eliminate a bunch for terms and simplify the above equation to

$$\alpha \approx \frac{\frac{1}{\sigma_s^2}}{(N-1)\alpha^2 + \frac{1}{\sigma_\mu^2} + \frac{1}{\sigma_s^2}}.$$

But because α shrinks, the term with N in the denominator must dominate so this simplifies further to

$$\alpha^3 \approx \frac{1}{\sigma_s^2(N-1)} \implies \alpha \approx \frac{1}{\sigma_s^{\frac{2}{3}}\sqrt[3]{N-1}}.$$

Thus $\frac{v_i}{k}$ has, conditional on μ, mean $\frac{1}{\sigma_s^{\frac{2}{3}}\sqrt[3]{N-1}}\mu$ and variance $\frac{\sigma_s^2}{\sigma_s^{\frac{2}{3}}\sqrt[3]{N-1}}+1$ and $\frac{V}{k}$ has mean $\frac{N}{N-1}\left(\frac{N-1}{\sigma_s}\right)^{\frac{2}{3}}\mu$ and variance, for large N approximately, N and standard deviation $\sqrt{N}$. Thus the mean grows relative to the standard deviation in N, implying that with probability approaching 1 for large N the decision is made in the direction of μ so that information is aggregated.

To finish, we need to show that the variance of total votes grows with N so that the approximation that everyone buys nearly linear votes is correct. Conditional on any s_i the variance of μ is $\frac{1}{\frac{1}{\sigma_\mu^2}+\frac{1}{\sigma_s^2}}$ and its mean is $\frac{\sigma_\mu^2}{\sigma_\mu^2+\sigma_s^2}s_i$.

$$V_{-i} = k\left[\alpha(N-1)\mu + \alpha\sum_{j\neq i} s_j - \mu + \sum_{j\neq i}\epsilon_j\right]$$

so the mean of V_{-i} conditional on s_i is $\frac{k\alpha(N-1)\sigma_\mu^2}{\sigma_\mu^2+\sigma_s^2}s_i$ and its conditional variance is $k^2\alpha^2(N-1)^2\sigma_\mu^2 + (N-1)\left(\sigma_s^2+1\right)$. Under my assumption that $\sigma_s^2 \gg \sigma_\mu^2$ the mean is approximately 0 for all but very unlikely and extreme signals. Furthermore for large N the leading term in the expression for the variant is $k^2\alpha^2(N-1)^2\sigma_\mu^2 = k^2\sigma_\mu^2\left[\frac{N-1}{\sigma_s^2}\right]^{\frac{4}{3}}$. Thus the density of a tie is approximately

$$\frac{\sigma_s^{\frac{8}{3}}}{\sqrt{2\pi}k\sigma_\mu^2\left(N-1\right)^{\frac{2}{3}}}.$$

This yields a matching-coefficients equation

$$k^2 = \frac{\sigma_s^{\frac{8}{3}}}{2\sqrt{2\pi}\sigma_\mu^2\left(N-1\right)^{\frac{2}{3}}} \implies k = \frac{\sigma_s^{\frac{4}{3}}}{2^{\frac{3}{4}}\sqrt[4]{\pi}\sigma_\mu\sqrt[3]{N-1}}.$$

Thus the variance of votes grows with $(N-1)^{\frac{2}{3}}$ and for large N no individual will try to buy the whole election.

Thus as $N \to \infty$ in any QV equilibrium the alternative is implemented if and only if $\mu = 1$, which is efficient. Under 1p1v the alternative is implemented with probability .5 regardless of μ.

The assumption that σ_μ^2 is small was useful here as it allowed us to find a solution where vote purchases were linear in signals because the chance of a tie is the same for all types. When σ_μ^2 is not small this would fail and those with extreme signals would buy fewer votes than those with signals near 0. This is closely related to the case of aggregate uncertainty discussed in Subsection 6.2 and Subappendix F.2. Like there, we are sure this would cause some slowing of information aggregation or even limit inefficiency. But we suspect that it would also leave the basic qualitative conclusions and comparisons to 1p1v in tact. We hope to analyze this in a future draft. We also plan to investigate a normally distributed example where information aggregates in both settings, but we believe it aggregates more quickly (in terms for rates, not just constants) under QV. We also hope to show a more general result, extending the conditions for limiting information aggregation beyond those of Feddersen and Pesendorfer (1997) under QV.

Why does QV outperform 1p1v here? The reason is that voting only allows individuals to express a binary directional preference. If most (here all) individuals care more about their preferences than their information, information will play little or no role in the social decision. On the other hand, under QV, information can be expressed through a more subtle shading of vote purchases. While on an individual level this expression is small, it cumulates throughout the market allowing information to be aggregated.