

# Speculative Equilibrium with Differences in Higher-Order Beliefs

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## Abstract

Modest differences in higher-order beliefs can have large price effects. To illustrate this, we generalize a single-period model of competitive trading with different information, like Hellwig [1980] with two types of symmetrically informed traders. We allow traders to have possibly different dogmatic beliefs about the mean, different dogmatic beliefs about other traders' beliefs, and so on for higher and higher orders of beliefs. Even when every trader's first-order expectations are unbiased, overvaluation results when traders have inconsistent higher-order beliefs that their own expectations are more optimistic than average. Lack of common knowledge destabilizes prices more as market liquidity disappears.

Keywords: higher-order beliefs, rational expectations, common knowledge, speculation, bubbles, heterogeneous beliefs, overconfidence, asymmetric information

JEL Classification: G14, D82, D83, D84

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# 1 Introduction

Bubbles are often attributed to overly optimistic beliefs by market participants. In standard noisy rational expectations equilibrium models, over-optimism cannot arise because the market price aggregates information in an unbiased manner. The rational expectations result that prices are unbiased is derived under the two assumptions that (1) agents have unbiased expectations and (2) this lack of bias is common knowledge. The literature on higher-order beliefs suggests that even small deviations from the common knowledge assumption can have large effects on equilibrium outcomes (e.g., Rubinstein [1989] and Weinstein and Yildiz [2007]), potentially making prices biased estimates of value.

This paper examines the following questions in the context of an analytically tractable, intuitive model that generalizes the standard rational expectations framework to include differences in higher-order beliefs: Can over-optimism (or over-pessimism) arise due to a lack of common knowledge? If so, how and why does it arise? Can small differences in higher-order beliefs have large price effects? Is over-optimism the cause or the consequence of asset overvaluation?

To answer these questions, we modify a standard noisy competitive rational expectations model, in which two groups of investors have different private information, by allowing the two groups of investors to have different higher-order beliefs about the mean of the asset's payoff. The corresponding noisy rational expectations model is a simple version of the model in Hellwig [1980]. Standard noisy rational expectations models assume that the structure of the model is common knowledge. We depart from this common knowledge assumption in only one way; we assume each group of investors to have possibly different dogmatic beliefs about the mean of the asset's payoff, possibly different dogmatic beliefs about the other traders' beliefs about the mean, possibly different dogmatic beliefs about the other traders' beliefs about his own group's beliefs about the mean, and so on up to higher and higher order. We find that modest differences in these higher-order beliefs may have large effects on prices in comparison with the otherwise equivalent rational expectations set-up. In the limit as the standard deviation of noise trading goes to zero, inconsistencies in beliefs may have an arbitrarily large positive or negative effect on prices, even if the differences in beliefs are small.

Our results provide an alternative interpretation for episodes of unusually high asset

valuations, such as occurred for dot-com stocks during the period 1999-2000 and for real estate assets during the period 2005-2007. These episodes are typically assumed to result from extremely inflated optimistic beliefs arising from irrational exuberance (e.g., Shiller [2000]). We show that extremely inflated asset values can result instead from small inconsistencies in higher-order beliefs which are not particularly optimistic. The misspecified beliefs lead to inflated prices, and traders become optimistic as a result of incorrectly inferring from the inflated prices that other traders in the market have extremely bullish private information. Thus, according to our interpretation, high prices do not necessarily result from *ex ante* optimism. Instead, optimism can result *ex post* from high prices which are themselves the consequence of non-optimistic misspecified *ex ante* beliefs. Our results analogously provide an interpretation for episodes of unusually low asset valuations.

Call the two types of investors  $A$  and  $B$ . Suppose that both  $A$  and  $B$  believe dogmatically that their own beliefs concerning an asset's payoffs are correct and unbiased but also believe incorrectly that the other group possesses beliefs which are slightly too pessimistic. As a result of this inconsistency between the first- and the second-order expectations, investors  $A$  and  $B$  both adjust the quantity demanded at each price upward by a small amount to correct for the perceived slight bearish effect of other investors' pessimism on prices. By pushing prices slightly higher, these demand adjustments bias prices in an upward direction by a small amount.

Now consider what happens when this small misspecification of beliefs propagates to higher-order beliefs. Suppose that  $A$  believes that  $B$  believes that  $A$  is slightly optimistic but  $B$  believes that  $A$  believes that  $B$  believes that  $A$  is slightly pessimistic. Suppose that  $A$  and  $B$  hold similarly inconsistent third- and fourth-order symmetric beliefs about  $B$ 's beliefs. As a result of this slight inconsistency between the third-order and the fourth-order expectations, both  $A$  and  $B$  further adjust their demands upward by an additional small amount to correct for the perceived pessimism in the other investors' higher-order beliefs. Prices are "too high" as a result two small adjustments made by  $A$  and two small adjustments made by  $B$ .

The logic extends to higher and higher orders of beliefs. Suppose that  $A$  believes that  $B$  believes that  $A$  believes that  $B$  believes that  $A$  is slightly optimistic but  $B$  believes that  $A$  believes that  $B$  believes that  $A$  believes that  $B$  believes that  $A$  is slightly pessimistic. Suppose that  $A$  and  $B$  hold similarly inconsistent symmetric fifth- and

sixth-order beliefs about  $B$ 's beliefs. This slight inconsistency between the fifth- and the sixth-order expectations results in yet another round of upward adjustments to price. We now have three upward adjustments by  $A$  and three by  $B$ . If this slight misspecification persists to higher and higher orders of beliefs, the result is an unbounded number of small upward adjustments to prices.

We show that the infinite sum of small adjustments to prices takes the form of a geometric series, in which weights on the  $n$ th round of adjustments are proportional to powers of some endogenously determined geometric decay coefficient which lies between zero and one. How close these geometric decay coefficients are to one depends on the parameters defining the investors' demand functions. In a competitive equilibrium, the investors' demands for the asset are linear functions of their private signals and the market price. The coefficients on the market price and the private signals are common knowledge, taking the same values in the model with disagreement as in the corresponding competitive noisy rational expectations model without disagreement. As the standard deviation of noise trading becomes smaller, traders put more weight on the market price and less weight on their private signals. The geometric decay coefficient in the geometric series describing the effect of misspecified beliefs on prices is a simple function of the coefficients in the investors' demand function. In the limit as noise trading goes to zero, the geometric decay coefficient goes to one, as a result of investors putting more and more weight on the price and less and less weight on their own private information. If the decay coefficient is close enough to one, a large enough finite sum of small adjustments in higher and higher order beliefs can have an arbitrarily large positive or negative effect on prices.

This result stands in a stark contrast to the conventional interpretation of standard noisy rational expectations models, in which a smaller standard deviation of noise trading leads to more informative and more "stable" prices. It is related instead to the conventional result that market liquidity disappears in the limit as the standard deviation of noise trading goes to zero, making the price highly sensitive to small shocks to supply. When the market is illiquid due to a small level of noise trading, the market price also becomes highly sensitive to small inconsistencies in beliefs, as a result of which small inconsistencies propagate into large price effects.

Investors  $A$  and  $B$  both correctly conjecture that the price is a linear combination of  $A$ 's private signal,  $B$ 's private signal, a noise trading term, and a constant term. When

traders have different beliefs about the first moment, they disagree only about the value of the constant term, which can be interpreted as the mean of the asset's payoff. Since traders agree about the values of parameters related to second moments, the coefficients on the private signals and noise trading are common knowledge, taking the same values as in the otherwise identical rational expectations equilibrium in which the mean is common knowledge; these coefficients related to second moments do not change when disagreement about first moments is added to the model.

For example, modest higher-order differences in beliefs about the mean asset payoff can make the constant term in the price have a very large value, as a result of which prices are likely to be inflated. Nevertheless, both  $A$  and  $B$  believe that the constant term has a much smaller value. When both investors  $A$  and  $B$  observe mediocre private information, they both submit demand schedules adjusted upward for the perceived excessive bearishness of the other group; they are both surprised to observe a very high price, from which they both incorrectly infer that the other investors have exceptionally bullish information. In this way, extreme asset overvaluation can happen when every investor inconsistently believes that his expectation is more optimistic than the average expectation.

Our results imply that market fragility may arise from a small degree of inconsistency in beliefs. Both groups of traders believe that the market is “almost efficient” in the sense that it almost aggregates information correctly. Significantly inefficient information aggregation results from mild inconsistencies in higher-order beliefs. In this sense, bubbles can result from a misplaced *ex post* dogmatic belief in market efficiency and not from misplaced *ex ante* optimism. Although our model is a static one, we believe its intuition explains escalating optimism associated with prices in bubble episodes, such as the internet stock bubble in 1999 or the mortgage debt bubble in 2006. As such episodes unfold, high prices may result not from exogenous optimism of traders but rather from an endogenous process of gradually inflating expectations due to a slight degree of inconsistency in higher-order beliefs.

Solving learning problems with disagreement in beliefs over infinite hierarchies of beliefs can lead to non-trivial technical difficulties. The usual approach in the noisy rational expectations literature—followed by Grossman and Stiglitz [1980], Hellwig [1980], and Diamond and Verrecchia [1981]—infers private information from a linear price function consisting of signals and noise. The rational expectations version of our model

converges to Grossman [1976] when noise trading vanishes. The usual approach does not work in our model of higher-order beliefs. Even though ours is a static one-period model, there is an “infinite regress” in beliefs based on the way in which traders with different beliefs learn from prices. Since heterogeneous beliefs at each level of an infinite hierarchy affect the price in a different way, the price function is determined by an infinite number of different sets of coefficients. This leads to an infinite number of simultaneous equations, which are potentially difficult to solve. We therefore develop a different method which collapses this infinite dimensional problem into a single dimensional one. We first construct a statistic that is a function of a conditional expectation of fundamentals and noise; the conditional expectation is already a consequence of an infinite number of adjustments for biases associated with higher-order beliefs. This statistic is not only a single dimensional signal, but also turns out to be a sufficient statistic for private information. This allows us to construct a linear equilibrium, from which we reverse-engineer the infinite number of terms in the price function by filtering each bias in the higher-order beliefs out from this sufficient statistic iteratively. Once we obtain an equilibrium price function using this technique, we can proceed to solve the equilibrium as in a standard noisy rational expectations model.

The organization of the paper is as follows. Section 2 discusses the related literature. Section 3 introduces the model and defines the concept of differences in higher-order beliefs. Section 4 states the main result characterizing the equilibrium and shows how the main result applies to four special cases. Section 5 provides the proofs of the results by developing a framework for solving belief-updating problems under differences in higher-order beliefs. Section 6 discusses the amplification mechanism arising from inconsistent beliefs. Section 7 discusses practical implications of the results. Section 8 concludes.

## 2 Higher Order Expectations and Higher Order Beliefs in the Literature

In order to understand the contribution of our paper, it is useful to make a distinction between “higher-order beliefs” and “higher-order expectations.” Higher-order expectations can become important when traders have different information, in which case trader  $A$ ’s expectation of trader  $B$ ’s expectation is generally different from trader  $B$ ’s expectation itself, even when traders share a common prior or agree to disagree about

the correct prior. When traders have different higher-order beliefs, they neither share a common prior nor “agree to disagree” about the correct prior. Instead, their disagreement propagates to beliefs of higher-order, e.g.,  $A$ ’s beliefs about  $B$ ’s prior distribution may differ from  $B$ ’s prior distribution itself.

Differences in higher-order beliefs propagate into higher-order expectations differently from the way in which differences in private information propagate into higher-order expectations when higher-order beliefs are the same. In particular, trader  $A$ ’s expectation of trader  $B$ ’s expectation is potentially affected both by differences in information sets and differences in higher-order beliefs. In this paper, differences in information (asymmetric information) lead to differences in higher-order *conditional* expectations. Difference in beliefs, by contrast, lead to differences in higher-order *unconditional* expectations.

Although Muth [1961] does not include a common-prior or common-knowledge assumption in his concept of rational expectations, standard noisy rational expectations models usually assume that investors have the same prior, which is also common knowledge among them. Consequently, there is no disagreement at any level in hierarchies of traders’ beliefs. Our setup generalizes a differential information model like Hellwig [1980] by adding disagreement concerning higher-order beliefs; disagreements in first- or the second-order beliefs nest into our framework as special cases. We also make a technical contribution to the literature by developing a signal extraction technique that allows us to solve the model in closed form. We demonstrate that price stability fails precisely due to learning from prices under differences in higher-order beliefs.

The law of iterated expectations applies in situations where traders have different information sets and the same higher-order beliefs, which do not necessarily require a common prior. The law of iterated expectations does not necessarily apply when traders have different higher-order beliefs, even when they share the same information. In our model, the law of iterated expectations does not apply because traders have different beliefs. In our model, traders also have different private information but, as in the rational expectations literature, they infer a noisy signal of other trader’s private information from prices. Intuitively, the weight that traders put on the price, versus their own private information, affects the extent to which differences in higher-order beliefs affect prices at higher and higher order. The higher the weight on the price, the greater the effects of modest differences in beliefs on prices.

In the financial economics literature, higher-order expectations arise when traders forecast the forecasts of others. Following Keynes [1936], researchers in financial economics have tried to use differences in higher-order expectations to explain asset prices as the result of “beauty contests” in which traders speculate based on their estimates of the valuations of other traders. Townsend [1983] points out that this kind of model often gives rise to an “infinite regress” problem in which each trader must keep track of an infinite number of state variables to implement an optimal trading strategy. Makarov and Rytchkov [2012] avoid the infinite regress problem by transforming the problem from the time domain into the frequency domain. Some models avoid the infinite regress problem because higher-order expectations collapse to first-order expectations, including He and Wang [1995], Allen, Morris and Shin [2006], Bacchetta and Wincoop [2008], and Kyle, Obizhaeva and Wang [2013]. Because the value of an asset depends on the average opinions of all the investors, investors tend to overweight public signals rather than their own private signals. This creates slow aggregation of information. Most of the papers find that the law of iterated expectations fails when applied to the average of expectations, i.e., the average of expectations fails to be a martingale even when each expectation included in the average is individually a martingale. In general, the amplification of biases through aggregation of information in the financial market does not arise in this line of literature because investors have either agreed to agree or agreed to disagree about each other’s beliefs. In our one-period model, an infinite regress problem arises due to differences in higher-order beliefs; this problem would not occur if traders shared a common prior or agreed to disagree.

Some papers—e.g., Banerjee, Kaniel and Kremer [2009], Cao and Ou-Yang [2009], and Banerjee [2011]—use differences (or heterogeneities) in the first- or second-order beliefs to explain anomalies in financial markets. These papers do not introduce differences of opinion over an infinite level of hierarchies of beliefs into a noisy rational expectations model. For example, Banerjee et al. [2009] find that higher-order disagreement leads to price drift in a dynamic trading model. Investors infer estimates of others’ private signals from prices but do not update their beliefs on fundamentals because they believe that others’ signals are uninformative. Our model allows the investors to learn about fundamentals from prices in the presence of heterogeneous beliefs.

Hassan and Mertens [2011] show how mispricing can arise from learning when traders make erroneous biased guesses which are magnified in prices. Although they do not formally model differences in higher-order beliefs, their model shares a mechanism similar



to ours. Harrison and Kreps [1978] and Scheinkman and Xiong [2003] generate inflated prices from resale options which arise in dynamic models with agreement to disagree and short-sale constraints. Similarly, Biais and Bossaerts [1998] generate inflated prices from resale options in a model where traders do not have common knowledge about the distribution of one another's private valuations. The mechanism is different in our model, where prices may become inflated due to the manner in which inconsistencies in higher-order beliefs affect learning from prices.

### 3 Model

#### 3.1 Basic Setup

Consider a competitive model of one-period trading based on differential private information similar to Hellwig [1980]. A risky asset with random payoff  $\tilde{v}$  is traded against a safe asset whose return is normalized to one. Random noise traders generate an exogenous supply of the risky asset  $\tilde{\epsilon}_x$  distributed  $N(\bar{x}, \frac{1}{\tau_x})$ . The supply of the risky asset is purchased by a continuum of two types of atomistic informed traders, labeled  $A$  and  $B$ , each with measure one-half. Each informed trader has exponential utility with constant absolute risk aversion parameter  $\gamma$ . The risky asset is in zero net supply; each investor's initial endowment is assumed to be zero. The risky asset's payoff is distributed  $N(\bar{v} + \mu, \frac{1}{\tau_v})$ . Each investor  $i \in \{A, B\}$  receives a private noisy signal  $\tilde{s}_i$  such that

$$\tilde{s}_i = \tilde{v} + \tilde{\epsilon}_i, \tag{1}$$

where  $\tilde{\epsilon}_i$  is noise that is distributed  $N(0, \frac{1}{\tau_s})$ . The random variables  $\tilde{v}$ ,  $\tilde{\epsilon}_A$ ,  $\tilde{\epsilon}_B$ , and  $\tilde{\epsilon}_x$  are independently distributed.

Except for beliefs about the parameter  $\mu$ , the structure of the economy is common knowledge. In particular, the investors all agree about the values of the parameters  $\bar{x}$ ,  $\gamma$ ,  $\tau_v$ ,  $\tau_s$ , and  $\tau_x$ . They agree about the covariance structure of all the random variables  $\tilde{v}$ ,  $\tilde{\epsilon}_A$ ,  $\tilde{\epsilon}_B$ , and  $\tilde{\epsilon}_x$ . They agree that each investor's initial endowment is zero, and they agree that the means  $\tilde{\epsilon}_A$  and  $\tilde{\epsilon}_B$  are zero. If the value of  $\mu$  were also common knowledge, the model would collapse to a special case of the competitive rational expectations equilibrium of Hellwig [1980].

### 3.2 Hierarchies of Beliefs

We study the implications of assuming that investors  $A$  and  $B$  may have *different higher-order beliefs* concerning the value of the parameter  $\mu$ . Let  $E^i[\cdot]$  denote the (subjective) expectation operator of investor  $i \in \{A, B\}$  on any random variable. Although investors might in principle hold beliefs about the value of  $\mu$  that are probability distributions, we assume it is common knowledge that first and higher-order beliefs are “dogmatic” point estimates, which can be represented by applying an expectation operator to a degenerate probability distribution. For example, investor  $A$  believes that  $\tilde{\mu}$  is a random variable which takes the value  $\mu^A$  with probability one. Using the notation  $\mu^i := E^i[\mu]$  for  $i, j \in \{A, B\}$  to denote  $i$ 's belief about  $\mu$ , we represent investor  $A$ 's and  $B$ 's “first-order beliefs” as conditional expectations:

$$E^A[\tilde{v}] = \bar{v} + \mu^A, \quad \text{and} \quad E^B[\tilde{v}] = \bar{v} + \mu^B. \quad (2)$$

Using the notation  $\mu^{ij} := E^i E^j[\mu]$  for  $i, j \in \{A, B\}$ , we represent investor  $A$ 's and  $B$ 's “second-order beliefs” as second-order expectations:

$$E^A E^B[\tilde{v}] = \bar{v} + \mu^{AB}, \quad \text{and} \quad E^B E^A[\tilde{v}] = \bar{v} + \mu^{BA}. \quad (3)$$

On the one hand, if  $\mu^{AB}$  is greater than  $\mu^A$ , investor  $A$  believes that investor  $B$  is more optimistic about the fundamental value than the investor  $A$  is. On the other hand, if  $\mu^{AB}$  is greater than  $\mu^B$ , investor  $A$  incorrectly believes that investor  $B$  is more optimistic about the fundamental value than the investor  $B$  actually is.

Following the same logic, we represent investor  $A$ 's and  $B$ 's “third-order beliefs” as

$$E^A E^B E^A[\tilde{v}] = \bar{v} + \mu^{ABA}, \quad \text{and} \quad E^B E^A E^B[\tilde{v}] = \bar{v} + \mu^{BAB}. \quad (4)$$

Because a given trader's expectation operator is a projection, higher-order beliefs only matter when the expectation operators of investor  $A$  and  $B$  are alternating, e.g.,  $E^A E^A E^B[\cdot] = E^A E^B[\cdot]$ . For notational convenience in expressing beliefs of arbitrarily high order, we define  $\mu^{A(1)} := \mu^A$ ,  $\mu^{A(2)} := \mu^{BA}$ ,  $\mu^{A(3)} := \mu^{ABA}$ , and so on. More generally we define  $A(n)$  recursively as  $A(1) = A$ ,  $A(2) = BA$ , and  $A(n+2) = A(n)BA$ . Similarly, we define  $B(1) = B$ ,  $B(2) = AB$ , and  $B(n+2) = B(n)AB$ . For example,  $B(3) = BAB$ ,  $B(4) = ABAB$ ,  $B(5) = BABAB$ , and so on.

If investor  $A$ 's first-order belief is correct, we have  $\mu = \mu^A$ . If investor  $B$ 's second-order belief about investor  $A$ 's first-order belief is correct, we have  $\mu^A = \mu^{BA}$ . If investor  $A$ 's third-order belief about investor  $B$ 's second-order belief is correct, then we have  $\mu^{BA} = \mu^{ABA}$ . In general, we say that  $n$ th-order beliefs about investor  $A$ 's or  $B$ 's beliefs are correct if  $\mu^{A(n-1)} = \mu^{A(n)}$  or  $\mu^{B(n-1)} = \mu^{B(n)}$ ; for  $n = 1$  we need the convention  $\mu^{A(0)} = \mu^{B(0)} = \mu$ .

We define “higher-order beliefs” as beliefs of degree two and higher. Investors have “agreement” in higher-order beliefs when  $n$ th-order beliefs about  $A$ 's and  $B$ 's beliefs are the same for all  $n = 2, 3, \dots$ . When investors have agreement in higher-order beliefs, beliefs of all orders  $n \in \mathbb{N} = \{1, 2, 3, \dots\}$  are common knowledge. This does not imply that beliefs are the same; if first-order beliefs are different ( $\mu^A \neq \mu^B$ ), even when beliefs are common knowledge, there is “agreement to disagree”.

When higher-order beliefs are different, we make a distinction between incorrect beliefs and inconsistent beliefs. For  $j \in \{A, B\}$  and  $n \in \mathbb{N}$ , define  $\Delta\mu^j(n)$  as

$$\Delta\mu^j(n) := \mu^{j(n)} - \mu^{j(n-1)}, \quad \text{where } \mu^{j(0)} := \mu. \quad (5)$$

For  $j \in \{A, B\}$ , we say that  $j$ 's beliefs of order  $n$  are “incorrect” if  $\Delta\mu^j(n) \neq 0$ , in which case  $\Delta\mu^j(n)$  quantifies the degree of incorrectness or “bias” in beliefs. We say that beliefs of order  $n$  are “inconsistent” if  $\Delta\mu^{A(n)} + \Delta\mu^{B(n)} \neq 0$ . Inconsistent beliefs are incorrect “on average.” If beliefs are inconsistent, then either  $A$ 's or  $B$ 's beliefs must be incorrect. If beliefs are “consistent,” i.e.,  $\Delta\mu^{A(n)} + \Delta\mu^{B(n)} = 0$ , both  $A$  and  $B$  may have incorrect beliefs, as long as they are “correct on average” in the sense that  $\Delta\mu^{A(n)} = -\Delta\mu^{B(n)} \neq 0$ . Our concept of consistent beliefs is consistent with Muth [1961]'s definition of rational expectations, which requires the price to incorporate beliefs which are correct on average.

We define “absolute optimism” to be inconsistent first-order beliefs such that the investors' beliefs about the mean of the fundamental value are, on average, higher than the correct mean, i.e.,  $\frac{1}{2}(\mu^A + \mu^B) > \mu$  (equivalent to inconsistently positive first-order beliefs  $\Delta\mu^{A(1)} + \Delta\mu^{B(1)} > 0$ ). We define “relative optimism” to be inconsistent second-order beliefs such that  $A$  and  $B$  believe, on average, that they are more optimistic than  $B$  and  $A$ , respectively, i.e.,  $\mu^{AB} + \mu^{BA} < \mu^A + \mu^B$  (equivalent to inconsistently negative second-order beliefs  $\Delta\mu^{A(2)} + \Delta\mu^{B(2)} < 0$ ). Absolute pessimism and relative pessimism are defined analogously by changing the inequality sign.

We define “perceived agreement” to be inconsistent higher-order beliefs such that  $A$  and  $B$  both incorrectly believe that their beliefs are common knowledge. Investor  $A$  and investor  $B$  may either agree or disagree about  $\mu$ . They incorrectly believe that they either agree to agree or agree to disagree, even though this belief is incorrect because their higher-order beliefs differ. This belief structure is characterized by only four constants:  $\mu^A$ ,  $\mu^B$ ,  $\mu^{AB}$ , and  $\mu^{BA}$ . Since traders’ beliefs about other traders’ beliefs are incorrect, we have  $\mu_A \neq \mu_{BA}$  and  $\mu_B \neq \mu_{AB}$ . Since  $A$  believes that his beliefs are common knowledge, we have  $\mu^A = \mu^{ABA} = \mu^{ABABA} = \mu^{ABABABA} = \dots$ . Since  $B$  believes that his beliefs about  $A$ ’s beliefs are common knowledge, we have  $\mu^{BA} = \mu^{BABA} = \mu^{BABABA} = \dots$ . Since the same applies symmetrically to  $B$ , perceived agreement implies that higher-order beliefs satisfy  $\mu^{A(2n-1)} = \mu^A$ ,  $\mu^{B(2n-1)} = \mu^B$ ,  $\mu^{A(2n)} = \mu^{BA}$  and  $\mu^{B(2n)} = \mu^{AB}$  for all  $n \in \mathbb{N}$ .

Perceived agreement defines a belief structure which has an infinite number of higher order inconsistencies. For higher-order beliefs of all orders higher than one, the errors have the same absolute magnitude but alternate in sign; i.e., we have  $\Delta\mu^{A(n)} = (-1)^n \Delta\mu^{A(2)}$  and  $\Delta\mu^{B(n)} = (-1)^n \Delta\mu^{B(2)}$ , for  $n = 3, 4, 5, \dots$

For mathematical simplicity, we assume that inconsistencies are uniformly bounded; i.e., there exists a positive constant  $c$  such that  $|\Delta\mu^{A(n)} + \Delta\mu^{B(n)}| < c$  for all  $n \in \mathbb{N}$ .

## 4 Equilibrium

An equilibrium is defined in a standard manner, so that it collapses to a competitive rational expectations equilibrium when the belief structure is characterized by rational expectations, but otherwise the equilibrium concept is modified to take account of investors’ higher-order beliefs. An equilibrium is described by three functions  $P$ ,  $X_A$ , and  $X_B$  such that the price is given by  $\tilde{p} = P(\tilde{s}_A, \tilde{s}_B, \tilde{\epsilon}_x)$ , and the quantities demanded by traders  $A$  and  $B$  are given respectively by  $\tilde{x}_A = X_A(\tilde{s}_A, \tilde{p})$  and  $\tilde{x}_B = X_B(\tilde{s}_B, \tilde{p})$ . The functions  $P$ ,  $X_A$ , and  $X_B$  define an equilibrium if (1) markets clear in the sense that  $\frac{1}{2}\tilde{x}_A + \frac{1}{2}\tilde{x}_B = \tilde{\epsilon}_x$ ; and (2) the demands  $\tilde{x}_A$  and  $\tilde{x}_B$  maximize investor  $A$ ’s and  $B$ ’s expected utility, taking the price as given (perfect competition) and *taking into account higher-order beliefs*.

As in the standard noisy rational expectations literature, our main result concerns existence and uniqueness of an equilibrium in which the price  $\tilde{p}$  is an affine function of

noise trading  $\tilde{\epsilon}_x$  and the two signals  $\tilde{s}_A, \tilde{s}_B$ , with symmetry suggesting equal weights on both signals such that

$$\tilde{p} = \pi_0 + \pi_s(\tilde{s}_A + \tilde{s}_B) - \pi_x\tilde{\epsilon}_x, \quad (6)$$

where  $\pi_0, \pi_s$  and  $\pi_x$  are constants. Symmetry also suggests that the demand functions  $X_A$  and  $X_B$  are the same.

Our main result states that there is a unique equilibrium in which the price has the symmetric linear form conjectured in (6). The price collapses to the standard competitive rational expectations price when the traders have rational expectations. Otherwise, the equilibrium price is the sum of the rational expectations price and a constant. The constant is defined by an infinite sum of terms representing the average degree of incorrectness or bias in first-order beliefs and inconsistencies in beliefs of arbitrarily higher order. The main innovation of this paper is our derivation of this infinite sum by repeated recursive substitutions. In equilibrium, investors  $A$  and  $B$  trade different quantities both because of stochastic differences in signals  $\tilde{s}_A, \tilde{s}_B$  and because of non-stochastic differences in beliefs.

The following lengthy theorem states our result:

**Theorem 1.** *There always exists an equilibrium, and it is unique among the class of affine price functions that satisfy (6). The equilibrium price function is the sum of two terms: (i) the rational expectations equilibrium price  $\tilde{p}^{REE}$  and (ii) an cumulative bias term  $\theta$ , i.e.,*

$$\tilde{p} = \underbrace{\tilde{p}^{REE}}_{\text{rational expectations equilibrium price}} + \underbrace{\theta}_{\text{cumulative bias}}, \quad (7)$$

where

$$\tilde{p}^{REE} := \pi_0^{REE} + \pi_s(\tilde{s}_A + \tilde{s}_B) - \pi_x\tilde{\epsilon}_x, \quad (8)$$

and

$$\theta =: \sum_{n=1}^{\infty} \theta_n, \quad (9)$$

and

$$\theta_n := \frac{1 - \beta_s - \beta_w}{2(1 - \beta_w)} (-\beta_w)^{n-1} (\Delta\mu^{A(n)} + \Delta\mu^{B(n)}). \quad (10)$$

The constants  $\beta_s, \beta_w$  and  $\tau$  are uniquely determined by the system of equations

$$\beta_s = \frac{\beta_s^2(1 - \beta_w) \frac{1}{\tau_s \tau_v} + \frac{4\gamma^2(1 + \beta_w)^2}{\tau^2} \frac{1}{\tau_x \tau_v}}{\beta_s^2 \left( \frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2} \right) + \frac{4\gamma^2(1 + \beta_w)^2}{\tau^2} \left( \frac{1}{\tau_x \tau_v} + \frac{1}{\tau_s \tau_x} \right)}, \quad (11)$$

$$\beta_w = \frac{\beta_s(1 - \beta_w^2) \frac{1}{\tau_s \tau_v}}{\beta_s^2 \left( \frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2} \right) + \frac{4\gamma^2(1 + \beta_w)^2}{\tau^2} \left( \frac{1}{\tau_x \tau_v} + \frac{1}{\tau_s \tau_x} \right)}, \quad (12)$$

$$\frac{1}{\tau} = \left( 1 - \beta_s - \frac{\beta_s}{1 - \beta_w} \beta_w \right) \frac{1}{\tau_v}, \quad (13)$$

in terms of which  $\pi_0^{REE}$ ,  $\pi_s$ , and  $\pi_x$  are given by

$$\pi_0^{REE} = \frac{1 - \beta_s - \beta_w}{1 - \beta_w} (\bar{v} + \mu) + \frac{2\gamma\beta_w}{\tau(1 - \beta_w)} \bar{x}, \quad (14)$$

$$\pi_s = \frac{\beta_s}{2(1 - \beta_w)}, \quad (15)$$

$$\pi_x = \frac{\gamma(1 + \beta_w)}{\tau(1 - \beta_w)}. \quad (16)$$

Let  $\tilde{x}_A$  and  $\tilde{x}_B$  denote the portfolio holdings by investor A and B, respectively. Then, the difference in portfolio holdings  $\Delta\tilde{x} := \tilde{x}_A - \tilde{x}_B$  is equal to the sum of two terms based on (i) differences in information and (ii) inconsistencies in higher-order beliefs:

$$\Delta\tilde{x} = \underbrace{\frac{\tau}{\gamma} \frac{\beta_s}{1 + \beta_w} (\tilde{s}_A - \tilde{s}_B)}_{\text{difference in information}} + \underbrace{\frac{\tau}{\gamma} \frac{1 - \beta_s - \beta_w}{1 - \beta_w} \sum_{n=1}^{\infty} \Delta x_n}_{\text{differences in higher-order beliefs}}, \quad (17)$$

where

$$\Delta x_n := \beta_w^{n-1} (\Delta\mu^{A(n)} - \Delta\mu^{B(n)}). \quad (18)$$

The first component  $\tilde{p}^{REE}$  of the price function (7) is the price in standard noisy rational expectations models (e.g., Hellwig [1980]), which does not feature any differences in first- and higher-order beliefs. The second component  $\theta$  adjusts for differences in first- or higher-order beliefs. Notice that  $\theta$  is a constant; its value is unaffected by the realizations of the random variables  $\tilde{v}, \tilde{\epsilon}_A, \tilde{\epsilon}_B$  and  $\tilde{\epsilon}_x$ .

Clearly,  $\theta_n$  becomes zero whenever the investors have correct  $n$ th-order beliefs about the other's beliefs, i.e. Furthermore,  $\theta_n = 0$  if and only if  $\Delta\mu^{A(n)} + \Delta\mu^{B(n)} = 0$ ; thus,  $\theta_n = 0$  for all  $n \in \mathbb{N}$  means that the investors beliefs are consistent for all orders of

belief  $n$ , even though each investor group may have biased beliefs offset by an equal and opposite bias of the other investor group. The hypothesis that consistent beliefs make prices unbiased is an extension of the original formulation of the rational expectations hypothesis by Muth [1961], who conjectured that the expectations of agents tend to be distributed about the objective distribution of true outcomes. This conjecture allows investors to be correct in aggregate even though each individual may be incorrect. Our approach makes precise how this way of thinking about rational expectations also applies to higher-order beliefs.

Equations (9) and (10) also show that the cumulative bias component takes the form of a geometric series. The weights alternate in sign, with weights on  $n$ th-order beliefs proportional to powers of the geometric decay coefficient  $\beta_w$ , which lies between zero and one. Therefore, the price impact of differences in higher-order beliefs decreases geometrically as the order of beliefs increases. As  $\beta_w$  becomes close to one, however, the decay rate becomes very small, thereby making the price impact significant even at very high order.

The impact of  $n$ th-order difference in higher-order beliefs on the portfolio holdings is zero (i.e.,  $\Delta x_n = 0$ ) whenever both investors  $A$  and  $B$  have correct  $n$ th-order beliefs for all  $n$  (i.e.,  $\Delta\mu^{A(n)} = \Delta\mu^{B(n)} = 0$ ). Moreover,  $\Delta x_n = 0$  whenever the  $n$ th-order beliefs of investors  $A$  and  $B$  are the same, even if they are incorrect ( $\Delta\mu^{A(n)} = \Delta\mu^{B(n)} \neq 0$ ). Higher-order beliefs affect trading volume only to the extent that the beliefs are different, regardless of whether the beliefs are correct or not.

Differences in beliefs, not inconsistency in beliefs, amplify trading volume. If all the investors have relative optimism (or pessimism) with similar magnitude, there will be a very small effect on trading volume but a large effect on prices. If higher-order beliefs are incorrect but consistent (i.e., the errors offset due to the same magnitude but opposite sign), there is no effect on prices but there is increased trading volume.

Standard rational expectations models, such as Grossman [1976] and Hellwig [1980], have the property that investors tend to ignore their own private signals and only learn from prices if the standard deviation of noise trading is close to zero. When the standard deviation of noise trading is large, the opposite happens; investors tend to ignore noisy prices and learn only from their own private signals. In this section, we study these limit cases when investors have differences in higher-order beliefs.

When there is little noise in the supply (i.e.,  $\tau_x \rightarrow \infty$ ), all the private information is

fully revealed through the price (e.g., Grossman [1976] and Milgrom and Stokey [1982]). In particular,  $\tilde{p}^{REE}$  converges to the price function in Grossman [1976].

**Corollary 1.** *As  $\tau_x \rightarrow \infty$  (noise trading vanishes), we have  $\beta_s \rightarrow 0, \beta_w \rightarrow 1$  and  $\frac{1}{\tau} \rightarrow \frac{1}{2\tau_s + \tau_v}$ . Furthermore, the rational expectation equilibrium price component converges to*

$$\tilde{p}^{REE} = \frac{\tau_v}{2\tau_s + \tau_v}(\bar{v} + \mu) - \frac{\gamma}{2\tau_s + \tau_v}\bar{x} + \frac{\tau_s}{2\tau_s + \tau_v}(\tilde{s}_A + \tilde{s}_B). \quad (19)$$

*If investors have perceived agreement and either relative optimism or relative pessimism, then  $\theta \rightarrow +\infty$  or  $\theta \rightarrow -\infty$ , respectively.*

*Proof.* See Appendix. □

As a result of  $\beta_w$  going to one,  $\theta$  can diverge even when there is very little deviation from the common knowledge assumption (i.e., even when  $\Delta\mu^{A(n)} + \Delta\mu^{B(n)}$  are very small for any  $n \in \mathbb{N}$ ). This result illustrates the more general point made by Rubinstein [1989] that an equilibrium under common knowledge can be very different from an equilibrium under “almost common knowledge”.

When noise trading intensity is close to zero, investors update expectations based only on prices. When there is inconsistency in higher-order beliefs, biases in prices are magnified because of a large “belief multiplier” effect. The intuition is that each investor attributes high prices to private information of the other investor, and this creates feedback effects which magnify the price effects of inconsistencies in beliefs.

Little private information is revealed through the price when there is a higher level of supply noise (i.e.,  $\tau_x \rightarrow 0$ ). Because the investors do not depend on prices for their belief updates, prices are not unaffected by differences in higher-order beliefs. The bias in prices depends only on first-order beliefs ( $\mu_A$  and  $\mu_B$ ), not on differences in higher-order beliefs, as the following Corollary states:

**Corollary 2.** *As  $\tau_x \rightarrow 0$  (noise becomes infinite), we have  $\beta_s \rightarrow \frac{\tau_s}{\tau_s + \tau_v}, \beta_w \rightarrow 0$  and  $\frac{1}{\tau} \rightarrow \frac{1}{\tau_s + \tau_v}$ . The rational expectations equilibrium price converges to*

$$\tilde{p}^{REE} = \frac{\tau_v}{\tau_s + \tau_v}(\bar{v} + \mu) + \frac{1}{2} \frac{\tau_s}{\tau_s + \tau_v}(\tilde{s}_A + \tilde{s}_B) - \frac{\gamma}{\tau_s + \tau_v}\tilde{\epsilon}_x. \quad (20)$$



and the cumulative bias component converges to

$$\theta = \frac{1}{2} \frac{\tau_v}{\tau_s + \tau_v} (\Delta\mu^{A(1)} + \Delta\mu^{B(1)}). \quad (21)$$

*Proof.* See Appendix. □

Our result appears to contrast starkly with standard noisy rational expectations models in which a smaller standard deviation of noise trading leads to more informative and stable prices. By pushing the weight on prices below one, a higher standard deviation of noise trading strengthens the contraction mapping property for belief updates when there are differences in higher-order beliefs. Our result is related to Nyarko [1997] and Weinstein and Yildiz [2007], who use a contraction mapping argument to show that uncertainties in higher-order beliefs have an effect on best response mappings which destabilizes the equilibrium. In Section 5, we show that a fixed point exists if  $\beta_w$  is less than one. Since errors in beliefs are assumed to be bounded, a resulting contraction property prevents the belief updates from exploding over the infinite iteration process. As noise trading vanishes, however, we have  $\beta_w \rightarrow 1$ , as a result of which the price effects of very small belief inconsistencies can become very large. In Section 6, we further discuss such “belief multiplier effect” on prices.

We apply our results to the following four different specifications for the higher-order beliefs of the investors:

## 4.1 Rational Expectations

In an equilibrium with “rational expectations,” all investors know the true value of  $\mu$ , and it is common knowledge, i.e.,  $\mu^{A(n)} = \mu^{B(n)} = \mu$  for all  $n \in \mathbb{N}$ . This is equivalent to standard noisy rational expectations equilibrium where beliefs are correct and therefore consistent for all orders  $n \in \mathbb{N}$ .

The cumulative bias component  $\theta$  is equal to zero because  $\theta_n = 0$  for all  $n \in \mathbb{N}$ . In particular,  $\theta_n = 0$  for all values of  $\tau_x$ , including when  $\tau_x$  is very large or very small.

We have  $\Delta x_n = 0$  for all  $n \in \mathbb{N}$  because  $\mu = \mu^A = \mu^B$ . Thus, difference in holdings  $\Delta\tilde{x}$  depends only on differences in information, not differences in beliefs:

$$\Delta\tilde{x} = \frac{\tau}{\gamma} \frac{\beta_s}{1 + \beta_w} (s_A - s_B). \quad (22)$$

For the limit cases, we have  $\Delta\tilde{x} \rightarrow 0$  as  $\tau_x \rightarrow \infty$  and  $\Delta\tilde{x} \rightarrow \frac{\tau_s}{\gamma}(s_A - s_B)$  as  $\tau_x \rightarrow 0$ .

## 4.2 Agreement to Agree

In an equilibrium with “agreement to agree,” investor  $A$  and investor  $B$  have identical beliefs on  $\mu$ , these beliefs are common knowledge, but the beliefs may be incorrect. Thus, there exists some constant  $\bar{\mu}$  such that  $\mu^{A(n)} = \mu^{B(n)} = \bar{\mu}$  for all  $n \in \mathbb{N}$ . Both investor  $A$  and investor  $B$  believe that they are participating in a rational expectations equilibrium. In the special case  $\bar{\mu} = \mu$ , beliefs are correct; agreement-to-agree becomes equivalent to rational expectations. When  $\bar{\mu} \neq \mu$ , the equilibrium is not a rational expectations equilibrium, even though both  $A$  and  $B$  believe they are participating in a rational expectations equilibrium.

Because higher-order beliefs are correct, the cumulative bias component depends only on the first-order error in beliefs. We obtain

$$\theta = \frac{1}{2}(1 - 2\pi_s)(\Delta\mu^{A(1)} + \Delta\mu^{B(1)}) = -(1 - 2\pi_s)(\mu - \bar{\mu}). \quad (23)$$

For the limit cases, we have  $\theta \rightarrow \frac{1}{2} \frac{\tau_v}{2\tau_s + \tau_v} (\Delta\mu^{A(1)} + \Delta\mu^{B(1)})$  as  $\tau_x \rightarrow \infty$ , and  $\theta \rightarrow \frac{1}{2} \frac{\tau_v}{\tau_s + \tau_v} (\Delta\mu^{A(1)} + \Delta\mu^{B(1)})$  as  $\tau_x \rightarrow 0$ .

Identically to the case of rational expectations, differences in holdings  $\Delta\tilde{x}$  depend only on differences in information, not differences in beliefs.

## 4.3 Agreement to Disagree (or Heterogeneous Beliefs)

In an equilibrium with “agreement to disagree,” investor  $A$  and investor  $B$  disagree about  $\mu$ , but they agree to disagree, in the sense that their disagreement is common knowledge. Their disagreement implies  $\mu^A \neq \mu^B$ ; the fact that their disagreement is common knowledge implies  $\mu^{A(n)} = \mu^A$  and  $\mu^{B(n)} = \mu^B$  for all  $n \in \mathbb{N}$ . Beliefs of order two and higher are correct and therefore consistent. Since  $A$ ’s and  $B$ ’s first-order beliefs differ, both cannot be correct.

Because higher-order beliefs are correct, the bias in prices depends only on the first-

order error in beliefs. We obtain

$$\theta = \frac{1}{2}(1 - 2\pi_s)(\Delta\mu^{A(1)} + \Delta\mu^{B(1)}) = -(1 - 2\pi_s)\left(\mu - \frac{1}{2}(\mu_A + \mu_B)\right). \quad (24)$$

For the limit cases, we have  $\theta \rightarrow \frac{1}{2}\frac{\tau_v}{2\tau_s + \tau_v}(\Delta\mu^{A(1)} + \Delta\mu^{B(1)})$  as  $\tau_x \rightarrow \infty$  and  $\theta \rightarrow \frac{1}{2}\frac{\tau_v}{\tau_s + \tau_v}(\Delta\mu^{A(1)} + \Delta\mu^{B(1)})$  as  $\tau_x \rightarrow 0$ .

Because  $\Delta x_1 = -(\mu^A - \mu^B)$  and  $\Delta x_n = 0$  for all  $n \geq 2$ , the difference in holdings depends both on differences in information and differences in beliefs:

$$\Delta\tilde{x} = \frac{\tau}{\gamma} \frac{\beta_s}{1 + \beta_w} (s_A - s_B) + \frac{\tau}{\gamma} \frac{1 - \beta_s - \beta_w}{1 - \beta_w} (\mu^A - \mu^B). \quad (25)$$

In limit cases,  $\Delta\tilde{x} \rightarrow \frac{\tau_v}{\gamma}(\mu^A - \mu^B)$  as  $\tau_x \rightarrow \infty$ , and  $\Delta\tilde{x} \rightarrow \frac{\tau_s}{\gamma}(s_A - s_B) + \frac{\tau_v}{\gamma}(\mu^A - \mu^B)$  as  $\tau_x \rightarrow 0$ .

#### 4.4 Perceived Agreement

In an equilibrium with “perceived agreement,” recall that investor  $A$  and investor  $B$  incorrectly believe that their beliefs are common knowledge. We consider two interesting subcases:

The first interesting subcase occurs when both  $A$  and  $B$  incorrectly believe they are participating in a rational expectations equilibrium. Even though  $\mu^A \neq \mu^B$ ,  $A$  believes that  $B$  shares his beliefs ( $\mu^{AB} = \mu^A$ ), and  $B$  believes that  $A$  shares his beliefs ( $\mu^{BA} = \mu^B$ ). Both  $A$  and  $B$  believe it is common knowledge that their beliefs are the same, even though they are different. This subcase is fully described by two parameters,  $\mu^A$  and  $\mu^B$ . Investor  $A$  believes he is participating in a rational expectations equilibrium with  $\mu = \mu^A$ . Investor  $B$  believes he is participating in a rational expectations equilibrium with  $\mu = \mu^B$ . Although higher-order beliefs are incorrect, higher-order beliefs are consistent because the errors in higher-order beliefs are of the same magnitude and have opposite signs. As we discuss below, these differences in higher-order beliefs magnify trading volume, but these differences in higher-order beliefs do not affect prices.

The second interesting subcase occurs when  $A$  and  $B$  incorrectly but symmetrically believe that they agree to disagree. Suppose that  $\mu^A = \mu^B > \mu^{AB} = \mu^{BA}$ . Here,  $A$  and  $B$  actually agree about the mean, but they inconsistently believe otherwise. There is relative optimism (i.e.,  $\Delta\mu^{A(2)} + \Delta\mu^{B(2)} < 0$ ):  $A$  believes he is more optimistic than  $B$

and  $B$  believes he is more optimistic than  $A$ . As we discuss below, this set of inconsistent beliefs, which is also defined by two parameters, leads to inflated prices which explode in the limit as noise trading vanishes, even though trading volume is not affected by these inconsistent higher-order beliefs.

In the first three cases—rational expectations, agreement-to-agree, and agreement-to-disagree—investors have consistent higher-order beliefs about others' beliefs, i.e., all beliefs are common knowledge. In this fourth case—perceived agreement—there are inconsistencies in the investors' beliefs about others' beliefs. Spelling out the consequences of these inconsistent beliefs is the main contribution of this paper.

With perceived agreement, we have  $\theta_1 = (1 - 2\pi_s)(\Delta\mu^{A(1)} + \Delta\mu^{B(1)})$  and  $\theta_n = -(1 - 2\pi_s)\beta_w^{n-1}(\Delta\mu^{A(2)} + \Delta\mu^{B(2)})$  for all  $n \geq 2$ . According to (10), as  $n$  increases, the belief updates  $\theta_n$  alternate in sign if the sign of the inconsistencies  $\Delta\mu^{A(n)} + \Delta\mu^{B(n)}$  are the same for all  $n = 2, 3, 4, \dots$ . With perceived agreement, however, the sign of the inconsistencies  $\Delta\mu^{A(n)} + \Delta\mu^{B(n)}$  also alternates as  $n$  increases. Thus, all of the belief updates  $\theta_n$  have both the same sign and the same absolute value. With relative optimism, the belief inconsistency  $\Delta\mu^{A(2)} + \Delta\mu^{B(2)}$  is negative and this makes all of the price adjustment terms  $\theta_n$  positive; with relative pessimism, the belief inconsistency  $\Delta\mu^{A(2)} + \Delta\mu^{B(2)}$  is positive and this makes all of the price adjustment terms  $\theta_n$  negative. The total adjustment  $\theta$  is a geometric series which sums to

$$\theta = (1 - 2\pi_s) \left[ (\Delta\mu^{A(1)} + \Delta\mu^{B(1)}) - \frac{\beta_w}{2(1 - \beta_w)} (\Delta\mu^{A(2)} + \Delta\mu^{B(2)}) \right]. \quad (26)$$

As the standard deviation of noise trading becomes arbitrarily small, the power series coefficient  $\beta_w$  becomes arbitrarily close to one. Thus, the factor  $\frac{1}{2}\beta_w/(1 - \beta_w)$ , which multiplies belief inconsistency, becomes arbitrarily large. This makes the price effect of a small second-order belief inconsistency  $\Delta\mu^{A(2)} + \Delta\mu^{B(2)}$  arbitrarily large! Prices become very high with a modest degree of relative optimism and very low with a modest degree of relative pessimism.

As  $\tau_x \rightarrow \infty$  (noise trading vanishes), we have

$$\theta \rightarrow \begin{cases} \infty, & \text{if } \Delta\mu^{A(2)} - \Delta\mu^{B(2)} < 0; \\ \frac{1}{2} \frac{\tau_v}{2\tau_s + \tau_v} (\Delta\mu^{A(1)} + \Delta\mu^{B(1)}), & \text{if } \Delta\mu^{A(2)} - \Delta\mu^{B(2)} = 0; \\ -\infty, & \text{if } \Delta\mu^{A(2)} - \Delta\mu^{B(2)} > 0; \end{cases}$$

As  $\tau_x \rightarrow 0$  (noise trading explodes), we have

$$\theta \rightarrow \frac{1}{2} \frac{\tau_v}{\tau_s + \tau_v} (\Delta\mu^{A(1)} + \Delta\mu^{B(1)}). \quad (27)$$

Since  $\Delta x_1 = \mu^A - \mu^B$  and  $\Delta x_n = -(-\beta_w)^{n-1}(\Delta\mu^{A(2)} - \Delta\mu^{B(2)})$  for all  $n \geq 2$ , the difference in holdings depend on differences in information, differences in first-order beliefs, and differences in higher-order beliefs:

$$\Delta\tilde{x} = \frac{\tau}{\gamma} \frac{\beta_s}{1 + \beta_w} (s_A - s_B) + \frac{\tau}{\gamma} \frac{1 - \beta_s - \beta_w}{1 - \beta_w^2} [(1 + \beta_w)(\Delta\mu^{A(1)} - \Delta\mu^{B(1)}) + \beta_w(\Delta\mu^{A(2)} - \Delta\mu^{B(2)})]. \quad (28)$$

As  $\tau_x \rightarrow \infty$ , we have

$$\Delta\tilde{x} \rightarrow -\frac{\tau_v}{2\gamma} [2(\Delta\mu^{A(1)} - \Delta\mu^{B(1)}) + (\Delta\mu^{A(2)} - \Delta\mu^{B(2)})]. \quad (29)$$

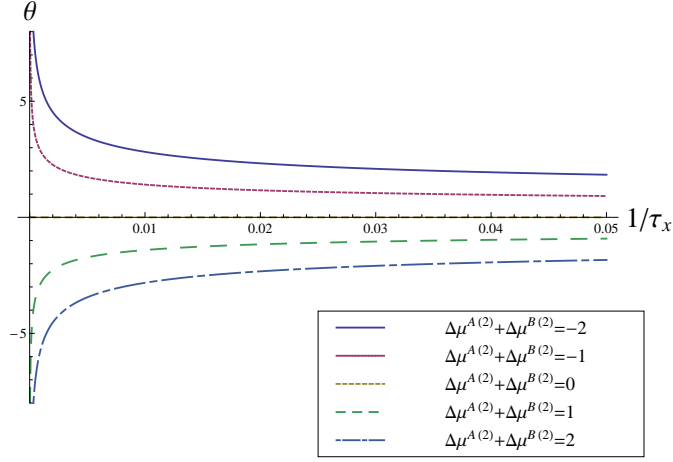
On the other hand, as  $\tau_x \rightarrow 0$ , we have

$$\Delta\tilde{x} \rightarrow \frac{\tau_s}{\gamma} (s_A - s_B) + \frac{\tau_v}{\gamma} (\mu^A - \mu^B). \quad (30)$$

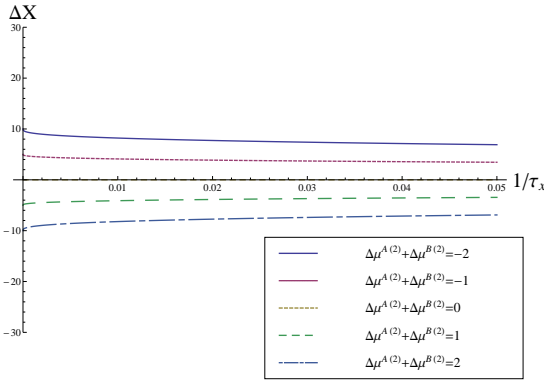
The results are graphically illustrated with numerical examples in Figure 1. Prices become arbitrarily high (or low) in the case of relative optimism (or pessimism) as the standard deviation of noise trading goes to zero, but trading volume stays stable even in that case. Therefore, the investors will not differ significantly in their holdings even when the cumulative bias component is more pronounced.

## 5 Proofs of the Results

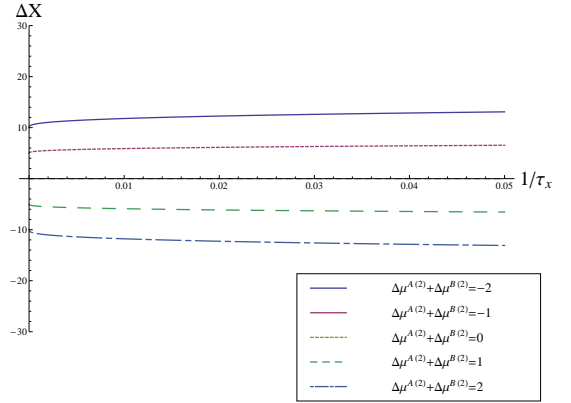
We prove theorem 1 in several steps over the next three subsections. First, we conjecture that the equilibrium price is given by (6). The price function is defined by a constant term  $\pi_0 = \pi_0^{REE} + \theta$  and two coefficients  $\pi_s$  and  $\pi_x$ . Since assumed disagreement concerns the mean and not the covariance structure of the model, we also conjecture that (i) the coefficients  $\pi_s$ ,  $\pi_x$  are common knowledge but (ii) investors may disagree about the value of the constant term  $\pi_0$ . The investors disagree about both  $\pi_0^{REE}$  and  $\theta$  because they disagree about  $\mu$ , thus also disagree about  $\pi_0$ . The price  $\tilde{p}^{REE}$  is equivalent to the



(a) Cumulative Bias Component



(b.i) Difference in Holdings when  $\mu^A = \mu^B$



(b.ii) Difference in Holdings when  $\mu^{AB} = \mu^{BA}$

Figure 1: The top panel plots the cumulative bias component ( $\theta$ ) with respect to the variance of noise trading ( $1/\tau_x$ ) under perceived agreement. The bottom panel plots difference in holdings between investor  $A$  and  $B$  ( $\Delta\tilde{x}$ ) with respect to the variance of noise trading ( $1/\tau_x$ ) under perceived agreement.

rational expectations equilibrium price as in Hellwig [1980] where  $\mu$  is assumed to be common knowledge among all the investors.

Dealing with potential disagreement about the constant term and the coefficients makes analysis of the equilibrium difficult. In standard rational expectations models with differential information—such as Grossman [1976], Hellwig [1980], and Diamond and Verrecchia [1981]—all of the parameters defining the price function are common knowledge. In standard rational expectations models, solving the investors' inference

problems and trading strategies leads to a system of equations which can be solved directly for the parameters defining the common price function. In our model, by contrast, traders disagree about the price function due to differences in higher-order beliefs; attempting to solve the investors' inference problems and to calculate trading strategies simultaneously leads an infinite system of equations resulting from an infinite regress of higher-order beliefs. Since this infinite system of equations is not easily solved, we adopt a solution strategy which is different from the strategy used to solve rational expectations models. Our strategy replaces an intractable infinite system of equations with an infinite sequence of recursive substitutions which converts the price effect of the infinite regress in higher-order beliefs into a tractable power series involving only exogenous constants.

In Section 5.1, we apply this recursive belief updating strategy to obtain the posterior belief of each investor under differences in higher-order beliefs. In Section 5.2, we prove the existence and uniqueness of equilibrium given the posterior beliefs of the investors. In Section 5.3, we characterize the equilibrium price and portfolios to finish proving Theorem 1.

## 5.1 Learning

### 5.1.1 Signal Extraction with both Differences in Information and Differences in Beliefs

Since (i) the random variables  $\tilde{v}$ ,  $\tilde{s}_A$ ,  $\tilde{s}_B$  and  $\tilde{\epsilon}_x$  are jointly normally distributed, (ii) the conjectured price  $\tilde{p}$  is a linear function of these random variables, and (iii) the investors have exponential utility, it follows that investor  $i \in \{A, B\}$  has a linear demand function given by

$$X_i(s_i, p) = \frac{E^i[\tilde{v}|s_i, p] - p}{\gamma \text{Var}^i[\tilde{v}|s_i, p]}. \quad (31)$$

Define the random variable  $\tilde{x}_i$  by  $\tilde{x}_i := X_i(\tilde{s}_i, \tilde{p})$ .

Symmetry implies that conditional variances are the same constant, which we denote  $1/\tau$ :

$$\frac{1}{\tau} := \text{Var}^A[\tilde{v}|s_A, p] = \text{Var}^B[\tilde{v}|s_B, p]. \quad (32)$$

Conditional on realizations of  $s_A$ ,  $s_B$  and  $\epsilon_x$ , the market clearing condition can be

written

$$\frac{1}{2}X_A(s_A, p) + \frac{1}{2}X_B(s_B, p) = \epsilon_x. \quad (33)$$

Substituting (31) into (33) shows that the equilibrium price  $p$  satisfies

$$p = \frac{1}{2}E^A[\tilde{v}|s_A, p] + \frac{1}{2}E^B[\tilde{v}|s_B, p] - \frac{\gamma}{\tau}\epsilon_x. \quad (34)$$

Since equation (34) is an implication of market clearing, it is common knowledge that this equation is satisfied in equilibrium.

Investors  $A$  and  $B$  interpret this equation differently because differences in information and differences in beliefs both affect the way in which they learn from the price  $\tilde{p}$  about the information of the other trader.

Differences in information lead to the same issues of asymmetric information as in a standard rational expectations equilibrium. Consider the perspective of investor  $A$ . Investor  $A$  observes  $\tilde{s}_A$  and  $\tilde{p}$ . He therefore observes perfectly the left side  $\tilde{p}$  and the first term on the right side  $\frac{1}{2}E^A[\tilde{v}|s_A, p]$ . Since  $A$  observes neither  $\tilde{s}_B$  nor  $\tilde{\epsilon}_x$ , the two terms  $\frac{1}{2}E^B[\tilde{v}|s_B, p]$  and  $\frac{\gamma}{\tau}\epsilon_x$  are, conditioning on the information of investor  $A$ , random variables which add up to the observed (non-random) quantity  $\tilde{p} - \frac{1}{2}E^A[\tilde{v}|s_A, p]$ . In a rational expectations equilibrium, investor  $A$  conditions on both price  $\tilde{p}$  and his private signal  $\tilde{s}_A$  to estimate the two terms which he does not observe.

Differences in beliefs lead to different issues of interpretation in (34). Since investors  $A$  and  $B$  have different beliefs about the unconditional mean of the fundamental value  $\tilde{v}$ , they have different *unconditional* expectations of the various terms in (34). Consider the term  $\frac{1}{2}E^B[\tilde{v}|s_B, p]$ . Investor  $B$ 's unconditional expectation its value is  $\frac{1}{2}E^B E^B[\tilde{v}|s_B, p]$ , which by the law of iterated expectations equals  $\frac{1}{2}E^B[\tilde{v}]$ . Investor  $A$ 's unconditional expectation is  $\frac{1}{2}E^A E^B[\tilde{v}|s_B, p]$ . With differences in beliefs, these expectations may be different, i.e.  $E^A E^B[\tilde{v}|s_B, p] \neq E^B[\tilde{v}]$ ; higher-order unconditional expectations may be different as well.

These differences in beliefs affect how  $A$  learns from prices in a complicated way.  $A$  knows that  $B$ 's demand function is based on  $B$ 's estimate of fundamental value. Since  $A$  believes that  $B$ 's estimate may be biased due to different higher-order beliefs about the mean  $\bar{v} + \mu$ ,  $A$  adjusts his own trading strategy to “undo” these biases. Investor  $A$  also believes that  $B$  has adjusted his demand schedule to take account of  $B$ 's perceptions of the biases in  $A$ 's demand schedule; therefore,  $A$  further adjusts his strategy to undo these



higher-order biases as well. These adjustments propagate to arbitrarily higher and higher orders of beliefs. Resolving these differences in unconditional expectations leads to an infinite regress in beliefs, which depends upon difference in beliefs about unconditional expectations of arbitrarily high order. This is the problem which we address next.

In standard rational expectations models where beliefs are the same—such as Grossman [1976] or Hellwig [1980]—it is common knowledge that these higher-order adjustments are all zero. The law of iterated expectations implies  $E^A E^B[\tilde{v}|s_B, p] = E^B[\tilde{v}] = E[\tilde{v}]$  because traders share a common prior which implies  $E^A[\cdot] = E^B[\cdot] = E[\cdot]$ . The problem of how investors learn from the price function is readily resolved because  $A$  and  $B$  share a common prior. When beliefs are different, investors do not share a common prior and the standard approach does not work in a straightforward manner. Therefore, we take a different approach which resolves simultaneously both differences in conditional expectations (due to asymmetric information) and differences in unconditional expectations (due to differences in beliefs).

The mathematical analysis is simplified if we multiply the sum  $\frac{1}{2}E^B[\tilde{v}|s_B, p] + \frac{\gamma}{\tau}\epsilon_x$  by two and demean  $\tilde{\epsilon}_x$ . Therefore, define the function  $W_B : \mathbf{R}^3 \rightarrow \mathbf{R}$  by

$$W_B(s_B, p, \epsilon_x) := E^B[\tilde{v}|s_B, p] - \frac{2\gamma}{\tau}(\epsilon_x - \bar{x}). \quad (35)$$

and define the random variable  $\tilde{w}_B$  by  $\tilde{w}_B := W_B(\tilde{s}_B, \tilde{p}, \tilde{\epsilon}_x)$ . This scaling implies  $E^A[\tilde{w}_B] = E^A E^B[\tilde{v}|\tilde{s}_B, \tilde{p}]$ . Therefore, if we can calculate the value of  $E^A[\tilde{w}_B]$ , we have solved the problem of how to calculate the value of  $E^A E^B[\tilde{v}|\tilde{s}_B, \tilde{p}]$ .

Observing the pair of random variables  $\tilde{s}_A, \tilde{w}_B$  is informationally equivalent to observing the pair  $\tilde{s}_A, \tilde{p}$  because we can use (34) and (35) to express the random variable  $\tilde{w}_B$  as a linear combination of  $p, E^A[\tilde{v}|s_A, p]$  and  $\frac{2\gamma}{\tau}\bar{x}$ , all of which are observed by investor  $A$ :

$$\tilde{w}_B = 2\tilde{p} - E^A[\tilde{v}|\tilde{s}_A, \tilde{p}] + \frac{2\gamma}{\tau}\bar{x}. \quad (36)$$

Since the right side of (36) is a linear combination of  $\tilde{s}_A, \tilde{p}$ , and a constant term, this informational equivalence implies  $E^A[\tilde{v}|s_A, w_B] = E^A[\tilde{v}|s_A, p]$ .

Analogously for investor  $B$ , we define

$$W_A(s_A, p, \epsilon_x) := E^A[\tilde{v}|s_A, p] - \frac{2\gamma}{\tau}(\epsilon_x - \bar{x}). \quad (37)$$

With  $\tilde{w}_A := W_A(\tilde{s}_A, \tilde{p}, \tilde{\epsilon}_x)$ , we obtain  $E^B[\tilde{v}|s_B, w_A] = E^B[\tilde{v}|s_B, p]$ .

Substituting (35) and (37) into the market clearing condition (34) yields a representation of the price with respect to  $\tilde{w}_A$  and  $\tilde{w}_B$  as follows:

$$\tilde{p} = \frac{1}{2}\tilde{w}_A + \frac{1}{2}\tilde{w}_B + \frac{\gamma}{\tau}\tilde{\epsilon}_x - \frac{2\gamma}{\tau}\bar{x}. \quad (38)$$

### 5.1.2 Iterated Belief Updates by Repeated Recursive Substitutions

Because  $\tilde{w}_B$  is a sufficient statistic for the price  $\tilde{p}$ , investor  $A$ 's conditional expectation  $E^A[\tilde{v}|s_A, p]$  can be expressed as a linear regression of  $\tilde{v}$  on  $\tilde{s}_A$  and  $\tilde{w}_B$  (instead of  $\tilde{s}_A$  and  $\tilde{p}$ ) given by

$$E^A[\tilde{v}|s_A, w_B] = \bar{v} + \mu^A + \beta_s^A(s_A - E^A[\tilde{s}_A]) + \beta_w^A(w_B - E^A[\tilde{w}_B]), \quad (39)$$

where  $\beta_s^A$  and  $\beta_w^A$  are endogenously-determined non-negative regression coefficients for the belief update of  $\tilde{v}$  given signals  $\tilde{s}_A$  and  $\tilde{w}_B$ . Analogously, for investor  $B$ , we have

$$E^B[\tilde{v}|s_B, w_A] = \bar{v} + \mu^B + \beta_s^B(s_B - E^B[\tilde{s}_B]) + \beta_w^B(w_A - E^B[\tilde{w}_A]). \quad (40)$$

The coefficients for the belief updates  $\beta_s^A, \beta_w^A, \beta_s^B$  and  $\beta_w^B$  in (39) and (40) are common knowledge. As we show in Section 5.2.1, symmetry makes the coefficients the same for both  $A$  and  $B$ , i.e.,  $\beta_s^A = \beta_s^B = \beta_s$  and  $\beta_w^A = \beta_w^B = \beta_w$ . As we also show in Section 5.2.2, our assumption that beliefs are dogmatic point estimates makes the coefficients the same as in an otherwise equivalent standard noisy rational expectations model. We conjecture that both  $\beta_s$  and  $\beta_w$  are between zero and one given that  $\tau_x$  is greater than zero and finite (since  $\tilde{s}_A, \tilde{s}_B, \tilde{w}_A$ , and  $\tilde{w}_B$  are all noisy signals of  $\tilde{v}$  with a coefficient of one on  $\tilde{v}$ ). We will prove in Lemma 2 that this is indeed the case in equilibrium.

Substituting the left-hand-side of (39) into (37), the random variable  $\tilde{w}_A$  can be written

$$\tilde{w}_A = \bar{v} + \mu^A + \beta_s^A(\tilde{s}_A - E^A[\tilde{s}_A]) + \beta_w^A(\tilde{w}_B - E^A[\tilde{w}_B]) - \frac{2\gamma}{\tau}(\tilde{\epsilon}_x - \bar{x}). \quad (41)$$

Similarly, substituting the left-hand-side of (40) into (35), the random variable  $\tilde{w}_B$  can

be written

$$\tilde{w}_B = \bar{v} + \mu^B + \beta_s^B (\tilde{s}_B - E^B[\tilde{s}_B]) + \beta_w^B (\tilde{w}_A - E^B[\tilde{w}_A]) - \frac{2\gamma}{\tau} (\tilde{\epsilon}_x - \bar{x}). \quad (42)$$

It is common knowledge that equations (41) and (42) hold.

In a rational expectations equilibrium, traders interpret both equations in the same way. They agree that  $\mu^A = \mu^B = \mu$ . Since they share a common prior with common unconditional expectation operators  $E[\cdot] = E^A[\cdot] = E^B[\cdot]$  both traders agree that  $E[\tilde{w}_A] = E[\tilde{w}_B] = \bar{v} + \mu$ .

When there are differences in beliefs, traders  $A$  and  $B$  interpret these equations differently, as a result of which the seemingly simple calculation of unconditional expectations of  $\tilde{w}_A$  and  $\tilde{w}_B$  becomes quite complicated. If we apply the unconditional expectation operator  $E^B[\cdot]$  to (41) or  $E^A[\cdot]$  to (42), the different interpretations do not go away; instead, the terms  $E^B E^A[\tilde{w}_B]$  or  $E^A E^B[\tilde{w}_A]$  appear. Since investors may interpret these higher-order expectations differently, differences in higher-order beliefs propagate into higher and higher orders. Repeated applications of expectation operators  $E^A[\cdot]$  and  $E^B[\cdot]$  generate an infinite number of differences in interpretation related to differences in beliefs of higher and higher order. Calculation of unconditional expectations of  $\tilde{w}_A$  and  $\tilde{w}_B$  appears to require solving an infinite system of equations for an infinite number of unknown unconditional expectations of arbitrarily high order.

The main technical contribution of this paper is to deal with this issue by applying the unconditional expectation operators  $E^{B(n)}[\cdot]$  and  $E^{A(n)}[\cdot]$  to both sides of (41) and (42), respectively, obtaining (for any  $n \in \mathbb{N}$ ):

$$E^{B(n)}[\tilde{w}_A] = \bar{v} + \mu^{A(n+1)} + \beta_s^A (\mu^{B(n)} - \mu^{A(n+1)}) + \beta_w^A (\bar{v} + \mu^{B(n)} - E^{A(n+1)}[\tilde{w}_B]); \quad (43)$$

$$E^{A(n)}[\tilde{w}_B] = \bar{v} + \mu^{B(n+1)} + \beta_s^B (\mu^{A(n)} - \mu^{B(n+1)}) + \beta_w^B (\bar{v} + \mu^{A(n)} - E^{B(n+1)}[\tilde{w}_A]). \quad (44)$$

We then recursively substitute the left-hand-side of (43) and (44) for  $n + 1$  into the right-hand-side of (44) and (43) for  $n$ , respectively, obtaining an infinite series of terms involving  $\mu^{A(n)}$  and  $\mu^{B(n)}$ . The resulting infinite series is a shortcut which makes it unnecessary to solve the infinite system of equations using more complicated methods. The shortcut exploits the tri-diagonal structure of the infinite system of equations.

First, applying investor  $A$ 's expectation operator to  $\tilde{w}_B$  in (42) yields

$$E^A[\tilde{w}_B] = \bar{v} + E^A[\mu^B] + \beta_s^B \left( E^A[\tilde{s}_B] - E^A E^B[\tilde{s}_B] \right) + \beta_w^B \left( E^A[\tilde{w}_A] - E^A E^B[\tilde{w}_A] \right), \quad (45)$$

or equivalently,

$$\underbrace{E^A[\tilde{w}_B]}_{A's \text{ belief about } \tilde{w}_B} = \underbrace{\bar{v} + \mu^{AB}}_{A's \text{ belief about } B's \text{ belief about the fundamental value}} + \underbrace{\beta_s^B (\mu^A - \mu^{AB})}_{\text{adjustment for disagreement about } \tilde{s}_B} + \underbrace{\beta_w^B (\bar{v} + \mu^A - E^A E^B[\tilde{w}_A])}_{\text{adjustment for disagreement about } \tilde{w}_A}. \quad (46)$$

Equation (46) shows that disagreement about  $\mu$  creates two additional effects that make investor  $A$ 's belief about  $\tilde{w}_B$  differ from investor  $B$ 's: (i) disagreement about  $\tilde{s}_B$ , and (ii) disagreement about  $\tilde{w}_A$ . To see this, note that investor  $A$  believes that investor  $B$  subtracts an incorrect amount of bias from  $\tilde{s}_B$ ; therefore, investor  $A$  adjusts for this disagreement about  $\tilde{s}_B$ . Investor  $A$  also realizes that investor  $B$  uses  $\tilde{w}_A$  to extract investor  $A$ 's private information; consequently investor  $B$  adjusts for disagreement about  $\tilde{s}_A$  in a similar manner. This results in disagreement about  $\tilde{w}_A$ , which affects  $\tilde{w}_B$  through investor  $B$ 's belief update. Therefore, investor  $A$  adjusts for the amount of disagreement about  $\tilde{w}_A$  accordingly.

Because of disagreement about  $\tilde{w}_A$ , evaluating  $E^A[\tilde{w}_B]$  requires us to solve for the second-order expectation of  $\tilde{w}_A$ . Similarly to (46), we have

$$E^B[\tilde{w}_A] = \bar{v} + \mu^{BA} + \beta_s^A (\mu^B - \mu^{BA}) + \beta_w^A \left( \bar{v} + \mu^B - E^B E^A[\tilde{w}_B] \right). \quad (47)$$

Now we can represent the second-order expectation of  $\tilde{w}_A$  with respect to the third-order expectation of  $\tilde{w}_B$  as follows:

$$E^A E^B[\tilde{w}_A] = \bar{v} + \mu^{ABA} + \beta_s^A (\mu^{AB} - \mu^{ABA}) + \beta_w^A \left( \bar{v} + \mu^{AB} - E^A E^B E^A[\tilde{w}_B] \right). \quad (48)$$

Using (46) and (48), we can represent  $E^A[\tilde{w}_B]$  with a term involving the third-order expectation of  $\tilde{w}_B$ :

$$E^A[\tilde{w}_B] = \bar{v} + \mu^{AB} + \beta_s^B (\mu^A - \mu^{AB}) + \beta_w^B (\mu^A - \mu^{ABA}) - \beta_s^A \beta_w^B (\mu^{AB} - \mu^{ABA}) - \beta_w^A \beta_w^B \left( \bar{v} + \mu^{AB} - E^A E^B E^A[\tilde{w}_B] \right). \quad (49)$$

The new terms  $\beta_w^B(\mu^A - \mu^{ABA})$  and  $-\beta_s^A\beta_w^B(\mu^{AB} - \mu^{ABA})$  are additional adjustments in investor  $A$ 's beliefs.

We can also obtain the third-order expectation of  $\tilde{w}_B$  in a similar manner to the second-order expectation of  $\tilde{w}_A$  in (48). Substituting (48) into (49) in turn allows us to represent  $E^B[\tilde{w}_A]$  with an expression involving the fourth-order expectation of  $\tilde{w}_A$  as follows:

$$\begin{aligned} E^A[\tilde{w}_B] &= \bar{v} + \mu^{AB} + \beta_s^B(\mu^A - \mu^{AB}) + \beta_w^B(\mu^A - \mu^{ABA}) - \beta_s^A\beta_w^B(\mu^{AB} - \mu^{ABA}) \\ &\quad - \beta_w^A\beta_w^B(\mu^{AB} - \mu^{ABAB}) + \beta_s^B\beta_w^A\beta_w^B(\mu^{ABA} - \mu^{ABAB}) \\ &\quad + \beta_w^A(\beta_w^B)^2(\bar{v} + \mu^{ABA} - E^A E^B E^A E^B[\tilde{w}_A]). \end{aligned} \quad (50)$$

The new terms  $-\beta_w^A\beta_w^B(\mu^{AB} - \mu^{ABAB})$  and  $\beta_s^B\beta_w^A\beta_w^B(\mu^{ABA} - \mu^{ABAB})$  are additional adjustments in investor  $A$ 's beliefs.

It is clear that the order of expectations needed to evaluate  $E^A[\tilde{w}_B]$  increases at every iteration. Therefore, recursively substituting higher-order expectations into  $E^A[\tilde{w}_B^A]$  yields the following infinite series of adjustments  $E^A[\tilde{w}_B]$  for differences in higher-order beliefs of arbitrarily high order:

$$\begin{aligned} \underbrace{E^A[\tilde{w}_B]}_{A's \text{ belief about } \tilde{w}_B} &= \underbrace{\bar{v} + \mu^{AB}}_{A's \text{ belief about } B's \text{ belief about the fundamental value}} \\ &\quad + \underbrace{\sum_{n=0}^{\infty} (\beta_w^A\beta_w^B)^n \left[ \beta_w^B(\mu^{A(2n+1)} - \mu^{A(2n+3)}) - \beta_w^A\beta_w^B(\mu^{B(2n+2)} - \mu^{B(2n+4)}) \right]}_{\text{adjustments for disagreement about the price function}} \\ &\quad + \underbrace{\sum_{n=0}^{\infty} (\beta_w^A\beta_w^B)^n \left[ \beta_s^B(\mu^{A(2n+1)} - \mu^{B(2n+2)}) - \beta_s^A\beta_w^B(\mu^{B(2n+2)} - \mu^{A(2n+3)}) \right]}_{\text{adjustments for disagreement about the signals}}. \end{aligned}$$

We can now represent investor  $A$ 's conditional expectation of  $\bar{v}$  as

$$E^A[\tilde{v}|s_A, w_B] = \bar{v} + \mu_\infty^A + \beta_s^A(s_A - \bar{v}) + \beta_w^A(w_B - \bar{v}), \quad (51)$$

where the constant term  $\mu_\infty^A$  is the following infinite sum of all the bias adjustment made

by investor  $A$ :<sup>1</sup>

$$\begin{aligned}
\mu_\infty^A &:= (1 - \beta_s^A)\mu^A - \beta_w^A\mu^{AB} \\
&- \beta_w^A \sum_{n=0}^{\infty} (\beta_w^A\beta_w^B)^n \left[ \beta_w^B(\mu^{A(2n+1)} - \mu^{A(2n+3)}) - \beta_w^A\beta_w^B(\mu^{B(2n+2)} - \mu^{B(2n+4)}) \right] \\
&- \beta_w^A \sum_{n=0}^{\infty} (\beta_w^A\beta_w^B)^n \left[ \beta_s^B(\mu^{A(2n+1)} - \mu^{B(2n+2)}) - \beta_s^A\beta_w^B(\mu^{B(2n+2)} - \mu^{A(2n+3)}) \right].
\end{aligned} \tag{53}$$

Likewise, we can represent investor  $B$ 's conditional expectation of  $\tilde{v}$  as

$$E^B[\tilde{v}|s_B, w_A] = \bar{v} + \mu_\infty^B + \beta_s^B(s_B - \bar{v}) + \beta_w^B(w_A - \bar{v}), \tag{54}$$

with  $\mu_\infty^B$  given by

$$\begin{aligned}
\mu_\infty^B &:= (1 - \beta_s^B)\mu^B - \beta_w^B\mu^{BA} \\
&- \beta_w^B \sum_{n=0}^{\infty} (\beta_w^A\beta_w^B)^n \left[ \beta_w^A(\mu^{B(2n+1)} - \mu^{B(2n+3)}) - \beta_w^A\beta_w^B(\mu^{A(2n+2)} - \mu^{A(2n+4)}) \right] \\
&- \beta_w^B \sum_{n=0}^{\infty} (\beta_w^A\beta_w^B)^n \left[ \beta_s^A(\mu^{B(2n+1)} - \mu^{A(2n+2)}) - \beta_s^B\beta_w^A(\mu^{A(2n+2)} - \mu^{B(2n+3)}) \right].
\end{aligned} \tag{55}$$

Finally, using  $\mu_\infty^A$  and  $\mu_\infty^B$ , equations (41) and (42) for the random variables  $\tilde{w}_A$  and  $\tilde{w}_B$  can be written

$$\tilde{w}_A = \bar{v} + \mu_\infty^A + \beta_s^A(\tilde{s}_A - \bar{v}) + \beta_w^A(\tilde{w}_B - \bar{v}) - \frac{2\gamma}{\tau}(\tilde{\epsilon}_x - \bar{x}); \tag{56}$$

$$\tilde{w}_B = \bar{v} + \mu_\infty^B + \beta_s^B(\tilde{s}_B - \bar{v}) + \beta_w^B(\tilde{w}_A - \bar{v}) - \frac{2\gamma}{\tau}(\tilde{\epsilon}_x - \bar{x}). \tag{57}$$

The above two equations can be solved for the two unknowns  $\tilde{w}_A$  and  $\tilde{w}_B$  as linear functions of  $\tilde{s}_A$ ,  $\tilde{s}_B$ ,  $\tilde{\epsilon}_x$ , and constant terms including  $\mu_\infty^A$  and  $\mu_\infty^B$ . In the next section, using (56) and (57), the proof characterizing equilibrium proceeds in the same way as a proof for the rational expectations equilibrium. The only difference is that the constant terms  $\mu_\infty^A$  and  $\mu_\infty^B$  adjust for differences in higher-order beliefs of arbitrarily high order.

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<sup>1</sup>Alternatively, we could represent  $\mu_\infty^A$  as

$$\mu_\infty^A := (1 - \beta_s^A)\mu^A - \beta_w^A(E^A[\tilde{w}_B] - \bar{v}). \tag{52}$$

This makes it possible to obtain the equilibrium price function under any specification of dogmatic higher-order beliefs, even in the absence of common knowledge (e.g., Case 4.).

## 5.2 Equilibrium Solution with Recursive Belief Updates

### 5.2.1 Equilibrium Beliefs and Price

In (56) and (57), we show how the terms  $\mu_\infty^A$  and  $\mu_\infty^B$  allow sufficient statistics  $\tilde{w}_A$  and  $\tilde{w}_B$  to be endogenously derived so that the market clears and each investor chooses demand functions consistent with his own beliefs of all orders. With these adjustments, the derivation of the equilibrium proceeds exactly as in the case of the standard rational expectations model, except for the constant terms  $\mu_\infty^A$  and  $\mu_\infty^B$  carried along in the calculations. By finding a fixed point for  $\tilde{w}_A$  and  $\tilde{w}_B$  in (56) and (57), we can pin down how the equilibrium expectations of investor  $A$  and  $B$  given  $\tilde{s}_A$ ,  $\tilde{s}_B$  and  $\tilde{\epsilon}_x$  are affected by differences in beliefs.

Solving (56) and (57) for  $\tilde{w}_A$  and  $\tilde{w}_B$  yields

$$\begin{aligned} \tilde{w}_A = \bar{v} &+ \frac{1}{1 - \beta_w^A \beta_w^B} (\mu_\infty^A + \beta_w^A \mu_\infty^B) + \frac{\beta_s^A}{1 - \beta_w^A \beta_w^B} (\tilde{s}_A - \bar{v}) + \frac{\beta_s^B \beta_w^A}{1 - \beta_w^A \beta_w^B} (\tilde{s}_B - \bar{v}) \\ &- \frac{1 + \beta_w^A}{1 - \beta_w^A \beta_w^B} \frac{2\gamma}{\tau} (\tilde{\epsilon}_x - \bar{x}). \end{aligned} \quad (58)$$

and

$$\begin{aligned} \tilde{w}_B = \bar{v} &+ \frac{1}{1 - \beta_w^A \beta_w^B} (\mu_\infty^B + \beta_w^B \mu_\infty^A) + \frac{\beta_s^B}{1 - \beta_w^A \beta_w^B} (\tilde{s}_B - \bar{v}) + \frac{\beta_s^A \beta_w^B}{1 - \beta_w^A \beta_w^B} (\tilde{s}_A - \bar{v}) \\ &- \frac{1 + \beta_w^B}{1 - \beta_w^A \beta_w^B} \frac{2\gamma}{\tau} (\tilde{\epsilon}_x - \bar{x}). \end{aligned} \quad (59)$$

Because we find that both  $\tilde{w}_A$  and  $\tilde{w}_B$  are normally distributed in equilibrium, this confirms that our initial conjecture that the equilibrium price is given by (6) is true.

The coefficients of the linear Bayesian update rules (39) and (40) only depend on the covariance structure of the signals  $\tilde{s}_A$ ,  $\tilde{s}_B$ ,  $\tilde{w}_A$  and  $\tilde{w}_B$ . Because investors  $A$  and  $B$  are symmetric (other than their signal realizations and higher-order beliefs), the covariance structures of both types of investors will be identical. Therefore, we can define  $\beta_s :=$

$\beta_s^A = \beta_s^B$  and  $\beta_w := \beta_w^A = \beta_w^B$ . Finally, we can solve for the equilibrium price by substituting (58) and (59) into the price function (38).

**Lemma 1.** *The equilibrium price is an affine function of  $(\tilde{s}_A, \tilde{s}_B, \tilde{\epsilon}_x)$ ,*

$$\tilde{p} = \pi_0 + \pi_s(\tilde{s}_A + \tilde{s}_B) - \pi_x \tilde{\epsilon}_x, \quad (60)$$

where

$$\pi_0 = \frac{1}{2(1 - \beta_w)}(\mu_\infty^A + \mu_\infty^B) + \frac{1 - \beta_s - \beta_w}{1 - \beta_w} \bar{v} + \frac{2\gamma\beta_w}{\tau(1 - \beta_w)} \bar{x}, \quad (61)$$

$$\pi_s = \frac{\beta_s}{2(1 - \beta_w)}, \quad (62)$$

$$\pi_x = \frac{\gamma(1 + \beta_w)}{\tau(1 - \beta_w)}. \quad (63)$$

Therefore, the initial conjecture that the price function has the symmetric linear form in (6) is indeed true; the coefficient on the two different signals  $\tilde{s}_A$  and  $\tilde{s}_B$  are identical due to the symmetry. While the values of the parameters  $\pi_s$  and  $\pi_x$  are common knowledge, investors  $A$  and  $B$  may not agree on the value of the parameter  $\pi_0$  because they may disagree about  $\mu_\infty^A$  and  $\mu_\infty^B$ .

Using symmetry, we can further simplify (53) and (55) to obtain the following expressions:

$$\mu_\infty^A = (1 + \beta_w)(1 - \beta_s - \beta_w) \sum_{n=0}^{\infty} \beta_w^{2n} (\mu^{A(2n+1)} - \beta_w \mu^{B(2n+2)}), \quad (64)$$

and

$$\mu_\infty^B = (1 + \beta_w)(1 - \beta_s - \beta_w) \sum_{n=0}^{\infty} \beta_w^{2n} (\mu^{B(2n+1)} - \beta_w \mu^{A(2n+2)}). \quad (65)$$

Therefore, we have

$$\mu_\infty^A + \mu_\infty^B = (1 + \beta_w)(1 - \beta_s - \beta_w) \sum_{n=1}^{\infty} (-\beta_w)^{n-1} (\mu^{A(n)} + \mu^{B(n)}). \quad (66)$$



## 5.2.2 Symmetric Solutions for Both Investors' Optimization Problems

By solving one investor's problem (e.g., investor  $A$ 's), we are able to obtain the solution to both  $A$ 's and  $B$ 's problems by symmetry. Note that investor  $A$  correctly perceives  $\mu_\infty^A$ , but misperceives  $\mu_\infty^B$  due to differences in higher-order beliefs.<sup>2</sup> From (55), we can derive investor  $A$ 's perception of  $\mu_\infty^B$ , which is denoted to be  $\mu_\infty^{AB}$ , as follows:

$$\mu_\infty^{AB} := (1 + \beta_w)(1 - \beta_s - \beta_w) \sum_{n=1}^{\infty} \beta_w^{2(n-1)} (\mu^{B(2n)} - \beta_w \mu^{A(2n+1)}). \quad (67)$$

Now the signal structure that is perceived by investor  $A$  can be represented as follows:<sup>3</sup>

$$\begin{pmatrix} \tilde{s}_A \\ \tilde{w}_B \end{pmatrix} = \begin{pmatrix} \mu^A & 1 \\ \bar{w}_B^A & \frac{\beta_s}{1-\beta_w} \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} \tilde{\epsilon}_A \\ \frac{\beta_s \beta_w}{1-\beta_w^2} \tilde{\epsilon}_A + \frac{\beta_s}{1-\beta_w^2} \tilde{\epsilon}_B - \frac{2\gamma}{(1-\beta_w)\tau} \tilde{\epsilon}_x \end{pmatrix}, \quad (69)$$

where

$$\bar{w}_B^A := \left( \frac{1 - \beta_s - \beta_w}{1 - \beta_w} \right) \bar{v} + \frac{1}{1 - \beta_w^2} (\mu_\infty^{AB} + \beta_w \mu_\infty^A) + \frac{2\gamma}{(1 - \beta_w)\tau} \bar{x}. \quad (70)$$

Therefore, investor  $A$ 's Bayesian updating rule on the expectation of  $\tilde{v}$  conditional on the pair  $(s_A, w_B)$  is given by

$$E^A[\tilde{v}|s_A, w_B] = E^A[\tilde{v}] + \Lambda \Omega^{-1} \begin{pmatrix} s_A - E^A[\tilde{s}_A] \\ w_B - E^A[\tilde{w}_B] \end{pmatrix}, \quad (71)$$

where

$$\Omega := \begin{pmatrix} \frac{1}{\tau_v} + \frac{1}{\tau_s} & \frac{\beta_s}{1-\beta_w} \frac{1}{\tau_v} + \frac{\beta_s \beta_w}{1-\beta_w^2} \frac{1}{\tau_s} \\ \frac{\beta_s}{1-\beta_w} \frac{1}{\tau_v} + \frac{\beta_s \beta_w}{1-\beta_w^2} \frac{1}{\tau_s} & \frac{\beta_s^2}{(1-\beta_w)^2} \frac{1}{\tau_v} + \frac{\beta_s^2 \beta_w^2}{(1-\beta_w^2)^2} \frac{1}{\tau_s} + \frac{\beta_s^2}{(1-\beta_w^2)^2} \frac{1}{\tau_s} + \frac{4\gamma^2}{\tau^2(1-\beta_w)^2} \frac{1}{\tau_x} \end{pmatrix}, \quad (72)$$

<sup>2</sup>Likewise, investor  $B$  correctly perceives  $\mu_\infty^B$ , but misperceives  $\mu_\infty^A$ .

<sup>3</sup>From (59), we can derive investor  $A$ 's perception of  $\tilde{w}_B$  as follows:

$$\tilde{w}_B = \bar{w}_B^A + \frac{\beta_s \beta_w}{1 - \beta_w^2} \tilde{s}_A + \frac{\beta_s}{1 - \beta_w^2} \tilde{s}_B - \frac{2\gamma}{(1 - \beta_w)\tau} \tilde{\epsilon}_x. \quad (68)$$

and

$$\Lambda := \begin{pmatrix} \frac{1}{\tau_v} \\ \frac{\beta_s}{1-\beta_w} \frac{1}{\tau_v} \end{pmatrix}^\top. \quad (73)$$

Investor  $A$ 's Bayesian updating rule on the variance of  $\tilde{v}$  conditional on the pair  $(s_A, w_B)$  is also given by

$$\text{Var}^A[\tilde{v}|s_A, w_B] = \left(1 - \beta_s - \frac{\beta_s \beta_w}{1 - \beta_w}\right) \frac{1}{\tau_v}. \quad (74)$$

Using (32), (39), (71) and (74), we can obtain the following nonlinear system of equations:

$$\beta_s = \frac{\frac{\beta_s^2}{(1-\beta_w)^2} (1 - \beta_w) \frac{1}{\tau_s \tau_v} + \frac{4\gamma^2 (1+\beta_w)^2}{\tau^2 (1-\beta_w)^2} \frac{1}{\tau_x \tau_v}}{\frac{\beta_s^2}{(1-\beta_w)^2} \left(\frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2}\right) + \frac{4\gamma^2 (1+\beta_w)^2}{\tau^2 (1-\beta_w)^2} \left(\frac{1}{\tau_x \tau_v} + \frac{1}{\tau_s \tau_x}\right)}; \quad (75)$$

$$\beta_w = \frac{\frac{\beta_s}{1-\beta_w} (1 + \beta_w) \frac{1}{\tau_s \tau_v}}{\frac{\beta_s^2}{(1-\beta_w)^2} \left(\frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2}\right) + \frac{4\gamma^2 (1+\beta_w)^2}{\tau^2 (1-\beta_w)^2} \left(\frac{1}{\tau_x \tau_v} + \frac{1}{\tau_s \tau_x}\right)}; \quad (76)$$

$$\frac{1}{\tau} = \left(1 - \beta_s - \frac{\beta_s}{1 - \beta_w} \beta_w\right) \frac{1}{\tau_v}. \quad (77)$$

We can show that a unique solution for this system of equations exists in a similar fashion as in Hellwig [1980].

**Lemma 2.** *There exists a unique solution  $\beta_s^*, \beta_w^*$  and  $\tau^*$  for the system of equations, and the solution is independent of differences in higher-order beliefs. Furthermore,  $\beta_s^*, \beta_w^* \in (0, 1)$  and  $\beta_s^* + \beta_w^* < 1$  if  $\tau_x \in (0, \infty)$ .*

*Proof.* See Appendix. □

Lemma 1 and Lemma 2 prove that our initial conjecture of the equilibrium price function (6) is true, and it is unique among the class of the affine price function (6). Lemma 1 also implies that  $\pi_s$  and  $\pi_x$  are unaffected by the values of  $\mu_\infty^A$  and  $\mu_\infty^B$ , and  $\pi_0$  is uniquely determined given  $\mu_\infty^A$  and  $\mu_\infty^B$ . Because the investors disagree about  $\mu_\infty^A$  and  $\mu_\infty^B$  in the presence of differences in higher-order beliefs, the investors agree about  $\pi_s, \pi_x$ , but disagree about  $\pi_0$  as is conjectured. This finishes the proof of existence and uniqueness of the equilibrium in Theorem 1.

## 5.3 Equilibrium Prices and Trading Volumes

### 5.3.1 Equilibrium Price

From Lemma 1, the equilibrium price can be decomposed into two components as follows:

$$\tilde{p} = \tilde{p}^{REE} + \theta, \quad (78)$$

where

$$\tilde{p}^{REE} := \frac{1 - \beta_s - \beta_w}{1 - \beta_w}(\bar{v} + \mu) + \frac{2\gamma\beta_w}{\tau(1 - \beta_w)}\bar{x} + \frac{\beta_s}{2(1 - \beta_w)}(\tilde{s}_A + \tilde{s}_B) - \frac{\gamma(1 + \beta_w)}{\tau(1 - \beta_w)}\tilde{\epsilon}_x, \quad (79)$$

$$\theta := \frac{1 - \beta_s - \beta_w}{1 - \beta_w} \left[ -\mu + \frac{1 + \beta_w}{2} \sum_{n=1}^{\infty} (-\beta_w)^{n-1} (\mu^{A(n)} + \mu^{B(n)}) \right]. \quad (80)$$

In order to understand how biases in beliefs affect prices, we express  $\theta$  as an infinite sum of terms, each of which quantifies the bias  $\Delta\mu^{A(n)}$  and  $\Delta\mu^{B(n)}$  in the beliefs of a different order  $n$ . This gives us the desired equations (9) and (10).

### 5.3.2 Equilibrium Trading Volumes

From (31) and (32), investor  $i$ 's optimal portfolio given  $\tilde{s}_i$  and  $\tilde{p}$  can be represented by

$$\tilde{x}_i = \frac{\tau}{\gamma} \left[ E^i[\tilde{v}|\tilde{s}_i, \tilde{p}] - \tilde{p} \right]. \quad (81)$$

Let  $\Delta\tilde{x}$  denote the difference between investor  $A$ 's and  $B$ 's holdings of the risky asset, i.e.,  $\Delta\tilde{x} := \tilde{x}_A - \tilde{x}_B$ . Notice that  $\tilde{x}_A$  and  $\tilde{x}_B$  can be expressed by  $\tilde{\epsilon}_x$  and  $\tilde{\epsilon}_x$  as follows:

$$\tilde{x}_A = \tilde{\epsilon}_x + \frac{1}{2}\Delta\tilde{x} \quad (82)$$

$$\tilde{x}_B = \tilde{\epsilon}_x - \frac{1}{2}\Delta\tilde{x}. \quad (83)$$

Using (81), we have

$$\Delta\tilde{x} = \frac{\tau}{\gamma} \left[ E^A[\tilde{v}|\tilde{s}_A, \tilde{p}] - E^B[\tilde{v}|\tilde{s}_B, \tilde{p}] \right]. \quad (84)$$

Substituting (51) and (54) into (84) yields

$$\Delta\tilde{x} = \frac{\tau}{\gamma} \left[ \beta_s(\tilde{s}_A - \tilde{s}_B) + \beta_w(\tilde{w}_B - \tilde{w}_A) + (\mu_\infty^A - \mu_\infty^B) \right]. \quad (85)$$

Finally, substituting (58) and (59) into (85) yields<sup>4</sup>

$$\begin{aligned} \Delta\tilde{x} &= \frac{\tau}{\gamma} \frac{1}{1 + \beta_w} \left[ \beta_s(\tilde{s}_A - \tilde{s}_B) + (\mu_\infty^A - \mu_\infty^B) \right] \\ &= \frac{\tau}{\gamma} \frac{\beta_s}{1 + \beta_w} (\tilde{s}_A - \tilde{s}_B) + \frac{\tau}{\gamma} (1 - \beta_s - \beta_w) \sum_{n=1}^{\infty} \beta_w^{n-1} (\mu^{A(n)} - \mu^{B(n)}). \end{aligned} \quad (86)$$

Analogously to the derivation of (10) (in Section 5.3.1), we can represent  $\Delta\tilde{x}$  as a function of difference in the signals and differences in the biases of higher-order beliefs, and this gives us the desired equations (17) and (18).

## 6 Belief Multiplier Effects

Let us take another look at the system of equations (41) and (42):

$$\tilde{w}_A = \bar{v} + \mu^A + \beta_s^A(\tilde{s}_A - E^A[\tilde{s}_A]) + \beta_w^A(\tilde{w}_B - E^A[\tilde{w}_B]) - \frac{2\gamma}{\tau}(\tilde{\epsilon}_x - \bar{x}), \quad (87)$$

$$\tilde{w}_B = \bar{v} + \mu^B + \beta_s^B(\tilde{s}_B - E^B[\tilde{s}_B]) + \beta_w^B(\tilde{w}_A - E^B[\tilde{w}_A]) - \frac{2\gamma}{\tau}(\tilde{\epsilon}_x - \bar{x}). \quad (88)$$

We can obtain expressions for  $E^B[\tilde{w}_A]$  and  $E^A[\tilde{w}_B]$  by applying the expectation operators  $E^B[\cdot]$  and  $E^A[\cdot]$  to both sides of (87) and (88), respectively, obtaining

$$E^B[\tilde{w}_A] = \bar{v} + \mu^{BA} + \beta_s^A(\mu^B - \mu^{BA}) + \beta_w^A(\bar{v} + \mu^B - E^{BA}[\tilde{w}_B]), \quad (89)$$

$$E^A[\tilde{w}_B] = \bar{v} + \mu^{AB} + \beta_s^B(\mu^A - \mu^{AB}) + \beta_w^B(\bar{v} + \mu^A - E^{AB}[\tilde{w}_A]). \quad (90)$$

Now, to evaluate  $E^B[\tilde{w}_A]$  and  $E^A[\tilde{w}_B]$  on the left-hand-sides of (89) and (90), we must evaluate  $E^{BA}[\tilde{w}_B]$  and  $E^{AB}[\tilde{w}_A]$  on the right-hand-sides, respectively. Evaluating  $E^{BA}[\tilde{w}_B]$  and  $E^{AB}[\tilde{w}_A]$  in turn requires the evaluation of  $E^{ABA}[\tilde{w}_B]$  and  $E^{BAB}[\tilde{w}_A]$ , and so on. Therefore, differences in higher-order beliefs can generate an infinite number of unknowns to solve for.

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<sup>4</sup>For this, we use the realizations of  $\tilde{w}_A$  and  $\tilde{w}_B$  given the realizations of  $\tilde{s}_A, \tilde{s}_B$  and  $\tilde{\epsilon}_x$ .

In Section 5.1, we resolved the issue by taking expectation operators recursively. Such a problem does not arise whenever the investors have agreement in higher-order beliefs, i.e., there is common knowledge about beliefs. If so, then the recursive substitutions can be stopped at some finite value of  $N$  for special cases when  $E^{A(N)}[\tilde{w}_B]$  and  $E^{B(N)}[\tilde{w}_A]$  become common knowledge, i.e., for all  $n \geq 1$ , we have  $E^{A(N+n)}[\tilde{w}_B] = E^{A(N)}[\tilde{w}_B]$  and  $E^{B(N+n)}[\tilde{w}_A] = E^{B(N)}[\tilde{w}_A]$ . The following are the examples of such cases:

1. “*Rational Expectations*”:  $E^{A(n)}[\tilde{w}_B] = E^{B(n)}[\tilde{w}_A] = \bar{v} + \mu$  for any  $n \in \mathbb{N}$ .
2. “*Agreement to Agree*”:  $E^{A(n)}[\tilde{w}_B] = E^{B(n)}[\tilde{w}_A] = \bar{v} + \bar{\mu}$  for some  $\bar{\mu}$  and any  $n \in \mathbb{N}$ .
3. “*Agreement to Disagree*”: Since  $\mu^{A(n)} = \mu^A$  and  $\mu^{B(n)} = \mu^B$  for all  $n \in \mathbb{N}$ , the values of  $\mu^A$  and  $\mu^B$  are common knowledge with  $\mu^{BA} = \mu^A$  and  $\mu^{AB} = \mu^B$ . Recursive substitution stops at  $n = 1$ , with equations (89) and (90) becoming

$$E^B[\tilde{w}_A] = \bar{v} + \mu^A + \beta_s^A(\mu^B - \mu^A) + \beta_w^A(\bar{v} + \mu^B - E^A[\tilde{w}_B]), \quad (91)$$

$$E^A[\tilde{w}_B] = \bar{v} + \mu^B + \beta_s^B(\mu^A - \mu^B) + \beta_w^B(\bar{v} + \mu^A - E^B[\tilde{w}_A]). \quad (92)$$

Notice that investors’ belief updates depend mutually on each other, with  $E^B[\tilde{w}_A]$  a function of  $E^A[\tilde{w}_B]$  in (91), and vice versa in (92). Solving (91) and (92) for  $E^A[\tilde{w}_B]$  and  $E^B[\tilde{w}_A]$  yields

$$E^B[\tilde{w}_A] = \bar{v} + \mu^A + \frac{\beta_s^A}{1 - \beta_w^A \beta_w^B}(\mu^B - \mu^A) - \frac{\beta_s^B \beta_w^A}{1 - \beta_w^A \beta_w^B}(\mu^A - \mu^B), \quad (93)$$

$$E^A[\tilde{w}_B] = \bar{v} + \mu^B + \frac{\beta_s^B}{1 - \beta_w^A \beta_w^B}(\mu^A - \mu^B) - \frac{\beta_s^A \beta_w^B}{1 - \beta_w^A \beta_w^B}(\mu^B - \mu^A). \quad (94)$$

The solution for the third case—agreement to disagree—in equations (93) and (94) can be interpreted as a shortcut for evaluating the infinite series that would have been obtained by an infinite number of recursive substitutions. The first-order disagreements are amplified due to the mutual dependence of the investors’ belief updates, with the initial impact of disagreement  $\beta_s^A(\mu^B - \mu^A)$  amplified by a multiple of  $\frac{1}{1 - \beta_w^A \beta_w^B}$  in (93). This multiplier reflects an infinite number of rounds of belief updates between the two investor groups. The key intuition is that any initial disagreement between them feeds back into each investor’s beliefs over and over again.

In the case of agreeing to disagree, just like in the case for the Keynesian multiplier, the intuition for the multiplier captures the way in which successive approximations can be used to solve linear equations by repeated substitutions at each iteration. In the first round, if investor  $B$  believes that investor  $A$  interprets his own private signal in a more optimistic way than investor  $B$  would interpret  $A$ 's signal, this induces  $B$  to believe that investor  $A$ 's sufficient statistic  $\tilde{w}_A$  has an optimistic bias. Because investor  $B$  wants to extract unbiased information about investor  $A$ 's private signal, he adjusts for investor  $A$ 's optimistic bias by subtracting the bias from  $\tilde{w}_A$ . As a result, investor  $A$  believes that investor  $B$ 's belief is affected in a pessimistic way, so the initial impact of disagreement feeds back into  $\tilde{w}_B$  with a factor of  $\beta_w^B$ . Investor  $A$  wants to extract unbiased information about investor  $B$ 's private information. Thus, he offsets investor  $B$ 's pessimistic bias by adding it back to  $\tilde{w}_B$ , with a factor of  $\beta_w^A$ . Now investor  $B$  believes that a second layer of optimistic bias has been added to  $\tilde{w}_A$ . The initial disagreement  $\beta_s^A(\mu^B - \mu^A)$  feeds back into  $\tilde{w}_A$ , but the magnitude of feedback effects has a factor of  $\beta_w^A\beta_w^B$ . In the second round, investor  $B$  further incorporates this feedback effect into his learning; thus, investor  $A$  also incorporates investor  $B$ 's reaction into investor  $A$ 's own learning. This second layer of optimistic bias again feeds back into  $\tilde{w}_A$  with a factor of  $\beta_w^A\beta_w^B$ , thereby adding a third layer of optimistic bias. This feedback is repeated in the third and fourth rounds, and so on. Consequently, the total impact of the initial disagreement about the signal is given by  $\beta_s^A(\mu^B - \mu^A) + \beta_s^A(\beta_w^A\beta_w^B)(\mu^B - \mu^A) + \beta_s^A(\beta_w^A\beta_w^B)^2(\mu^B - \mu^A) + \dots = \frac{\beta_s^A}{1 - \beta_w^A\beta_w^B}(\mu^B - \mu^A)$ . The method converges to a solution in a stable manner because the linear system has a contraction property.

Now suppose that there exist higher-order inconsistencies in beliefs which go beyond agreeing to disagree. These inconsistencies in beliefs are also amplified with a multiplier effect. When second-order, third-order, and higher-order disagreements propagate into higher-order beliefs with multiplier effects, the result can be a substantial effect on prices. Therefore, we interpret the cumulative bias term  $\theta$  as the accumulation of multiplier effects of inconsistent belief in each order as is demonstrated in (9). In the next section, we study the implications of such multiplier effects on prices.

## 7 Implications

So far, we have demonstrated that differences in higher-order beliefs can lead to amplification of biases along with extremely inflated or deflated prices. This leads naturally to the following questions: What makes higher-order beliefs differ? What prevents the formation of common knowledge in investors' beliefs? Aumann [1976] shows that the common prior assumption makes agents agree to agree publicly when they can communicate directly. Communication thus tends to make beliefs become common knowledge. This can still be true even using alternative ways of communication such as price mechanisms (e.g., Sebenius and Geanakoplos [1983]). In the presence of heterogeneous beliefs, the possibility of direct communication thus allows beliefs to become common knowledge, even when investors agree to disagree.<sup>5</sup> In the absence of common priors and direct communication, however, there is no guarantee that agents will form common knowledge on each other's beliefs. Forming higher-order beliefs through alternative methods such as collecting signals on others' beliefs than direct communication would be insufficient for forming common knowledge if such signals are imperfect. Consequently, differences in higher-order beliefs in speculative markets may naturally arise whenever common priors and direct communications are not viable.

Remember that the mean of the risky asset's payoff  $\tilde{v}$  is given by  $\bar{v} + \mu$ , and the investors may disagree about the second component of the mean  $\mu$ . There are multiple ways of interpreting  $\mu$ . One potential interpretation could be the bias stemming from using subjectively-chosen "valuation benchmarks". In practice, valuing an asset in absolute terms is often very difficult. However, relative valuation based on the prices of other assets is much easier. For example, investors may use valuation multiples by choosing a certain set of comparable assets and their characteristics such as earnings. The lack of consensus on the benchmarks would easily create disagreement about the valuation. Alternatively, disagreement about  $\mu$  may also arise due to different interpretations of public signals such as credit ratings, analysts forecasts or even rumors on social networks. Disagreement would be even stronger when there are dispersions in such public signals. Publicly available signals also serve as tools of learning others' beliefs. For example, if there exist two competing views on the fundamentals (e.g., bullish and bearish views), one type of investors would expect that there are others who hold an opposite views.

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<sup>5</sup>Furthermore, they would eventually agree to agree publicly but perhaps only after an arbitrarily long time (e.g., Geanakoplos and Polemarchakis [1982]).

Competing views are more likely to arise when there is lack of consensus. Likewise, the lack of past data could also easily make beliefs differ among investors.

We have seen that when higher-order beliefs are consistent—as the first three cases of standard noisy rational expectations, agreement-to-agree, and agreement-to-disagree—the cumulative bias component is always finite because feedback effects do not occur in the investors’ learning process. On the other hand, when higher-order beliefs are inconsistent—as in the fourth case of perceived agreement—the cumulative bias component is unbounded when there is relative optimism or pessimism. Trading volumes (or difference in holdings) remain bounded, even when prices are arbitrarily inflated or deflated. In summary, relative optimism that is combined with perceived agreement leads to inflated prices by creating extreme optimism. For example, if each individual’s opinion is more bullish than the consensus (i.e., relative optimism), and each believes that the consensus is how the others are understanding the market but they are aware of the bullish view (i.e., perceived agreement), then this would be sufficient to create bubbly prices in the market as a result of amplified optimism. Similarly, relative pessimism with perceived agreement can generate crashes in the market as a result of amplified pessimism. Therefore, extreme asset overvaluation may not result from optimism itself (i.e., absolute optimism) but may result from traders’ incorrect, inconsistent higher-order beliefs that they are more optimistic than average (i.e., relative optimism).

Although our model is a static one, our results can be applied to explain escalating optimism in bubble or mania episodes. Suppose that some set of investors acquires some good information about an asset, then buys some shares of the asset. This leads to an increase in the price, but also leads other investors infer that there must have been a good signal. In turn, this leads the first set of investors who bought the asset to believe that the others must have gotten even better information and bid up the price again, and this also leads to the second set of investors bid up the price for the same reason, and so on. Because the mistaken inferences of the investors are inconsistent, they are reinforcing. They may, however, get into such a situation when they believe that they are only modestly more bullish than others (i.e., modest relative optimism). Because they believe that the consensus is too bearish, their valuation is always higher than the consensus. This would lead to naive investing behavior of buying assets at any level of prices whenever good news is given. Such seemingly puzzling behavior may come from inconsistent higher-order beliefs.



In practice, the situations we have described by relative optimism can arise in various contexts of trading financial securities. For example, we can apply our theory to understand the internet bubble (1997-2000) to some degree. The lack of data on earnings of internet start-up companies may have created differing views on the fundamentals among investors. Suppose that investors in general held mildly optimistic views about the general prospects of the economy during the internet bubble. At the same time, investors may have perceived themselves to be slightly more optimistic about such prospects, on average, than other investors. Such a discrepancy in the higher-order beliefs can lead to unbounded price increases that incur the pattern of feedback effects such that price increases and belief updates are reinforcing each other.

Another example may be the real estate debt bubble of 2004-2006. The average market participant may have believed that the market used complex but correct mathematical models to value real estate and real-estate-backed debt, as a function of expectations of real estate price appreciation. Suppose the average market participants believed themselves to be more optimistic about real estate prices than the average market participant. This relative optimism could lead to a bubble in both real estate and real-estate-backed debt prices. We show that such a bubble becomes more pronounced when market participants place a greater weight on market prices and a smaller weight on their own private information.

In standard noisy rational expectations models, a smaller standard deviation of noise trading leads to more informative and more stable prices. When market liquidity disappears in the limit as the standard deviation of noise trading goes to zero, this makes the price highly sensitive to small shocks. Our results show that price stability and informativeness are in fact extremely fragile with respect to even small biases in beliefs when market liquidity disappears. In the examples provided in the previous section, we show that disagreement can create trading volume without too much price impact but inconsistency in higher-order beliefs can create price instability without too much trading volume. Therefore, our result suggests that lack of common knowledge may explain some episodes of price overreaction to new information (e.g., De Bondt and Thaler [1985]).

Misplaced confidence in persistently incorrect higher-order beliefs is consistent with heuristic ways of thinking documented in the behavioral literature. The “false consensus effect”—the belief that others hold the same view as oneself—is a well-documented (e.g.,

Ross, Greene and House [1977]). Mullen et al. [1985] state that “False consensus refers to an egocentric bias that occurs when people estimate consensus for their own behaviors. Specifically, the false consensus hypothesis holds that people who engage in a given behavior will estimate that behavior to be more common than it is estimated to be by people who engage in alternative behaviors”.<sup>6</sup> Egan, Merkle and Weber [forthcoming] find that investors’ second-order beliefs tend to be too negative compared to actual beliefs of others (i.e., relative optimism). There is also other evidence on the “bias blind spot” related to the tendency of ignoring one’s own bias (e.g., Pronin, Lin and Ross [2002]), which may contribute to misplaced confidence in inaccurate higher-order beliefs.

## 8 Concluding Remarks

In financial markets, prices play a role of clearing the market as well as aggregating information. In general, the literature on noisy rational expectations model finds that prices efficiently aggregate diverse information in the market. In this paper, we demonstrate that such informational efficiency may be fragile in the presence of lack of common knowledge about investors’ beliefs. If there exists inconsistency in higher-order beliefs, investors’ collective learning through the price mechanism may lead to amplification of their relative biases. In particular, price impacts are greater when the standard deviation of noise trading gets smaller. Consequently, prices may be greatly inflated or deflated as a result of inflated optimism or pessimism that results from learning under differences in higher-order beliefs. Our results shed a light on understanding asset overvaluation apparently driven by extreme optimism. We argue that overvaluation may not be the result of optimism itself, but rather optimism may result from overvaluation which occurs as a result of misspecified higher-order beliefs.

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<sup>6</sup>See Dawes [1990] for a survey on this.

# Appendices

## Proof of Lemma 2

*Proof.* Define  $Q := \frac{\tau_s}{\tau_x}$ . From Lemma 1, we obtain

$$Q = \frac{\tau\beta_s}{2\gamma(1 + \beta_w)}. \quad (\text{A.1})$$

From (75) and (76), we derive

$$\beta_s = \frac{\beta_w}{1 + \beta_w}\beta_s + \frac{\frac{1}{\tau_x\tau_v}}{Q^2\left(\frac{2}{\tau_s\tau_v} + \frac{1}{\tau_s^2}\right) + \left(\frac{1}{\tau_x\tau_v} + \frac{1}{\tau_s\tau_x}\right)}. \quad (\text{A.2})$$

Therefore, we have

$$\frac{\tau\beta_s}{2\gamma(1 + \beta_w)} = \frac{\tau\frac{1}{\tau_x\tau_v}}{2\gamma\left[Q^2\left(\frac{2}{\tau_s\tau_v} + \frac{1}{\tau_s^2}\right) + \left(\frac{1}{\tau_x\tau_v} + \frac{1}{\tau_s\tau_x}\right)\right]}. \quad (\text{A.3})$$

By substituting (75) and (76) into (77), we obtain

$$\frac{1}{\tau} = \frac{Q^2\frac{1}{\tau_s^2\tau_v} + \frac{1}{\tau_x\tau_s\tau_v}}{Q^2\left(\frac{2}{\tau_s\tau_v} + \frac{1}{\tau_s^2}\right) + \left(\frac{1}{\tau_x\tau_v} + \frac{1}{\tau_s\tau_x}\right)}. \quad (\text{A.4})$$

From (A.3) and (A.4), the value of  $Q$  satisfies

$$Q = \frac{\frac{1}{\tau_x}}{2\gamma\left(Q^2\frac{1}{\tau_s^2} + \frac{1}{\tau_x\tau_s}\right)} = \frac{\tau_s}{2\gamma}\left[1 - \frac{Q^2}{Q^2 + \frac{\tau_s}{\tau_x}}\right]. \quad (\text{A.5})$$

Notice that  $Q < 0$  and  $Q > \frac{\tau_s}{2\gamma}$  are infeasible as a solution for equation (A.5). Let  $Y = [0, \frac{\tau_s}{2\gamma}]$ , and define the mapping  $T : Y \rightarrow Y$  by

$$T(Q) := \frac{\tau_s}{2\gamma}\left[1 - \frac{Q^2}{Q^2 + \frac{\tau_s}{\tau_x}}\right]. \quad (\text{A.6})$$

Notice that  $T$  is continuous in  $Q$ . When  $\tau_x = \infty$ , it is trivial that  $Q^* = 0$  is a unique solution for equation (A.5). Now, suppose that  $\tau_x < \infty$ . Because  $T(0) - 0 = \frac{\tau_s}{2\gamma} > 0$  and

$T\left(\frac{\tau_s}{2\gamma}\right) - \frac{\tau_s}{2\gamma} = -\frac{\left(\frac{\tau_s}{2\gamma}\right)^2}{\left(\frac{\tau_s}{2\gamma}\right)^2 + \frac{\tau_s}{\tau_x}} < 0$ , there exists a fixed point  $Q^* \in Y$  such that  $T(Q^*) = Q^*$ . Furthermore, the solution  $Q^*$  is unique because  $T$  is a strictly decreasing function, i.e.,  $T'(\cdot) < 0$ .

From (A.4),  $\tau^*$  is uniquely determined by  $Q^*$ . From (75), we also derive

$$\beta_w = \frac{\tau Q(1 - \beta_w) \frac{1}{\tau_s \tau_v}}{2\gamma \left[ Q^2 \left( \frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2} \right) + \left( \frac{1}{\tau_x \tau_v} + \frac{1}{\tau_s \tau_x} \right) \right]} \quad (\text{A.7})$$

$$= \frac{\tau Q \frac{1}{\tau_s \tau_v}}{\tau Q \frac{1}{\tau_s \tau_v} + 2\gamma \left[ Q^2 \left( \frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2} \right) + \left( \frac{1}{\tau_x \tau_v} + \frac{1}{\tau_s \tau_x} \right) \right]},$$

$$\beta_s = \frac{Q^2(1 - \beta_w) \frac{1}{\tau_s \tau_v} + \frac{1}{\tau_x \tau_v}}{Q^2 \left( \frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2} \right) + \left( \frac{1}{\tau_x \tau_v} + \frac{1}{\tau_s \tau_x} \right)}. \quad (\text{A.8})$$

Therefore,  $\beta_w^*$  is uniquely determined by  $Q^*$  and  $\tau^*$  from (A.7), and  $\beta_s^*$  is uniquely determined by  $Q^*$ ,  $\tau^*$  and  $\beta_w^*$  from (A.8). Therefore, the solution for the system of equations exists and it is unique.

Notice that the solution  $(\beta_s^*, \beta_w^*, \tau^*)$  is unaffected by the values of  $\mu_\infty^A$  and  $\mu_\infty^B$  because the system of equations (75), (76) and (77) does not include any of them as parameters. Furthermore, (A.7) implies that  $\beta_w$  is between zero and one given  $\tau_x \in (0, \infty)$ . Then, (A.8) implies that  $\beta_s$  is also between zero and one. Finally, we prove that  $\beta_s^* + \beta_w^* < 1$  when  $\tau_x \in (0, \infty)$ . Suppose not. Then, (77) implies that  $\tau$  should be less than zero. However, (A.4) implies that  $\tau$  is positive. The result follows by contradiction.  $\square$

## Proof of Corollary 1

*Proof.* In the proof of Theorem 1, it has already been shown that  $Q^* \rightarrow 0$  as  $\tau_x \rightarrow \infty$ . Furthermore, (A.5) implies that  $\tau_x Q^{*2} \rightarrow \infty$  as  $\tau_x \rightarrow \infty$  because  $Q^*$  will not converge to zero otherwise. Then, (A.4) implies  $\frac{1}{\tau^*} \rightarrow \frac{1}{2\tau_s + \tau_v}$  as  $\tau_x \rightarrow \infty$ . We also find that  $\tau_x Q^* \rightarrow \infty$  as  $\tau_x \rightarrow \infty$  because  $\tau_x Q^* > \tau_x Q^{*2}$  when  $\tau_x$  is sufficiently large. From (A.7), we have

$$\beta_w^* = \frac{\tau^* \frac{1}{\tau_s \tau_v}}{\tau^* \frac{1}{\tau_s \tau_v} + 2\gamma \left[ Q^* \left( \frac{2}{\tau_s \tau_v} + \frac{1}{\tau_s^2} \right) + \left( \frac{1}{Q^* \tau_x \tau_v} + \frac{1}{Q^* \tau_s \tau_x} \right) \right]}. \quad (\text{A.9})$$

Therefore,  $\beta_w^* \rightarrow 1$  as  $\tau_x \rightarrow \infty$  because  $Q^* \rightarrow 0$  and  $Q^* \tau_x \rightarrow \infty$ . Then, (A.8) implies

that  $\beta_s^* \rightarrow 0$  as  $\tau_x \rightarrow \infty$ . We can obtain the following from (77):

$$\frac{\beta_s}{1 - \beta_w} = \frac{1 - \beta_s}{\beta_w} - \frac{\tau_v}{\tau\beta_w}. \quad (\text{A.10})$$

This implies that  $\frac{\beta_s}{1 - \beta_w} \rightarrow \frac{2\tau_s}{2\tau_s + \tau_v}$  as  $\tau_x \rightarrow \infty$ . Finally,  $\tilde{p}^{REE}$  converges to the following as  $\tau_x \rightarrow \infty$ :

$$\frac{1 - \beta_s - \beta_w}{1 - \beta_w}(\bar{v} + \mu) - \frac{\gamma}{\tau}\bar{x} + \frac{\beta_s}{2(1 - \beta_w)}(\tilde{s}_A + \tilde{s}_B). \quad (\text{A.11})$$

From (9), it is clear that  $\theta$  diverges to infinity (or negative infinity) if  $\liminf_{n \rightarrow \infty} \theta_n > 0$  (or  $\limsup_{n \rightarrow \infty} \theta_n < 0$ ), which is the case with relative optimism (or pessimism) and perceived agreement.  $\square$

## Proof of Corollary 2

*Proof.* From (A.5), we find  $Q^* \rightarrow \frac{\tau_s}{2\gamma}$  as  $\tau_x \rightarrow 0$ . The results follow immediately from (A.4), (A.7), (A.8) and Lemma 1.  $\square$

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