‘Friendship-based’ Games*

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Abstract

This paper analyzes a model of games played on a social network that allows for local correlations in players’ strategies. The model employs a ‘friendship-based’ approach that keeps track of local correlations in agents’ behavior. The model is applied to two specific classes of games, games of strategic complements and strategic substitutes. We also examine the dependence of diffusion on network clustering – the probability that two individuals with a mutual friend are friends of each other – which is not possible with a mean-field approach.

1 Introduction

People rarely make decisions in isolation. Often, the choices and experiences of friends, family, and acquaintances shape our beliefs and behavior. This observation motivates a stream of recent research that addresses the effects of social network structure on behavior (Jackson and Yariv, 2005, 2007; Jackson and Rogers, 2007; López-Pintado, 2008; Lamberson, 2009, 2010; Galeotti et al., 2010). All of these articles employ a “mean-field analysis” borrowed from physics to understand how equilibrium diffusion levels depend on the structure of social interactions.

The mean-field approach requires one of two interpretations. Either, the analysis is thought of as an approximation to the true diffusion dynamics, or agents are assumed to have limited information about their social contacts: the agents act as if the behavior of their neighbors matches the behavior of the population as a whole. In order to gain analytic tractability, the method discards much of the connectivity information of an actual network, retaining only the underlying degree distribution. In particular, correlation in neighboring agents’ behavior is lost. For example, in an epidemic sick people are more likely to be connected to other sick people, and when a new technology spreads technology adopters are more likely to be connected to other adopters. These effects are lost in the mean-field approach.

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Biologists have developed an alternative approximation approach for modeling disease spread known as a "pair approximation," or "correlation model" (Matsuda et al., 1992; Keeling et al., 1997; Morris, 1997; Van Baalen, 2000). This method keeps track of local correlations in behavior and consequently better approximates simulated diffusion patterns. Here, we adapt the pair approximation approach to understanding the spread of behaviors in a model of games played in a social network. In this model, the focus is on friendship ties rather than individual actors, so we refer to it as a 'friendship-based' game.

Our paper is most closely related to the "network games" framework of Galeotti et al. (2010). The partial information structure of the network games framework corresponds with a mean-field approximation rather than the pair approximation that we employ. From an information perspective the friendship-based game allows us to capture a richer information structure – rather than assume that an agent expects her neighbors to play like the population as a whole, we assume that the agent expects her neighbors to play like the population *conditional on her own behavior*. If we think of both the network games and friendship-based games as approximations to a process occurring in a fixed social network, simulations demonstrate that the additional information regarding local correlations included in the friendship-based model lead to more accurate results.

Alternatively, one can conceive of both models as representing a process in which agents are randomly matched with other players at each time step. In the network games approach, agents are matched at random, while the analogous process in the friendship-based model has agents matched with other agents randomly conditional on their current strategy. The friendship-based approach also provides additional information about the resulting patterns of play : we can observe not only the fraction of agents using a particular strategy, but also the local correlations in strategies. This additional information comes at some cost – in many cases we are not able to analytically solve for equilibria – however, using numeric methods we can not only find equilibria, but can also solve for the entire trajectory of play over time.

We apply the model to two classes of games, games of strategic complements and games of strategic substitutes. We find that games of strategic complements tend to exhibit multiple adoption equilibria separated by a tipping point analogous to the "epidemic threshold" in disease spread models. Locally, agents rapidly split into clusters using competing strategies. Games of strategic substitutes tend towards a unique equilibrium. Agents strategies are locally dissociative. For example, if we think of the model as capturing provision of a public good, a few agents serve the role of local providers of the good while their neighbors free ride.

Our paper is also related to work in evolutionary game theory (Ohtsuki et al., 2006; Ohtsuki and Nowak, 2006). As in the evolutionary game theory literature, we impose a specific dynamic on agents' strategy updating to help solve the equilibrium selection problem (Foster and Young, 1990; Kandori et al., 1993), but the dynamic we choose is more economically motivated. Typical dynamics in evolutionary game theory assume that the probability that an agent switches strategies is related to the payoffs (i.e. fitness) of *other agents*. Intuitively, when an agent dies more fit strategies are more likely to invade the resulting opening. In our model, an agent only switches strategies if doing so is a best response *for them*. In other words, an agent's strategy choice is based on her own payoffs rather than those of her neighbors.

2 The Model

We consider a finite population of n agents. For each agent i there is a collection of other agents $N(i) \subset \{1, \ldots, n\} \setminus \{i\}$, which we call the *friends* or *neighbors* of i. Friendship is reciprocal, so $j \in N(i)$ implies $i \in N(j)$. This information defines a *network* with nodes the agents $i = 1, \ldots, n$, and a link from i to j if $j \in N(i)$ (Jackson, 2008; Newman, 2010).

At each time step $t = 1, 2, 3, \ldots$ each agent plays one of two strategies, x or y. Payoffs are a function of a player's strategy and those played by her neighbors. We assume that an agent's payoffs depend only the number of neighbors playing each of the two strategies, not on their specific identities. For an agent i, let $k_x(i)$ and $k_y(i)$ denote the number of neighbors playing x and y, respectively. When the focal agent is clear or unimportant, we drop the dependence on i, and simply write k_x or k_y (note we have also suppressed dependence on time). Then the payoffs can be specified by two payoff functions: $\pi_x(k_x, k_y)$ and $\pi_y(k_x, k_y)$.

3 Friendship-based Games

Even in small populations with simple network structures, the game described in the previous section often admits a multitude of equilibria. To help solve the equilibrium selection problem we employ two common strategies. First, we impose a specific dynamic on the game. Second, we specify a partial information structure.

Before turning to the game dynamics, we introduce some notation. To simplify the analysis, assume that each agent has a fixed number of neighbors, k (or equivalently, each agent plays with the expectation of meeting a fixed number of agents on average). Under this assumption, we can write the payoffs as functions of the number of neighbors playing strategy x: $\pi_x(k_x)$ and $\pi_y(k_x)$. This is a significant assumption, as several studies have shown that the network degree distribution impacts diffusion levels (Jackson and Yariv, 2005, 2007; Jackson and Rogers, 2007;

López-Pintado, 2008; Lamberson, 2009, 2010; Galeotti et al., 2010). Degree information can be incorporated in the friendship-based model (for one related approach see Eames and Keeling, 2002); however, it significantly complicates the notation and analysis. Since degree distribution effects have been well studied, here we put them aside and focus on other factors.

We will use the first three greek letters as variables to denote either of the two strategies: $\alpha, \beta, \gamma \in \{x, y\}$. Let p_α denote the proportion of agents in the population playing strategy α. Let $p_{\alpha\beta}$ denote the proportion of connected ordered pairs of agents in the network with the first agent playing α and the second agent playing β. Because we use ordered pairs, $p_{xy} = p_{yx}$, and $p_{\alpha\alpha}$ is even. Let $p_{\alpha|\beta}$ denote the probability that a random neighbor of an agent playing strategy β is playing strategy α. Note that the following relationships hold among these variables:

$$p_x + p_y = 1 \tag{1}$$

$$p_{xx} + 2p_{xy} + p_{yy} = 1 \tag{2}$$

$$p_{x|\alpha} + p_{y|\alpha} = 1 \tag{3}$$

$$p_{\alpha\beta} = p_{\alpha|\beta}\, p_\beta. \tag{4}$$

The last equation follows from Baye's rule.

At each time step an agent is chosen uniformly at random to update his strategy. We assume that agents are myopic and only attempt to maximize their next period payoffs. Thus, because only one agent updates at a time, it is rational for the agent to choose the best response to the play of his neighbors in the previous period. Since payoffs are determined entirely by the number of neighbors playing x, we can define the best response function, $BR : \{0, \ldots, n\} \to \{x, y\}$ so that $BR(k_x)$ is the best response for an agent with k_x neighbors playing x. When the payoffs to both strategies are equal, we arbitrarily designate x as the best response.

In a given time step only one agent updates her strategy, so if p_x changes it increases or decreases by $1/N$. The probability that p_x increases equals the probability that the agent selected to update his strategy is currently playing y, p_y, times the probability that the best response for that agent is x, $P[k_x \in BR^{-1}(x)]$. We need to calculate the probability that a random agent playing y has k_x neighbors playing x. Let i be the agent chosen to update her strategy and suppose that i is playing strategy y. Then the probability that a random neighbor of i is playing x is $p_{x|y}$, and the probability that a random neighbor of i plays y is $p_{y|y}$. If (conditional on i playing y) the strategies of i's neighbors are independent then the probability that i has k_x neighbors playing x and $k_y = k - k_x$ neighbors playing

y is

$$\frac{k!}{k_x!k_y!}p_{x|y}^{k_x}p_{y|y}^{k_y}. \tag{5}$$

In general i's neighbors will not be (conditionally) independent. In empirical social networks, two individuals with a common friend are much more likely to be friends with each other than two randomly chosen individuals (Newman and Park, 2003). This feature, known as *clustering* or *transitivity* (Jackson, 2008), implies that often the state of two neighbors of a given node will affect one another directly. Thus, calculating the probability that i has k_x neighbors playing x and k_y neighbors playing y requires knowing the probability of larger configurations such as $p_{x|yx}$ – the probability that an agent plays x given that they neighbor an agent playing y that has another neighbor playing x. To keep track of these probabilities requires tracking the frequency of triple configurations in the network, which in turn requires knowing the frequency of quadruples and so on. Rather than continue this expansion, we stop here and make the assumption that the agents behave as if the actions of their neighbors are independent conditional on their own actions. Thus, $p_{\alpha|\beta\gamma}$ is simply $p_{\alpha|\beta}$. We call this the *conditionally independent neighbors* (CIN) assumption.

If the network is a tree, this is not an assumption, but instead follows from the fact that the only path between two neighbors of an agent i goes through i. The assumption is also approximately true in large Erdös-Rényi graphs, because these graphs have low clustering (Newman, 2010). In general, however, this will not be the case. Nevertheless, the CIN assumption has proven to lead to highly accurate approximations in other models, even in clustered networks (Morris, 1997; Newman, 2010). Under this assumption, equation (5) gives the probability that an agent playing y has k_x neighbors playing x and k_y neighbors playing y.

The probability that p_x decreases equals the probability that the agent selected to update her strategy is currently playing x, p_x, times the probability that the best response for that agent is y, $P[k_x \in BR^{-1}(y)]$. The probability that a random agent playing x has k_x neighbors playing x and k_y neighbors playing y is

$$\frac{k!}{k_x!k_y!}p_{x|x}^{k_x}p_{y|x}^{k_y}. \tag{6}$$

Thus, the net rate of change for p_x is

$$\dot{p}_x = \sum_{BR^{-1}(x)} \frac{p_y}{N}\frac{k!}{k_x!k_y!}p_{x|y}^{k_x}p_{y|y}^{k_y} - \sum_{BR^{-1}(y)} \frac{p_x}{N}\frac{k!}{k_x!k_y!}p_{x|x}^{k_x}p_{y|x}^{k_y}. \tag{7}$$

Equation (7) does not fully specify the game dynamics because the right hand side has the terms $p_{\alpha|\beta}$, which by equation (4) depend on the pair frequencies $p_{\alpha\beta}$.

We can write down an equation describing the rate of change of the pairs $p_{\alpha\beta}$ in the same way that we did for the singletons p_x. The identities in equations (1) through (4) imply that it is sufficient to consider any one of the four pair types. Consider the xx pairs. Let i be the agent randomly selected to update her strategy. If i is playing strategy y and has k_x neighbors playing strategy x, where $k_x \in BR^{-1}(x)$, then p_{xx} increases by $\frac{2k_x}{kN}$ (there are kN total pairs and the change from playing y to x creates $2k_x$ xx pairs since pairs are counted in both directions). Making use of the CIN assumption, the probability that a randomly chosen agent is of type y and has k_x neighbors playing x equals the probability that a random agent plays y times the probability that an agent playing y has k_x neighbors playing x:

$$p_y \frac{k!}{k_x!k_y!} p_{x|y}^{k_x} p_{y|y}^{k_y}. \tag{8}$$

Thus, the rate of increase of p_{xx} is

$$\frac{2k_x}{kN} p_y \frac{k!}{k_x!k_y} p_{x|y}^{k_x} p_{y|y}^{k_y}. \tag{9}$$

A similar calculation gives the rate of decrease of p_{xx} leading to the net rate of change

$$\dot{p}_{xx} = \sum_{BR^{-1}(x)} \frac{2k_x}{kN} p_y \frac{k!}{k_x!k_y!} p_{x|y}^{k_x} p_{y|y}^{k_y} - \sum_{BR^{-1}(y)} \frac{2k_x}{kN} p_x \frac{k!}{k_x!k_y!} p_{x|x}^{k_x} p_{y|x}^{k_y}. \tag{10}$$

Equations (7) and (10) completely describe the strategy dynamics of the game.

4 Strategic Complements and Substitutes

In this section we examine two specific classes of games that have also been considered in the network games framework of Galeotti et al. (2010): games of strategic complements and strategic substitutes.

4.1 Strategic Complements

Following Galeotti et al. (2010), we say that the payoffs exhibit strategic complements if

$$\pi_x(k_x) - \pi_y(k_x) \geq \pi_x(k'_x) - \pi_y(k'_x) \tag{11}$$

for any k_x and k'_x with $k_x \geq k'_x$. In other words, as the number of neighbors playing x increases, the change in benefits to playing x exceeds the change in benefits to playing y.

4.1.1 Example: Standards competition with positive externalities

Games of strategic complements provide a simple model of agents choosing between two standards with positive externalities. For example, suppose the population of agents represents the population of economists in the United States and each of the two strategies corresponds to the choice of a statistical software package. Links join economists who collaborate with one another. Economists can only use one software package at a time, and because they share files with their collaborators, the benefits to using a particular package increase with the number of colleagues that use the software. Specifically, we might have

$$\pi_x(k_x) = f(k_x) - c_x \tag{12}$$

and

$$\pi_y(k_x) = f(k - k_x) - c_y, \tag{13}$$

where c_x and c_y are the costs of using x and y, respectively, and f is a non-decreasing function capturing the increasing benefits of using a package used by your collaborators.

4.1.2 Dynamics with strategic complements

In games of strategic complements, the best response function follows a threshold rule: if $k_x \geq \tau$ play x, if $k_x < \tau$ play y.[1] Equations (7) and (10) reduce to

$$\dot{p}_x = \sum_{k_x=\tau}^{k} \frac{p_y}{N} \frac{k!}{k_x!k_y!} p_{x|y}^{k_x} p_{y|y}^{k_y} - \sum_{k_x=0}^{\tau-1} \frac{p_x}{N} \frac{k!}{k_x!k_y!} p_{x|x}^{k_x} p_{y|x}^{k_y}, \tag{14}$$

and

$$\dot{p}_{xx} = \sum_{k_x=\tau}^{k} \frac{2k_x}{kN} p_y \frac{k!}{k_x!k_y!} p_{x|y}^{k_x} p_{y|y}^{k_y} - \sum_{k_x=0}^{\tau-1} \frac{2k_x}{kN} p_x \frac{k!}{k_x!k_y!} p_{x|x}^{k_x} p_{y|x}^{k_y}. \tag{15}$$

For a given set of initial conditions, equations (14) and (15) can be numerically integrated to give the predicted evolution of play in the population over time. For example, Figure 1 plots the fraction of agents playing strategy x over time in a Bernoulli random network (Erdös and Rényi, 1960) with 1000 nodes and tie probability .01 ($k \approx 10$). The lines show results from numerical solutions to the friendship-based model specified by equations (14) and (15) for a range of initial

[1] Here we assume that either the exact threshold is a non-integer, or that ties are always broken by playing x.

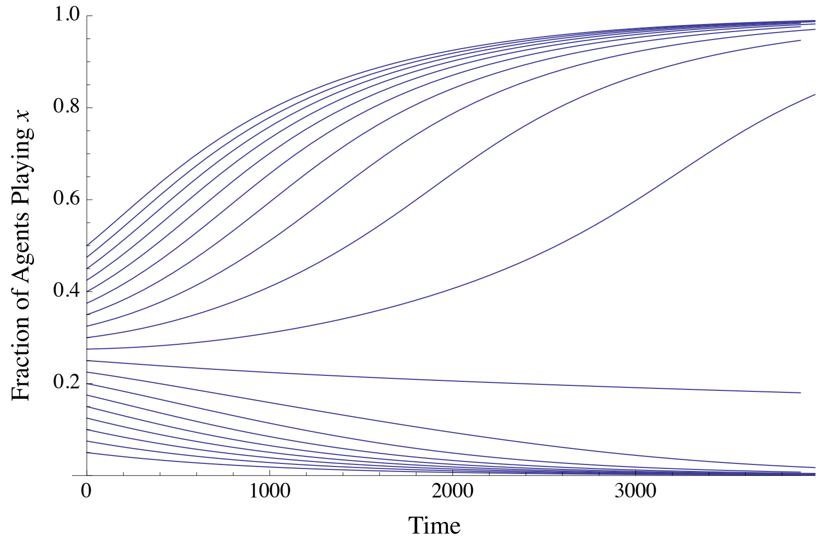

Figure 1: The friendship-based model for a game of strategic complements under a range of initial conditions.

conditions.[2] The best response threshold for playing strategy x is $\tau = 4$. In other words, x is the best response for any agent with four or more neighbors playing x.

In this example, depicted in Figure 1, there is a critical threshold of initial strategy x players required in order for strategy x to diffuse throughout the population. This threshold is analogous to the "epidemic threshold" in compartmental models of disease spread.

4.2 Strategic Substitutes

Galeotti et al. (2010) also consider games of strategic substitutes, which are particularly interesting for understanding the provision of public goods. In many cases it may only be necessary that the public good be provided locally. For example, in the case of information provision if one person in our network neighborhood does the work of gathering and disseminating important information, then there is no need for us to duplicate their effort. The payoff inequalities for games of strategic substitutes follow the opposite relationship as for games of strategic complements

[2]Numerical solutions to the friendship-based differential equations were obtained using Mathematica (Wolfram Research, Inc., 2008).

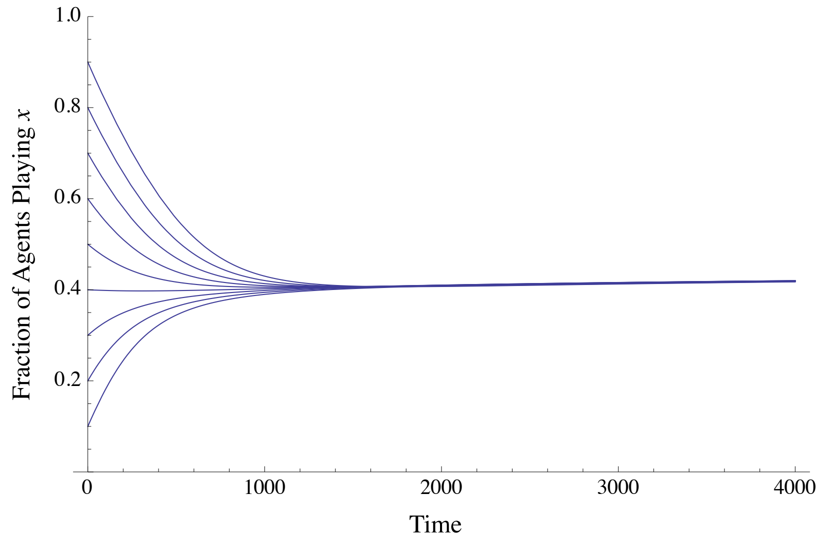

Figure 2: The friendship-based model for a game of strategic substitutes under a range of initial conditions.

studied in the previous section:

$$\pi_x(k_x) - \pi_y(k_x) \leq \pi_x(k_x') - \pi_y(k_x') \tag{16}$$

for any k_x and k_x' with $k_x \geq k_x'$.

4.2.1 Example: Best shot public goods game

For example, consider the best shot public goods game described by Galeotti et al. (2010). Playing strategy x corresponds to providing a public good at a cost $0 < c_x < 1$ and strategy y corresponds to doing nothing. Payoffs are

$$\pi_x(k_x) = 1 - c_x \tag{17}$$

and

$$\pi_y(k_x) = \begin{cases} 1 & k_x \geq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{18}$$

4.2.2 Dynamics with strategic substitutes

With strategic substitutes the best response function again satisfies a threshold rule, but in this case the inequality is reversed: if $k_x \leq \tau$ play x, if $k_x > \tau$ play y. The analogous versions of equations (14) and (15) are:

$$\dot{p}_x = \sum_{k_x=0}^{\tau} \frac{p_y}{N} \frac{k!}{k_x! k_y!} p_{x|y}^{k_x} p_{y|y}^{k_y} - \sum_{k_x=\tau+1}^{k} \frac{p_x}{N} \frac{k!}{k_x! k_y!} p_{x|x}^{k_x} p_{y|x}^{k_y}, \tag{19}$$

and

$$\dot{p}_{xx} = \sum_{k_x=0}^{\tau} \frac{2k_x}{kN} p_y \frac{k!}{k_x! k_y!} p_{x|y}^{k_x} p_{y|y}^{k_y} - \sum_{k_x=\tau+1}^{k} \frac{2k_x}{kN} p_x \frac{k!}{k_x! k_y!} p_{x|x}^{k_x} p_{y|x}^{k_y}. \tag{20}$$

As with strategic complements, equations (19) and (20) can be numerically integrated to determine the diffusion pattern. Figure 2 plots the fraction of strategy x players over time for a game of strategic substitutes with $\tau = 4$ for a range of initial conditions. Unlike the strategic complements case, with strategic substitutes the long run outcome is the same regardless of the initial conditions.

5 Local Correlation

Besides the diffusion curve, the friendship-based model provides additional information about the correlation in neighbors behavior that is unavailable in a mean-field analysis. For example, Figure 3 plots the conditional probability $p_{x|x}$ that a given neighbor of an agent playing x is also playing x in the friendship-based model for strategic substitutes (solid red) and strategic complements (solid blue) ($\tau = 4$ in both cases).

For comparison, we also examine the analogous mean-field model.[3] Since the mean-field model assumes that neighboring agents' strategies are independent, p_x is the best estimate for $p_{x|x}$ available. The dashed lines in Figure 3 display the predicted fraction of agents playing strategy x from the mean-field model.

In the game of strategic substitutes (red), the mean-field model predicts that any given agent will play strategy x with probability .465, and regardless of her strategy, any neighbor of a given agent will also play x with probability .465. The

[3] In this model, regardless of their own play, agents assume that the behavior of their neighbors equals the average behavior in the population. Thus, $p_{x|y} = p_{x|x} = p_x$ and $p_{y|y} = p_{y|x} = p_y$. Equation (7) becomes:

$$\dot{p}_x = \sum_{BR^{-1}(x)} \frac{p_y}{N} \frac{k!}{k_x! k_y!} p_x^{k_x} p_y^{k_y} - \sum_{BR^{-1}(y)} \frac{p_x}{N} \frac{k!}{k_x! k_y!} p_x^{k_x} p_y^{k_y}. \tag{21}$$

This is a closed differential equation, so there is no need to formulate the analog of equation (10).

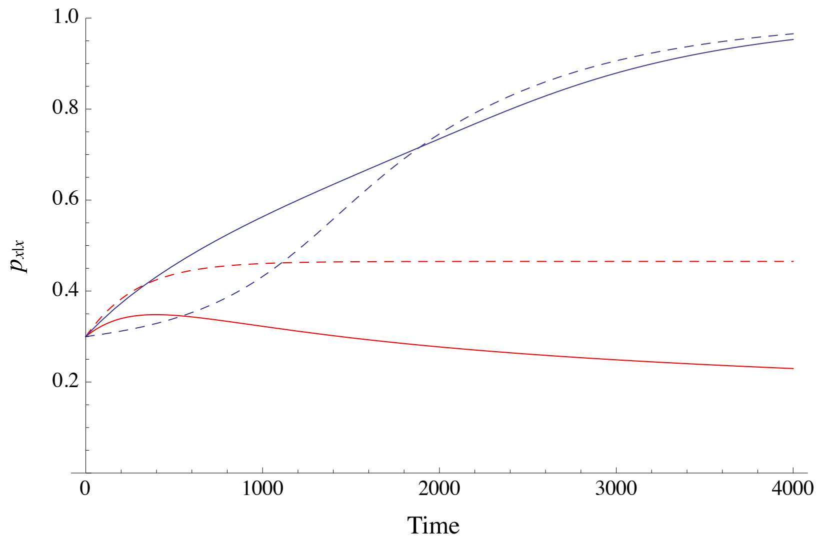

Figure 3: The conditional probability $p_{x|x}$ from the friendship-based model (solid lines), and the fraction of agents playing strategy x from a mean-field model (dashed lines). The blue lines are for a game of strategic complements and the red lines are for a game of strategic substitutes.

probability that an agent playing x has more than four of her neighbors playing x is

$$\sum_{k_x=5}^{10} \frac{10!}{k_x!(10-k_x)!} .465^{k_x}(1-.465)^{10-k_x} = .534. \tag{22}$$

Thus, more than half of the agents playing x would be better off playing y. If we think of playing x as providing a public good, then the good is over provided in the mean-field model.

As the figure shows, the friendship-based model predicts a dissociative relationship in neighboring agents' strategies (solid red). The probability that a random agent plays x is .473, but the probability that the neighbor of an agent playing x also plays x is only .230. The probability that an agent playing x has more than four neighbors playing x is only

$$\sum_{k_x=5}^{10} \frac{10!}{k_x!(10-k_x)!} .230^{k_x}(1-.230)^{10-k_x} = .057. \tag{23}$$

Agents in the friendship-based model fare better because they take their own behavior into account when calculating the expected behavior of their neighbors. For example, consider the case of information provision. If I have provided information to my friends in the past, then I know that in the future my friends are less likely to seek out information because they may count on free riding off of me. Therefore, I should be more likely to continue to provide information because I know that my friends are unlikely to provide it.

With strategic complements (blue lines), players coordinate locally. Ultimately the mean-field model and the friendship-based model make similar predictions: both models predict that nearly all of the agents play x and therefore the probability that a neighbor of an agent playing x also plays x approaches one. However, the friendship-based model results in much more rapid local coordination.

6 Dependence on Parameters

We now examine how agents' strategies depend on the model parameters. In particular, we examine how strategies depend on the number of neighbors of each agent, k, and the threshold for taking action x, τ.

Several studies have demonstrated that average degree shapes behavior in networks. Ohtsuki et al. (2006) show that when the average degree is sufficiently high, cooperation can be an evolutionary stable strategy in a prisoner's dilemma game. Jackson and Rogers (2007) and López-Pintado (2008) show that increasing degree increases diffusion when a behavior spreads through a network by contact, as in models of disease spread. Similarly, Jackson and Yariv (2007) prove that increasing degree increases diffusion in a threshold model. Lamberson (2010) demonstrates that in a model of social learning, the effect of changes in degree depends on agents' prior beliefs and the payoffs to using a new technology.

Increasing an agent's degree increases the expected number of neighbors of that agent playing x. If the best response threshold, τ, remains constant, then this increases the likelihood that a particular agent has sufficiently many neighbors playing x to make playing y a best response in a game of strategic substitutes. Therefore, increasing k tends to decrease the fraction of agents playing x with strategic substitutes. Conversely, with strategic complements increasing degree increases the likelihood that an agent has sufficiently many neighbors playing x to making playing x the best response. Therefore, for a fixed τ increasing degree in a game of strategic complements increases the fraction of agents playing x (i.e. lowers the epidemic threshold). Numerical solutions support these arguments. Similar arguments show that increasing τ increases the play of x with strategic substitutes and raises the epidemic threshold for x under strategic complements.

7 Clustering

Up to this point we have always made the conditionally independent neighbors assumption; however, many empirical social networks exhibit significant clustering (Newman and Park, 2003; Watts, 2004), casting doubt on this assumption. In this section we describe a refinement to the friendship-based game that incorporates clustering. We measure clustering using the well-known *clustering coefficient*:

$$C = \frac{\text{(number of closed paths of length two)}}{\text{(number of paths of length two)}} \tag{24}$$

(p. 199 Newman, 2010). The clustering coefficient captures the probability that any two neighbors of a given node are themselves neighbors.

When C is non-zero, we expect that some of the neighboring nodes are neighbors of each other, and thus their strategies are not independent. Consider three nodes, a central node i with two neighbors, j and k. Let $p_{\alpha|\beta\gamma}$ denote the probability that j uses strategy α when i and k play strategies β and γ, respectively. We can split this probability into two components, the probability when j and k are connected forming a closed triangle, $p^{\bigtriangledown}_{\alpha|\beta\gamma}$, and the probability when j and k are not connected, $p^{\vee}_{\alpha|\beta\gamma}$. Then

$$p_{\alpha|\beta\gamma} = C p^{\bigtriangledown}_{\alpha|\beta\gamma} + (1-C) p^{\vee}_{\alpha|\beta\gamma}. \tag{25}$$

The probability $p_{\alpha|\beta\gamma}$ is the easier of the two to deal with. Since in this case j and k are not connected, we assume that their strategies are independent conditional on the strategy of i, and thus $p^{\vee}_{\alpha|\beta\gamma} = p_{\alpha|\beta}$. To estimate $p^{\triangledown}_{\alpha|\beta\gamma}$, we use a strategy suggested by Morris (1997) (see also Rand, 1999). First we assume that

$$\frac{p^{\triangledown}_{\alpha|\beta\gamma}}{p^{\triangledown}_{\alpha|\cdot\gamma}} = \frac{p^{\vee}_{\alpha|\beta\gamma}}{p^{\vee}_{\alpha|\cdot\gamma}}. \tag{26}$$

The idea behind this assumption is that the strategy of i, given the strategies of j and k, should not be affected much by whether or not j and k are also neighbors. In the right hand side denominator, since j and k are not connected, we should have $p^{\vee}_{\alpha|\cdot\gamma} = p_{\alpha|\cdot} = p_{\alpha}$. For the left hand side denominator, the probability that j uses strategy α given that j is connected to i using an unknown strategy and k using strategy γ is simply $p_{\alpha|\gamma}$. Substituting into equation (26) we obtain the approximation

$$p^{\triangledown}_{\alpha|\beta\gamma} = \frac{p_{\alpha|\beta}\, p_{\alpha|\gamma}}{p_{\alpha}}. \tag{27}$$

Finally, substituting back into equation (25) gives

$$p_{\alpha|\beta\gamma} = C\,\frac{p_{\alpha|\beta}\, p_{\alpha|\gamma}}{p_{\alpha}} + (1 - C)\, p_{\alpha|\beta}. \tag{28}$$

To see how we incorporate this new conditional probability into the dynamic equations (7) and (10), it helps to rewrite the equations slightly. We start with equation (10) for $\dot{p}_{xx}$. Rather than thinking of randomly selecting a node to update, we could randomly selected an edge and then select a node at one end of that edge (since all nodes have the same degree, these are equivalent). The probability p_{xx} can only increase if we select the y end of either an xy edge or a yx edge, which occurs with probability $\frac{1}{2}(p_{xy} + p_{yx}) = \frac{1}{2}(2p_{xy}) = p_{xy}$.

Call the selected agent i and the opposite end of the selected edge j. Agent i will switch to playing x if $k_x \in BR^{-1}(x)$. Since we already know that j is playing x, the probability that i has exactly k_x neighbors playing x is equal to the probability that i has $k_x - 1$ additional neighbors playing x. The probability that any one of i's neighbors (besides j) plays x is $p_{x|yx}$. Now, rather than the CIN assumption, we assume that i's neighbors are independent conditional on i and j's strategies. Thus, the probability that i has k_x neighbors playing x and $k_y = k - k_x$ neighbors playing y is

$$\frac{(k-1)!}{(k_x - 1)!k_y!}\, p_{x|yx}^{k_x - 1}\, p_{y|yx}^{k_y}. \tag{29}$$

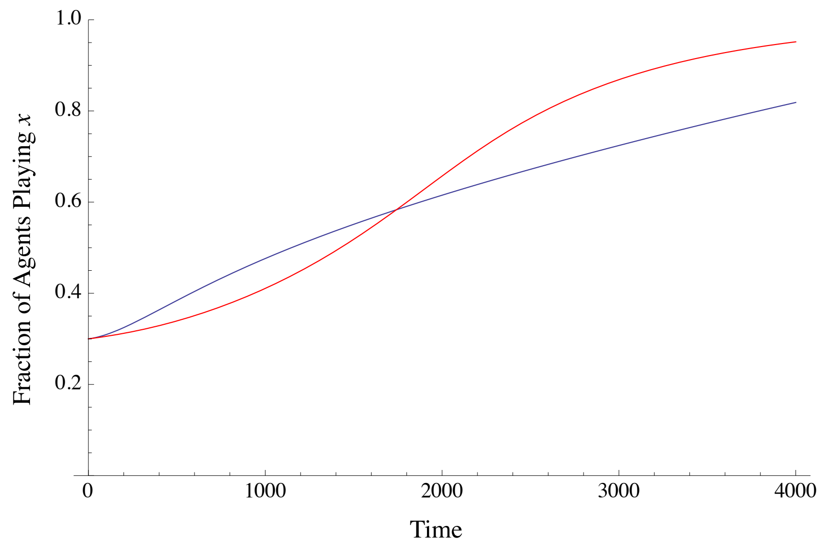

Figure 4: The effect of clustering under strategic complements. Clustering: $C = 1$ blue, $C = 0$ red.

If i changes from y to x then p_{xx} increases by $2/N$. Performing a similar calculation for decreases in p_{xx} we obtain

$$\dot{p}_{xx} = \sum_{BR^{-1}(x)} \frac{2}{N} p_{xy} \frac{(k-1)!}{(k_x-1)!k_y!} p_{x|yx}^{k_x-1} p_{y|yx}^{k_y} - \sum_{BR^{-1}(y)} \frac{2}{N} p_{xx} \frac{(k-1)!}{(k_x-1)!k_y!} p_{x|xx}^{k_x-1} p_{y|xx}^{k_y} \tag{30}$$

$$= \sum_{BR^{-1}(x)} \frac{2k_x}{kN} p_{xy} \frac{k!}{k_x!k_y!} p_{x|yx}^{k_x-1} p_{y|yx}^{k_y} - \sum_{BR^{-1}(y)} \frac{2k_x}{kN} p_{xx} \frac{k!}{k_x!k_y!} p_{x|xx}^{k_x-1} p_{y|xx}^{k_y}. \tag{31}$$

Under the CIN assumption this equation is equivalent to (10). To see this, note that under CIN $p_{\alpha|\beta\gamma}$ is replaced by $p_{\alpha|\beta}$ giving

$$\dot{p}_{xx} = \sum_{BR^{-1}(x)} \frac{2k_x}{kN} p_{xy} \frac{k!}{k_x!k_y!} p_{x|y}^{k_x-1} p_{y|y}^{k_y} - \sum_{BR^{-1}(y)} \frac{2k_x}{kN} p_{xx} \frac{k!}{k_x!k_y!} p_{x|x}^{k_x-1} p_{y|x}^{k_y}. \tag{32}$$

Substituting $p_{xy} = p_{x|y} p_y$ and $p_{xx} = p_{x|x} p_x$, we recover (10).

Now, instead of replacing $p_{\alpha|\beta\gamma}$ by $p_{\alpha|\beta}$, we use (28) for $p_{\alpha|\beta\gamma}$ in equation

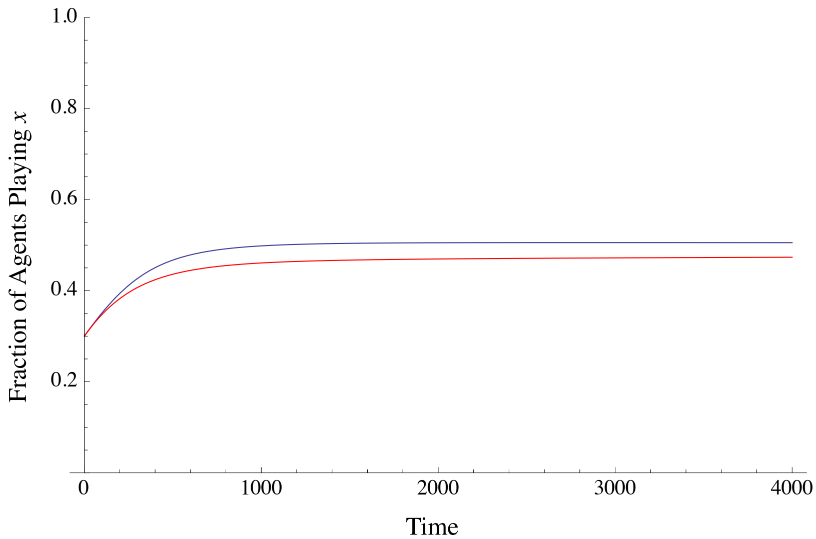

Figure 5: The effect of clustering under strategic substitutes. Clustering: $C = 1$ blue, $C = 0$ red.

(31). A similar argument applies to $\dot{p}_x$ giving

$$\dot{p}_x = \sum_{BR^{-1}(x)} \frac{1}{N} p_{xy} \frac{k!}{k_x! k_y!} p_{x|yx}^{k_x - 1} p_{y|yx}^{k_y} - \sum_{BR^{-1}(y)} \frac{1}{N} p_{xx} \frac{k!}{k_x! k_y!} p_{x|xx}^{k_x - 1} p_{y|xx}^{k_y}, \tag{33}$$

where again (28) is used for the $p_{\alpha|\beta\gamma}$ terms. When $C = 0$, we obtain the same equations as in the original friendship-based model.

Figure 4 depicts the effect of clustering under strategic complements. Initially, clustering speeds the spread of strategy x. However, as time passes, strategy x spreads more slowly in the clustered network than the non-clustered network.

Figure 5 depicts the effect of clustering under strategic substitutes. When the network is highly clustered, more agents play strategy x. In the case of public goods provision, this means that more agents have to play the role of the public good provider in a highly clustered network than in a non-clustered network.

8 Conclusion

Local correlation is likely to arise in many empirical situations where individuals' behavior is influenced by that of their social contacts. This paper develops a framework that allows for local correlation in players' strategies. The framework not only provides a more realistic model of many empirical situations, but also provides predictions about local correlation that have been unavailable in previous models. For example, we find that agents' strategies tend to be locally associative in games of strategic substitutes and locally dissociative in games of strategic complements. We also extend the framework to account for network clustering.

Several simplifying assumptions are made to make the analysis more tractable. In particular, we have only considered binary action games and we have assumed that all agents have the same degree. Examining richer action spaces and general degree distributions within the friendship-based framework would be interesting directions for future work.

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