Efficient Risk Sharing with Limited Commitment and Storage*

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December 31, 2013

Abstract

We extend the model of risk sharing with limited commitment (Kocherlakota, 1996) by introducing both a public and a private (non-contractible and/or non-observable) storage technology. Positive public storage relaxes future participation constraints and may hence improve risk sharing, contrary to the case where hidden income or effort is the deep friction. The characteristics of constrained-efficient allocations crucially depend on the storage technology's return. In the long run, if the return on storage is (i) moderately high, both assets and the consumption distribution may remain time-varying; (ii) sufficiently high, assets converge almost surely to a constant and the consumption distribution is time-invariant; (iii) equal to agents' discount rate, perfect risk sharing is self-enforcing. Agents never have an incentive to use their private storage technology, i.e., Euler inequalities are always satisfied, at the constrained-efficient allocation of our model, while this is not the case without optimal public asset accumulation.

Keywords: risk sharing, limited commitment, hidden storage, dynamic contracts

JEL codes: E20

*We thank Andy Atkeson, Hugo Hopenhayn, Yang K. Lu, Alessandro Mennuni, Nicola Pavoni, Raffaele Rossi, and seminar and conference participants at SAET in Faro, Central Bank of Hungary, ESEM in Oslo, Universitat Autònoma de Barcelona, UCLA, University of Bonn, Institut d'Anàlisi Econòmica (IAE-CSIC), RES PhD Meeting in London, Universitat Pompeu Fabra, Paris School of Economics, Cardiff Business School, Federal Reserve Bank of St. Louis, University of Rochester, University of Edinburgh, SED in Limassol, the Symposium on Economics and Institutions in Anacapri, CEF in Prague, NBER Summer Institute, EEA in Málaga, Barcelona GSE Trobada, University of Girona, SITE in Stockholm, University of Amsterdam, IHS in Vienna, University of Uppsala, CEAR workshop on Income Dynamics, Income Risk, and Income Risk-Sharing in Atlanta, and IIES in Stockholm for useful comments and suggestions. All errors are our own.

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1 Introduction

The literature on incomplete markets either exogenously restricts asset trade, most prominently by allowing only a risk-free bond to be traded (Huggett, 1993; Aiyagari, 1994), or considers a deep friction which limits risk sharing endogenously. With private information as the friction, a few papers (Allen, 1985; Cole and Kocherlakota, 2001; Ábrahám, Koehne, and Pavoni, 2011) have integrated these two strands of literature by introducing a storage technology. This paper considers limited commitment (Kocherlakota, 1996), and makes a similar contribution by introducing both a public and a private storage technology.

Storage potentially affects the constrained-efficient allocation through several channels. First, it allows the social planner to shift resources intertemporally. Second, it makes agents' outside option more attractive as it serves as an instrument to smooth consumption in autarky. Third, if storage is not observable (and/or not contractible), it increases considerably the agents' set of possible deviations. We provide a thorough analytical characterization of an environment where risk sharing arrangements are subject to limited commitment and both public and private storage are available.[1]

In several economic contexts where the model of risk sharing with limited commitment has been applied, agents are likely to have a way to transfer resources intertemporally. In the context of village economies (Ligon, Thomas, and Worrall, 2002), households may keep grain or cash around the house for self-insure purposes, and there also exist community grain storage facilities. Households in the United States (Krueger and Perri, 2006) may keep savings in cash or 'hide' their assets abroad. Spouses within a household (Mazzocco, 2007) accumulate both joint assets and savings for personal use. Partners in a law firms have both common and private assets. Countries (Kehoe and Perri, 2002) may also have joint savings (in a stability fund, such as the European Stability Mechanism, for example) in addition to their individual asset balances. Consequently, when we study self-enforcing risk sharing in these environments, we need to take into account private and public technologies which make it possible to transfer resources from today to the future. The insights we derive in this paper can be useful for all these applications.

Our starting point is the two-sided lack of commitment framework of Kocherlakota (1996), which we often refer to as the basic model. Agents are infinitely lived, risk averse, and ex-

[1] In the existing models of risk sharing with limited commitment, only public and/or observable and contractible individual intertemporal technologies have been considered (Marcet and Marimon, 1992; Ligon, Thomas, and Worrall, 2000; Kehoe and Perri, 2002; Ábrahám and Cárceles-Poveda, 2006; Krueger and Perri, 2006). Moreover, the above papers do not provide a thorough analysis of the effects of storage opportunities on the constrained-efficient allocation.

ante identical. They receive a risky endowment each period. We assume that there is no aggregate uncertainty in the sense that the aggregate endowment is constant. Agents may make transfers to each other in order to smooth their consumption. These transfers are subject to limited commitment, i.e., each agent must be at least as well off as in autarky at each time and state of the world. The storage technology we introduce allows the planner and the agents to transfer resources from one period to the next and earn a net return r, $-1 \leq r \leq 1/\beta - 1$, where β is agents' subjective discount factor.

We first introduce only public storage. We assume that agents are excluded from the returns of the publicly accumulated assets, a(n endogenous) Lucas tree, when they default, as in Krueger and Perri (2006).[2] This implies that the higher the level of public assets is, the lower the incentives for default are in this economy. We show that public storage is used in equilibrium as long as its return is sufficiently high and risk sharing is partial in the basic model. The characteristics of constrained-efficient allocations, such as long-run asset and consumption dynamics, will crucially depend on the return on storage. We show that, in the long run, if the return on storage is moderately high, assets remain stochastic and the consumption distribution varies over time. If the return on storage is sufficiently high, assets converge almost surely to a constant and the consumption distribution is time-invariant. Risk sharing remains partial as long as the storage technology is inefficient, i.e., $r < 1/\beta - 1$, and perfect risk sharing is self-enforcing in the long run if the return on storage is equal to agents' discount rate.

To understand how public storage matters, note that limited commitment makes markets endogenously incomplete, i.e., individual consumptions are volatile over time. This market incompleteness triggers precautionary saving/storage motives for the agents and the planner. This motive is stronger when cross-sectional income and consumption inequality are higher. At the same time, higher public assets reduce default incentives, thereby reducing consumption dispersion and the precautionary motive for saving. Further, agents would like to front load consumption as long as $\beta(1 + r) < 1$, i.e., if they are impatient relative to the return on storage. Optimal asset accumulation is determined by these conflicting forces. If $\beta(1 + r) = 1$, it is optimal for the planner to fully complete the market by storage in the long run. This is because the trade-off between imperfect insurance and an inefficient intertemporal technology is no longer present.

[2] Publicly stored assets may be protected by community enforcement (guards for public grain storage facilities in villages), by contracts (divorce law for couples), or by international organizations (for countries). Alternatively, one may think of an outside financial intermediary implementing public storage, as in our decentralization, see Section 2.5. Note that Karaivanov and Townsend (2013) also assume the presence of a financial intermediary for Thai villages.

The introduction of public storage has new qualitative implications for the dynamics of consumption predicted by the model when assets are stochastic in the long run. First, the amnesia property, i.e., the property that the consumption allocation only depends on the current income of the agent with a binding participation constraint and is independent of the past history of shocks whenever a participation constraint binds (Kocherlakota, 1996), does not hold. Second, the persistence property of the basic model, i.e., that the consumption allocation does not change for 'small' changes in the income distribution, does not hold either. There is a common intuition behind these results: the past history of shocks affects current consumptions through aggregate assets. Data on household income and consumption support neither the amnesia, nor the strong persistence property of the basic model (see Broer, 2012, for an extensive analysis). Hence, these differences are steps in the right direction for the limited commitment framework to explain consumption dynamics.

We also show that constrained-efficient allocations can be decentralized as competitive equilibria with endogenous borrowing constraints (Alvarez and Jermann, 2000) and a competitive financial intermediation sector which runs the storage technology (Ábrahám and Cárceles-Poveda, 2006). In this environment, equilibrium asset prices take into account the externality of aggregate storage on default incentives. In this sense, our paper provides a joint theory of endogenous borrowing constraints and endogenously growing (and shrinking) asset/Lucas trees in equilibrium.

We then consider hidden (non-contractible and/or non-observable) storage as well. Access to hidden storage not only changes the value of autarky, but it may also enlarge the set of possible deviations along the equilibrium path. That is, agents could default and store in every period either simultaneously or subsequently. This implies that, in principle, we need to consider a model where agents' incentive to default on transfers and their incentive to store, as well as their incentive to store in autarky, are taken into account. Indeed, we show that whenever the return on storage is high enough and the basic limited commitment model exhibits relatively little risk sharing, the constrained-efficient allocation in the basic model without public storage is not incentive compatible if agents have access to hidden storage.[3] This is because the constrained-efficient level of consumption dispersion triggers a precautionary saving motive whenever an agent has high consumption and the return on storage is high enough.

In contrast to the basic model, at the constrained-efficient allocation in our model with public storage agents no longer have an incentive to store. In other words, with optimal public

[3]Note that this result does not hinge on how agents' outside option is specified precisely: they may or may not be allowed to store in autarky, and they may or may not face additional punishment for defaulting.

asset accumulation the social planner can preempt the agents' storage incentives, or, hidden storage no longer matters. This is true because the planner has more incentive to store than the agents. First, the planner stores for the agents, because she inherits their consumption smoothing preferences. Second, storage by the planner makes it easier to satisfy agents' participation constraints in the future. In other words, the planner internalizes the positive externality generated by accumulated assets on future risk sharing.

This result means that the characteristics of constrained-efficient allocations in a model with both public and private storage and a model with only public storage are the same. They correspond exactly as long as agents' outside option is the same. This result also means that in our model with limited commitment and public storage agents' Euler inequalities are always satisfied. The Euler inequality cannot be rejected in micro data from developed economies, once labor supply decisions and demographics are appropriately accounted for (Attanasio, 1999). Therefore, we bring limited commitment models in line with this third observation about consumption dynamics as well.

Public and private storage have been considered in a private information environment with full commitment by Cole and Kocherlakota (2001). They show that public storage is never used and agents' private saving incentives are binding in equilibrium, eliminating any risk sharing opportunity beyond self-insurance.[4] When the deep friction is limited commitment as opposed to private information, the results are very different: first, public storage is used in equilibrium, and second, private storage incentives do not bind. The main difference between the two environments is that in our environment more public storage helps to reduce the underlying limited commitment friction, while with private information public asset accumulation would make incentive provision for truthful revelation more costly.

We finally ask: what is the overall effect of access to storage on consumption dispersion and welfare? This crucially depends on the return on storage. The availability of storage increases the value of autarky, which increases consumption dispersion and reduces welfare, while accumulated public assets decrease consumption dispersion and increase available resources, hence improve long-run welfare. When the return on storage is sufficiently high, there are welfare gains in the long run, because the economy gets close to perfect risk sharing and aggregate consumption is higher than in the basic model. When the return on storage is lower, the negative effect of a better outside option dominates the positive effect of public assets on welfare. In the short run, public asset accumulation also has costs in terms of foregone consumption. Hence, it is a quantitative question whether access to storage improves

[4] See also Allen (1985) and Ábrahám, Koehne, and Pavoni (2011).

welfare taking into account the transition from the moment storage becomes available. For this reason, we propose an algorithm to solve the model numerically, and present some computed examples to illustrate the effects of the availability of storage and its return on asset accumulation, risk sharing, and welfare. For the parametrizations we have considered, the short-term losses dominate the long-run gains for all returns on storage. However, given private storage, public asset accumulation always improves welfare.

The rest of the paper is structured as follows. Section 2 introduces and characterizes our model with public storage. Section 3 shows that agents' hidden storage incentives are eliminated under optimal public asset accumulation. We also show that this is not the case in the basic model. Section 4 presents some computed examples. Section 5 concludes.

2 The model with public storage

We consider an endowment economy with two types of agents, $i = \{1, 2\}$, each of unit measure, who are infinitely lived and risk averse. All agents are ex-ante identical in the sense that they have the same preferences and are endowed with the same exogenous random endowment process. Agents in the same group are ex-post identical as well, meaning that their endowment realizations are the same at each time t.[5] Let $u()$ denote the utility function. We assume that it is characterized by harmonic absolute risk aversion (HARA), i.e., $u'(c) = (a + c)^{-\sigma}$, where $a \geq 0$ and $\sigma > 0$.[6] Note that HARA utility functions satisfy prudence, i.e., $u'''() > 0$. We further assume that inverse marginal, ${}^{1}\!/\!{}_{u'()}$, is convex, that is, $\sigma \geq 1$.[7] The common discount factor is denoted by β.

Let s_t denote the state of the world realized at time t and s^t the history of endowment realizations, that is, $s^t = (s_1, s_2, ..., s_t)$. Given s_t, agent 1 has income $y(s_t)$, while agent 2 has income equal to $(Y - y(s_t))$, where Y is the aggregate endowment. Note that there is no aggregate uncertainty in the sense that the aggregate endowment is constant. However, the distribution of income varies over time. We further assume that income has a discrete support with N elements, that is, $y(s_t) \in \left\{y^1, \ldots, y^j, \ldots, y^N\right\}$ with $y^j < y^{j+1}$, and is independently and identically distributed (i.i.d.) over time, that is, $\Pr\left(y(s_t) = y^j\right) = \Pr\left(y^j\right) = \pi^j$, $\forall t$. The assumptions that there are two types of agents and no aggregate uncertainty impose some symmetry on both the income realizations and the probabilities. In particular,

[5] We will refer to agent 1 and agent 2 below. Equivalently, we could say type-1 and type-2 agents, or agents belonging to group 1 and group 2.

[6] Note that relative risk aversion is constant for $a = 0$, and we get exponential utility with $a = 1$ and σ approaching infinity.

[7] Some of our results hold under weaker assumptions.

$y^j = Y - y^{N-j+1}$ and $\pi^j = \pi^{N-j+1}$. The i.i.d. assumption can be relaxed, we only need weak positive dependence, i.e., that expected future lifetime utility is weakly increasing in current income.

Suppose that risk sharing is limited by two-sided lack of commitment to risk sharing contracts, i.e., insurance transfers have to be voluntary, or, self-enforcing, as in Thomas and Worrall (1988), Kocherlakota (1996), and others. Each agent may decide at any time and state to default and revert to autarky. This means that only those risk sharing contracts are sustainable which provide a lifetime utility at least as great as autarky after any history of endowment realizations for each agent. We assume that the punishment for deviation is exclusion from risk sharing arrangements in the future. This is the most severe subgame-perfect punishment in this context. In other words, it is an optimal penal code in the sense of Abreu (1988) (Kocherlakota, 1996). Note that so far our setting is identical to that of Kocherlakota, 1996, which we extend next.

We introduce a storage technology, which makes it possible to transfer resources from today to tomorrow. Assets stored earn a net return r, with $-1 \leq r \leq 1/\beta - 1$. Note that if $r = -1$ we are back to the basic limited commitment model of Kocherlakota (1996). In this section we only allow for public storage, to which defaulting agents do not have access (as in Krueger and Perri, 2006). In Section 3 we also allow agents to store both in autarky and potentially in equilibrium in a hidden way.

The constrained-efficient risk sharing contract is the solution to the following optimization problem:

$$\max_{c_i(s^t)} \sum_{i=1}^{2} \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) u\left(c_i\left(s^t\right)\right), \tag{1}$$

where λ_i is the (initial) Pareto-weight of agent i, $\Pr(s^t)$ is the probability of history s^t occurring, and $c_i(s^t)$ is the consumption of agent i at time t when history s^t has occurred; subject to the resource constraints,

$$\sum_{i=1}^{2} c_i\left(s^t\right) \leq \sum_{i=1}^{2} y_i\left(s_t\right) + (1+r)B\left(s^{t-1}\right) - B\left(s^t\right), \quad B\left(s^t\right) \geq 0, \quad \forall s^t, \tag{2}$$

where $B(s^t)$ denotes public storage when history s^t has occurred, with $B(s^0)$ given; and the participation constraints,

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr\left(s^r \mid s^t\right) u\left(c_i\left(s^r\right)\right) \geq U_i^{au}\left(s_t\right), \quad \forall s^t, \forall i, \tag{3}$$

where $\Pr(s^r \mid s^t)$ is the conditional probability of history s^r occurring given that history s^t occurred up to time t, and $U_i^{au}(s_t)$ is the expected lifetime utility of agent i when in autarky if state s_t has occurred today. In mathematical terms,

$$U_1^{au}\left(s_t\right) = u\left(y(s_t)\right) + \frac{\beta}{1-\beta}\sum_{j=1}^{N}\pi^j u\left(y^j\right) \tag{4}$$

and

$$U_2^{au}\left(s_t\right) = u\left(Y - y(s_t)\right) + \frac{\beta}{1-\beta}\sum_{j=1}^{N}\pi^j u\left(y^j\right).$$

The above definition of autarky assumes that agents cannot use the storage technology in autarky. Note, however, that the qualitative results remain the same under different outside options as long as the strict monotonicity of the autarky value in current income is maintained. For example, agents could save in autarky (as in Krueger and Perri, 2006 and in Section 3), or they might endure additional punishments from the community for defaulting (as in Ligon, Thomas, and Worrall, 2002).

2.1 Characterization

We focus on the characteristics of constrained-efficient allocations. Our characterization is based on the recursive Lagrangian approach of Marcet and Marimon (2011). However, the same results can be obtained using the promised utility approach (Abreu, Pearce, and Stacchetti, 1990).

Let $\beta^t \Pr\left(s^t\right)\mu_i\left(s^t\right)$ denote the Lagrange multiplier on the participation constraint, (3), and let $\beta^t \Pr\left(s^t\right)\gamma\left(s^t\right)$ be the Lagrange multiplier on the resource constraint, (2), when history s^t has occurred. The Lagrangian is

$$\begin{aligned}\mathscr{L} = &\sum_{t=1}^{\infty}\sum_{s^t}\beta^t \Pr\left(s^t\right)\left\{\sum_{i=1}^{2}\Bigg[\lambda_i u\left(c_i\left(s^t\right)\right)\right.\\ &+\mu_i\left(s^t\right)\left(\sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\Pr\left(s^r \mid s^t\right)u\left(c_i\left(s^r\right)\right) - U_i^{au}\left(s_t\right)\right)\Bigg]\\ &\left.+\gamma\left(s^t\right)\left(\sum_{i=1}^{2}\left(y_i\left(s_t\right) - c_i\left(s^t\right)\right) + (1+r)B\left(s^{t-1}\right) - B\left(s^t\right)\right)\right\},\end{aligned}$$

with $B\left(s^t\right) \geq 0$. Using the ideas of Marcet and Marimon (2011), we can write the Lagrangian in the form

$$\begin{aligned}\mathscr{L} = &\sum_{t=1}^{\infty}\sum_{s^t}\beta^t \Pr\left(s^t\right)\left\{\sum_{i=1}^{2}\left[M_i\left(s^t\right)u\left(c_i\left(s^t\right)\right) - \mu_i\left(s^t\right)U_i^{au}\left(s_t\right)\right]\right.\\ &\left.+\gamma\left(s^t\right)\left(\sum_{i=1}^{2}\left(y_i\left(s_t\right) - c_i\left(s^t\right)\right) + (1+r)B\left(s^{t-1}\right) - B\left(s^t\right)\right)\right\},\end{aligned}$$

where $M_i(s^t) = M_i(s^{t-1}) + \mu_i(s^t)$ and $M_i(s^0) = \lambda_i$.

The necessary first-order condition[8] with respect to agent i's consumption when history s^t has occurred is

$$\frac{\partial \mathscr{L}}{\partial c_i(s^t)} = M_i(s^t)\, u'(c_i(s^t)) - \gamma(s^t) = 0. \tag{5}$$

Combining such first-order conditions for agent 1 and agent 2, we have

$$x(s^t) \equiv \frac{M_1(s^t)}{M_2(s^t)} = \frac{u'(c_2(s^t))}{u'(c_1(s^t))}. \tag{6}$$

Here $x(s^t)$ is the temporary Pareto weight of agent 1 relative to agent 2.[9] Defining

$$\upsilon_i(s^t) = \frac{\mu_i(s^t)}{M_i(s^t)}$$

and using the definitions of $x(s^t)$ and $M_i(s^t)$, we can obtain the law of motion of x as

$$x(s^t) = x(s^{t-1}) \frac{1 - \upsilon_2(s^t)}{1 - \upsilon_1(s^t)}. \tag{7}$$

The planner's Euler inequality, i.e., the optimality condition for $B(s^t)$, is

$$\gamma(s^t) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t)\, \gamma(s^{t+1}), \tag{8}$$

which, using (5), can also be written as

$$M_i(s^t)\, u'(c_i(s^t)) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1} \mid s^t)\, M_i(s^{t+1})\, u'(c_i(s^{t+1})).$$

Then, using (6) and (7), the planner's Euler becomes

$$u'(c_i(s^t)) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) \frac{u'(c_i(s^{t+1}))}{1 - \upsilon_i(s^{t+1})}, \tag{9}$$

where $0 \leq \upsilon_i(s^{t+1}) \leq 1$, $\forall s^{t+1}, \forall i$. Given the definition of $\upsilon_i(s^{t+1})$ and equation (7), it is easy to see that (8) represents exactly the same mathematical relationship for both agents.

Equation (9) determines the choice of public storage, $B(s^t)$. It is clear that, first, the higher the return on storage is, the more incentive the planner has to store. Second, whenever we do not have perfect risk sharing, that is, $c_i(s^{t+1})$ varies over s^{t+1} for a given s^t, the

[8]Under general conditions, these conditions are also sufficient together with the participation and resource constraints.

[9]To reinforce this interpretation, notice that if no participation constraint binds in history s^t for either agent, i.e., $\mu_1(s^\tau) = \mu_2(s^\tau) = 0$ for all subhistories $s^\tau \subseteq s^t$, then $x(s^t) = \lambda_1/\lambda_2$, the initial relative Pareto weight of agent 1.

planner has a precautionary motive for storage, a typical motive for saving in models with (endogenously) incomplete markets. Third, the new term compared to standard models is $1/\left(1-\upsilon_i\left(s^{t+1}\right)\right) \geq 1$. This term is strictly bigger than 1 for states when agent i's participation constraint is binding. Hence, future binding participation constraints amplify the return on storage. This is the case, because higher storage will make the participation constraints looser in the future by reducing the relative attractiveness of default. The planner internalizes this effect when choosing the level of public storage.

Next, we introduce some useful notation and show more precisely the recursive formulation of our problem. This recursive formulation is going to be the basis for both the analytical characterization and the numerical solution procedure. Let c and y denote the current consumption and income, respectively, of agent 1, and $V()$ denote his value function. The following system is recursive with $X = (y, B, x)$ as state variables:

$$x'(X) = \frac{u'\left(Y + (1+r)B - B'(X) - c(X)\right)}{u'\left(c(X)\right)} \tag{10}$$

$$x'(X) = x\frac{1-\upsilon_2(X)}{1-\upsilon_1(X)} \tag{11}$$

$$u'\left(c(X)\right) \geq \beta(1+r)\sum_{y'} \Pr\left(y'\right)\frac{u'\left(c(X')\right)}{1-\upsilon_1(X')} \tag{12}$$

$$u\left(c(X)\right) + \beta\sum_{y'} \Pr\left(y'\right) V\left(X'\right) \geq U^{au}\left(y\right) \tag{13}$$

$$u\left(Y + (1+r)B - B'(X) - c(X)\right) + \beta\sum_{y'} \Pr\left(y'\right) V\left(Y - y', B'(X), 1/x'(X)\right) \geq U^{au}\left(Y - y\right) \tag{14}$$

$$B'(X) \geq 0. \tag{15}$$

The first equation, (10), where we have used the resource constraint to substitute for $c_2(X)$, says that the ratio of marginal utilities between the two agents has to be equal to the current relative Pareto weight. Equation (11) is the law of motion of the co-state variable, x. Equation (12) is the social planner's Euler inequality, which we have derived above. Equations (13) and (14) are the participation constraints of agent 1 and agent 2, respectively. Finally, equation (15) makes sure that storage is never negative.

Given the recursive formulation above, and noting that the outside option $U^{au}()$ is monotone in current income and takes a finite set of values, the solution can be characterized by a set of state-dependent intervals on the temporary Pareto weight. This is analogous to the basic model, where public storage is not considered (see Ljungqvist and Sargent, 2004, for a textbook treatment). The key difference is that these optimal intervals on the relative Pareto weight depend not only on current endowment realizations but also on B. The following lemma will be useful for specifying the optimal state-dependent intervals, and hence for characterizing the dynamics of our model.

Lemma 1. *$c(\tilde{y}, B, \tilde{x}) = c(\hat{y}, B, \hat{x})$, $B'(\tilde{y}, B, \tilde{x}) = B'(\hat{y}, B, \hat{x})$, and $V(\tilde{y}, B, \tilde{x}) = V(\hat{y}, B, \hat{x})$ for all $(\tilde{y}, \tilde{x}), (\hat{y}, \hat{x})$ such that $x'(\tilde{y}, B, \tilde{x}) = x'(\hat{y}, B, \hat{x})$. That is, for determining consumptions, public storage, and agents' expected lifetime utility, the current relative Pareto weight, x', is a sufficient statistic for the current income state, y^j, and last period's relative Pareto weight, x.*

Proof. Once we know x', equations (10) and (12), which do not depend on x, give c and B'. Then, the left hand side of (13) gives V. □

Lemma 1 implies that, with some abuse of notation, we can express consumptions, storage, and agents' lifetime utility in terms of accumulated assets and the current Pareto weight. That is, we can write $c(B, x')$, $B'(B, x')$, and $V(B, x')$.

The following conditions define the lower and upper bound of the optimal intervals in state y^j as a function of B:

$$V(B, \underline{x}^j(B)) = U^{au}\left(y^j\right) \text{ and } V\left(B, \frac{1}{\overline{x}^j(B)}\right) = U^{au}\left(Y - y^j\right). \tag{16}$$

Hence, given the inherited Pareto weight, x_{t-1}, and accumulated assets, B, the updating rule is

$$x_t = \begin{cases} \overline{x}^j(B) & \text{if} \quad x_{t-1} > \overline{x}^j(B) \\ x_{t-1} & \text{if} \quad x_{t-1} \in \left[\underline{x}^j(B), \overline{x}^j(B)\right] \\ \underline{x}^j(B) & \text{if} \quad x_{t-1} < \underline{x}^j(B) \end{cases}. \tag{17}$$

The ratio of marginal utilities is kept constant whenever this does not violate the participation constraint of either agent. When the participation constraint binds for agent 1, the relative Pareto weight moves to the lower limit of the optimal interval, just making sure that this agent is indifferent between staying and defaulting. Similarly, when agent 2's participation constraint binds, the relative Pareto weight moves to the upper limit of the optimal interval. Thereby, it is guaranteed that, ex ante, as much risk sharing as possible is achieved while satisfying the participation constraints.

Note that, given that the value of autarky is strictly increasing in current income and the value function is strictly increasing in the current Pareto weight, $\overline{x}^j(B) > \overline{x}^{j-1}(B)$ and $\underline{x}^j(B) > \underline{x}^{j-1}(B)$ for all $N \geq j > 1$ and B. It is easy to see that, unless autarky is the only implementable allocation, we have that $\overline{x}^j(B) > \underline{x}^j(B)$ for some j.

Given the utility function, the income process, and B, the intervals for different states may or may not overlap depending on the discount factor, β. The higher β is, the wider these intervals are. By a standard folk theorem (Kimball, 1988), for β sufficiently high all intervals overlap, that is, $\overline{x}^1(B) \geq \underline{x}^N(B)$, hence perfect risk sharing is implementable at the given asset level. At the other extreme, when β is sufficiently low, agents stay in autarky.

As public assets are accumulated (or decumulated) these optimal intervals change. The intervals are wider when B is higher. This is easy to see from (16). Take the first equality. The right hand side is independent of B, and the value function on the left hand side is increasing in both its arguments (available resources and own relative Pareto weight), hence as B increases $\underline{x}^j(B)$, the lower limit of the optimal interval in state y^j. must decrease. Similarly, from the second inequality in (16), $1/\overline{x}^j(B)$ must decrease, hence $\overline{x}^j(B)$, the upper limit, must increase. Moreover, $\overline{x}^j(B)$ is strictly increasing and $\underline{x}^j(B)$ is strictly decreasing in B for all j, as long as the length of the j-interval is not zero.

We can describe the dynamics of the model with similar optimal intervals and updating rule on consumption as on the relative Pareto weight. Using (10), we can now implicitly define the limits of the optimal intervals on consumption as

$$\begin{aligned} &\overline{c}^j(B): \ \overline{x}^j(B) = \frac{u'\left(Y + (1+r)\,B - B'(\overline{x}^j(B), B) - \overline{c}^j(B)\right)}{u'\left(\overline{c}^j(B)\right)} \\ \text{and} \quad &\underline{c}^j(B): \ \underline{x}^j(B) = \frac{u'\left(Y + (1+r)\,B - B'(\underline{x}^j(B), B) - \underline{c}^j(B)\right)}{u'\left(\underline{c}^j(B)\right)}. \end{aligned} \tag{18}$$

Symmetry implies that $\overline{c}^j(B) = Y + (1+r)\,B - B'(\overline{x}^j(B), B) - \underline{c}^{N-j+1}(B)$. Further, whenever public assets are constant over time, $B^* \equiv B' = B$, we can implicitly define the limits of the optimal consumption intervals as

$$\overline{c}^j\left(B^*\right): \overline{x}^j\left(B^*\right) = \frac{u'\left(Y + rB^* - \overline{c}^j\left(B^*\right)\right)}{u'\left(\overline{c}^j\left(B^*\right)\right)} \ \text{ and } \ \underline{c}^j\left(B^*\right): \underline{x}^j\left(B^*\right) = \frac{u'\left(Y + rB^* - \underline{c}^j\left(B^*\right)\right)}{u'\left(\underline{c}^j\left(B^*\right)\right)}.$$

It is easy to see that consumption is monotone in the end-of-period relative Pareto weight in the constant assets case, because aggregate resources are constant at $Y + rB^*$. However, in general, aggregate consumption varies a $(1+r)\,B - B'(x', B)$ varies of time, which depends on x'. Hence, an increase in the current relative Pareto weight, in principle, may imply a sufficiently large decrease in aggregate consumption so that agent 1's consumption decreases. For now, we assume that this is not the case.

Assumption 1. *If $\tilde{x}' > \hat{x}'$ then $c\left(B, \tilde{x}'\right) > c\left(B, \hat{x}'\right)$, $\forall B$. That is, consumption by agent 1 is strictly increasing in his current relative Pareto weight.*

We show below in Proposition 1 that the characterization which we derive using Assumption 1 implies that Assumption 1 must hold. That is, we use a 'guess and verify' approach.

In order to better understand some key characteristics of the dynamics of this model, we now focus on the case where public storage is constant over time. Then, from the next section, we study in detail the joint dynamics of consumption dispersion and assets. However, as we show later, under some conditions the economy will converge (almost surely) to a constant level of public assets. Further, the basic model is a special case of this economy with $B' = B = 0$.

We consider scenarios where the long-run equilibrium is characterized by imperfect risk sharing. That is, we assume from now on that $\overline{x}^1(B^*) < \underline{x}^N(B^*)$, or, equivalently, that $\overline{c}^1(B^*) < \underline{c}^N(B^*)$. We do this both because there is overwhelming evidence from several applications (households in a village or in the United States, spouses in a household, countries) about less than perfect risk sharing, and because that case is theoretically not interesting, as it is equivalent to the well-known (unconstrained-)efficient allocation of constant individual consumptions over time. It is not difficult to see that for a constant B the law of motion described by (17) implies that, in the long run, risk sharing arrangements subject to limited commitment are characterized by a finite set of consumption values determined by the limits of the optimal consumption intervals. It turns out that considering two scenarios is enough to describe the general picture: (i) each agent's participation constraint is binding only when his income is highest, and (ii) each agent's participation constraint is binding in more than one state.[10] Given this, to describe the constrained-efficient allocations in these two scenarios, it is sufficient to consider three income states, i.e., $N = 3$. Hence, for all our graphical and numerical examples, we set $N = 3$.

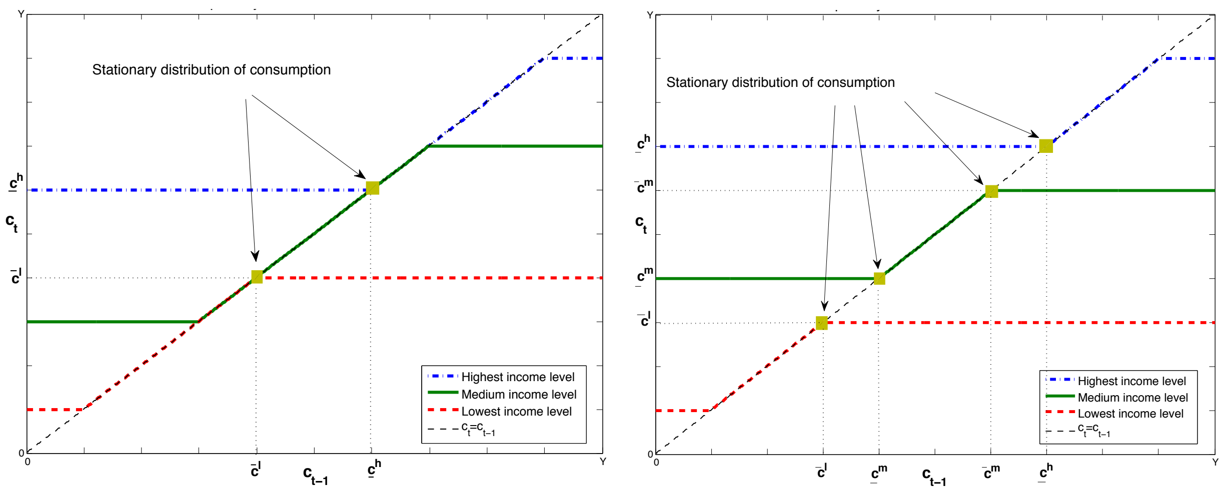
Consider an endowment process where each agent gets y^h, y^m, or y^l units of the consumption good, with $y^h > y^m > y^l$, with probabilities π^h, π^m, and π^l, respectively. Symmetry implies that $y^m = (y^h + y^l)/2$ and $\pi^h = \pi^l = (1 - \pi^m)/2$.

Given constant assets in the long run, the consumption intervals become wider if either β increases for a given B^*, as in the basic model, or B^* increases for a given β. Both changes make autarky less attractive. This is true in the former case because agents put higher weight on insurance in the future, and in the latter because agents are excluded from the benefits of more public assets upon default. If partial insurance occurs, there are two possible scenarios depending on the level of the discount factor and public assets. For higher levels of β and/or B^*, $\overline{c}^m(B^*) \geq \underline{c}^h(B^*) > \overline{c}^l(B^*) \geq \underline{c}^m(B^*)$. This means that the consumption

[10] It will become clear below that assets can only be optimally constant in this case if they are zero.

interval for state y^m overlaps with both the interval associated with state y^h and the one association with state y^l. This is the case where each agent's participation constraint binds for the highest income level only. Panel (a) in Figure 1 presents an example satisfying these conditions.

Figure 1: Consumption dynamics in the long run



(a) Overlapping intervals

(b) Disjunct intervals

Notes: In panel (a) the interval for state y^m overlaps with the intervals for state y^h and state y^l. In panel (b) all three state-dependent intervals are disjunct.

Suppose that the initial consumption level of agent 1 is below $\underline{c}^h(B^*)$. When agent 1 draws a high income realization (which occurs with probability 1 in the long run), his consumption jumps to $\underline{c}^h(B^*)$. Then it stays at that level until his income jumps to the lowest level. At that moment, agent 2's participation constraint binds, because he has high income, and consumption of agent 1 drops to $\overline{c}^l(B^*)$. Then we are back to where we started from. A very similar argument holds whenever agent 1's initial consumption is above $\underline{c}^h(B^*)$. This implies that consumption takes only two values, $\underline{c}^h(B^*)$ and $\overline{c}^l(B^*)$, in the long run. When consumption changes, it always moves between these two levels, and the past history of income realizations does not matter. This is the amnesia property of the basic model (Kocherlakota, 1996). When state y^m occurs after state y^h or state y^l, the consumption allocation remains unchanged. That is, consumption does not react at all to this 'small' change in income. This is the persistence property of the basic model. Note that consumption also remains unchanged over time if the sequence (h, m, h) or the sequence (l, m, l) takes place.

The key observation here is that, although individuals face consumption changes over time, the consumption distribution is time-invariant. In every period, half of the agents consume $\underline{c}^h(B^*)$ and the other half consume $\overline{c}^l(B^*)$. Finally, note that this happens for any N as long as $\overline{c}^2(B^*) \geq \underline{c}^N(B^*) > \overline{c}^1(B^*) \geq \underline{c}^{N-1}(B^*)$.

For lower levels of β and/or B^*, none of the three intervals overlap, i.e., $\underline{c}^h(B^*) > \overline{c}^m(B^*) > \underline{c}^m(B^*) > \overline{c}^l(B^*)$. Panel (b) in Figure 1 shows an example of this second case. When all three intervals are disjunct, consumption takes four values in the long run. Notice that the participation constraint of agent 1 may bind for both the medium and the high level of income. That is, whenever his income changes his consumption changes as well, and similarly for agent 2.

In this second case, in state y^m the past history determines which agent's participation constraint binds, therefore consumption is Markovian. Current incomes and the identity of the agent with a binding participation constraint fully determine the consumption allocation. The dynamics of consumption exhibit amnesia in this sense here. Further, consumption responds to every income change, hence the persistence property does not manifest itself.

The key observation for later reference is that the consumption distribution changes between $\{\underline{c}^m(B^*), \overline{c}^m(B^*)\}$ and $\left\{\overline{c}^l(B^*), \underline{c}^h(B^*)\right\}$. That is, the cross-sectional distribution of consumption is different whenever state y^m occurs from when an unequal income state, y^h or y^l, occurs. If there are $N > 3$ income states, the cross-sectional consumption distribution changes over time whenever $\overline{c}^2(B^*) < \underline{c}^N(B^*)$ and $\overline{c}^1(B^*) < \underline{c}^{N-1}(B^*)$.[11]

2.2 The dynamics of public assets and the consumption distribution

The next proposition provides a key property of the aggregate storage decision rule and characterizes the short-run dynamics of assets. It shows how public storage varies with the consumption and income distribution.

Proposition 1. *$B'(B, x')$ is strictly increasing in x' for $x' \geq 1$ and $B'(B, x') > 0$. That is, the higher cross-sectional consumption inequality is, the higher public asset accumulation is. $B'(y^j, B, x) \geq B'(y^k, B, x)$, $\forall (B, x)$, where $j \geq N/2 + 1$, $k \geq N/2$, and $j > k$. The inequality is strict, i.e., $B'(y^j, B, x) > B'(y^k, B, x)$, if the optimal intervals for states y^j and y^k do not overlap given B. That is, aggregate asset accumulation is weakly increasing with cross-sectional income inequality.*

Proof. In Appendix A.

The intuition for Proposition 1 is coming from two related observations. Higher inequality in the current period implies higher expected consumption inequality/risk next period. Under

[11]The number of income states and the number of states where a participation constraint binds determine the possible number of long-run consumption levels, and consequently the persistence property may appear.

convex inverse marginal utility, the planner has a higher precautionary motive for saving whenever she faces more risk tomorrow.[12]

We are now ready to characterize the long-run behavior of public assets and the consumption distribution.

Proposition 2. *Assume that β is such that agents obtain low risk sharing in the sense that the consumption distribution is time-varying without public storage.*

(i) There exists r_1 such that for all $r \in [-1, r_1]$, public storage is never used in the long run.

(ii) There exists a strictly positive $r_2 > r_1$ such that for all $r \in (r_1, r_2)$, B remains stochastic but bounded, and the consumption distribution is time-varying in the long run.

(iii) For all $r \in [r_2, 1/\beta - 1)$, B converges almost surely to a strictly positive constant, B^*, which is independent of the initial level of assets, and where the consumption distribution is time-invariant, but perfect risk sharing is not achieved.

(iv) Whenever $r = 1/\beta - 1$, B converges almost surely to a strictly positive constant and perfect risk sharing is self-enforcing.

If β is such that the consumption distribution is time-invariant without public storage, then $r_1 = r_2$, hence only (i), (iii), and (iv) occur.

Proof. In Appendix A.

The intuition behind Proposition 2 is that the social planner trades off two effects of increasing aggregate storage: it is costly as long as $\beta(1+r) < 1$, but less so the higher r is, and it is beneficial because it reduces consumption dispersion in the future. The level of public assets chosen just balances these two opposing forces. The relative strength of these two forces naturally depends on the return to storage, r. When the cross-sectional consumption distribution is time-varying (case (ii)), the relative strength of the two forces determining asset accumulation changes over time, as we have shown in Proposition 1. This implies that assets cannot settle at a constant level in this case. When the return on storage is sufficiently high (case (iii)), assets are accumulated so that participation constraints are only binding for agents with the highest income in the long run, and the consumption distribution becomes time-invariant. In this case, there is a unique constant level of assets, B^*, which exactly balances the trade-off between impatience and the risk sharing gains of storage.

[12] Note that for log() utility, B' is weakly increasing in x', i.e., in cross-sectional consumption inequality, since $1/u'$ is linear in this case, while for CRRA utility functions with a coefficient of relative risk aversion strictly greater than 1, the empirically more plausible range, $1/u'$ is strictly convex.

Decreasing public assets by a small amount would decrease future risk sharing more than the gain coming from the decrease in the inefficient transfer of resources to the future. Note that this is in contrast to an individual saving problem à la Huggett (1993). Finally, in the limiting case of $\beta(1+r)=1$ (case (iv)), there is no trade-off in the long run, hence assets are accumulated until the level where full insurance is enforceable.

We illustrate the dynamics of assets in our model on two figures. First, Figure 2 shows the short-run dynamics of assets in the case where they converge to a constant in the long run (case (iii) of Proposition 2). We assume further that we are already in the range of aggregate assets where the participation constraint binds only when an agent has the highest possible income. The solid (blue) line represents $B'\left(B, \underline{x}^N(B)\right)$, i.e., we compute B' assuming that the relevant participation constraint is binding. It is easy to see from the figure that at $B=B^*$ assets remain constant in the long run, since $B'=B=B^*$.

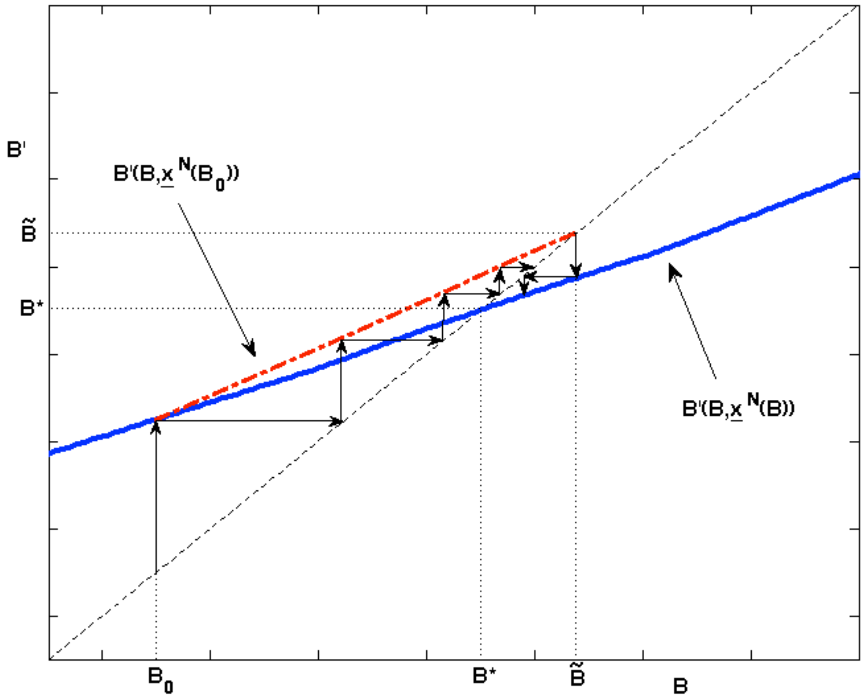
Now, we explain how assets converge to B^*. Suppose that state y^N occurs when inherited assets are at the initial level $B_0<B^*$. Then public storage is $B'\left(B_0, \underline{x}^N(B_0)\right)$. Next period, if any state y^j with $j \geq 2$ occurs, no participation constraint is binding, hence, according to Proposition 1, assets are $B'\left(B, \underline{x}^N\left(B_0\right)\right) > B'\left(B, \underline{x}^N\left(B\right)\right)$, because given $B>B_0$ we have $\underline{x}^N\left(B\right) < \underline{x}^N\left(B_0\right)$. The dynamics of asset in states y^j with $j \geq 2$, i.e., when no participation constraint binds, is represented by the dot-dashed (red) line. As long as state y^1 does not occur, assets are determined by this line and would eventually converge to the level $\widetilde{B}>B^*$. However, state y^1 occurs almost surely before $\widetilde{B}$ is reached. If the level of assets when y^1 occurs is above B^*, then assets are determined by the solid (blue) line, and they have to decline. If a participation constraint continues to bind, which happens in both state y^1 and state y^N, assets converge to B^* along the solid (blue) line. If no participation constraint binds, then according to Proposition 1 assets decline even more. This may result in the asset level dropping below B^*, but it remains above B_0. Then the same dynamics start again but in a tighter neighborhood around B^*. This argument implies that, although almost-sure convergence is guaranteed, it does not happen in a monotone way generically.

Before describing the dynamics of assets when they are stochastic in the long run (case (ii)), we characterize the bounds of the stationary distribution of assets. Let $\underline{B}$ $\left(\overline{B}\right)$ denote the lower (upper) limit of the stationary distribution of assets. Let an upper index m refer to the least unequal income state(s).[13]

Proposition 3. *The lower limit of the stationary distribution of public assets, $\underline{B}$, is either*

[13] Note that m refers to one state if N is odd, state $y^{N/2+1}$, and two states when N is even, $y^{N/2}$ and $y^{N/2+1}$.

Figure 2: Short-run asset dynamics when assets are constant in the long run



strictly positive and is implicitly given by

$$u'\left(\overline{c}^{m}\left(\underline{B}\right)\right)=\beta(1+r)\sum_{j=1}^{N}\pi^{j}\frac{u'\left(c\left(y^{j},\underline{B},\overline{x}^{m}\left(\underline{B}\right)\right)\right)}{1-\upsilon\left(y^{j},\underline{B},\overline{x}^{m}\left(\underline{B}\right)\right)},\tag{19}$$

or is zero and (19) *holds as strict inequality. The upper limit of the stationary distribution of public assets,* $\overline{B}$*, is implicitly given by*

$$u'\left(c\left(y^{N},\overline{B},\underline{x}^{N}\left(\underline{B}\right)\right)\right)=\beta(1+r)\sum_{j=1}^{N}\pi^{j}u'\left(c\left(y^{j},\overline{B},\underline{x}^{N}\left(\underline{B}\right)\right)\right).\tag{20}$$

Proof. In Appendix A.

Figure 3 illustrates both the short- and long-run dynamics of public assets in the case where they are stochastic in the long run. For simplicity, we consider three income states. This means that there are two types of states: two with high income and consumption inequality (states y^h and y^l) and one with low income and consumption inequality (state y^m). The solid (red) line represents $B'\left(B,\underline{x}^{h}(B)\right)$, i.e., storage in state y^h (or y^l) when the relevant participation constraint is binding. Similarly, the dot-dashed (blue) line represents $B'\left(B,\overline{x}^{m}(B)\right)$, i.e., storage in state y^m when the relevant participation constraint is binding. Starting from B_0, if state y^m occurs repeatedly, assets converge to the lower limit of their stationary distribution, $\underline{B}$. The relevant participation constraint is always binding along this path, because inherited assets keep decreasing.

The dashed (green) line represents the scenario where state y^h (or state y^l) occurs when inherited assets are at the lower limit of the stationary distribution, $\underline{B}$, and then the same state occurs repeatedly. This is when assets approach the upper limit of their stationary distribution, $\overline{B}$. The relevant participation constraint is not binding from the period after the switch to y^h, therefore storage given inherited assets is described by the function $B'\left(B, \underline{x}^h(\underline{B})\right)$.

Finally, assume, without loss of generality, that state y^l occurred many times while approaching $\overline{B}$, and suppose that state y^h occurs when inherited assets are (close to) $\overline{B}$. In this case, $x' = \underline{x}^h\left(\overline{B}\right) < \underline{x}^h\left(\underline{B}\right)$, and assets decrease. They then converge to a level $\widetilde{\underline{B}}$ from above with the relevant participation constraint binding along this path. The same happens whenever $B > \widetilde{\underline{B}}$ when we switch to state y^h (or y^l). $\widetilde{\underline{B}}$ is implicitly given by

$$u'\left(\underline{c}^h\left(\widetilde{\underline{B}}\right)\right) = \beta(1+r) \sum_{j=\{l,m,h\}} \pi^j u'\left(c\left(y^j, \widetilde{\underline{B}}, \underline{x}^h\left(\widetilde{\underline{B}}\right)\right)\right).$$

Note that as long as only state y^h and y^l occur, assets remain constant at $\widetilde{\underline{B}}$, similarly as in the previous figure. The key difference is that when the income distribution switches to the most equal one (y^m), a participation constraint binds, triggering a move in x toward 1, hence assets drop according to Proposition 1.

The characterization results of the previous three propositions imply that Assumption 1 holds, that is, consumption must be monotone in the current relative Pareto weight. We formally state this in the next proposition.

Conjecture 1. *Propositions 1, 2, and 3 imply that if $\tilde{x}' > \hat{x}'$ then $c\left(B, \tilde{x}'\right) > c\left(B, \hat{x}'\right)$, $\forall B$. That is, consumption by agent 1 is strictly increasing in his current relative Pareto weight.*

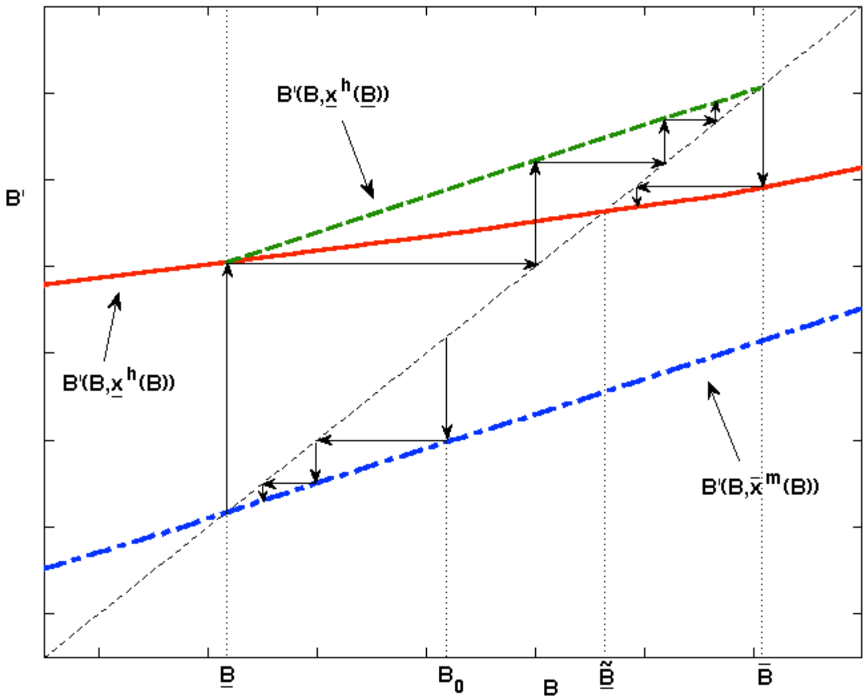
We provide a partial proof in Appendix A. In numerical examples we have considered this property holds.

The intuition for Conjecture 1 is simple. As a response to increasing inequality, it cannot be optimal to increase public storage so much that both agents have lower consumption. That would contradict the optimal intertemporal smoothing behavior of the planner.

2.3 The dynamics of individual consumptions

Having characterized assets, we now turn to the dynamics of consumption. One key property of the basic model is that whenever either agent's participation constraint binds ($\upsilon_1(X) > 0$

Figure 3: Asset dynamics when assets are stochastic in the long run



or $\upsilon_2(X) > 0$), the resulting allocation is independent of the preceding history. In our formulation, this implies that x' is only a function of y^j and the identity of the agent with a binding participation constraint. This is often called the amnesia property (Kocherlakota, 1996), and typically data do not support this pattern, see Broer (2012) for the United States and Kinnan (2012) for Thai villages. Allowing for storage helps to bring the model closer to the data in this respect.

Proposition 4. *The amnesia property does not hold when public assets are stochastic in the long run.*

Proof. x' and hence current consumption depend on both current income and inherited assets, B, when a participation constraint binds. This implies that the past history of income realizations affects current consumptions through B. $\square$

Another property of the basic model is that whenever neither participation constraint binds ($\upsilon_1(X) = \upsilon_2(X) = 0$), the consumption allocation is constant and hence exhibits an extreme form of persistence. This can be seen easily: (11) gives $x' = x$, and the consumption allocation is only a function of x' with constant aggregate income. This implies that for 'small' income changes which do not trigger a participation constraint to bind, we do not see any change in individual consumptions. It is again not easy to find evidence for this pattern in the data, see Broer (2012). In our model, even if the relative Pareto weight does not change, (10) does not imply that individual consumptions will be the same next period as in the current period. This is because $(1+r)B - B'(X)$ is generically not equal to $(1+r)B' - B''(X')$ when assets are stochastic in the long run. The only exceptions are asset levels $\underline{B}$, $\underline{\widetilde{B}}$, and $\overline{B}$ on Figure 3 with the appropriate income states occurring. However, the probability that assets settle at these points in the stationary distribution is zero.

Proposition 5. *The persistence property does not hold generically when public assets are stochastic in the long run.*

Proof. Even though $x' = x$, when neither participation constraint binds, consumption is only constant if net savings are identical in the past and the current period. This is generically not the case when B is stochastic. □

The last two propositions imply that the dynamics of consumption in the our model are richer and closer to the data than in the basic model in a qualitative sense. We leave the study of the quantitative implications of storage on consumption dynamics to future work.

2.4 Welfare

It is clear that access to public storage cannot reduce welfare, because zero assets can always be chosen. Along the same lines, if public storage is positive for at least the most unequal income state, then welfare strictly improves. Proposition 2 implies that this is the case whenever the basic model does not display perfect risk sharing and the return on storage is higher than $r_1 < 1/\beta - 1$.

2.5 Decentralization

Ábrahám and Cárceles-Poveda (2006) show how to decentralize a limited commitment economy with capital accumulation and production. That economy is similar to the current one in one important aspect: agents are excluded from receiving capital income after default. They introduce competitive intermediaries and show that a decentralization with endogenous debt constraints which are 'not too tight' (which make the agents just indifferent between participating and defaulting), as in Alvarez and Jermann (2000), is possible. However, Ábrahám and Cárceles-Poveda (2006) use a neoclassical production function where wages depend on aggregate capital. This implies that there the value of autarky depends on aggregate capital as well.[14] They show that if the intermediaries are subject to endogenously determined

[14]This is also the case in the two-country production economy of Kehoe and Perri (2004).

capital accumulation constraints, then this externality can be taken into account, and the constrained-efficient allocation can be decentralized as a competitive equilibrium.[15]

Public storage can be thought of as a form of capital, B units of which produce $Y + (1+r)B$ units of output tomorrow and which fully depreciates. Hence, the results above directly imply that a competitive equilibrium corresponding to the constrained-efficient allocation exists. In particular, households trade Arrow securities subject to endogenous borrowing constraints which prevent default, and the intermediaries also sell these Arrow securities to build up public storage. The key intuition is that equilibrium Arrow security prices take into account binding future participation constraints, as these prices are given by the usual pricing kernel. Moreover, agents do not hold any 'shares' in public storage, hence their autarky value is not affected. Finally, no arbitrage or perfect competition guarantees that the intermediaries make zero profits in equilibrium. As opposed to Ábrahám and Cárceles-Poveda (2006), capital accumulation constraints are not necessary, because in our model public storage does not affect agents' outside option.

3 The model with both public and private storage

So far, we have assumed that storage is available to the social planner, but agents can use it neither in autarky nor while in the risk sharing arrangement. In this section, we allow agents to use the same storage technology as the social planner. This both affects their autarky value and enlarges the set of possible actions (and deviations). In practice, allowing for private storage requires adding agents' Euler inequalities as constraints to the problem given by the objective function (1) and the constraints (2) and (3), and modifying the participation constraints, (3).

The social planner's problem becomes

$$\max_{\{c_i(s^t),B(s^t)\}} \sum_{i=1}^{2} \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) u\left(c_i\left(s^t\right)\right) \tag{21}$$

$$\text{s.t.} \quad \sum_{i=1}^{2} c_i\left(s^t\right) \leq \sum_{i=1}^{2} y_i\left(s_t\right) + (1+r)B\left(s^{t-1}\right) - B\left(s^t\right), \quad B\left(s^t\right) \geq 0, \quad \forall s^t, \tag{22}$$

$$(P1) \qquad \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr\left(s^r \mid s^t\right) u\left(c_i\left(s^r\right)\right) \geq \tilde{U}_i^{au}\left(s_t\right), \quad \forall s^t, \forall i, \tag{23}$$

$$u'\left(c_i\left(s^t\right)\right) \geq \beta(1+r) \sum_{s^{t+1}} \Pr\left(s^{t+1} \mid s^t\right) u'\left(c_i\left(s^{t+1}\right)\right), \quad \forall s^t, \forall i. \tag{24}$$

[15] Chien and Lee (2010) achieve the same objective by taxing capital instead of using a capital accumulation constraint.

The objective function, the resource constraint and the non-negativity of storage restriction remain the same as before. The participation constraints, (23), change slightly, since $\tilde{U}_i^{au}(s_t)$ (to be defined precisely below) is the value function of autarky when storage is allowed. Agents' Euler inequalities, equation (24), guarantee that agents have no incentive to deviate from the proposed allocation by storing privately. Note that we implicitly assume that private storage is zero at the initial period.

A few remarks are in order about this structure before we turn to the characterization of constrained-efficient allocations. First, agents can store in autarky, but they lose access to the benefits of the public asset.[16] This implies that $\tilde{U}_i^{au}(y^j) = V_i^{au}(y^j, 0)$, where $V_i^{au}(y^j, b)$ is defined as

$$V_i^{au}\left(y^j, b\right) = \max_{b'} \left\{ u(y^j + (1+r)b - b') + \beta \sum_{k=1}^{N} \pi^k V_i^{au}\left(y^k, b'\right) \right\}, \tag{25}$$

where b denotes private savings. Since $V_i^{au}(y^j, 0)$ is increasing (decreasing) in j for agent 1 (2), it is obvious that if we replace the autarky value in the model of Section 2 (or in the basic model) with the one defined here, the same characterization holds. Note that, unlike in Bulow and Rogoff (1989), saving in autarky is not state-contigent. [17]

Second, we use a version of the first-order condition approach (FOCA) here. That is, these constraints only cover a subset of possible deviations. In particular, we verify that the agent is better off staying in the risk arrangement rather than defaulting and possible storing (constraint (23), see also (25)), and that he has no incentive to store given that he does not ever default (constraint (24), agents' first-order condition). It is not obvious whether these constraints are sufficient to guarantee incentive compatibility,[18] because multiple and multi-period deviations are not considered by these constraints. In particular, an agent can store in the current period (to increase his value of autarky in future periods) and default in a later period. For now, we assume that these deviations are not profitable given the contract which solves Problem *P1*. We first characterize the solution under this assumption. Then, we apply a numerical verification strategy, which is discussed in Appendix C, to show that agents indeed have no incentive to use these more complex deviations.

Third, both the participation constraints, (23), and the Euler constraints, (24), involve

[16]This is the same assumption as in Krueger and Perri (2006), where agents lose access to the benefits of a tree after defaulting. In our model the 'tree' is endogenous.

[17]Bulow and Rogoff (1989) find that access to state-contingent "cash-in-advance contracts" in autarky prevents risk sharing in equilibrium. However, "[t]his conclusion does depend on a sovereign's ability to reproduce any risk-sharing advantages of loan contracts by holding a portfolio of foreign assets" (p. 49).

[18]In fact, Kocherlakota (2004) shows that in an economy with private information and hidden storage the first-order condition approach can be invalid.

future decision variables. Given these two types of forward-looking constraints, a recursive formulation using either the promised utilities approach (Abreu, Pearce, and Stacchetti, 1990) or the Lagrange multipliers approach (Marcet and Marimon, 2011) is difficult. Euler constraints have been dealt with using the agent's marginal utility as a co-state variable in models with moral hazard and hidden storage, see Werning (2001) and Ábrahám and Pavoni (2008). In our environment, this could raise serious tractability issues, since we would need two more continuous co-state variables, in addition to the state variable to make the participation constraints recursive.

In this paper, we follow a different approach that avoids these complications. In particular, we show that the solution of a simplified problem where agents' Euler inequalities are ignored satisfies those Euler constraints. That is, instead of Problem *P1*, we consider the following simpler problem:

$$
(P2) \qquad \begin{aligned}
&\max_{\{c_i(s^t),B(s^t)\}} \sum_{i=1}^{2} \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) u\left(c_i\left(s^t\right)\right) \\
&\text{s.t.} \quad \sum_{i=1}^{2} c_i\left(s^t\right) \leq \sum_{i=1}^{2} y_i\left(s_t\right) + (1+r)B\left(s^{t-1}\right) - B\left(s^t\right), \;\; B\left(s^t\right) \geq 0, \;\; \forall s^t, \\
&\qquad \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr\left(s^r \mid s^t\right) u\left(c_i\left(s^r\right)\right) \geq \tilde{U}_i^{au}\left(s_t\right), \;\; \forall s^t, \forall i,
\end{aligned}
$$

This is the problem we studied in Section 2, the only difference being that the autarky value is different. Now, we are ready to state the main result of this section.

Proposition 6. *The solution of the model with hidden storage, P1, corresponds to the solution of the simplified problem, P2.*

Proof. We prove this proposition by showing that the allocation which solves *P2* satisfies agents' Euler inequalities (24), the only additional constraints. Note that the planner's Euler, (9), is a necessary condition for optimality for *P2*. It is clear that the right hand side of (9) is bigger than the right hand side of (24), for $i = \{1, 2\}$, since $0 \leq \upsilon_i\left(s^{t+1}\right) \leq 1$, $\forall s^{t+1}$. Therefore, (9) implies (24). □

This result implies that the characteristics of the constrained-efficient allocation of Problem *P1* are the same as those of Problem *P2*, which is the problem we studied in Section 2. Proposition 6 also means that private storage does not matter as long as public asset accumulation is optimal. We have to emphasize, however, that the result that no private storage occurs hinges on the assumption of optimal public asset accumulation with the same return.

Therefore, Proposition 6 does not imply that we expect no hidden storage to occur in the data.

The intuition behind this result is that the planner has more incentive to store than the agents. She stores for the agents, because she inherits their consumption smoothing preferences. Thereby she can eliminate the agents' incentive to store in a hidden way. Further, comparing (9) and (24) again, it is obvious that the planner has more incentive to store than the agents in all but the most unequal states. In particular, the presence of $1/\left(1-\upsilon_i\left(s^{t+1}\right)\right)>1$ in the planner's Euler indicates how public asset accumulation helps the planner to relax future participation constraints, and thereby improve risk sharing, or, make markets more complete. In other words, the planner internalizes the positive externality of public asset accumulation on future risk sharing.

Next, we relate the case with both private and public storage to the case with private storage in autarky but without public storage. The following result follows from Proposition 6.

Corollary 1. *The planner stores in equilibrium whenever an agent's Euler inequality is violated at the constrained-efficient allocation of the basic model with no public storage and private storage only in autarky.*

Corollary 1 says that whenever agents have private storage incentives in the basic model, public storage is used in equilibrium. However, this result is only interesting if private storage matters, i.e., agents' Euler inequalities are violated, in the basic model under general conditions. This is what we establish next.

3.1 Does hidden storage matter in the basic model?

In this section, we identify conditions under which agents would store at the constrained-efficient solution of the basic model without public storage. We assume that partial insurance occurs at the solution, because otherwise it is trivial that private storage is never used. If agents' Euler inequalities are violated, the solution is not robust to deviations when private storage is available. Further, Corollary 1 implies that public storage is going to be positive, at least under some histories, whenever this technology is available.

We first consider the benchmark case where agents have access to an efficient intertemporal technology, i.e., storage earns a return r such that $\beta(1+r)=1$. Afterwards, we study the general case. We only examine whether agents would use the available hidden intertemporal technology at the constrained-efficient allocation of the basic model. We do not make any assumption about the number of income states, except that income may take a finite number of values and the support of the income distribution is bounded.

Lemma 2. *Suppose that partial insurance occurs and the hidden storage technology yields a return r such that $\beta(1+r) = 1$. Then agents' Euler inequalities are violated at the constrained-efficient allocation of the basic model.*

Proof. We show that the Euler inequality is violated at the constrained-efficient allocation at least when an agent receives the highest possible income, y^N, hence his participation constraint is binding. By the characterization in Section 2.1, it is clear that for all future income levels his consumption will be no greater than his current consumption, i.e., $c\left(y^j, 0, \underline{x}^N(0)\right) \leq \underline{c}^N(0)$. If partial insurance occurs, then it must be that there exists some state y^k where the agent consumes $c\left(y^k, 0, \underline{x}^N\right) < \underline{c}^N(0)$. Then,

$$u'(\underline{c}^N(0)) < \sum_{y^j} \pi^j u'\left(c\left(y^j, 0, \underline{x}^N(0)\right)\right),$$

that is, the Euler inequality is violated. □

It is obvious that if the return on storage is low, the constrained-efficient allocation of the basic model satisfies agents' Euler inequalities. The following proposition shows that for all economies with partial insurance one can find a threshold return on storage above which agents' storage incentives bind in the basic model.

Proposition 7. *There exists $\tilde{r} < 1/\beta - 1$ such that for all $r > \tilde{r}$ agents' Euler inequalities are violated at the constrained-efficient allocation of the basic model.*

Proof. $\tilde{r}$ is defined as the solution to

$$u'(\underline{c}^N(0)) = \beta(1+\tilde{r}) \sum_{y^j} \Pr\left(y^j\right) u'\left(c\left(y^j, 0, \underline{x}^N(0)\right)\right). \tag{26}$$

For $\tilde{r}$ close to -1, the right hand side is close to zero. By Lemma 2, the right hand side is greater than the left hand side for $\tilde{r} = 1/\beta - 1$. It is obvious that the right hand side is continuous and increasing in $\tilde{r}$. Therefore, there is a unique $\tilde{r}$ that solves equation (26), and agents' Euler inequalities are violated for higher values of r. □

The intuition behind this result is that whenever partial insurance occurs, the agent enjoying high consumption in the current period faces a weakly decreasing consumption path. Therefore, if a storage technology with sufficiently high return is available, the agent uses it for self-insurance purposes. We can also show that the threshold $\tilde{r}$ in Proposition 7 can be negative. In particular, we have shown that agents would use a storage technology with $r = 0$ under non-restrictive conditions. A necessary condition is that the consumption distribution is time-varying in the long run. The proofs of these results are available upon request.

3.2 The dynamics of individual consumptions revisited

We have shown in Section 2.3 that, introducing public storage, we overturn two counterfactual properties of consumption dynamics in the basic model, the amnesia and persistence properties. We can improve on the basic model with respect to a third aspect of the dynamics of consumption. In particular, the Euler inequality cannot be rejected in household survey data from developed economies, once household demographics and labor supply are appropriately accounted for (see Attanasio, 1999, for a comprehensive review of the literature). Since in our model with public storage agents' Euler inequalities are satisfied, while they are violated in the basic model, we bring limited commitment models in line with this third observation as well.

We have shown that two counterfactual properties of the dynamics of consumption in the basic limited commitment no longer hold in our model, namely, the amnesia and persistence properties. In addition, agents' Euler inequalities hold, as in the data. Would other extensions of the basic model yield the same improvements? Two components are necessary: (i) allowing for private storage, which makes sure that the Euler inequalities are satisfied, and (ii) an endogenous aggregate variable which makes aggregate consumption vary without aggregate income changing, for which the natural candidate is aggregate saving. Hence ours is the simplest extension to the basic limited commitment framework which delivers all three properties.

3.3 Welfare revisited

In Section 2.4 we have argued that access to public storage unambiguously reduces consumption dispersion and improves welfare. It is clear that hidden storage counteracts these benefits of storage, because it increases the value of agents' outside option, which in itself increases consumption dispersion and reduces welfare. The overall effects of access to both public and private storage are hence ambiguous in general, and depend on the return to storage, r. We first compare welfare at the long-run stationary distribution of our model with both public and private storage and the basic model without storage. Afterwards, we discuss the effects of the transition from the moment when storage becomes available.

In the following proposition we compare consumption dispersion and (equal-weighted) social welfare in the long-run steady state in two economies. In the first economy neither public nor private storage is available, in the second one both are available.

Proposition 8.

(i) There exists $\tilde{r}_1$ such that for all $r \in [-1, \tilde{r}_1]$ storage is not used even in autarky, therefore access to storage leaves consumption dispersion unchanged and is welfare neutral.

(ii) There exists $\tilde{r}_2 > \tilde{r}_1$ such that for all $r \in (\tilde{r}_1, \tilde{r}_2]$ storage is used in autarky but not in equilibrium, therefore consumption dispersion increases and welfare deteriorates as a result of access to storage.[19]

(iii) There exists $\tilde{r}_3 > \tilde{r}_2$ such that for all $r \in (\tilde{r}_2, \tilde{r}_3)$ public storage is (at least sometimes) positive, but access to storage is still welfare reducing and consumption dispersion is higher than in the basic model without storage. Access to storage is welfare neutral in the long run at the threshold $r = \tilde{r}_3$.

(iv) There exists $\tilde{r}_4 > \tilde{r}_3$ such that for all $r \in (\tilde{r}_3, \tilde{r}_4)$ access to storage is welfare improving in the long run, but consumption dispersion is still higher than in the basic model. Consumption dispersion is the same at the threshold $r = \tilde{r}_4$.

(v) For all $r \in (\tilde{r}_4, 1/\beta - 1]$ access to storage is welfare improving in the long run, and consumption dispersion is lower than in the basic model.

Proof. (i) It is easy to see that storage is never used when its return is close to -1, i.e., as long as it is below some threshold $\tilde{r}_1$. (ii) It is similarly easy to see that storage in equilibrium implies storage in autarky. This follows from the fact that the planner's and the agents' saving incentives are the same when income inequality is highest, i.e., when the incentive to store is highest, and agents' Euler inequality is more stringent in autarky than in equilibrium with some risk sharing. Then, if storage only takes place in autarky, the only effect of storage is that the value of agents' outside option increases, which reduces risk sharing and welfare. (iii) As r further increases to above the threshold $\tilde{r}_2$, according to Proposition 2 the planner finds public storage optimal. However, by continuity, at this point the negative effect of the increase in the value of autarky dominates the positive effect of the (small) stock of public assets on risk sharing. Therefore, welfare still goes down as a result of access to storage. (iv)-(v) If $r = 1/\beta - 1$, perfect risk sharing occurs and aggregate consumption is $Y + rB^*$

[19] Nothing changes as long as perfect risk sharing is self-enforcing. This happens for r sufficiently small when perfect risk sharing occurs without storage.

rather than Y, therefore welfare is strictly higher in the long run. Further, consumption dispersion is zero. Then, for any r in a small neighborhood of $1/\beta - 1$, the positive effect of the increase in aggregate consumption dominates the negative effect of the increase in the value of autarky, hence welfare improves. For such r, consumption dispersion is small. By continuity there exists $\tilde{r}_2 < \tilde{r}_3 < 1/\beta - 1$ where the two welfare levels are equalized. At this level of storage return, aggregate consumption has to be higher than in the basic model (at least after some histories). Hence, welfare can be the same only if consumption dispersion is higher than in the basic model. By continuity this should hold above $\tilde{r}_3$ as well until the threshold $\tilde{r}_4 \leq 1/\beta - 1$. □

Even when welfare improves in the long run, accumulating public assets has short-run costs, since it reduces aggregate consumption in the short run. This implies that the total gains (losses) from gaining access to storage are lower (higher) than those we have considered in Proposition 8. However, it is not clear whether access to both private and public storage will improves welfare. For this reason, we will explore this issue using numerical examples in Section 4.

4 Computed examples

In this section we solve for the constrained-efficient allocation in economies with limited commitment and access to public and private storage. As in Section 3, agents are allowed to store in autarky. We describe the algorithm we have applied in more detail in Appendix B. We show that aggregate storage can be significant in magnitude. We also illustrate how risk sharing, welfare, and the dynamics of consumption are affected by the availability of storage with different returns $-1 \leq r \leq 1/\beta - 1$.

We assume that agents' per-period utility function is of the CRRA form with a coefficient of relative risk aversion equal to 1, i.e., $u() = \ln()$. Income of both agents is i.i.d. over time, and may take three values, $\{0.2, 0.5, 0.8\}$, with equal probabilities. Income is perfectly negatively correlated across the two agents, hence aggregate income is 1 in all three states. We consider two discount factors, low ($\beta = 0.7$) and high ($\beta = 0.8$). In the former case risk sharing is partial without storage, however, the consumption distribution is time-invariant (i.e., the participation constraint of each agent binds only for the highest income level). In the latter case, perfect risk sharing occurs without access to storage. Note that this does not imply that public and private storage cannot be relevant as access to private storage increases the autarky values and may prevent full insurance with zero public assets. This triggers public asset accumulation if the return on storage is sufficiently high. In turn, public assets may bring the allocation close to perfect risk sharing again at a higher level of aggregate consumption. At the limit, when the return is as high as the discount rate, perfect risk occurs in the long run for any set of parameter values, see Proposition 2.

We present the simulation results on a few figures. First, let us look at the behavior of assets in the long run. Figure 4 shows the limits of the stationary distribution of assets, the first panel for $\beta = 0.7$ and the second for $\beta = 0.8$. Note the difference in scales in the two panels. Assets in the long run naturally increase with r. When the storage technology is efficient ($r = 1/\beta - 1$), assets reach at least 35.7 (38.2) percent of aggregate (non-asset) income in the long run when $\beta = 0.7$ ($\beta = 0.8$) (not represented). Depending on the history of shocks, assets may reach a higher level even if their initial level is zero.[20]

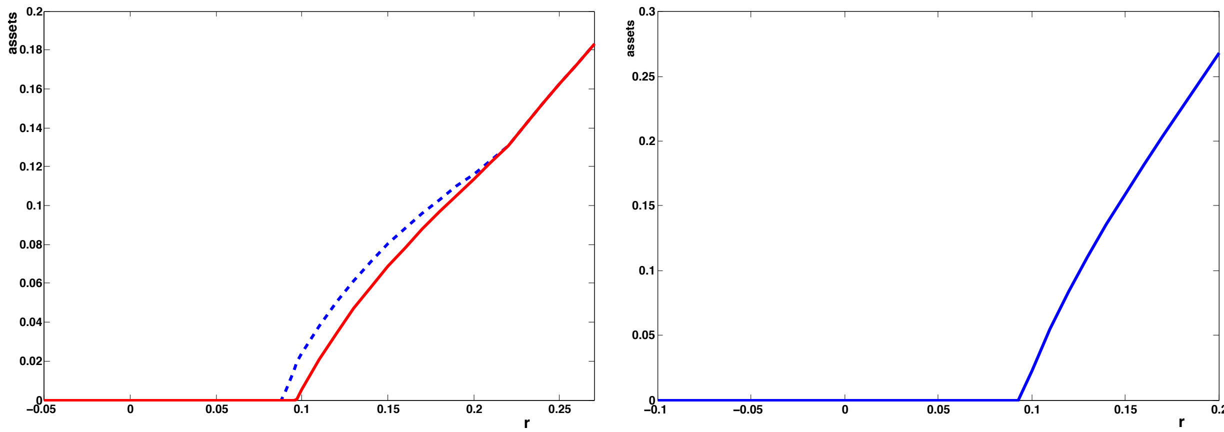
When the discount factor is high ($\beta = 0.8$), the participation constraints in state y^m do not bind in the long run, and assets always converge to a constant for any return on storage (case (iii) in Proposition 2). Public storage is positive for $r \geq 0.094$. For example, with $r = 0.16$ the planner's savings amount to 18.21 percent of aggregate (non-asset) income, while with $r = 0.11$ they are 5.49 percent.

When $\beta = 0.7$, for intermediate values of r the participation constraints bind in all three states, and assets remain stochastic in the long run (case (ii) in Proposition 2). Public storage is (sometimes) positive for $r \geq 0.089$. For example, with $r = 0.14$ public assets vary between 5.81 and 7.13 percent of aggregate (non-asset) income. When the interest rate is $r = 0.095$, assets vary between 0 and 1.47 percent. This last example shows that 0 can be part of the stationary distribution of assets when they are stochastic in the long run (see Proposition 3).

Figure 5 shows the possible long-run consumption values. Together with Figure 4, this figure reflects the different cases described in Propositions 2 and 8. If $\beta = 0.7$ ($\beta = 0.8$) for returns below $\tilde{r}_1 = -0.304$ ($\tilde{r}_1 = -0.416$) storage does not even affect the value of autarky and hence it is not used in equilibrium either. In this case, the allocation is not affected by the availability of storage. Given our parametrization, this implies that in the low patience case ($\beta = 0.7$) the consumption distribution has two values, while in the high patience case ($\beta = 0.8$) full risk sharing is enforceable. In fact, for $\beta = 0.8$, perfect risk sharing occurs in the long run for $r < -0.077$. As long as r is below $\tilde{r}_2 = 0.089$ ($\tilde{r}_2 = 0.094$) for β low (high), public storage is still not used, but storage increases the value of autarky, so consumption

[20] The lowest possible level of assets in the long run when $r = 1/\beta - 1$ is reached if $x = 1$. Depending on the history of shocks, perfect risk sharing may occur at a different x, and the higher cross-sectional consumption inequality is, the higher assets are in the long run. Assets reach the highest possible level if one of the unequal income states, y^l or y^h, occurs in every period starting from zero assets till perfect risk sharing becomes self-enforcing. This follows from Proposition 3.

Figure 4: Assets in the long run



(a) $\beta = 0.7$ (b) $\beta = 0.8$

Notes: The lower and upper limits of the stationary distribution of public assets. The two coincide when $\beta = 0.8$. The aggregate endowment is 1 in each period. Note the difference in scales in the two panels.

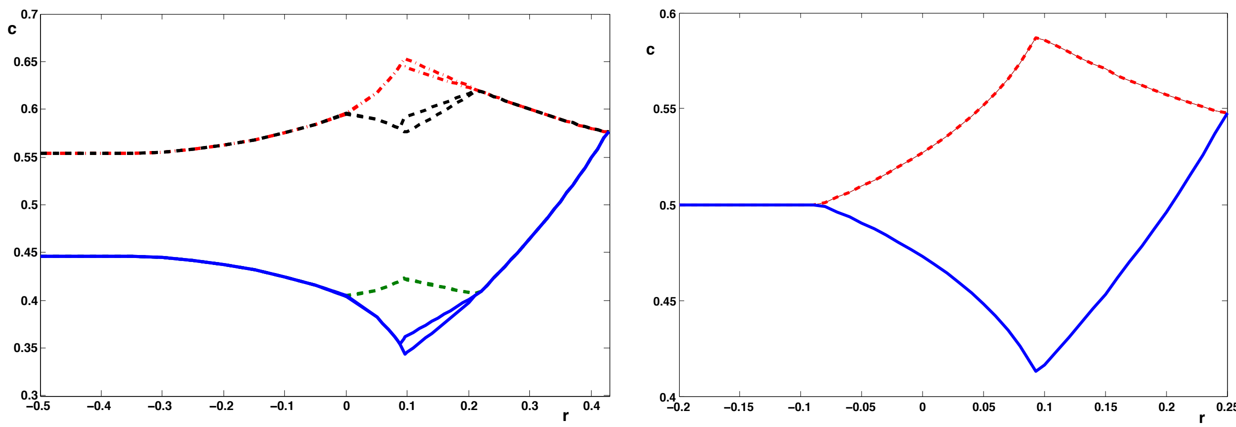
dispersion increases with the rate of return on storage.[21] For $r \geq \tilde{r}_2$, as r and aggregate asset accumulation increases, consumption dispersion declines until full risk sharing is achieved when $\beta(1+r) = 1$.

One important difference between the two cases is that with the low beta, at $r = 0.01$ the autarky values become such that a participation constraint binds in state y^m as well. For this reason, in Panel (a) of Figure 5, we see four consumption levels (as in Panel (b) of Figure 1) as long as public storage is not used. As the return reaches $r_1 = \tilde{r}_2 = 0.089$ public storage is used, and assets remains stochastic in the long run until $r_2 = 0.216$ (case (ii) in Proposition 2). This implies that in this case, even in the long run, consumptions not only depend on current income but also on the level of assets. For this reason, in Panel (a) of Figure 5, we have displayed the maximum and minimum levels of consumption for a given income state. Remember that in state y^m individual consumptions depend on which asymmetric state occurred last. Notice that for this parametrization, the stochasticity of assets has small effects on the levels and dispersion of consumption. At $r = 0.216$ the participation constraints stop binding in state y^m, and hence the consumption distribution becomes time-invariant and assets converge to a constant level.

Figure 6 shows long-run welfare expressed in per-period consumption equivalents. We have characterized long-run welfare in Proposition 8. When storage only increases the value

[21] For $\beta = 0.8$, the autarky value is affected already for a lower storage return, however at these levels full insurance is still enforceable.

Figure 5: Consumption in the long run



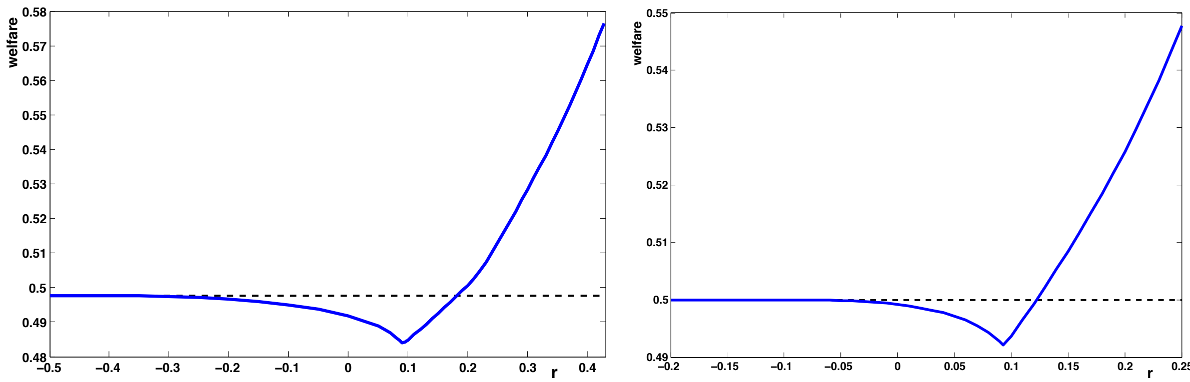
(a) $\beta = 0.7$ (b) $\beta = 0.8$

Notes: The lower and upper limits of the stationary distribution of consumption in different states. For $\beta = 0.7$, the solid (blue) lines are the limits for state y^l, the dashed (green) lines are the limits for state y^m when the last asymmetric state that occurred was state y^l (the difference between the lower and upper limit is too small to be visible on this figure), the higher dashed (black) lines are the limits for state y^m when the last asymmetric state that occurred was state y^h, and the dot-dashed (red) lines are the limits for state y^h. For $\beta = 0.8$, assets are never stochastic in the long run and consumption may take two values at most for all r. Note the difference in scales for the two panels.

of autarky, it decreases welfare. However, when the return is high enough so that it is used by the planner in equilibrium, it may increase welfare in the long run. When $\beta = 0.7$ the threshold return above which long-run welfare improves is $\tilde{r}_3 = 0.186$, when $\beta = 0.8$ it is 0.122. Note that at these thresholds, consumption dispersion is higher than in the case without storage, however aggregate consumption is also higher. As we approach the efficient level of storage, consumption dispersion disappears, hence welfare is always higher with than without storage in the long run. The welfare gain is equal to a 15 percent increase in consumption when $\beta = 0.7$, and is close to a 10 percent increase when $\beta = 0.8$.

Finally, we compute average welfare from the moment the storage technology becomes available. We do this to take into account the costs of asset accumulation. Figure 7 shows the results. In these two examples, access to both public and private storage lowers welfare for all r. The reason is that there are large welfare costs associated with the build-up of aggregate assets, and in our two examples these costs dominate the long-run gains. It is not clear how general this result is, and we leave this investigation to future work due to high computational costs. We know, however, that if perfect risk sharing is self-enforcing without private storage (as with $\beta = 0.8$), public storage is never positive even when it is

Figure 6: Welfare in the long run



(a) $\beta = 0.7$

(b) $\beta = 0.8$

Notes: The solid (blue) line shows long-run welfare per period in consumption-equivalent terms with both public and private storage. The dashed (black) line shows long-run welfare per period in consumption-equivalent terms without storage for reference. Note the difference in scales in the two panels.

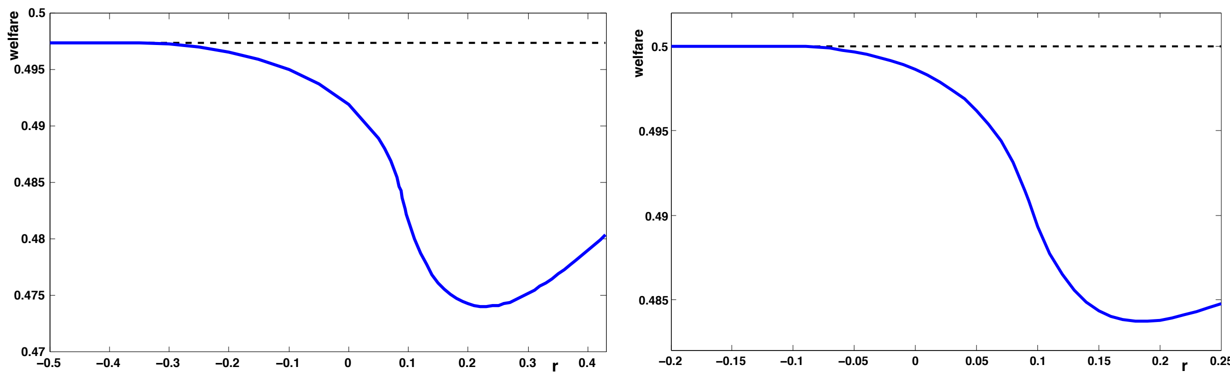
available. This implies that when we allow for private storage, the feasible set shrinks, and hence welfare deteriorates. Panel (b) of Figure 7 confirms this. With $\beta = 0.7$ risk sharing is partial without private storage. Here, public storage would be used and would surely improve welfare if private storage were not allowed. However, private storage reduces risk sharing by improving the outside options of agents. Hence, the overall effect could go either way. We do not see these results as a case against improving storage technologies. If we take hidden private storage unavoidable, then our results in Section 2 indicate that public storage certainly improves welfare.

5 Concluding remarks

This paper has shown that some implications of the basic limited commitment model with no private or public storage are not robust to hidden storage. When public storage is allowed though, the incentive for private storage is eliminated in the constrained-efficient allocation. The intertemporal technology is used in equilibrium even though the aggregate endowment is constant and the return is lower than the discount rate, i.e., $\beta(1 + r) < 1$. Further, when income inequality is not the highest, the planner has more incentive to store than the agents. The reason for additional storage by the planner is that public assets relax future participation constraints and hence improve risk sharing.

The effects of the availability of both public and private storage on asset accumulation,

Figure 7: Welfare including transition



(a) $\beta = 0.7$ (b) $\beta = 0.8$

Notes: The solid (blue) line shows expected lifetime utility in per-period consumption-equivalent terms from the moment when (both public and private) storage becomes available. The dashed (black) line shows expected lifetime utility in per-period consumption-equivalent terms without storage for reference. Note the difference in scales in the two panels.

consumption dispersion, and welfare depend on its return. In the long run, (i) for low r, access to storage is welfare neutral, because it is not used, hence we are back to the basic model of Kocherlakota (1996); (ii) for higher r, storage happens only in autarky, therefore, consumption dispersion increases and welfare decreases, but storage does not matter otherwise; (iii) for yet higher r, hidden storage matters in equilibrium in the basic model, public storage is (sometimes) positive, stochastic, and depends positively on consumption inequality as long as inverse marginal utility is convex, the consumption distribution is time-varying, and many consumption values occur;[22] (iv) for yet higher r, public storage becomes positive and constant in the long run, and only two consumption levels occur, i.e., the consumption distribution is time-invariant; (v) for $r = 1/\beta - 1$, public storage is positive and constant, and perfect risk sharing occurs. Long-run welfare improves above some threshold return, which is less than the discount rate. At the same time, there are short-run costs to accumulating assets. However, given access to private storage, public asset accumulation always reduces consumption dispersion and improves welfare.

The dynamics of individual consumptions are richer in our model compared to the basic model when assets are stochastic in the long run. In particular, the amnesia and persistence properties do not hold in general, which brings limited commitment models closer to the data (Broer, 2012). Further, in our model agents' Euler inequalities hold, which is consistent with

[22] This third case only occurs for some set of parameter values.

empirical evidence from developed countries (Attanasio, 1999).

Comparing our model with limited commitment and storage to models with hidden income or effort and storage (Allen, 1985; Cole and Kocherlakota, 2001; Ábrahám, Koehne, and Pavoni, 2011) points to some similarities and remarkable differences. In both models, hidden storage reduces welfare by imposing tighter constraints on risk sharing. In private information models, public storage cannot mitigate this effect and hence it is never used in equilibrium. In contrast, in our model public storage is used in equilibrium and welfare improves if its return is sufficiently high. This is because in our model storage by the planner relaxes the incentive problem, by relaxing future participation constraints; while in the hidden income/effort context aggregate asset accumulation makes incentive provision more expensive.

Our model could be applied in several economic contexts. The model predicts that risk sharing among households in villages can be improved by a public grain storage facility. Cooperation among partners in a law firm, for example, can be facilitated by common assets that someone quitting the partnership has no access to. Our model also provides a rationale for marriage contracts to specify that some commonly held assets are lost by the spouse who files for divorce. Finally, supranational organizations may help international risk sharing by simply having a jointly held stock of assets. The European Stability Mechanism may serve this purpose. Future work should study the quantitative implications of storage using some of these applications.

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Appendices

A Proofs

Proof of Proposition 1. We consider three income states for expositional reasons. Generalizing the proof to more income states is straightforward. Assume indirectly that $B'(B,\tilde{x}') = B'(B,\hat{x}') \equiv B'$.[23] This assumption and (10) imply that $u'(c(B,\tilde{x}')) < u'(c(B,\hat{x}'))$.

First, consider $\tilde{x}'$ and $\hat{x}'$ such that $\min\left\{\underline{x}^h(B'), \overline{x}^m(B')\right\} \geq \tilde{x}' > \hat{x}' \geq 1$. Let us rewrite (12) as

$$1 \geq \beta(1+r)\sum_{y'} \Pr(y') \frac{u'(c(y',B',x'))}{u'(c(B,x'))(1-\upsilon_1(y',B',x'))}. \tag{27}$$

We now detail what happens next period, so that we can compare the right hand side of (30) for $\tilde{x}'$ and $\hat{x}'$.

- If state y^h occurs, then the participation constraint of agent 1 is binding. Given that B' is the same for $\tilde{x}'$ and $\hat{x}'$ under our indirect assumption, x'' will equal $\underline{x}^h(B')$ for both and c' will equal $\underline{c}^h(B')$ for both. However, the ratio on the right hand side of (30) differs because $\upsilon_1(y',B',\tilde{x}') < \upsilon_1(y',B',\hat{x}')$. For $x' = \{\hat{x}',\tilde{x}'\}$ we obtain

$$\frac{u'\left(\underline{c}^h(B')\right)}{u'(c(B,x'))(1-\upsilon_1(y',B',x'))} = \frac{u'\left(\overline{c}^l(B')\right)}{u'(c_2(B,x'))}.$$

  where we have combined (10) and (11).

- If state y^m occurs, then no participation constraint is binding, hence the relative Pareto weight does not change. For HARA utility functions, it can be shown using simple algebra that each agent's marginal utility grows at the rate $((2a+c'+c_2')/(2a+c+c_2))^{-\sigma}$, hence we know that in this case

$$\frac{u'(c(B',\tilde{x}'))}{u'(c(B,\tilde{x}'))} = \frac{u'(c(B',\hat{x}'))}{u'(c(B,\hat{x}'))}.$$

- If state y^l occurs, then the participation constraint of agent 2 is binding. Given that B' is the same for $\tilde{x}'$ and $\hat{x}'$, x'' will equal $\overline{x}^l(B')$ and c' will equal $\overline{c}^l(B')$ for both. Thus for $x' = \{\hat{x}',\tilde{x}'\}$, we have

$$\frac{u'\left(\overline{c}^l(B')\right)}{u'(c(B,x'))}.$$

[23] If we assume indirectly that $B'(B,\tilde{x}') \leq B'(B,\hat{x}')$ for $\tilde{x}' > \hat{x}' \geq 1$ and $B'(B,\tilde{x}') \geq B'(B,\hat{x}')$ for $1 \geq \tilde{x}' > \hat{x}$, the steps of the proof are the same, but the algebra is more tedious.

In summary, for $x' = \{\hat{x}', \tilde{x}'\}$ on the right hand side of (30) we have

$$\beta(1+r)\left[\pi^e \frac{u'\left(\overline{c}^l(B')\right)}{u'\left(c_2\left(B,x'\right)\right)} + \pi^m \frac{u'\left(c\left(B',x'\right)\right)}{u'\left(c\left(B,x'\right)\right)} + \pi^e \frac{u'\left(\overline{c}^l(B')\right)}{u'\left(c\left(B,x'\right)\right)}\right],$$

where $\pi^e = \pi^h = \pi^l$. If this expression is greater for $\tilde{x}'$ than for $\hat{x}'$, then our indirect assumption is invalidated and B' has to be greater for $\tilde{x}'$ than for $\hat{x}'$ to satisfy (30). The second term is the same in the two expressions. Therefore, the sign of the difference is the sign of

$$\Delta_1\left(B,\tilde{x}',\hat{x}'\right) \equiv \frac{1}{u'\left(c_2\left(B,\tilde{x}'\right)\right)} + \frac{1}{u'\left(c\left(B,\tilde{x}'\right)\right)} - \left(\frac{1}{u'\left(c_2\left(B,\hat{x}'\right)\right)} + \frac{1}{u'\left(c\left(B,\hat{x}'\right)\right)}\right).$$

Given that $\tilde{x}' > \hat{x}' \geq 1$ implies $c_2\left(B,\tilde{x}'\right) < c_2\left(B,\hat{x}'\right) \leq c\left(B,\hat{x}'\right) < c\left(B,\tilde{x}'\right)$ by Assumption 1, this difference is strictly positive if $1/u'$ is strictly convex. So under this condition, B' is strictly increasing in x' in the case where $\min\left\{\underline{x}^h\left(B'\right), \overline{x}^m\left(B'\right)\right\} \geq x' \geq 1$.

Second, consider $\tilde{x}'$ and $\hat{x}'$ such that $\tilde{x}' > \hat{x}' \geq \overline{x}^m$.

- If state y^l occurs next period, nothing changes compared to the previous case, where $\min\left\{\underline{x}^h\left(B'\right), \overline{x}^m\left(B'\right)\right\} \geq \tilde{x}' > \hat{x}' \geq 1$.
- For state y^m the difference between the ratio on the right hand side of (30) for $\tilde{x}'$ and $\hat{x}'$ is

$$\Delta_2\left(B,B',\tilde{x}',\hat{x}'\right) \equiv \frac{u'\left(\overline{c}^m\left(B'\right)\right)}{u'\left(c\left(B,\tilde{x}'\right)\right)} - \frac{u'\left(\overline{c}^m\left(B'\right)\right)}{u'\left(c\left(B,\hat{x}'\right)\right)} > 0.$$

- In state y^h three cases are possible.
  - The participation constraint of agent 1 is binding for both $\tilde{x}'$ and $\hat{x}'$. Then we can use the previous case. Note that the difference between the right hand sides of (30) for $\tilde{x}'$ and $\hat{x}'$ is given by $\Delta_1\left(B,\tilde{x}',\hat{x}'\right) + \Delta_2\left(B,B',\tilde{x}',\hat{x}'\right) > 0$.
  - The participation constraint of agent 1 is not binding for either x'. Then the growth rate of marginal utility is the same for $\tilde{x}'$ and $\hat{x}'$. In this case, the difference between the right hand sides of (30) for $\tilde{x}'$ and $\hat{x}'$ is given by

$$\pi^e u'\left(\overline{c}^l\left(B'\right)\right)\left(\frac{1}{u'\left(c\left(B,\tilde{x}'\right)\right)} - \frac{1}{u'\left(c\left(B,\hat{x}'\right)\right)}\right) + \Delta_2\left(B,B',\tilde{x}',\hat{x}'\right) > 0.$$

  - The participation constraint of agent 1 is binding for $\hat{x}'$, but not for $\tilde{x}'$. Then $c_2\left(B',\tilde{x}'\right) < \overline{c}^l(B')$. Therefore, the difference between the right hand sides of (30) for $\tilde{x}'$ and $\hat{x}'$ is given by

$$\Delta_1\left(B,\tilde{x}',\hat{x}'\right) + \Delta_2\left(B,B',\tilde{x}',\hat{x}'\right) + \frac{u'\left(c_2\left(B',\tilde{x}'\right)\right) - u'\left(\overline{c}^l(B')\right)}{u'\left(c_2\left(B,\tilde{x}'\right)\right)} > 0.$$

Finally, consider $\tilde{x}'$ and $\hat{x}'$ such that $\tilde{x}' \geq \overline{x}^m(B') > \hat{x}'$. The only difference compared to the previous case is in state y^m. We have $c(B', \hat{x}') < \overline{c}^m(B')$. This implies that

$$\Delta_3(B, B', \tilde{x}', \hat{x}') = \frac{u'(\overline{c}^m(B'))}{u'(c(B, \tilde{x}'))} - \frac{u'(c(B', \hat{x}'))}{u'(c(B, \hat{x}'))} > 0.$$

Hence the same argument as in the previous case follows replacing $\Delta_2(B, B', \tilde{x}', \hat{x}')$ with $\Delta_3(B, B', \tilde{x}', \hat{x}')$.

Since the problem is symmetric, to establish the relationship between B' and $x' \leq 1$, we can consider $1/x' \geq 1$. This means that B' increases as $x' \leq 1$ decreases, i.e., as cross-sectional consumption inequality increases.

From Lemma 1 we know that $B'(y^j, B, x) = B'(B, x')$. If $j > k$, and the optimal intervals for these two states do not overlap given B, then x' must be higher in state y^j than in state y^k, and we have already shown that assets depend positively on cross-sectional consumption inequality. If the optimal intervals overlap given B, then there exists x for which $x' = x$ in both states y^j and y^k. Aggregate savings are identical in the two states in this case. $\square$

Proof of Proposition 2. Part (i). It is easy to see that r_1 is implicitly defined by the planner's Euler, (12), with equality when agent 1 has the highest possible income. That is, r_1 is implicitly given by

$$u'\left(c\left(y^N, 0, \underline{x}^N(0)\right)\right) = \beta(1+r_1)\sum_j \pi^j \frac{u'\left(c\left(y^j, 0, \underline{x}^N(0)\right)\right)}{1-\upsilon_1\left(y^j, 0, \underline{x}^N(0)\right)},$$

If $r > r_1$ public assets will be positive at least when income inequality is highest, while if $r \leq r_1$ public assets will be zero in the long run.

Next, we show that assets are bounded in the long run, which we need for parts (ii)-(iv). It is easy to see that there exists a high level of inherited assets, denoted $\widehat{B}$, such that perfect risk sharing is at least temporarily enforceable, that is, $\overline{x}^1\left(\widehat{B}\right) \geq \underline{x}^N\left(\widehat{B}\right)$. Therefore, if $r < 1/\beta - 1$, $B'(B, x') < B$ for all $B \geq \widehat{B}$ and $\overline{x}^1(B) \geq x' \geq \underline{x}^N(B)$, i.e., assets optimally decrease; and assets stay constant if $r = 1/\beta - 1$. This implies that assets are bounded above in the long run.

We now turn to parts (ii) and (iii). We first show that if the consumption distribution is time-invariant, then there exists a unique constant level of assets, B^*, such that all the conditions of constrained-efficiency are satisfied. Afterwards, we show that assets converge almost surely to B^* starting from any initial level, B_0. Then, we establish that assets remain stochastic when the consumption distribution is time-varying (case (ii)). Finally, we show that case (iii) occurs when the return on storage is high but less than the discount rate, while assets remain stochastic when the return is below some threshold, denoted r_2.

Recall that if aggregate assets are constant, the optimal intervals for the relative Pareto weight are time-invariant. Given that each agent's participation constraint binds only for the highest income level in the long run, the optimality condition (10) and $\underline{x}^N(B^*)$ $(\overline{x}^1(B^*))$ uniquely determine $\underline{c}^N(B^*)$ $(\overline{c}^1(B^*))$, the time-invariant high (low) consumption level. Then, using the planner's Euler, we can determine the unique level of B^* such that all optimality conditions are satisfied. The planner's Euler is

$$u'\left(\underline{c}^N(B^*)\right) = \beta(1+r)\left[(1-\pi^e)\,u'\left(\underline{c}^N(B^*)\right) + \pi^e u'\left(\overline{c}^1(B^*)\right)\right],$$

where $\pi^e = \pi^N = \pi^1$. Dividing both sides by $u'\left(\underline{c}^N(B^*)\right)$, we obtain

$$\begin{aligned} 1 &= \beta(1+r)\left[(1-\pi^e) + \pi^e \frac{u'\left(c^l(B^*)\right)}{u'\left(c^h(B^*)\right)}\right] \\ &= \beta(1+r)\left[(1-\pi^e) + \pi^e \underline{x}^N(B^*)\right], \end{aligned} \tag{28}$$

where we have used (10). Note that $\underline{x}^N(B^*)$ is monotone and continuous in B^*. Further, at $B^* = 0$ the right hand side of equation (28) is larger than 1 by assumption, and at $B^* = \widehat{B}$ the right hand side of (28) is smaller than 1, because $\underline{x}^N(\widehat{B}) = 1$ and $B^* < \widehat{B}$. Therefore, we know that there exists a unique B^* where the planner's Euler holds with equality by setting $B' = B = B^*$.

Next, we show that assets converge almost surely to B^* starting from any initial level, B_0. We already know that $B'(B_0, x') < B_0$ for the ergodic range of x' when $B_0 \geq \widehat{B}$, i.e., when perfect risk sharing is (temporarily) self-enforcing, and $B'(0, x') > 0$ for some x' in the ergodic range of x', since $r > r_1$ by assumption. Consider $B^* < B_0 < \widehat{B}$ first, and assume that state y^N occurs and agent 1's participation constraint is binding. This is without loss of generality, because this happens with probability 1 in the long run, and the problem is symmetric across the two agents. We know that the right hand side of (28) is smaller than 1, because $\underline{x}^N(B_0) < \underline{x}^N(B^*)$. Therefore, marginal utility tomorrow has to increase relative to marginal utility today to satisfy the planner's Euler, therefore $B'(B_0) < B_0$. What happens next period? The participation constraint will bind again even if the same state occurs.[24] This is because $B'(B_0) < B_0$ implies $\underline{x}^N(B'(B_0)) > \underline{x}^N(B_0)$. Then assets will decrease again. What if some state y^j with $2 \leq j \leq N-1$ occurs? We know that the participation constraints in these states are not binding for any $B \geq B^*$, because they are not binding for B^*. This means that now $x' = x = \underline{x}^N(B_0) < \underline{x}^N(B'(B_0))$. Then, by Proposition 1, storage is lower than when the participation constraint is binding. Note that if states $y^2, ..., y^{N-1}$ occur

[24] Note that this never happens in the basic model.

repeatedly, assets converge to a level below B^*. Then we are in the case where $B_0 < B^*$, which we now turn to.

Consider $0 \leq B_0 < B^*$, and suppose again that state y^N occurs and agent 1's participation constraint is binding. We know that $\underline{x}^N(B_0) > \underline{x}^N(B^*)$ in this case. Using (28) again, it follows that $B'(B_0) > B_0$. Now, if the same state occurs next period (in fact, any state y^j with $j \geq 2$), then the participation constraint is slack. This means that now $x' = x = \underline{x}^N(B_0) > \underline{x}^N(B'(B_0))$. Then, by Proposition 1, storage is higher than when the participation constraint is binding. This also implies that if state y^1 does not occur for many periods, assets converge to a level above B^*. Then once y^1 occurs, which happens with probability 1 in the long run, we are back to the case $B_0 > B^*$, and assets start decreasing.[25]

So far, we have shown that when $B_0 < B^*$, assets increase. Unless we are on a path when agents are get the highest income shock exactly in turns, assets converge towards a level higher than B^*. We have also shown that whenever $B_0 > B^*$ and an agent's participation constraint binds, asset decrease. Again, unless one of the agents always receives the highest shock, assets converge to a value lower than B^*. This implies that assets oscillate around B^*. Almost sure convergence is guaranteed because these oscillations shrink whenever a participation constraint binds in the increasing and/or decreasing part. This happens with probability one.

Part (ii). Consider the case where in the long run there is a third state in which a participation constraint binds. In this case, each agent's consumption takes at least four different values in the long run. These have to satisfy an additional participation constraint, an additional resource constraint, and an additional Euler, which is generically impossible for constant B.

Finally, we have to show that case (ii) occurs if $r_1 < r \leq r_2$, while case (iii) occurs if $r_2 < r < 1/\beta - 1$. It is easy to see that B^* is lower if r is lower, where B^* can be computed for any r ignoring the participation constraints of states y^j with $2 \leq j \leq N-1$. However, as assets decrease, the optimal intervals become narrower, and eventually $\overline{c}^2(B) < \underline{c}^N(B)$ and $\overline{c}^1(B) < \underline{c}^{N-1}(B)$. Hence, r_2 is implicitly given by (28) such that B^* is such that $\overline{x}^2(B^*) = \underline{x}^N(B^*)$ (and $\overline{x}^1(B^*) = \underline{x}^{N-1}(B^*)$).

Part (iv). If $\beta(1+r) = 1$, the only way to satisfy agents' Euler inequalities in all states is to provide them with a perfectly smooth consumption stream over time. Further, as long as a participation constraint binds given B, the planner has an incentive to store more,

[25] Participation constraints in more states may be binding when B is low, even if they only bind in states y^1 and y^N for B^*. However, with probability 1 assets will reach a level where the participation constraints of the other states are no longer binding.

because she does not face a trade-off between improving risk sharing and using an inefficient intertemporal technology. $\square$

Proof of Proposition 3. From Proposition 1 it is clear that $\underline{B}$ is approached if a least unequal income state, denoted y^m, happens repeatedly, while $\overline{B}$ is approached with state y^N (or y^1) happening many times in a row.

If B is part of the stationary distribution, then it must be that $B \geq \underline{B}$. This means that there are less and less resources available over time while assets approach $\underline{B}$, hence the relevant participation constraint always binds along this path. The planner's Euler,

$$u'\left(\overline{c}^m\left(\underline{B}\right)\right) \geq \beta(1+r)\left[\pi^e u'\left(c\left(y^l, \underline{B}, \overline{x}^m(\underline{B})\right)\right) + (1-2\pi^e)\, u'\left(\overline{c}^m\left(\underline{B}\right)\right) \right.$$
$$\left. + \pi^e u'\left(c\left(y^h, \underline{B}, \overline{x}^m(\underline{B})\right)\right)\right],$$

as equality defines $\underline{B}$ if $\underline{B} > 0$. If at $\underline{B} = 0$ this Euler is satisfied as a strict inequality, then the lower bound is 0.

The upper limit of the stationary distribution, $\overline{B}$, is approached from below, hence, along that path, the highest shock (state y^N or y^1) happens repeatedly and no participation constraint binds. Let B_1 denote the level of inherited assets when we switch to state y^N (or y^1), and let $\widetilde{B}$ denote the level of assets to where B converges. Note that along this path the relative Pareto weight is constant at $\underline{x}^N(B_1)$. Given B_1, $\widetilde{B}$ is the solution to the following system:

$$\frac{u'\left(c_2\left(y^N, \widetilde{B}, \underline{x}^N(B_1)\right)\right)}{u'\left(c\left(y^N, \widetilde{B}, \underline{x}^N(B_1)\right)\right)} = \underline{x}^N(B_1)$$

$$c\left(y^N, \widetilde{B}, \underline{x}^N(B_1)\right) + c_2\left(y^N, \widetilde{B}, \underline{x}^N(B_1)\right) = Y + r\widetilde{B}$$

$$u'\left(c\left(y^N, \widetilde{B}, \underline{x}^N(B_1)\right)\right) = \beta(1+r)\sum_{j=1}^{N} \pi^j u'\left(c\left(y^j, \widetilde{B}, \underline{x}^N(B_1)\right)\right). \tag{29}$$

We have to find B_1 such that $\widetilde{B}$ is equal to $\overline{B}$, the upper limit of the stationary distribution. Using Proposition 1, we know that $B'(B, \underline{x}^N(B_1))$ is highest when $\underline{x}^N(B_1)$ is highest. In turn, $\underline{x}^N(B_1)$ is highest when B_1 is lowest, i.e., when B_1 is equal to the lower limit of the stationary distribution of assets, $\underline{B}$. Then, replacing $\underline{x}^N(B_1)$ with $\underline{x}^N(\underline{B})$ and $\widetilde{B}$ with $\overline{B}$ in (29) gives (20). $\square$

Partial proof of Conjecture 1. If $B'(B, \tilde{x}') \leq B'(B, \hat{x}')$ then this is trivial from (10). Consider now the case where $B'(B, \tilde{x}') > B'(B, \hat{x}')$. Given $^1/_{u'}$ convex, by Proposition 1 this can only happen if $\tilde{x}' > 1$ and $\hat{x}' > 1/\tilde{x}'$. We first show that if c' is weakly increasing in x'' next period, then c is strictly increasing in x' in the current period.

Let us rewrite (12) as equality and make explicit the dependence on x':

$$u'\left(c\left(x', B\right)\right) = \beta(1+r)\sum_{y'} \Pr\left(y'\right) \frac{u'\left(c\left(x''(y', x', B'), B'(x', B)\right)\right)}{1-\upsilon_1\left(y', x', B'(x', B)\right)}. \tag{30}$$

Taking derivatives with respect to x' and dropping function arguments for readability give

$$u''(c)\frac{\partial c}{\partial x'} = \beta(1+r)\sum_{y'} \Pr\left(y'\right)\left[\frac{u''(c')\left(\frac{\partial c'}{\partial x''}\frac{\Delta x''}{\Delta x'} + \frac{\partial c'}{\partial B'}\frac{\partial B'}{\partial x'}\right)}{1-\upsilon_1} + \frac{u'(c')}{(1-\upsilon_1)^2}\left(\frac{\partial \upsilon_1}{\partial x'} + \frac{\partial \upsilon_1}{\partial B'}\frac{\partial B'}{\partial x'}\right)\right].$$

We have assumed that $\frac{\partial B'}{\partial x'} > 0$, else the result is trivial. υ_1 is decreasing with both x' and B'. Also, $\frac{\Delta x''}{\Delta x'} \geq 0$. Hence, if $\frac{\partial c'}{\partial x''} > 0$, then $\frac{\partial c}{\partial x'} > 0$.

Proposition 2 shows that assets converge to a constant level in the long run almost surely if r is higher than some threshold r_2. That is, in the long run the characteristics of allocations are the same as in the basic model (while aggregate consumption is $Y + rB$ rather than Y), in particular, c strictly increases with x'. Then, moving backwards in time, c must strictly increase with x' in all periods.

Now, consider an $r < r_2$ in a small neighborhood of r_2. Since c is strictly increasing in x' for r_2, c must be at least weakly increasing in x' for r sufficiently close to r_2 by continuity. Then we know that in the previous period c is strictly increasing in x'. Now if the original B is part of the stationary distribution, then it will occur other times as well, so c must be strictly increasing in x' there too. Similarly, we can consider $r > r_1$ in a small neighborhood of r_1, where r_1 is the threshold below which zero public storage is optimal.

We then verify numerically that c is strictly increasing in x' for all $r \in (r_1, r_2)$ as well. $\square$

B Computation

We use the recursive system given by equations (10)-(15) to solve the model numerically. We discretize x and B (y is assumed to take a finite number of values). We have to determine x' and B' on a 3-dimensional grid on $X = (y, B, x)$. The initial values for $V(X')$, $c(X')$, and $\upsilon_1(X')$ are from the solution of a model where the participation constraints are ignored. We iterate until the value and policy functions converge.

As we proceed, we use the characteristics of the solution. In particular, we know that if agent 1's participation constraint binds at $\tilde{x}$, it also binds at all $x < \tilde{x}$. Similarly, if agent 2's participation constraint binds at $\hat{x}$, it also bind at all $x > \hat{x}$. At each iteration, at each income state and for each B, we solve directly for the limits $\tilde{x}$ and $\hat{x}$ using (13) and (14) with equality, respectively, first assuming that $B' = 0$. Afterwards, we check whether the planner's Euler is satisfied at the limits. If not, we solve a 2-equation system of (12) and (13) (or (14)), with unknowns B' and x'. Finally, we solve for a new B' at points on the x grid where neither participation constraint binds, i.e., at the interior of the optimal interval for (y, B) of the current iteration.

C Validity of the first-order condition approach

In order to verify that agents have no incentive to use 'store first and default later'-type of double deviations, we verify that given any level of hidden assets, public assets, incomes, and the inherited relative Pareto weight, agents are better off receiving as endowment the consumptions assigned by the constrained-efficient risk sharing contract rather than their own incomes today and in the future. In order to see this, along with the autarky consumption-saving problem, we solve the consumption-saving problem of an agent who receives the constrained-efficient consumption process as 'income.' Having computed the constrained-efficient policy functions as described in Appendix B, this is without conceptual difficulty, however, the computational cost is rather high, given that there are four state variables, three of which are continuous. We again exploit the characteristics of the solution, namely that the current Pareto weight takes values within an optimal state-dependent interval, in order to shorten computation time.

In examples we have studied, we find that agents are always better off receiving the constrained-efficient consumptions given any level of already accumulated private assets rather than the autarky incomes. Hence, they will never revert to autarky, as long as the first-order conditions are satisfied.