DISCRETE CHOICE CANNOT GENERATE DEMAND THAT IS ADDITIVELY SEPARABLE IN OWN PRICE

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ABSTRACT. We show that in a unit demand discrete choice framework with at least three goods, demand cannot be additively separable in own price. This result sharpens the analogous result of Jaffe and Weyl (2010), which ruled out linear demand. It has implications for testing of the discrete choice assumption, out-of-sample prediction, and welfare analysis.

1. INTRODUCTION

Demand is frequently an aggregation of "discrete choices" in which each consumer chooses at most one good from among a set of available options. Economists sometimes microfound aggregate demand in individual choice models.[1] However, because the distribution of individual consumers' preferences is typically unknown, the functional form of demand is often not based upon an aggregation over individuals.

In this note, we show that abstracting away from the discrete choice basis of demand is not justified, as a large class of functional forms cannot be generated by an aggregation of individuals' discrete choices. Specifically, extending results of Jaffe and Weyl (2010) for the case of linear demand, we show in Section 3 that if individual-level choice is discrete among more than two options, then demand cannot be additively separable in own price. Thus in addition to linear demand, we rule out demand forms of the type used by Bulow et al. (1985).

Our main theorem sharpens the Jaffe and Weyl (2010) answer to the Anderson et al. (1989) question of whether discrete choice can generate linear demand: we show that discrete choice demand must exhibit interaction between own price and other prices.[2] Additionally, as we discuss in Section 4, our results have implications for testing of the discrete choice assumption, out-of-sample prediction, and welfare analysis.

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[1] Commonly-used microfounded discrete choice models include logit and BLP; see Anderson et al. (1992) and Berry et al. (1995).

[2] While Jaffe and Weyl (2010) ruled out the possibility that demand for product i could take the form $D^i(p) = \beta_0 - \beta_i p_i + \sum_{k \neq i} \beta_k p_k$, they left open the possibility that demand for product i could take the form $D^i(p) = \beta_0 - \beta_i p_i + H(p_{-i})$, which we rule out.

2. Intuition

To see the intuition behind our result, consider Figure 1. In this figure, consumers with valuations in the lower-right region ($\mathfrak{D}^1$) demand good 1 and those in the upper-left region ($\mathfrak{D}^2$) demand good 2.

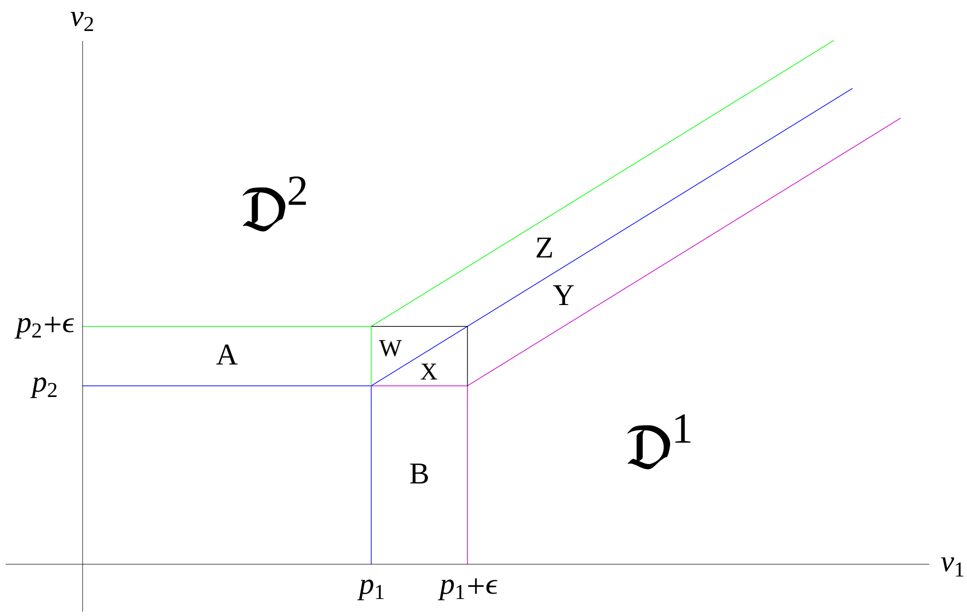

Figure 1. The case of two goods.

When prices are (p_1, p_2) and firm 1 raises its price by ϵ, it loses the demand of consumers with valuations in the regions B, X, and Y. Meanwhile, when prices are $(p_1, p_2+\epsilon)$ and firm 1 raises its price by ϵ, it loses the regions B, X, W, and Z. If demand is additively separable, the induced changes in demand for good 1 following an increase in p_1 of ϵ must be independent of p_2. Thus, denoting by $\tilde{R}$ the mass of consumers in region R, we have

$$0 = \tilde{B} + \tilde{X} + \tilde{W} + \tilde{Z} - (\tilde{B} + \tilde{X} + \tilde{Y}) = \tilde{W} + \tilde{Z} - \tilde{Y}. \tag{1}$$

Analogously, if firm 2 raises its price by ϵ when prices are (p_1, p_2) it loses regions A, W, and Z; when prices are $(p_1+\epsilon, p_2)$ it loses regions A, W, X, and Y. We therefore see that

$$0 = \tilde{A} + \tilde{W} + \tilde{X} + \tilde{Y} - (\tilde{A} + \tilde{W} + \tilde{Z}) = \tilde{X} + \tilde{Y} - \tilde{Z}. \tag{2}$$

Adding equations (1) and (2) gives $\tilde{W} + \tilde{X} = 0$. As $\epsilon \to 0$, this corresponds to the requirement that $f(p_1, p_2) = 0$. Thus, demand for goods 1 and 2 cannot be additively separable at (p_1, p_2) unless there is a "gap" in the distribution of valuations at those prices.

3. Main Result

We consider a market with N goods indexed $i = 1, \ldots, N$ and a unit mass of consumers. The prices of these goods are $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$. Each consumer has valuations for the N goods specified by the vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$; valuations v are distributed according to a continuously differentiable density function f. An *outside option* good, for which all consumers have value 0, is available; the price of the outside option is fixed at 0.[3] Each consumer purchases the good i which maximizes $v_i - p_i$, or chooses the outside option if $v_i - p_i$ is negative for all $i = 1, \ldots, N$. Demand for good i is therefore given by

$$D^i(p) = \int_{p_i}^{\infty} \int_{\prod_{j \neq i} S_{j,i}} f(v) dv_{-i} dv_i,$$

where $S_{j,i} = (-\infty, p_j + v_i - p_i]$ is the domain of valuations v_j for which $v_i - p_i > v_j - p_j$, i.e. the domain for which good i is superior to good j.

We say that demand for good i is *additively separable in own price* if there exist functions G and H such that $D^i(p)$ takes the form $D^i(p) = G(p_i) + H(p_{-i})$. Equivalently, demand for good i is additively separable in own price if $\frac{\partial^2 D^i(p)}{\partial p_i \partial p_k} = 0$ everywhere, for all $k \neq i$. We say the demand system is additively separable in own price if, for each i, demand for good i is additively separable in own price.

Theorem. *Suppose that $N \geq 2$ and that f has full support. Then, the demand system cannot be additively separable in own price.*

As we show in Appendix A,

$$2 \sum_i \sum_{k \neq i} \frac{\partial^2 D^i(p)}{\partial p_k \partial p_i} \leq - \sum_i \sum_{k \neq i} \int_{\prod_{j \neq i,k} S_{j,i}} f(v_{-i,k}, p_i, p_k) dv_{-i,k}, \tag{3}$$

which is strictly negative (implying non-separability) whenever the integral is non-zero for some i and k. The assumption that f have full support is sufficient for this conclusion, but is clearly not necessary.[4]

To outline our approach and expand on the intuition presented above, we now prove our theorem in the the case $N = 2$.

Proof in the case $N = 2$. Demand for good 1 is $D^1(p) = \int_{p_1}^{\infty} \int_{-\infty}^{p_2 + v_1 - p_1} f(v_1, v_2) dv_2 dv_1$; its cross-price derivative is $\frac{\partial D^1(p)}{\partial p_2} = \int_{p_1}^{\infty} f(v_1, p_2 + v_1 - p_1) dv_1$. Analogous expressions apply for $D^2(p)$. This gives the following second derivatives:

$$\frac{\partial^2 D^1(p)}{\partial p_1 \partial p_2} = -f(p_1, p_2) - \int_{p_1}^{\infty} f_2(v_1, p_2 + v_1 - p_1) dv_1 \tag{4}$$

$$= -f(p_1, p_2) - \int_{p_2}^{\infty} f_2(p_1 + v_2 - p_2, v_2) dv_2 \tag{5}$$

$$\frac{\partial^2 D^2(p)}{\partial p_2 \partial p_1} = -f(p_1, p_2) - \int_{p_2}^{\infty} f_1(p_1 + v_2 - p_2, v_2) dv_2 \tag{6}$$

$$= -f(p_1, p_2) - \int_{p_1}^{\infty} f_1(v_1, p_2 + v_1 - p_1) dv_1. \tag{7}$$

[3] As we demonstrate in Appendix B, if consumers are required to buy exactly one good $i \in \{1, \ldots, N\}$, an analog of our main theorem holds when $N > 2$. Thus, our result does not rely on the existence of the outside option except in the case $N = 2$. (As we discuss in Section 4, the well-known result of Hotelling (1929) relies on the fact that $N = 2$ and there is no outside option.)

[4] For example, if f has compact support, then the theorem applies to prices in the interior of $\operatorname{supp}(f)$.

Summing expressions (4)–(7) and rearranging terms, we obtain

$$\begin{aligned}
2\left(\frac{\partial^2 D^1(p)}{\partial p_1 \partial p_2}+\frac{\partial^2 D^2(p)}{\partial p_2 \partial p_1}\right) &= -4f(p_1,p_2) - \int_{p_1}^{\infty}(f_1(v_1,p_2+v_1-p_1)+f_2(v_1,p_2+v_1-p_1))dv_1 \\
&\qquad - \int_{p_2}^{\infty}(f_1(p_1+v_2-p_2,v_2)+f_2(p_1+v_2-p_2,v_2))dv_2 \\
&= -4f(p_1,p_2) - \int_{p_1}^{\infty}\frac{\partial}{\partial v_1}(f(v_1,p_2+v_1-p_1))dv_1 \\
&\qquad - \int_{p_2}^{\infty}\frac{\partial}{\partial v_2}(f(p_1+v_2-p_2,v_2))dv_2 \\
&= -2f(p_1,p_2). \qquad (8)
\end{aligned}$$

As long as there is not a gap in the distribution of valuations at (p_1, p_2), $-2f(p_1, p_2)$ is strictly negative. In that case, then, $\frac{\partial^2 D^1(p)}{\partial p_1 \partial p_2}+\frac{\partial^2 D^2(p)}{\partial p_2 \partial p_1}$ cannot vanish, hence demand for goods 1 and 2 cannot be additively separable. □

The $N > 2$ case is more complicated than the proof above, because of an additional term that vanishes when $N = 2$. However, that term is of the same sign as the term that is the direct analog to equation (8), so the proof is similar.

When there is no outside option, demand takes the form

$$D^i(p) = \int_{-\infty}^{\infty}\int_{\prod_{j\neq i} S_{j,i}} f(v)dv_{-i}dv_i.$$

We cannot simply normalize the price and valuation of a given good and call it the outside option because additive separability in the normalized prices would not imply additive separability in the original prices.[5] Nevertheless, with slight modifications shown in Appendix B, our proof extends to the case without an outside option.

4. DISCUSSION

Our results do not rely on global properties of the distribution of valuations or on boundary conditions – demand cannot be even *locally* additively separable unless there is a gap in the distribution of valuations. Previous theoretical work that has generated linear demand from aggregation of discrete choices has either had fewer than three options (as in Hotelling, 1929) or a highly restricted space of possible valuations (as in Salop, 1979).

Equation (3) implies that (on average) the cross-partial derivatives $\frac{\partial^2 D^i(p)}{\partial p_k \partial p_i}$ are negative. Thus, ignoring interactions between prices by assuming additively separable demand will lead to systematic underestimation of the change in demand that occurs when prices move in the same direction. This bias shows the importance of microfounding demand systems in individual choice models. Below, we discuss a few specific implications.

4.1. **Testing of Discrete Choice.**

Our results present a simple test of whether a market is well-modeled as a discrete choice setting with unit demand. Specifically, if $\sum_i \sum_{k\neq i} \frac{\partial^2 D^i(p)}{\partial p_k \partial p_i} \geq 0$ for all i and k, then demand does not arise from an aggregation of individual discrete choices.

[5] If good 1, say, is selected as the outside option, then normalized prices take the form $\tilde{p}_i \equiv p_i - p_1$; additive separability of demand with respect to the vector of normalized prices, $\tilde{p} \equiv (0, \tilde{p}_2, \ldots, \tilde{p}_N)$, is not equivalent to additive separability with respect to true prices $p = (p_1, \ldots, p_N)$.

4.2. **Out-of-Sample Prediction.** In a discrete choice market, if a firm experiments with variation in its own price p_i when other prices are stable at p^0_{-i}, then its estimates of the price-sensitivity of demand are only valid for $p_{-i} = p^0_{-i}$. If, for example, a cost shock causes other firms to raise their prices to $p^1_{-i} > p^0_{-i}$, then demand would be more price-sensitive than estimates conducted with $p_{-i} = p^0_{-i}$ would imply:

$$\left|\frac{\partial D^i(p_i, p^1_{-i})}{\partial p_i}\right| > \left|\frac{\partial D^i(p_i, p^0_{-i})}{\partial p_i}\right|.$$

The same logic applies to economists' estimates of demand with only limited variation: estimates based on data in which only one price varies at a time are systematically biased.

4.3. **Welfare Analysis.** The bias we have observed is relevant not just for predicting a price change's effect on demand, but also for predicting its effect on welfare. The decrease in welfare from a given change in p_i is greater when p_{-i} is higher.

This leads us to observe that there may be some situations in which assuming aditively separable demand is (approximately) appropriate. For example, consider the case of two single-product firms merging. Assuming additively separable demand will over-estimate the extent to which prices increase post-merger (since raising one price increases the demand elasticity for the other product, decreasing the incentive to raise its price), but for a given price change assuming additive separability will lead to under-estimation of the welfare effect. These two biases work in opposite directions and could, in theory, cancel out.[6] Unfortunately, however, such cancelation – and hence the appropriateness of the additive separability assumption – would be difficult to verify without a clear understanding of the microfoundations of the demand system.

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[6] In that case, although estimated price effects would be biased, welfare effect estimates would nonetheless be correct.

APPENDIX A. PROOF OF THE MAIN THEOREM

If each $D^i(p)$ is additively separable in own price, then

$$\frac{\partial^2 D^i(p)}{\partial p_k \partial p_i} = 0$$

for all i, k. Thus, in particular, additive separability in own price implies that

$$-2 \sum_i \sum_{k \neq i} \frac{\partial^2 D^i(p)}{\partial p_k \partial p_i} = 0. \tag{9}$$

To prove the theorem, it suffices to bound the left side of (9) strictly above 0, as such a bound guarantees that (9) cannot hold. We now derive such a bound.

Claim. *The left side of* (9) *is bounded below by*

$$\sum_i \sum_{k \neq i} \int_{\prod_{j \neq i,k} S_{j,i}} f(v_{-i,k}, p_i, p_k) dv_{-i,k} > 0. \tag{10}$$

Proof. As $N \geq 2$ and f is nonnegative with full support, it is clear that the inequality in (10) holds.[7] Thus, we need only prove the validity of the claimed bound.

Now, recall that for $k \neq i$, we have

$$\frac{\partial D^i(p)}{\partial p_k} = \int_{p_i}^{\infty} \left(\int_{\prod_{j \neq i,k} S_{j,i}} f(v_{-k}, p_k + v_i - p_i) dv_{-i,k} \right) dv_i. \tag{11}$$

Summing (11) across $k \neq i$ and then differentiating with respect to p_i, we compute that

$$\begin{aligned} \frac{\partial}{\partial p_i} \left(\sum_{k \neq i} \frac{\partial D^i(p)}{\partial p_k} \right) = &- \sum_{k \neq i} \int_{\prod_{j \neq i,k} S_{j,i}} f(v_{-i,k}, p_i, p_k) dv_{-i,k} \\ &- \int_{p_i}^{\infty} \left(\sum_{k \neq i} \int_{\prod_{j \neq i,k} S_{j,i}} f_k(v_{-k}, p_k + v_i - p_i) dv_{-i,k} \right) dv_i \\ &- \int_{p_i}^{\infty} \left(\sum_{k \neq i} \sum_{\ell \neq i,k} \int_{\prod_{j \neq i,k,\ell} S_{j,i}} f(v_{-k,\ell}, p_k + v_i - p_i, p_\ell + v_i - p_i) dv_{-i,k,\ell} \right) dv_i. \end{aligned} \tag{12}$$

Following a change of variables taking $v_\ell \mapsto p_\ell + v_i - p_i$, we see that

$$\int_{p_\ell}^{\infty} \int_{\prod_{j \neq \ell,i} S_{j,\ell}} f_i(v_{-i,\ell}, p_i + v_\ell - p_\ell, v_\ell) dv_{-i,\ell} dv_\ell = \int_{p_i}^{\infty} \int_{\prod_{j \neq i,\ell} S_{j,i}} f_i(v_{-i,\ell}, v_i, p_\ell + v_i - p_i) dv_{-i,\ell} dv_i.$$

[7] Note that in the case $N = 2$ (where there are no $j \neq i, k$), we use the convention that the integral in (10) is "empty":

$$\int_{\prod_{j \neq i,k} S_{j,i}} f(v_{-i,k}, p_i, p_k) dv_{-i,k} = f(p_i, p_k) > 0.$$

It then follows that

$$\begin{aligned}\sum_{\ell}\sum_{i\neq \ell}\int_{p_\ell}^{\infty}\int_{\prod_{j\neq \ell,i} S_{j,\ell}} f_i(v_{-i,\ell},p_i+v_\ell-p_\ell,v_\ell)dv_{-i,\ell}dv_\ell \\ = \sum_{\ell}\sum_{i\neq \ell}\int_{p_i}^{\infty}\int_{\prod_{j\neq i,\ell} S_{j,i}} f_i(v_{-i,\ell},v_i,p_\ell+v_i-p_i)dv_{-i,\ell}dv_i \\ = \sum_{i}\sum_{\ell\neq i}\int_{p_i}^{\infty}\int_{\prod_{j\neq i,\ell} S_{j,i}} f_i(v_{-i,\ell},v_i,p_\ell+v_i-p_i)dv_{-i,\ell}dv_i. \quad (13)\end{aligned}$$

Summing (12) across i and applying the identity (13) shows that[8]

$$\begin{aligned}-2\sum_i\sum_{k\neq i}\frac{\partial^2 D^i(p)}{\partial p_k\partial p_i} = 2\sum_i\sum_{k\neq i}\int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-i,k},p_i,p_k)dv_{-i,k} \\ +\sum_i\int_{p_i}^{\infty}\sum_{k\neq i}\Bigg[\int_{\prod_{j\neq i,k} S_{j,i}} \left(f_k(v_{-k},p_k+v_i-p_i)+f_i(v_{-k},p_k+v_i-p_i)\right)dv_{-i,k} \\ +2\sum_{\ell\neq i,k}\int_{\prod_{j\neq i,k,\ell} S_{j,i}} f(v_{-k,\ell},p_k+v_i-p_i,p_\ell+v_i-p_i)dv_{-i,k,\ell}\Bigg]dv_i. \quad (14)\end{aligned}$$

Finally, we observe that for any i, we have

$$\begin{aligned}\frac{\partial}{\partial v_i}\sum_{k\neq i}\int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-i,k},v_i,p_k+v_i-p_i)dv_{-i,k} \\ =\sum_{k\neq i}\Bigg[\int_{\prod_{j\neq i,k} S_{j,i}} \left(f_k(v_{-i,k},v_i,p_k+v_i-p_i)+f_i(v_{-i,k},v_i,p_k+v_i-p_i)\right)dv_{-i,k} \\ +\sum_{\ell\neq i,k}\int_{\prod_{j\neq i,k,\ell} S_{j,i}} f(v_{-k,\ell},p_k+v_i-p_i,p_\ell+v_i-p_i)dv_{-i,k,\ell}\Bigg]. \quad (15)\end{aligned}$$

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[8]Before obtaining (14) from (13), we must observe that, by relabeling,

$$2\sum_i\sum_{k\neq i}\int_{p_i}^{\infty}\int_{\prod_{j\neq i,k} S_{j,i}} f_k(v_{-k},p_k+v_i-p_i)dv_{-i,k}dv_i$$

is equal to

$$\sum_i\sum_{k\neq i}\int_{p_i}^{\infty}\int_{\prod_{j\neq i,k} S_{j,i}} f_k(v_{-k},p_k+v_i-p_i)dv_{-i,k}dv_i+\sum_{\ell}\sum_{i\neq \ell}\int_{p_\ell}^{\infty}\int_{\prod_{j\neq \ell,i} S_{j,\ell}} f_i(v_{-i,\ell},p_i+v_\ell-p_\ell,v_\ell)dv_{-i,\ell}dv_\ell.$$

Upon discarding one copy of the summation over $\ell \neq i, k$ in (14),[9] and collecting terms, we may use observation (15) to show that

$$
\begin{aligned}
-2\sum_i \sum_{k\neq i} \frac{\partial^2 D^i(p)}{\partial p_k \partial p_i} &\geq 2\sum_i \sum_{k\neq i} \int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-i,k}, p_i, p_k) dv_{-i,k} \\
&\quad + \sum_i \int_{p_i}^{\infty} \frac{\partial}{\partial v_i} \left[\sum_{k\neq i} \int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-i,k}, v_i, p_k + v_i - p_i) dv_{-i,k} \right] dv_i \\
&= 2\sum_i \sum_{k\neq i} \int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-i,k}, p_i, p_k) dv_{-i,k} - \left[\sum_i \sum_{k\neq i} \int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-i,k}, p_i, p_k) dv_{-i,k} \right]. \qquad (16)
\end{aligned}
$$

As the right side of (16) simplifies to (10), we have proven the claimed bound. □

APPENDIX B. DISCRETE CHOICE WITHOUT AN OUTSIDE OPTION

In this appendix, we consider a model of discrete choice *without* an outside option. We show that an analog of our main theorem holds whenever at least three goods are available in the market.

Goods $i = 1, \ldots, N$, prices $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$, and valuations $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ are as specified in Section 3. Again, there is a unit mass of consumers and valuations v are assumed to be distributed according to a continuously differentiable density function f.

No outside option is available – each consumer must purchase exactly one good $i \in \{1, \ldots, N\}$. Thus, each consumer purchases the good i which maximizes $v_i - p_i$, and so demand for good i is given by

$$D^i(p) = \int_{-\infty}^{\infty} \int_{\prod_{j\neq i} S_{j,i}} f(v) dv_{-i} dv_i.$$

Theorem. *Suppose that $N > 2$ and that f has full support. Then, the demand system cannot be additively separable in own price.*

Proof. As we observed in Appendix A, own-price additive separability of demand for good i implies that

$$-2\sum_i \sum_{k\neq i} \frac{\partial^2 D^i(p)}{\partial p_k \partial p_i} = 0. \qquad (17)$$

Now, for $k \neq i$, we have

$$\frac{\partial D^i(p)}{\partial p_k} = \int_{-\infty}^{\infty} \left(\int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-k}, p_k + v_i - p_i) dv_{-i,k} \right) dv_i. \qquad (18)$$

[9] We may do this (at the cost of introducing the inequality in (16)) because f is nonnegative.

Summing (18) across $k \neq i$ and then differentiating with respect to p_i, we compute that

$$\frac{\partial}{\partial p_i}\left(\sum_{k\neq i}\frac{\partial D^i(p)}{\partial p_k}\right) = -\int_{-\infty}^{\infty}\left(\sum_{k\neq i}\int_{\prod_{j\neq i,k} S_{j,i}} f_k(v_{-k}, p_k + v_i - p_i)dv_{-i,k}\right)dv_i$$
$$-\int_{-\infty}^{\infty}\left(\sum_{k\neq i}\sum_{\ell\neq i,k}\int_{\prod_{j\neq i,k,\ell} S_{j,i}} f(v_{-k,\ell}, p_k + v_i - p_i, p_\ell + v_i - p_i)dv_{-i,k,\ell}\right)dv_i. \quad (19)$$

The change of variables taking $v_\ell \mapsto p_\ell + v_i - p_i$ shows that

$$\int_{-\infty}^{\infty}\int_{\prod_{j\neq \ell,i} S_{j,\ell}} f_i(v_{-i,\ell}, p_i + v_\ell - p_\ell, v_\ell)dv_{-i,\ell}dv_\ell = \int_{-\infty}^{\infty}\int_{\prod_{j\neq i,\ell} S_{j,i}} f_i(v_{-i,\ell}, v_i, p_\ell + v_i - p_i)dv_{-i,\ell}dv_i.$$

This observation allows us to make a transformation analogous to (13), from which we obtain

$$-2\sum_i\sum_{k\neq i}\frac{\partial^2 D^i(p)}{\partial p_k \partial p_i}$$
$$= \sum_i\int_{-\infty}^{\infty}\left(\sum_{k\neq i}\sum_{\ell\neq i,k}\int_{\prod_{j\neq i,k,\ell} S_{j,i}} f(v_{-k,\ell}, p_k + v_i - p_i, p_\ell + v_i - p_i)dv_{-i,k,\ell}\right)dv_i$$
$$+ \sum_i\int_{-\infty}^{\infty}\sum_{k\neq i}\Bigg[\int_{\prod_{j\neq i,k} S_{j,i}} \left(f_k(v_{-k}, p_k + v_i - p_i) + f_i(v_{-k}, p_k + v_i - p_i)\right)dv_{-i,k}$$
$$+ \sum_{\ell\neq i,k}\int_{\prod_{j\neq i,k,\ell} S_{j,i}} f(v_{-k,\ell}, p_k + v_i - p_i, p_\ell + v_i - p_i)dv_{-i,k,\ell}\Bigg]dv_i. \quad (20)$$

Observation (15) shows that the second term of (20) is equal to

$$\sum_i\int_{-\infty}^{\infty}\frac{\partial}{\partial v_i}\left[\sum_{k\neq i}\int_{\prod_{j\neq i,k} S_{j,i}} f(v_{-i,k}, v_i, p_k + v_i - p_i)dv_{-i,k}\right]dv_i,$$

which vanishes because f is a density function. Thus, we see that

$$-2\sum_i\sum_{k\neq i}\frac{\partial^2 D^i(p)}{\partial p_k \partial p_i}$$
$$= \sum_i\int_{-\infty}^{\infty}\left(\sum_{k\neq i}\sum_{\ell\neq i,k}\int_{\prod_{j\neq i,k,\ell} S_{j,i}} f(v_{-k,\ell}, p_k + v_i - p_i, p_\ell + v_i - p_i)dv_{-i,k,\ell}\right)dv_i. \quad (21)$$

When $N > 2$, the right side of (21) is bounded strictly above 0,[10] hence (17) cannot hold; this proves the theorem. □

[10] As expected, the right side of (21) vanishes in the case $N = 2$, for which the Hotelling (1929) model shows that discrete choice without an outside option can generate linear demand.