

Online Appendix of
 “A Model of the Attention Economy:
 Data, Platforms, and Discovery”

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B Supplemental Material for Section 4

Appendix B.1 gives the formal statement of a firm’s bidding problem. Appendix B.2 proves the auxiliary lemmas used in the proof of Theorem 1.

B.1 Formal Statement of Firms’ Bidding Problem

Let τ_z denote the z th time of entry into an auction for consumer i . That is, τ_z is the z th arrival of a Poisson process that ticks at rate $\lambda_{at} + \lambda_f \mathbb{1}_{\{j \in \Omega_{it}\}}$. Taking the bidding strategy $B_t(\cdot)$ of its rivals as given, firm j sets bids to maximize the present value of flow profits (including the costs of advertising) from selling to a typical consumer i :

$$\Pi_{\mathbb{J}} = \max_{\{b_z\}} \mathbb{E} \left[\int_0^\infty e^{-\rho s} \pi_{\mathbb{J}s}(\hat{v}_{ij}) \mathbb{1}_{\{j \in \Omega_{is}\}} ds - \sum_{z=1}^\infty e^{-\rho \tau_z} B_{\tau_z}(\hat{v}_z^{(1)}) \mathbb{1}_{\{b_z > B_{\tau_z}(\hat{v}_z^{(1)})\}} \right]$$

where $\hat{v}_z^{(1)}$ is the highest expectation of the $N - 1$ other bidders in the z th auction: $\hat{v}_z^{(1)} \sim (H_{\tau_z}^c)^{N-1}$ conditional on τ_z . Above, the bid b_z in the z th auction is a measurable function of the expectation $\hat{v}_{ij}$ and time t .

B.2 Auxiliary Lemmas for Theorem 1

This subsection proves Lemmas 2–5, used in the proof of Theorem 1.

Lemma 1. *In an equilibrium, the bidding function $B_t(\cdot)$ is Lipschitz at each t .*

Proof. Since $B_t = V_t^{\text{In}} - V_t^{\text{Out}}$, it suffices to show that V_t^{In} and V_t^{Out} are Lipschitz. I argue for V_t^{In} ; the argument for V_t^{Out} is analogous.

Fix two expected values $\hat{v} > \tilde{v}$. Consider a firm with expected value $\tilde{v}$ that deviates by bidding as if its expected value were $\hat{v}$. The deviating firm’s bidding induces the same dynamics for $\mathbb{1}_{\{j \in \Omega_{is}\}}$ as a firm actually with expected value

$\hat{v}$, but its true flow profit at time s is $\pi_{\mathbb{J}s}\tilde{v}$ rather than $\pi_{\mathbb{J}s}\hat{v}$. Letting $\hat{V}_t^{\text{In}}(\tilde{v})$ denote the value of this deviation,

$$V_t^{\text{In}}(\hat{v}) - \hat{V}_t^{\text{In}}(\tilde{v}) = (\hat{v} - \tilde{v}) \mathbb{E} \left[\int_t^\infty e^{-\rho(s-t)} \pi_{\mathbb{J}s} \mathbb{1}_{\{j \in \Omega_{is}\}} ds \right],$$

where the expectation is taken under the $\hat{v}$ -bidding law. Note that ad costs cancel since both firms bid identically.

Optimality of $V_t^{\text{In}}(\tilde{v})$ implies $V_t^{\text{In}}(\tilde{v}) \geq \hat{V}_t^{\text{In}}(\tilde{v})$, so

$$V_t^{\text{In}}(\hat{v}) - V_t^{\text{In}}(\tilde{v}) \leq (\hat{v} - \tilde{v}) \mathbb{E} \left[\int_t^\infty e^{-\rho(s-t)} \pi_{\mathbb{J}s} \mathbb{1}_{\{j \in \Omega_{is}\}} ds \right] \leq \frac{\sup_{s \geq t} \pi_{\mathbb{J}s}}{\rho} (\hat{v} - \tilde{v}).$$

Since $V_t^{\text{In}}(\hat{v}) \geq V_t^{\text{In}}(\tilde{v})$ by monotonicity (a firm with higher expected value can always replicate the strategy of a firm with lower expected value), the difference is non-negative, and

$$|V_t^{\text{In}}(\hat{v}) - V_t^{\text{In}}(\tilde{v})| \leq \frac{\sup_{s \geq t} \pi_{\mathbb{J}s}}{\rho} |\hat{v} - \tilde{v}|.$$

Since $\sup_{s \geq t} \pi_{\mathbb{J}s} < \infty$, V_t^{In} is Lipschitz. □

Lemma 2. *If $\epsilon - 1 < 1/\varphi$, the platform problem (12) is concave and thus any solution is unique. If $\epsilon - 1 > 1/\varphi$, the platform's problem does not have a solution.*

Proof. Solving out the ODE for q_t yields

$$q_{kt} = e^{-\delta t} \int_0^t e^{\delta s} \ell_{ks}^\varphi ds + e^{-\delta t} q_0.$$

The flow utility of platform k is then

$$\pi_{\mathbb{K}t} \frac{\left(e^{-\delta t} \int_0^t e^{\delta s} \ell_{ks}^\varphi ds + e^{-\delta t} q_0 \right)^{\epsilon-1}}{\int_{\mathbb{K}} q_{zt}^{\epsilon-1} dz} - \ell_{kt}.$$

Suppose that $\epsilon - 1 < 1/\varphi$. Then we observe that the flow utility must be concave in $\{\ell_{kt}\}$ —The second term is linear; the first term's concavity is determined by the numerator, which is a CES aggregator with share weights determined by the exponential function. It is well known that this aggregator is strictly concave as long as $\epsilon - 1 < 1/\varphi$.

Since the flow utility is concave in $\{\ell_{kt}\}$ at each t , the objective function must also be concave in $\{\ell_{kt}\}$. As a result, any solution to platform k 's problem must be unique.

Now suppose that $\epsilon - 1 > 1/\varphi$. I claim that there can not exist an equilibrium. Fix an arbitrary strategy $\{\ell_{kt}\}$ and then consider scaling up by some factor χ . For large χ , flow utility is determined primarily by the term

$$\chi^{\varphi(\epsilon-1)} \pi_{\mathbb{K}t} \frac{\left(e^{-\delta t} \int_0^t e^{\delta s} \ell_{ks}^\varphi ds \right)^{\epsilon-1}}{\int_{\mathbb{K}} q_{zt}^{\epsilon-1} dz} - \chi \ell_{kt}.$$

Thus if $\varphi(\epsilon - 1) > 1$, it is possible for the platform to achieve an arbitrarily high value for the objective simply by scaling up χ . $\square$

Lemma 3. *There exists a unique solution to the ODE system*

$$\begin{aligned} \dot{\ell}_{kt} &= \frac{\rho + \delta}{1 - \varphi} \ell_{kt} - \frac{\varphi}{1 - \varphi} \frac{\pi_{\mathbb{K}t} A(\epsilon - 1)}{K q_{kt}} \ell_{kt}^\varphi \\ \dot{q}_{kt} &= \ell_{kt}^\varphi - \delta q_{kt} \end{aligned} \tag{36}$$

for the whole domain $t \in [0, \infty)$ for any positive initial conditions (ℓ_{k0}, q_{k0}) . Moreover, the solution for investment $\{\ell_{kt}\}$ is weakly increasing and continuous in its initial condition ℓ_{k0} . Moreover, if $\ell_{10} > \ell_{20}$, then $\ell_{1t} > \ell_{2t}$ for every t such that $\ell_{2t} > 0$.

Proof. Since the system can be written in standard form as $[\dot{\ell}_{kt} \ \dot{q}_{kt}] = f(t, [\ell_{kt} \ q_{kt}])$, where f is continuous in t and locally Lipschitz in $[\ell_{kt} \ q_{kt}]$ on $(0, \infty)^2$, the existence and uniqueness of local positive solutions follows from Picard–Lindelöf. These properties also imply that solutions are continuous in initial conditions. Observe that the ODE for ℓ_{kt} has an absorbing point at zero and by Gronwall’s inequality

$$\ell_{kt} \leq \ell_{k0} e^{\frac{\rho+\delta}{1-\varphi} t}$$

whenever a solution exists. If ℓ_{kt} reaches zero, we continue the solution at the absorbing boundary $\ell_{kt} = 0$, which yields the unique nonnegative continuation. Similarly from the ODE for q_{kt} we have

$$q_{kt} = e^{-\delta t} q_0 + \int_0^t e^{-\delta(t-s)} \ell_{ks}^\varphi ds \leq e^{-\delta t} q_0 + \int_0^t e^{-\delta(t-s)} \ell_{k0}^\varphi e^{\varphi \frac{\rho+\delta}{1-\varphi} s} ds.$$

But then the solution of the ODE system can not explode in finite time or become negative and so the maximal domain of existence must be all of $t \in [0, \infty)$.

To see that $\{\ell_{kt}\}$ is monotone in ℓ_{k0} divide both sides of the ODE for ℓ_{kt} by ℓ_{kt} . Then we have

$$\frac{\dot{\ell}_{kt}}{\ell_{kt}} = \frac{\rho + \delta}{1 - \varphi} - \frac{\varphi}{1 - \varphi} \ell_{kt}^{\varphi-1} \frac{\pi_{\mathbb{K}t} A(\epsilon - 1)}{K q_{kt}}.$$

Let $f_t = \ln(\ell_{kt})$. It suffices to show $\{f_t\}$ is monotone in f_0 . We have

$$\dot{f}_t = \frac{\rho + \delta}{1 - \varphi} - \frac{\varphi}{1 - \varphi} (e^{f_t})^{\varphi-1} \frac{\pi_{\mathbb{K}t} A(\epsilon - 1)}{K q_{kt}}$$

with

$$\dot{q}_{kt} = e^{\varphi f_t} - \delta q_{kt}.$$

Now I observe that $\dot{f}$ is increasing in both f_t and q_{kt} with the former being true since $\varphi < 1$. Moreover, any solution for q_{kt} is monotone in $\{f_s, s \leq t\}$.

Let $f_{10} > f_{20}$ be two initial conditions, with corresponding solutions $\{f_{1t}\}$ and $\{f_{2t}\}$. Suppose, toward a contradiction, that the two solutions meet at some time before the lower investment path reaches zero, and let τ be the first such time. Then $f_{1t} > f_{2t}$ on $[0, \tau)$ and $f_{1\tau} = f_{2\tau}$. Since q_{kt} is strictly increasing in the history $\{f_s : s \leq t\}$, it follows that $q_{1\tau} > q_{2\tau}$. Evaluating the differential equations for f_{1t} and f_{2t} at τ , the equality $f_{1\tau} = f_{2\tau}$ and the strict monotonicity of $\dot{f}_t$ in q_{kt} imply $\dot{f}_{1\tau} > \dot{f}_{2\tau}$. On the other hand, since $f_{1t} - f_{2t} > 0$ for $t < \tau$ and $f_{1\tau} - f_{2\tau} = 0$, we must have $\dot{f}_{1\tau} - \dot{f}_{2\tau} \leq 0$, a contradiction. Hence $\ell_{1t} > \ell_{2t}$ as long as the lower solution remains positive. If the lower solution reaches zero, then $\ell_{1t} \geq \ell_{2t} = 0$ thereafter by nonnegativity and the absorbing-boundary convention. Therefore ℓ_{kt} is weakly increasing in ℓ_{k0} . $\square$

Lemma 4. *Any solution of the ODE system (36) for investment ℓ_{kt} for any initial conditions either diverges, vanishes, or converges to steady state.*

Proof. In what follows, let (ℓ_{SS}, q_{SS}) denote the unique steady state of the ODE system (36).

Step 1: If $q_{kt} \rightarrow q_{SS}$, then $\ell_{kt} \rightarrow \ell_{SS}$.

Suppose $q_{kt} \rightarrow q_{SS}$ and fix $\alpha > 0$. There exists $\zeta > 0$ sufficiently small such that

$$\frac{\rho + \delta}{1 - \varphi} - \frac{\varphi}{1 - \varphi} \frac{(\pi_{\mathbb{K}} + \zeta) A(\epsilon - 1)}{K (q_{SS} - \zeta)} (\ell_{SS} + \alpha)^{\varphi-1} > 0. \quad (37)$$

Note that if α and ζ were zero, the left-hand side above would be zero, since (ℓ_{SS}, q_{SS}) is by definition a steady-state solution to (36).

Since $\pi_{\mathbb{K}t} \rightarrow \pi_{\mathbb{K}}$ and $q_{kt} \rightarrow q_{SS}$, for all t sufficiently large and any $\ell_{kt} \geq \ell_{SS} + \alpha$ we have

$$\begin{aligned} \frac{\dot{\ell}_{kt}}{\ell_{kt}} &= \frac{\rho + \delta}{1 - \varphi} - \frac{\varphi}{1 - \varphi} \frac{\pi_{\mathbb{K}t} A(\epsilon - 1)}{K q_{kt}} \ell_{kt}^{\varphi-1} \\ &\geq \frac{\rho + \delta}{1 - \varphi} - \frac{\varphi}{1 - \varphi} \frac{(\pi_{\mathbb{K}} + \zeta) A(\epsilon - 1)}{K (q_{SS} - \zeta)} (\ell_{SS} + \alpha)^{\varphi-1} \\ &> 0, \end{aligned}$$

where the first inequality uses $\pi_{\mathbb{K}t} < \pi_{\mathbb{K}} + \zeta$, $q_{kt} > q_{SS} - \zeta$, and (since $\varphi - 1 < 0$) $\ell_{kt}^{\varphi-1} \leq (\ell_{SS} + \alpha)^{\varphi-1}$ for $\ell_{kt} \geq \ell_{SS} + \alpha$.

Because $\ell^{\varphi-1}$ is decreasing, the inequality above shows that $\dot{\ell}_{kt}/\ell_{kt} > 0$ at every point with $\ell_{kt} \geq \ell_{SS} + \alpha$; hence the region $\{\ell \geq \ell_{SS} + \alpha\}$ is forward-invariant. Consequently, if $\ell_{kt} > \ell_{SS} + \alpha$ at some large t , then $\ell_{ks} > \ell_{SS} + \alpha$ for all $s \geq t$, the bound (37) applies throughout, and ℓ_{ks} diverges. But then $q_{kt} \rightarrow q_{SS}$ cannot hold. By the symmetric argument, $\dot{\ell}_{kt}/\ell_{kt} < 0$ whenever $\ell_{kt} \leq \ell_{SS} - \alpha$, so $\{\ell \leq \ell_{SS} - \alpha\}$ is forward-invariant; if $\ell_{kt} < \ell_{SS} - \alpha$ at some large t , then ℓ_{ks} vanishes and again $q_{kt} \rightarrow q_{SS}$ cannot hold. Therefore, for all large t , $\ell_{kt} \in [\ell_{SS} - \alpha, \ell_{SS} + \alpha]$. Since $\alpha > 0$ was arbitrary, $\ell_{kt} \rightarrow \ell_{SS}$.

Step 2: If $q_{kt} \nrightarrow q_{SS}$, then ℓ_{kt} diverges or vanishes.

Since $q_{kt} \nrightarrow q_{SS}$, there exists $\alpha > 0$ such that either $q_{kt} \geq q_{SS} + \alpha$ infinitely often or $q_{kt} \leq q_{SS} - \alpha$ infinitely often (or both). Fix such an α and take $\zeta > 0$ small and t large enough that $|\pi_{\mathbb{K}t} - \pi_{\mathbb{K}}| < \zeta$. We treat the two cases in turn.

1. **Case 1: $q_{kt} \geq q_{SS} + \alpha$ infinitely often.** Let ℓ^* solve $(\ell^*)^\varphi = \delta(q_{SS} + \alpha)$; this is the investment level that holds quality steady at $q_{SS} + \alpha$. Consider the first time τ at which $q_{k\tau} = q_{SS} + \alpha$. Since q reaches this level from below, $\dot{q}_{k\tau} \geq 0$, i.e. $\ell_{k\tau}^\varphi \geq \delta(q_{SS} + \alpha) = (\ell^*)^\varphi$, so $\ell_{k\tau} \geq \ell^*$.

For ζ sufficiently small, at any point with $\ell \geq \ell^*$ and $q \geq q_{SS} + \alpha$,

$$\begin{aligned} \frac{\dot{\ell}_{kt}}{\ell_{kt}} &= \frac{\rho + \delta}{1 - \varphi} - \frac{\varphi}{1 - \varphi} \frac{\pi_{\mathbb{K}t} A(\epsilon - 1)}{K q_{kt}} \ell_{kt}^{\varphi-1} \\ &> \frac{\rho + \delta}{1 - \varphi} - \frac{\varphi}{1 - \varphi} \frac{(\pi_{\mathbb{K}} + \zeta) A(\epsilon - 1)}{K (q_{SS} + \alpha)} (\ell^*)^{\varphi-1} \\ &> 0, \end{aligned}$$

where the inequality uses $q_{kt} \geq q_{SS} + \alpha$ and, since $\varphi - 1 < 0$, $\ell_{kt}^{\varphi-1} \leq (\ell^*)^{\varphi-1}$ for $\ell_{kt} \geq \ell^*$; both move the expression above its steady-state value of zero.

Two consequences follow. First, $\dot{\ell} > 0$ on this region, so ℓ cannot fall through ℓ^* . Second, $\ell \geq \ell^*$ gives $\dot{q} = \ell^\varphi - \delta q \geq 0$ at $q = q_{SS} + \alpha$, so q cannot fall through $q_{SS} + \alpha$. Hence the region $\{\ell \geq \ell^*, q \geq q_{SS} + \alpha\}$ is forward-invariant. The trajectory enters it at τ and remains, so the growth bound holds for all $s \geq \tau$, and $\ell_{ks} \rightarrow \infty$.

2. **Case 2: $q_{kt} \leq q_{SS} - \alpha$ infinitely often.** The argument mirrors Case 1. Let ℓ^{**} solve $(\ell^{**})^\varphi = \delta(q_{SS} - \alpha)$. At the first time q falls to $q_{SS} - \alpha$, q reaches the level from above, so $\dot{q} \leq 0$, giving $\ell^\varphi \leq \delta(q_{SS} - \alpha) = (\ell^{**})^\varphi$ and hence $\ell \leq \ell^{**}$. For ζ small, at any point with $\ell \leq \ell^{**}$ and $q \leq q_{SS} - \alpha$ the analogous bound gives $\dot{\ell}_{kt}/\ell_{kt} < 0$, so ℓ cannot rise through ℓ^{**} ; and $\ell \leq \ell^{**}$ gives $\dot{q} \leq 0$ at $q = q_{SS} - \alpha$, so q cannot rise through $q_{SS} - \alpha$. Thus $\{\ell \leq \ell^{**}, q \leq q_{SS} - \alpha\}$ is forward-invariant; the trajectory enters it and remains, and $\ell_{ks} \rightarrow 0$.

In either case ℓ_{kt} diverges or vanishes, completing the proof. $\square$

Lemma 5. *There exists a unique solution to the boundary-value problem (33).*

Proof. Uniqueness. Suppose for contradiction there are two solutions ℓ_{kt} and $\hat{\ell}_{kt}$ with initial conditions ℓ_{k0} and $\hat{\ell}_{k0}$, and suppose $\hat{\ell}_{k0} > \ell_{k0}$. By Lemma 3, since $\hat{\ell}_{kt} > \ell_{kt}$, it follows that $\dot{\hat{\ell}}_{kt} > \dot{\ell}_{kt}$ and therefore

$$\ln(\hat{\ell}_{kt}) - \ln(\ell_{kt}) \geq \ln(\hat{\ell}_{k0}) - \ln(\ell_{k0}) > 0$$

for all t . If both solutions converged to $\ell_{\mathbb{K}}$, this difference would tend to zero, contradicting the strictly positive lower bound. Hence at most one solution satisfies the terminal condition.

Existence. Consider a version of (33) in which the boundary at ∞ is replaced by a boundary at some finite $T > 0$. If a solution to this finite-horizon problem exists it is unique, by an argument analogous to the one just given. It is straightforward from (36) that increasing ℓ_{k0} makes ℓ_{kT} arbitrarily large, while decreasing ℓ_{k0} makes ℓ_{kT} arbitrarily close to zero: the right-hand side of (36) grows without bound as ℓ_{kt} diverges and is negative for all ℓ_{kt} in a neighborhood of zero. Thus, by continuity in the initial condition (Lemma 3), there is an initial condition, which we denote $\ell_{k0}(T)$, for which $\ell_{kT} = \ell_{SS}$; the corresponding solution to the initial value problem solves the boundary value problem with boundary at T .

Now take a sequence $\{T_n\}$ with $\lim_{n \rightarrow \infty} T_n = \infty$, and consider the corresponding solutions $\{\ell_{kt}(T_n)\}$, each extended to all t . By Lemma 4, each

such solution diverges, vanishes, or converges to steady state. Hence there are either infinitely many that diverge or infinitely many that vanish.

Suppose first that infinitely many diverge. Define

$$\ell_{k0}^* = \inf\{\ell_{k0}(T_n) \mid \text{the extended solution with this initial condition diverges}\},$$

and let ℓ_{kt}^* be the solution with initial condition ℓ_{k0}^* . We claim ℓ_{kt}^* converges to steady state, and therefore solves the original boundary value problem with boundary at ∞ . Suppose not. By Lemma 4, ℓ_{kt}^* then either diverges or vanishes.

1. **Suppose ℓ_{kt}^* diverges.** Let $T^* = \sup\{t \mid \ell_{kt}^* = \ell_{SS}\}$. Since ℓ_{kt}^* diverges, it eventually exceeds ℓ_{SS} permanently, so $\{t \mid \ell_{kt}^* = \ell_{SS}\}$ is bounded above; hence if nonempty, T^* is finite. To see it is nonempty, suppose instead it is empty, so ℓ_{kt}^* never equals ℓ_{SS} ; since it diverges, $\ell_{kt}^* > \ell_{SS}$ for all t . By monotonicity in the initial condition (Lemma 3), every solution with initial condition $\geq \ell_{k0}^*$ then also remains above ℓ_{SS} for all t . But each $\ell_{k0}(T_n)$ in the infimum defining ℓ_{k0}^* satisfies $\ell_{k0}(T_n) \geq \ell_{k0}^*$ and is, by construction, the initial condition of a solution with $\ell_{kT_n} = \ell_{SS}$ — a contradiction. Hence the set is nonempty and T^* is finite.

Thus ℓ_{kt}^* solves the boundary value problem with boundary at T^* . Now choose $T_n > T^*$ such that the extended solution to the boundary problem at T_n diverges; this is possible since these solutions, being at ℓ_{SS} but generically not at q_{SS} at time T_n , diverge or vanish by Lemma 4, and infinitely many diverge by assumption. This solution reaches ℓ_{SS} at $T_n > T^*$, that is, later than ℓ_{kt}^* does. By monotonicity in the initial condition, a later crossing of ℓ_{SS} corresponds to a smaller initial condition, so $\ell_{k0}(T_n) < \ell_{k0}^*$. This contradicts $\ell_{k0}^* = \inf_n \ell_{k0}(T_n)$.

2. **Suppose ℓ_{kt}^* vanishes.** Inspecting (36), there exist $\underline{\ell} > 0$ and $\underline{q} > 0$ such that if $\ell_{kt} < \underline{\ell}$ and $q_{kt} < \underline{q}$ at any time, then ℓ_{kt} vanishes; that is, the region $\{\ell < \underline{\ell}, q < \underline{q}\}$ is absorbing. Since ℓ_{kt}^* vanishes, fix $\alpha > 0$ and let T^* be the first time at which $\ell_{kT^*}^* < \underline{\ell} - \alpha$ and $q_{kT^*}^* < \underline{q} - \alpha$. Perturb the initial condition ℓ_{k0}^* upward by an arbitrarily small amount. By the definition of ℓ_{k0}^* , the perturbed solution diverges. Since any trajectory entering $\{\ell < \underline{\ell}, q < \underline{q}\}$ vanishes, a diverging trajectory never enters this region; in particular, at the fixed time T^* the perturbed trajectory satisfies $\ell_{kT^*} \geq \underline{\ell}$ or $q_{kT^*} \geq \underline{q}$, whereas $\ell_{kT^*}^* < \underline{\ell} - \alpha$ and $q_{kT^*}^* < \underline{q} - \alpha$. This jump of at least α , produced by an arbitrarily small perturbation of the initial condition, contradicts continuity of solutions in the initial condition at the fixed finite time T^* (Lemma 3).

In both cases we reach a contradiction, so ℓ_{kt}^* converges to steady state and solves the boundary value problem.

If instead infinitely many of the extended solutions vanish, the symmetric argument applies, with ℓ_{k0}^* redefined as the supremum of the initial conditions whose extended solutions vanish; the analogous contradictions show ℓ_{kt}^* converges to steady state. Since one of the two cases must hold, existence follows. $\square$

C Supplemental Material for Section 5

Appendix C.1 proves Proposition 2. Appendix C.2 proves Proposition 3. Appendix C.4 presents additional comparative statics omitted from the main text.

C.1 Proof of Proposition 2

Proof of Proposition 2. The proof proceeds via two Lemmas.

Lemma 6 proves that product consumption is monotone in data informativeness.

Lemma 6. *An increase in G in the mean-preserving spread order leads to a weak increase in the steady state value of $M\mu_H$ and product consumption C and a decrease in the cdf H^c in second-order stochastic dominance.*

Proof. Suppose that G increases in the mean-preserving spread order to $\hat{G}$. Let $\hat{H}^c$ denote the steady state cdf under $\hat{G}$. Define

$$\gamma(y) = \hat{G}(y) - G(y) \quad \text{and} \quad \nu(y) = \hat{H}^c(y) - H^c(y).$$

By Part 3 of Theorem 1,

$$J\hat{G}(y) = M\hat{H}^c(y)^N + (J - M)\hat{H}^c(y)$$

and

$$JG(y) = MH^c(y)^N + (J - M)H^c(y).$$

Subtracting the second equation from the first gives

$$J\gamma(y) = M \left[\hat{H}^c(y)^N - H^c(y)^N \right] + (J - M)\nu(y).$$

Using the identity

$$a^N - b^N = (a - b) \sum_{k=0}^{N-1} a^{N-1-k} b^k,$$

it follows that

$$\gamma(y) = Q(y)\nu(y),$$

where

$$Q(y) = \frac{J-M}{J} + \frac{M}{J} \sum_{k=0}^{N-1} \hat{H}^c(y)^{N-1-k} H^c(y)^k.$$

Observe that Q is nondecreasing and satisfies

$$Q(y) \geq \frac{J-M}{J} > 0.$$

Define

$$V(s) = \int_0^s \nu(y) \, dy.$$

Since $\hat{G}$ is a mean-preserving spread of G ,

$$\int_0^s \gamma(y) \, dy \geq 0$$

for every $s \geq 0$. Integrating by parts yields

$$Q(s)V(s) - \int_0^s V(y) \, dQ(y) \geq 0. \quad (38)$$

We claim that $V(s) \geq 0$ for every $s \geq 0$. Suppose for contradiction that there exists $s_1 > 0$ such that

$$V(s_1) < 0.$$

Since V is continuous and $V(0) = 0$, there exists $s^* \in (0, s_1]$ at which V attains its minimum on $[0, s_1]$. Hence,

$$V(s^*) < 0 \quad \text{and} \quad V(y) \geq V(s^*)$$

for every $y \in [0, s^*]$. Since Q is nondecreasing,

$$\int_0^{s^*} V(y) \, dQ(y) \geq V(s^*) [Q(s^*) - Q(0)].$$

Therefore, the left-hand side of (38) evaluated at s^* is at most

$$Q(s^*)V(s^*) - V(s^*) [Q(s^*) - Q(0)] = Q(0)V(s^*) < 0,$$

contradicting (38). Thus,

$$\int_0^s \nu(y) \, dy = V(s) \geq 0$$

for every $s \geq 0$. Strict inequality holds for some s , since otherwise $\nu \equiv 0$, which together with $\gamma(y) = Q(y)\nu(y)$ implies $\gamma \equiv 0$, contradicting the assumption that $\hat{G}$ is a nontrivial mean-preserving spread of G .

Since $V \geq 0$, it follows that $\hat{H}^c$ is lower than H^c in second order stochastic dominance and thus $\mu_{\hat{H}^c} \leq \mu_{H^c}$. This in turn implies $\mu_{\hat{H}} \geq \mu_H$. $\square$

Next, Lemma 7 shows that ad revenue is maximal in a limiting sense when data is uninformative. From Part 5 of Theorem 1, platform quality is then also maximal. To make the notation explicit, write $\pi_{\mathbb{K}}(G)$ for the steady-state average ad price as a function of the cdf G of expected values.

Lemma 7. *Let $\{G_n\}$ be a sequence of continuous cdfs converging pointwise to the Heaviside function centered at μ_F : $\lim_{n \rightarrow \infty} G_n(\hat{v}) = \mathbb{1}_{\{\hat{v} \geq \mu_F\}}$ for each $\hat{v} \in [0, \infty)$. Then $\lim_{n \rightarrow \infty} \pi_{\mathbb{K}}(G_n) = \sup_G \pi_{\mathbb{K}}(G)$ where the supremum is over all continuous cdfs G supported on $[0, \infty)$.*

Proof. Let G be arbitrary. Using (2) and (31), both evaluated in steady state, we have

$$\begin{aligned} \pi_{\mathbb{K}}(G) &= \mathbb{E}[B(\hat{v}^{(2)})] \\ &= \mathbb{E} \left[\pi_{\mathbb{J}} \int_0^{\hat{v}^{(2)}} \frac{1}{\rho + \lambda_f + \lambda_e(s)} ds \right] \\ &\leq \frac{I \mathbb{E}[\hat{v}^{(2)}]}{\sigma M \mu_H (\rho + \lambda_f)} \\ &\leq \frac{I \mathbb{E}[\hat{v}^{(1)}]}{\sigma M \mu_H (\rho + \lambda_f)} \\ &= \frac{I}{\sigma M (\rho + \lambda_f)} \end{aligned}$$

where above $\hat{v}^{(2)} \sim (H^c)^N + N(H^c)^{(N-1)}(1 - H^c)$ and $\hat{v}^{(1)} \sim (H^c)^N$. In the fourth line we use the fact that in steady state $H = (H^c)^N$. The notation has suppressed the dependency of H^c and H on G .

Using (17) in Theorem 1 we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \pi_{\mathbb{K}}(G_n) &= \lim_{n \rightarrow \infty} \frac{I}{\sigma M \int_0^\infty [1 - H(s, G_n)] ds} \\
&\times \int_0^\infty \frac{1 - NH^c(s, G_n)^{N-1} + (N-1)H^c(s, G_n)^N}{\rho + \lambda_f + \lambda_a H^c(s, G_n)^{N-1}} ds \\
&= \frac{I}{\sigma M \mu_F} \int_0^\infty \frac{1 - N\mathbb{1}_{s \geq \mu_F} + (N-1)\mathbb{1}_{s \geq \mu_F}}{\rho + \lambda_f + \lambda_a \mathbb{1}_{s \geq \mu_F}} ds \\
&= \frac{I}{\sigma M \mu_F} \int_0^{\mu_F} \frac{1}{\rho + \lambda_f} ds \\
&= \frac{I}{\sigma M(\rho + \lambda_f)}
\end{aligned}$$

where in the second equality I have used the dominated convergence theorem to pass the limit through the integral. Above I have made explicit the dependency of H^c on G_n in the notation. $\square$

The proof of Proposition 2 is complete. $\square$

C.2 Proof of Proposition 3

Proof of Proposition 3. The proof proceeds via a series of lemmas.

Lemma 8. *An increase in ϵ leads to a decrease in the ad display rate A .*

Proof. By Theorem 1, A is equal to

$$\arg \max_a a \nu(a)^{\epsilon-1} = \arg \max_a \ln(a) + (\epsilon - 1) \ln(\nu(a)). \quad (39)$$

Pick $\epsilon_2 > \epsilon_1$, and let a_1 solve (39) when $\epsilon = \epsilon_1$ and define a_2 analogously. By unique optimality,

$$\ln a_1 + (\epsilon_1 - 1) \ln \nu(a_1) > \ln a_2 + (\epsilon_1 - 1) \ln \nu(a_2)$$

and

$$\ln a_2 + (\epsilon_2 - 1) \ln \nu(a_2) > \ln a_1 + (\epsilon_2 - 1) \ln \nu(a_1).$$

Add these two equations and cancel terms:

$$(\epsilon_2 - \epsilon_1)(\ln \nu(a_2) - \ln \nu(a_1)) > 0.$$

Since $\epsilon_2 - \epsilon_1 > 0$ and $\ln \nu(\cdot)$ is decreasing, this implies $a_2 < a_1$. $\square$

Combining Lemma 8 and the following shows that an increase in ϵ leads to a decrease in product consumption.

Lemma 9. *Starting from steady state, an increase in A leads to a decrease in H_t and H_t^c in first-order stochastic dominance and an increase in $M_t\mu_{H_t}$ and C_{it} at all t .*

Proof. Recall from Step 3 of Appendix A.3 that

$$\dot{H}_t^c(\cdot) = \lambda_f \frac{JG(\cdot) - MH_t^c(\cdot)^N - (J - M)H_t^c(\cdot)}{J - M_t}.$$

Suppose the economy begins in the steady state associated with M_0 . Then

$$JG(\cdot) - M_0H_0^c(\cdot)^N - (J - M_0)H_0^c(\cdot) = 0.$$

Following an increase in A , we have $M > M_0$, so

$$\begin{aligned} JG(\cdot) - MH_0^c(\cdot)^N - (J - M)H_0^c(\cdot) \\ = (M - M_0) [H_0^c(\cdot) - H_0^c(\cdot)^N] \geq 0. \end{aligned}$$

Moreover,

$$\frac{\partial}{\partial H_t^c} (JG - M(H_t^c)^N - (J - M)H_t^c) = -MN(H_t^c)^{N-1} - (J - M) < 0,$$

so the drift is strictly decreasing in H_t^c and has a unique zero, namely the new steady state. Since $J - M_t > 0$, the drift is positive whenever H_t^c lies below the new steady state and vanishes at the new steady state. Hence H_t^c increases pointwise over time following the increase in A . Equivalently, H_t^c decreases in the sense of first-order stochastic dominance.

Since

$$M_t\mu_{H_t} = J\mu_G - (J - M_t)\mu_{H_t^c},$$

and both M_t increases and $\mu_{H_t^c}$ decreases, it follows that $M_t\mu_{H_t}$ increases. By Part 7 of Theorem 1,

$$C_{it} = \frac{\sigma - 1}{\sigma} I(M_t\mu_{H_t})^{\frac{1}{\sigma-1}},$$

so C_{it} also increases.

Finally, recall that

$$\dot{H}_t = \frac{A}{M_t} [(H_t^c)^N - H_t].$$

Because the economy is initially in steady state,

$$H_0 = (H_0^c)^N.$$

Moreover, H_t^c is nondecreasing over time, so $(H_t^c)^N$ is a nondecreasing target. It follows from the differential equation that

$$H_t \leq (H_t^c)^N$$

at every t .

Now consider two post-change values $A' > A$. Since

$$M_t(A) = M_0 e^{-\lambda_f t} + \frac{A}{\lambda_f} (1 - e^{-\lambda_f t}),$$

we have

$$\frac{A}{M_t(A)} = \frac{A}{M_0 e^{-\lambda_f t} + \frac{A}{\lambda_f} (1 - e^{-\lambda_f t})},$$

which is increasing in A . We have also shown that

$$H_t^c(A') \geq H_t^c(A)$$

pointwise.

Suppose the two solutions for H_t coincide at some value h . Since $h \leq (H_t^c(A))^N$, their drifts satisfy

$$\begin{aligned} & \frac{A'}{M_t(A')} [(H_t^c(A'))^N - h] - \frac{A}{M_t(A)} [(H_t^c(A))^N - h] \\ &= \left[\frac{A'}{M_t(A')} - \frac{A}{M_t(A)} \right] [(H_t^c(A))^N - h] \\ & \quad + \frac{A'}{M_t(A')} [(H_t^c(A'))^N - (H_t^c(A))^N] \geq 0. \end{aligned}$$

A standard scalar comparison argument therefore implies that $H_t(A') \geq H_t(A)$ pointwise at every t . Thus H_t decreases in the sense of first-order stochastic dominance. $\square$

Combining Lemma 8 and the following lemma shows that an increase in ϵ leads to an increase in ad revenue $\pi_{\mathbb{K}} A$.

Lemma 10. *An increase in A leads to a decrease in steady-state ad revenue $\pi_{\mathbb{K}} A$ holding H and H^c fixed while allowing M to vary according to $M = A/\lambda_f$.*

Proof. From Part 4 of Theorem 1, steady-state ad revenue is

$$\pi_{\mathbb{K}}A = \frac{IA}{\sigma M \mu_H} \int_0^\infty \frac{1 - NH^c(s)^{N-1} + (N-1)H^c(s)^N}{\rho + \lambda_f + \lambda_e(s)} ds,$$

where $\lambda_e(s) = \lambda_a H^c(s)^{N-1}$.

Moreover, Parts 2 and 3 of Theorem 1 imply

$$M = \frac{A}{\lambda_f}, \quad H(s) = H^c(s)^N, \quad \mu_H = \int_0^\infty [1 - H^c(s)^N] ds.$$

Therefore,

$$\pi_{\mathbb{K}}A = \frac{I\lambda_f}{\sigma} \frac{\int_0^\infty \frac{1 - NH^c(s)^{N-1} + (N-1)H^c(s)^N}{\rho + \lambda_f + \lambda_a H^c(s)^{N-1}} ds}{\int_0^\infty [1 - H^c(s)^N] ds}.$$

Since an increase in A leads to an increase in $\lambda_a = NA/(J-M)$, $\pi_{\mathbb{K}}A$ decreases. $\square$

This completes the proof of the proposition. $\square$

C.3 Sufficient Condition and Convex-Transform Order

I develop a sufficient condition for an increase in ϵ to lead to an increase in ad revenue and platform quality in steady state taking into account the equilibrium effects on H and H^c .

I begin by defining the convex-transform order following Shaked and Shanthikumar (2007, Sec. 4.B).

Definition 3 (Convex-transform order). *Let F and G be continuous cdfs that are strictly increasing on their respective interval supports, which are contained in $[0, \infty)$. Then F is smaller than G in the convex-transform order, written $F \leq_c G$, if $G^{-1} \circ F$ is convex on the support of F .*

The relation $F \leq_c G$ formalizes the statement that G is at least as right-skewed as F . When both distributions are smooth, convexity of the transform is equivalent to monotonicity of the ratio of quantile densities. The equivalence, stated in Lemma 11 and proven formally below, is asserted without proof in the proof of Theorem 4.B.8 of Shaked and Shanthikumar (2007, p. 217).

Lemma 11 (Quantile-density characterization). *Let F and G be cdfs supported on intervals in $[0, \infty)$, with densities f and g that are continuous and strictly positive on the interiors of their supports. Write $Q_F := F^{-1}$ and $Q_G := G^{-1}$ for the quantile functions, and let*

$$q_F(p) := Q'_F(p) = \frac{1}{f(Q_F(p))}, \quad q_G(p) := Q'_G(p) = \frac{1}{g(Q_G(p))}, \quad p \in (0, 1),$$

denote the associated quantile densities. Then

$$F \leq_c G \iff p \mapsto \frac{q_G(p)}{q_F(p)} \text{ is nondecreasing on } (0, 1),$$

and $G^{-1} \circ F$ is strictly convex if and only if q_G/q_F is strictly increasing.

Proof. Under the stated assumptions, F maps the interior I_F of its support one-to-one onto $(0, 1)$, and likewise G ; hence Q_F and Q_G are continuously differentiable with the stated derivatives, and $\varphi := Q_G \circ F$ is continuously differentiable on I_F with

$$\varphi'(x) = q_G(F(x)) f(x) = \frac{f(x)}{g(Q_G(F(x)))} > 0, \quad x \in I_F. \quad (40)$$

A continuously differentiable function on an interval is convex if and only if its derivative is nondecreasing, and strictly convex if and only if its derivative is strictly increasing. Since $Q_F : (0, 1) \rightarrow I_F$ is an increasing bijection, φ' is nondecreasing on I_F if and only if $p \mapsto \varphi'(Q_F(p))$ is nondecreasing on $(0, 1)$. Substituting $x = Q_F(p)$ into (40) and using $F(Q_F(p)) = p$ gives

$$\varphi'(Q_F(p)) = q_G(p) f(Q_F(p)) = \frac{q_G(p)}{q_F(p)},$$

which proves the claim. $\square$

I now state a sufficient condition for monotone comparative statics in Proposition 3 to hold taking into account the full equilibrium feedback.

For each $m \in (0, 1)$, let H_m^c be the unique cdf satisfying

$$m H_m^c(\hat{v})^N + (1 - m) H_m^c(\hat{v}) = G(\hat{v}), \quad \hat{v} \in \mathbb{R}. \quad (41)$$

In equilibrium, the steady-state distribution satisfies $H^c = H_{M/J}^c$ since $M(H^c)^N + (J - M)H^c = JG$. The following is a condition on the primitives N and G .

Condition 1. The cdf G is continuous with a strictly positive density on the interior of its support, and $\inf \text{supp } G = 0$. Moreover, for all $m_0 < m_1$ in $(0, 1)$, $H_{m_0}^c \leq_c H_{m_1}^c$.

Proposition 6. Under Condition 1, an increase in ϵ leads to an increase in ad revenue and platform quality in steady state.

Proof. From Part 4 of Theorem 1, steady-state ad revenue is

$$\pi_{\mathbb{K}} A = \frac{IA}{\sigma M \mu_H} \int_0^\infty \frac{1 - NH^c(s)^{N-1} + (N-1)H^c(s)^N}{\rho + \lambda_f + \lambda_e(s)} ds,$$

where $\lambda_e(s) = \lambda_a H^c(s)^{N-1}$.

Moreover, Parts 2 and 3 of Theorem 1 imply

$$M = \frac{A}{\lambda_f}, \quad H(s) = H^c(s)^N, \quad \mu_H = \int_0^\infty [1 - H^c(s)^N] ds.$$

Therefore,

$$\pi_{\mathbb{K}} A = \frac{I\lambda_f}{\sigma} \frac{\int_0^\infty \frac{1 - NH^c(s)^{N-1} + (N-1)H^c(s)^N}{\rho + \lambda_f + \lambda_a H^c(s)^{N-1}} ds}{\int_0^\infty [1 - H^c(s)^N] ds}.$$

Define

$$r_{\lambda_a}(u) \equiv \frac{1}{\rho + \lambda_f + \lambda_a u^{N-1}} \frac{1 - Nu^{N-1} + (N-1)u^N}{1 - u^N}.$$

Equivalently,

$$r_{\lambda_a}(u) = \frac{1}{\rho + \lambda_f + \lambda_a u^{N-1}} \left[N \frac{1 - u^{N-1}}{1 - u^N} - (N-1) \right].$$

Both factors on the right-hand side are nonnegative and decreasing in u . Hence r_{λ_a} is decreasing in u . It is also decreasing in λ_a .

Let Q denote the quantile function of H^c . Using that $\inf \text{supp } G = 0$, it follows that $\inf \text{supp } H^c = 0$ and therefore $Q(0) = 0$, so the change of variables $s = Q(u)$ gives

$$\pi_{\mathbb{K}} A = \frac{I\lambda_f}{\sigma} \frac{\int_0^1 r_{\lambda_a}(u)(1 - u^N)Q'(u) du}{\int_0^1 (1 - u^N)Q'(u) du}.$$

Define the probability measure μ_Q on $[0, 1]$ by

$$d\mu_Q(u) = \frac{(1 - u^N)Q'(u)}{\int_0^1 (1 - s^N)Q'(s) ds} du.$$

Then

$$\pi_{\mathbb{K}}A = \frac{I\lambda_f}{\sigma} \mathbb{E}_{\mu_Q} [r_{\lambda_a}(U)].$$

Now consider an increase in ϵ , and use hats to denote the resulting steady-state objects. By Lemma 8, $\hat{A} < A$. Because $M = A/\lambda_f$, it follows that $\hat{M} < M$. Furthermore, $\lambda_a = \frac{NA}{J-M}$, so the decline in A , together with the decline in M , implies $\hat{\lambda}_a < \lambda_a$. Consequently,

$$r_{\hat{\lambda}_a}(u) \geq r_{\lambda_a}(u) \quad \text{for every } u \in [0, 1].$$

Now compare the probability measures induced by Q and $\hat{Q}$. Their likelihood ratio satisfies

$$\frac{d\mu_{\hat{Q}}}{d\mu_Q}(u) = C \frac{\hat{Q}'(u)}{Q'(u)}$$

for a positive constant C . Since $\hat{M} < M$ while J is unchanged, $\hat{M}/J < M/J$; as $H^c = H_{M/J}^c$ and $\hat{H}^c = H_{\hat{M}/J}^c$, Condition 1 gives $\hat{H}^c \leq_c H^c$, so the right-hand side is nonincreasing in u by Lemma 11. Thus $\mu_{\hat{Q}}$ is smaller than μ_Q in the monotone-likelihood-ratio order. Because r_{λ_a} is decreasing,

$$\mathbb{E}_{\mu_{\hat{Q}}} [r_{\lambda_a}(U)] \geq \mathbb{E}_{\mu_Q} [r_{\lambda_a}(U)].$$

Combining this distributional effect with the pointwise effect of the decline in λ_a , we obtain

$$\mathbb{E}_{\mu_{\hat{Q}}} [r_{\hat{\lambda}_a}(U)] \geq \mathbb{E}_{\mu_Q} [r_{\lambda_a}(U)].$$

Therefore,

$$\hat{\pi}_{\mathbb{K}}\hat{A} \geq \pi_{\mathbb{K}}A.$$

The inequality is strict whenever either the likelihood-ratio shift or the decline in λ_a is nondegenerate. Hence an increase in ϵ raises steady-state ad revenue.

Finally, steady-state platform investment and quality are increasing in ad revenue by Theorem 1. It follows that platform quality also increases. $\square$

Lemma 12. *If $G(x) = (x/b)^\alpha$ on $[0, b]$ with $\alpha \in (0, 1]$, then Condition 1 is satisfied.*

Proof. The cdf G is continuous with density $g(x) = (\alpha/b)(x/b)^{\alpha-1}$, strictly positive on $(0, b)$, and $\inf \text{supp } G = 0$. It remains to verify the ordering. Fix $m_0 < m_1$ in $(0, 1)$ and write

$$\psi_m(u) := mu^N + (1 - m)u, \quad \psi'_m(u) = Nmu^{N-1} + (1 - m), \quad u \in [0, 1].$$

Step 1: Quantile functions. From equation (41), $u = H_m^c(\hat{v})$ if and only if $\psi_m(u) = G(\hat{v})$. Hence, at the quantile level u , the quantile function $Q_m := (H_m^c)^{-1}$ satisfies

$$G(Q_m(u)) = \psi_m(u) \iff \left(\frac{Q_m(u)}{b} \right)^\alpha = \psi_m(u).$$

Solving for $Q_m(u)$ yields

$$Q_m(u) = b \psi_m(u)^{1/\alpha}.$$

Step 2: Quantile densities. Differentiating by the chain rule,

$$Q'_m(u) = \frac{b}{\alpha} \psi_m(u)^{\frac{1}{\alpha}-1} \psi'_m(u) > 0 \quad \text{on } (0, 1).$$

Taking the ratio across m_1 and m_0 :

$$\frac{Q'_{m_1}(u)}{Q'_{m_0}(u)} = \underbrace{\left(\frac{\psi_{m_1}(u)}{\psi_{m_0}(u)} \right)^{\frac{1}{\alpha}-1}}_{\text{(I)}} \cdot \underbrace{\frac{\psi'_{m_1}(u)}{\psi'_{m_0}(u)}}_{\text{(II)}}. \quad (42)$$

By Lemma 11, it suffices to show that (42) is strictly increasing on $(0, 1)$. I will show that both factors (I) and (II) are monotone with factor (II) strictly monotone.

Step 3: Verify monotonicity. For a constant $c > 0$, consider on $t \geq 0$ the function

$$\rho_c(t) := \frac{cm_1t + (1 - m_1)}{cm_0t + (1 - m_0)}.$$

Numerator and denominator are positive for $t \geq 0$ (as $0 < m_i < 1$), and by the quotient rule,

$$\rho'_c(t) = \frac{cm_1(1 - m_0) - cm_0(1 - m_1)}{(cm_0t + (1 - m_0))^2} = \frac{c(m_1 - m_0)}{(cm_0t + (1 - m_0))^2} > 0.$$

So ρ_c is strictly increasing on $[0, \infty)$ for every $c > 0$. Substitute $t := u^{N-1}$, which is strictly increasing on $(0, 1)$ since $N \geq 2$. For factor (I):

$$\frac{\psi_{m_1}(u)}{\psi_{m_0}(u)} = \frac{m_1 u^{N-1} + (1 - m_1)}{m_0 u^{N-1} + (1 - m_0)} = \rho_1(t).$$

Since $1/\alpha \geq 1$, factor (I) is therefore weakly increasing in t and thus u . For factor (II),

$$\frac{\psi'_{m_1}(u)}{\psi'_{m_0}(u)} = \frac{N m_1 u^{N-1} + (1 - m_1)}{N m_0 u^{N-1} + (1 - m_0)} = \rho_N(t),$$

which is strictly increasing in t and thus u . Hence (42), the product of a positive nondecreasing factor and a positive strictly increasing factor, is strictly increasing on $(0, 1)$, and Lemma 11 yields $H_{m_0}^c \leq_c H_{m_1}^c$. $\square$

C.4 Additional Comparative Statics Omitted From the Main Text

Lemma 13. *Starting from steady state, an increase in J leads to an increase in H_t^c and H_t in the sense of first-order stochastic dominance and thus to an increase in $M_t \mu_{H_t}$ and C_{it} at all t .*

Proof. Fix an argument of the cdfs and suppress it throughout. Recall that

$$\dot{H}_t^c = \lambda_f \frac{J(G - H_t^c) + M(H_t^c - H_t^{cN})}{J - M_t}.$$

Suppose the economy initially lies in the steady state associated with J_0 . Then

$$J_0(G - H_0^c) + M(H_0^c - H_0^{cN}) = 0.$$

Following an increase from J_0 to $J > J_0$, the numerator of the post-change drift, evaluated at the initial steady state, is

$$J(G - H_0^c) + M(H_0^c - H_0^{cN}) = (J - J_0)(G - H_0^c).$$

Since

$$G - H_0^c = -\frac{M}{J_0} (H_0^c - H_0^{cN}) \leq 0,$$

it follows that the initial post-change drift satisfies

$$\dot{H}_0^c \leq 0.$$

Now define

$$Q_J(h) = J(G - h) + M(h - h^N).$$

Then

$$Q'_J(h) = -J + M(1 - Nh^{N-1}) \leq -J + M < 0,$$

so Q_J is strictly decreasing and has a unique zero, corresponding to the new steady state. Since $Q_J(H_0^c) \leq 0$, the solution decreases monotonically toward the new steady state. Hence H_t^c decreases pointwise. Equivalently, the distribution associated with H_t^c increases in the sense of first-order stochastic dominance.

Next, recall that

$$\dot{H}_t = \frac{A}{M_t} [(H_t^c)^N - H_t].$$

Neither A nor M_t changes with J . Moreover, because the economy begins in steady state,

$$H_0 = (H_0^c)^N.$$

Since H_t^c decreases pointwise, $(H_t^c)^N$ also decreases pointwise. Thus, at any point where two solutions for H_t coincide, the one associated with the larger value of J has a weakly smaller drift. A standard comparison argument for scalar differential equations therefore implies that H_t also decreases pointwise. Equivalently, the distribution associated with H_t increases in the sense of first-order stochastic dominance.

Since M_t is unaffected by J while μ_{H_t} increases, $M_t \mu_{H_t}$ increases. Finally, by Part 7 of Theorem 1,

$$C_{it} = \frac{\sigma - 1}{\sigma} I (M_t \mu_{H_t})^{\frac{1}{\sigma-1}},$$

so C_{it} also increases. □

Lemma 14. *Under Condition 1, an increase in J leads to an increase in steady-state ad revenue $\pi_{\mathbb{K}} A$.*

Proof. Let $\hat{J} > J$ and use hats to denote steady-state objects under $\hat{J}$. The ad display rate A maximizes $a \nu(a)^{\epsilon-1}$ and is therefore independent of J (Part 1 of Theorem 1), so $\hat{A} = A$ and, since $M = A/\lambda_f$, also $\hat{M} = M$. Two consequences. First,

$$\hat{\lambda}_a = \frac{NA}{\hat{J} - M} < \frac{NA}{J - M} = \lambda_a,$$

so $r_{\hat{\lambda}_a}(u) \geq r_{\lambda_a}(u)$ for every $u \in [0, 1]$, since r_{λ_a} is decreasing in λ_a . Second,

$$\hat{m} = \frac{M}{\hat{J}} < \frac{M}{J} = m,$$

so Condition 1 applied to the pair $\hat{m} < m$ gives $\hat{H}^c = H_{\hat{m}}^c \leq_c H_m^c = H^c$.

The representation

$$\pi_{\mathbb{K}}A = \frac{I\lambda_f}{\sigma} \mathbb{E}_{\mu_Q}[r_{\lambda_a}(U)]$$

derived in the proof of Proposition 6 holds at both J and $\hat{J}$: its derivation uses J only through H^c and λ_a , and $Q(0) = \hat{Q}(0) = 0$ since G is unchanged. By Lemma 11, $\hat{Q}'/Q'$ is nonincreasing, so $\mu_{\hat{Q}}$ is smaller than μ_Q in the monotone-likelihood-ratio order; because r_{λ_a} is decreasing,

$$\mathbb{E}_{\mu_{\hat{Q}}}[r_{\lambda_a}(U)] \geq \mathbb{E}_{\mu_Q}[r_{\lambda_a}(U)].$$

Combining this distributional effect with the pointwise effect of the decline in λ_a ,

$$\hat{\pi}_{\mathbb{K}}\hat{A} = \frac{I\lambda_f}{\sigma} \mathbb{E}_{\mu_{\hat{Q}}}[r_{\hat{\lambda}_a}(U)] \geq \frac{I\lambda_f}{\sigma} \mathbb{E}_{\mu_Q}[r_{\lambda_a}(U)] = \pi_{\mathbb{K}}A.$$

The inequality is strict: $\hat{\lambda}_a < \lambda_a$ strictly, $r_{\lambda_a}(u)$ is strictly decreasing in λ_a for $u \in (0, 1)$, and $\mu_{\hat{Q}}$ assigns positive mass to $(0, 1)$. $\square$

Lemma 15. *An increase in K has no effects on ad revenue $\pi_{\mathbb{K}t}A$ at any time t and leads to an increase in steady state platform consumption X .*

Proof. As seen from (31), the equilibrium ad revenue $\pi_{\mathbb{K}t}A$ does not depend on K . In steady state, using (18)

$$X = K^{\frac{1}{\epsilon-1}} \frac{\ell_{\mathbb{K}}^{\varphi}}{\delta} = \frac{1}{\delta} K^{\frac{1}{\epsilon-1}-\varphi} \left(\frac{\varphi \delta \pi_{\mathbb{K}}A(\epsilon-1)}{(\delta+\rho)} \right)^{\varphi}.$$

This is increasing in K if the coefficient $\epsilon-1 < 1/\varphi$. $\square$

D Supplemental Material for Section 6

Appendix D.1 presents proofs of the results in Section 6. Appendix D.2 presents general equilibrium analogs of Propositions 2 and 3.

D.1 Proofs of main results

Proof of Proposition 4. Product market clearing implies that $I_t = \frac{\sigma}{\sigma-1}(L - K\ell_{\mathbb{K}t})$. That is, income is equal to the markup times the quantity of labor left over for production.

To compute (19), let $\hat{\pi}_{\mathbb{K}} = \pi_{\mathbb{K}}/I$ be the average ad price per unit of income. From (17),

$$\hat{\pi}_{\mathbb{K}} = \frac{1}{\sigma M \mu_H} \int_0^\infty \frac{1 - NH^c(s)^{N-1} + (N-1)H^c(s)^N}{\rho + \lambda_f + \lambda_e(s)} ds. \quad (43)$$

Then, rewriting (18), we have

$$\ell_{\mathbb{K}} = \frac{\varphi \delta \hat{\pi}_{\mathbb{K}} A(\epsilon - 1)}{K(\rho + \delta)} I. \quad (44)$$

Given investment,

$$I = \frac{\sigma}{\sigma - 1} (L - K \ell_{\mathbb{K}}).$$

Solving this linear system for $\ell_{\mathbb{K}}$ yields (19). The rest of the proof is analogous to that of Theorem 1. $\square$

Proof of Theorem 2. To characterize the planner's steady state investment, it suffices to take as given steady-state ad display A^* and suppose that we have already reached steady state for H_t and M_t . The Hamiltonian for the planner's problem for investment is

$$\begin{aligned} \mathcal{H}(t, q_t, \lambda_t, \ell_{\mathbb{K}t}) = & \left[(L - K \ell_{\mathbb{K}t}) (M \mu_H)^{\frac{1}{\sigma-1}} \right]^{1-\tau} \left[K^{\frac{1}{\epsilon-1}} \nu(A^*) q_t \right]^\tau \\ & + \lambda_t (\ell_{\mathbb{K}t}^\varphi - \delta q_t) \end{aligned}$$

where λ_t satisfies

$$\rho \lambda_t - \dot{\lambda}_t = \left[(L - K \ell_{\mathbb{K}t}) (M \mu_H)^{\frac{1}{\sigma-1}} \right]^{1-\tau} \tau K^{\frac{\tau}{\epsilon-1}} \nu(A^*)^\tau q_t^{\tau-1} - \delta \lambda_t.$$

Maximizing the Hamiltonian with respect to the control yields

$$\lambda_t \varphi \ell_{\mathbb{K}t}^{\varphi-1} = (1 - \tau) K [L - K \ell_{\mathbb{K}t}]^{-\tau} \left[(M \mu_H)^{\frac{1}{\sigma-1}} \right]^{1-\tau} \left[K^{\frac{1}{\epsilon-1}} \nu(A^*) q_t \right]^\tau.$$

In steady state, λ_t must therefore be a constant. We have

$$\lambda_t = \frac{\left[(L - K \ell_{\mathbb{K}t}) (M \mu_H)^{\frac{1}{\sigma-1}} \right]^{1-\tau} \tau K^{\frac{\tau}{\epsilon-1}} \nu(A^*)^\tau q_t^{\tau-1}}{\delta + \rho}.$$

Substituting this into the previous equation, we have

$$\tau \frac{(L - K \ell_{\mathbb{K}t}) \varphi \ell_{\mathbb{K}t}^{\varphi-1}}{\delta + \rho} = (1 - \tau) K q_t.$$

In steady state, $q_t = \ell_{\mathbb{K}t}^\varphi / \delta$ so then

$$\tau \frac{\varphi (L - K \ell_{\mathbb{K}t})}{\delta + \rho} = (1 - \tau) K \frac{\ell_{\mathbb{K}t}}{\delta}.$$

Rearranging gives

$$\ell_{\mathbb{K}t} = \frac{\varphi \delta^{\frac{\tau}{1-\tau}}}{\rho + \delta + \varphi \delta^{\frac{\tau}{1-\tau}}} \frac{L}{K}.$$

The Hamiltonian is jointly concave in state and control and therefore satisfies the Mangasarian sufficient conditions for an optimal control.

To characterize the limiting ad display rate, let $U_t(a)$ denote flow utility under the constant ad display rate a , and let $U^{ss}(a)$ denote its steady-state value. Because $M_t \mu_{H_t}$ converges to its steady-state value at a rate independent of ρ , for every fixed ad display rate a ,

$$\int_0^\infty e^{-\rho t} U_t(a) dt = \frac{U^{ss}(a)}{\rho} + O(1),$$

as $\rho \rightarrow 0$, where the remainder is independent of ρ .

Suppose, by contradiction, that the limit of the planner's ad display rate does not maximize U^{ss} . Then some alternative $\tilde{a}$ yields a strictly larger steady-state flow payoff. By the expansion above, the transitional payoff difference is bounded, whereas the gain in discounted steady-state utility is of order $1/\rho$. Thus, for all sufficiently small ρ , switching to $\tilde{a}$ strictly increases welfare, contradicting optimality.

Since the planner's steady-state investment is independent of a , the terms in steady-state flow utility that depend on a are proportional to

$$[a \mu_H(a)]^{\frac{1-\tau}{\sigma-1}} \nu(a)^\tau.$$

Therefore, in the limit as $\rho \rightarrow 0$, the planner's ad display rate satisfies (23). $\square$

Proofs of Proposition 5 and Corollary 2.2. These results follow immediately from Proposition 4 and Theorem 2. $\square$

D.2 Comparative Statics Omitted from the Main Text

The following results show how the conclusions of Propositions 2 and 3 extend to the general equilibrium setting.

Lemma 16. *In steady state, as data becomes uninformative (i.e., along any sequence of continuous distributions G converging pointwise to the degenerate distribution at μ_F), ad revenue and platform quality converge to their suprema, while aggregate product consumption converges to its infimum, over all continuous distributions G with mean μ_F .*

Proof. I first show that ad revenue is maximal in the limiting sense as data becomes uninformative. By an analogous argument to Lemma 7, $\hat{\pi}_{\mathbb{K}}$ is maximal as data becomes uninformative. Total ad revenue is $\pi_{\mathbb{K}}A = I\hat{\pi}_{\mathbb{K}}A$. Suppose for contradiction that ad revenue is not maximal in this limit; then under some informative data, I must be high enough that ad revenue is higher. From the proof of Proposition 4, $I = \frac{\sigma}{\sigma-1}(L - K\ell_{\mathbb{K}})$, so a higher I requires lower investment $\ell_{\mathbb{K}}$. But (44) implies that lower investment requires lower ad revenue—a contradiction.

For product consumption, recall $C = \frac{\sigma-1}{\sigma}I(M\mu_H)^{1/(\sigma-1)}$. As data becomes uninformative, μ_H is minimal (Lemma 6). Ad revenue is maximal in this limit, so by (44) investment is maximal, which makes $I = \frac{\sigma}{\sigma-1}(L - K\ell_{\mathbb{K}})$ minimal. All three factors push C to its infimum.

Platform quality is maximal in the limit by Part 5 of Theorem 1, since investment is maximal. $\square$

Lemma 17. *Under Condition 1, an increase in ϵ raises ad revenue per unit of income ($\hat{\pi}_{\mathbb{K}}A$) and platform quality, but reduces product consumption, in steady state.*

Proof. By an analogous argument to that in the proof of Proposition 6, $\hat{\pi}_{\mathbb{K}}A$ rises with ϵ .

I next show that platform quality rises with ϵ . Suppose for contradiction that platform quality (and hence investment $\ell_{\mathbb{K}}$) falls when ϵ rises. By (44), lower investment implies lower ad revenue. Total ad revenue is $\pi_{\mathbb{K}}A = I\hat{\pi}_{\mathbb{K}}A$, so since $\hat{\pi}_{\mathbb{K}}A$ has risen, lower ad revenue requires lower I . But $I = \frac{\sigma}{\sigma-1}(L - K\ell_{\mathbb{K}})$, so lower I requires higher $\ell_{\mathbb{K}}$ —a contradiction.

For product consumption, recall $C = \frac{\sigma-1}{\sigma}I(M\mu_H)^{1/(\sigma-1)}$. In general equilibrium, ϵ directly affects C only through I , since M and μ_H do not depend on ϵ . Since platform quality rises, investment $\ell_{\mathbb{K}}$ rises as well, so $I = \frac{\sigma}{\sigma-1}(L - K\ell_{\mathbb{K}})$ falls. The indirect effect of a decrease in A further causes C to fall. Therefore C falls. $\square$

E Supplemental Material for Section 7

This appendix presents supplemental material for the duopoly model of Section 7. Appendix E.1 provides details of the algorithm to compute an equilibrium. Appendix E.2 describes a reformulation in which there is a single state variable and uses it to reveal the connection between my model and that of Budd et al. (1993).

E.1 Details of the Algorithm to Compute Equilibrium

I use an implicit upwind finite difference scheme to solve (26). Here, I provide details of this finite-difference scheme. In particular, I describe the iterative procedure (used in Steps 1 and 2 outlined in Section 7) to calculate the value function at an earlier time step given the value function at a later time step. At the end of this subsection, I discuss properties of the algorithm and mathematical foundations.

Discretize the State Space I first compactify and then discretize each dimension of the state space in a standard way using evenly spaced grids. The two quality dimensions share a common grid with gap Δ_q , and the time dimension uses gap Δ_t between grid points.

Finite Difference Scheme Throughout, I suppress the index k on the value function V_k for ease of notation. In the remaining parts of this document, let

$$V_{q_1}^+(q_1, q_2, t) \equiv \frac{V(q_1 + \Delta_q, q_2, t) - V(q_1, q_2, t)}{\Delta_q}$$

denote the forward difference and

$$V_{q_1}^-(q_1, q_2, t) \equiv \frac{V(q_1, q_2, t) - V(q_1 - \Delta_q, q_2, t)}{\Delta_q}$$

the backward difference approximation of $\partial V / \partial q_1$. Define V_t^+ analogously as the forward difference approximation of $\partial V / \partial t$.

Let

$$V_{q_1, q_1}^c(q_1, q_2, t) = \frac{V(q_1 + \Delta_q, q_2, t) - 2V(q_1, q_2, t) + V(q_1 - \Delta_q, q_2, t)}{\Delta_q^2}$$

denote the central difference approximation of $\partial^2 V / \partial q_1^2$. Define V_{q_2, q_2}^c analogously.

To implement the (semi-)implicit upwind scheme, I work with the following discretized version of (26):

$$\begin{aligned}
\rho V(q_1, q_2, t) = & \sup_{\ell_{k1}} V_t^+(q_1, q_2, t) + V_{q_1}^+(q_1, q_2, t)(\ell_{k1}^\varphi - \delta q_1)^+ \\
& + V_{q_1}^-(q_1, q_2, t)(\ell_{k1}^\varphi - \delta q_1)^- + V_{q_2}^+(q_1, q_2, t)(\ell_{k2}^\varphi - \delta q_2)^+ \\
& + V_{q_2}^-(q_1, q_2, t)(\ell_{k2}^\varphi - \delta q_2)^- - \ell_{k1} + \pi_{\mathbb{K}t} A \frac{q_1^{\epsilon-1}}{q_1^{\epsilon-1} + q_2^{\epsilon-1}}. \\
& + \frac{1}{2} \eta^2 q_1^2 V_{q_1, q_1}^c(q_1, q_2, t) + \frac{1}{2} \eta^2 q_2^2 V_{q_2, q_2}^c(q_1, q_2, t).
\end{aligned} \tag{45}$$

Thus, the finite-difference scheme is *upwind* in the sense that it uses the forward difference approximation of $\partial V / \partial q_1$ ($\partial V / \partial q_2$) whenever the drift of q_1 (q_2) is positive and the backward difference whenever the drift is negative. The scheme is *implicit* in the sense that it uses the forward-difference approximation of $\partial V / \partial t$. Note that (45) is a relationship between $V(\cdot, t + \Delta)$ and $V(\cdot, t)$ and thus we can use it to calculate $V(\cdot, t)$ given $V(\cdot, t + \Delta)$. The rest of this subsection details how this is done. I note that because the scheme is both upwind and implicit, $V(\cdot, t)$ depends *monotonically* on $V(\cdot, t + \Delta)$, which is valuable for the algorithm's *stability*.

Optimal Controls Using (45) to compute the value functions backward requires solving the optimization problem on the right-hand side. Using first-order conditions, we can show that there are only two candidates for the optimal controls: $\max\{\ell_k^+, \ell_k^c\}$ and $\min\{\ell_k^-, \ell_k^c\}$, where ℓ_k^+ , ℓ_k^- , and ℓ_k^c are defined as follows.

Define ℓ_k^+ by

$$V_{q_1}^+(q_1, q_2, t) \varphi(\ell_k^+)^{\varphi-1} = 1,$$

ℓ_{k1}^- by

$$V_{q_1}^-(q_1, q_2, t) \varphi(\ell_k^-)^{\varphi-1} = 1,$$

and ℓ_k^c by

$$(\ell_k^c)^\varphi = \delta q_1.$$

Construct the M matrix Consider the following term which appears on the right-hand side of (45):

$$\begin{aligned}
V_{q_1}^+(q_1, q_2, t)(\ell_{k1}^\varphi - \delta q_1)^+ & + V_{q_1}^-(q_1, q_2, t)(\ell_{k1}^\varphi - \delta q_1)^- \\
& + V_{q_2}^+(q_1, q_2, t)(\ell_{k2}^\varphi - \delta q_2)^+ + V_{q_2}^-(q_1, q_2, t)(\ell_{k2}^\varphi - \delta q_2)^- \\
& + \frac{1}{2} \eta^2 q_1^2 V_{q_1, q_1}^c(q_1, q_2, t) + \frac{1}{2} \eta^2 q_2^2 V_{q_2, q_2}^c(q_1, q_2, t).
\end{aligned} \tag{46}$$

I express the operator above in terms of a matrix M . Specifically, let N denote the number grid points for both quality dimensions. The expression (45) is to be expressed as MV where V is a $N^2 \times 1$ vector representing the value function at each quality (q_1, q_2) grid point and M is a $N^2 \times N^2$ matrix.

The rows of V are partitioned into N groups of N rows. Within each group, the state q_1 is the same and only q_2 varies.

More specifically, if⁴⁰

$$V(q_1, q_2, t) \triangleq k$$

where k is the index of the vector V , then the corresponding indices of the other value function terms in (46) are:

$$V(q_1, q_2 + \Delta_q, t) \triangleq k + 1,$$

$$V(q_1, q_2 - \Delta_q, t) \triangleq k - 1,$$

$$V(q_1 + \Delta_q, q_2, t) \triangleq k + N,$$

$$V(q_1 - \Delta_q, q_2, t) \triangleq k - N.$$

Given the vector V , the rows and columns of M are organized accordingly. Namely, M is populated along the main diagonal (which contains the coefficients in (46) on $V(q_1, q_2, t)$), the +1 and -1 diagonals (which contain the coefficients on $V(q_1, q_2 + \Delta_q, t)$ and $V(q_1, q_2 - \Delta_q, t)$ respectively), and the + N and - N diagonals (contain the coefficients on $V(q_1 + \Delta_q, q_2, t)$ and $V(q_1 - \Delta_q, q_2, t)$) respectively. All other entries in M are zero. Thus, M is a *sparse matrix* with only 5 populated diagonals. This sparsity is *critical* for computational speed and a key advantage of the continuous-time formulation. I take advantage of it in the MATLAB code using the `spdiags` function.

Calculate Value Functions at Previous Time Step Using the M matrix (constructed using the *optimal controls at time t* in (46)) together with (45), we can succinctly express $V(\cdot, t - \Delta_t)$ in terms of $V(\cdot, t)$ via the equation

$$V(\cdot, t - \Delta_t) = [I(1 + \rho\Delta_t) - \Delta_t M_t]^{-1} (\Delta_t g_t + V(\cdot, t))$$

where I is the identity matrix and $g_t = -\ell_{kt}^* + \pi_{\mathbb{K}t} A \frac{q_1^{\epsilon-1}}{q_1^{\epsilon-1} + q_2^{\epsilon-1}}$ is the flow profit from the optimal control ℓ_{kt}^* . For clarity, I have explicitly indexed the M matrix and optimal control by time t .

⁴⁰Recall that $\triangleq$ means “corresponds.”

Discussion of the Algorithm Though I have not proven convergence theoretically, the algorithm is informed by established mathematical foundations. The upwind finite-difference scheme used here falls within the class of monotone, stable, and consistent schemes that Barles and Souganidis (1991) prove converge to viscosity solutions of elliptic PDEs under suitable conditions. While their result does not directly apply to *coupled* HJB systems like the one studied here, the structural properties of the scheme offer reassurance about the reliability of the numerical method. The computation is also fast—though there are two state variables, with minimal optimization and no parallelization, on a 2021 base model 16 inch M1 Macbook pro, the MATLAB code calculates steady state in roughly 2 minutes and the full equilibrium dynamics in roughly 10 minutes for reasonably fine time and quality grids.

Furthermore, the approach taken here has precedent in economics. For example, Brunnermeier and Sannikov (2017) use an analogous procedure to solve a coupled HJB system in a macrofinance model. Brunnermeier and Sannikov (2017) consider a setting with a single state variable and atomistic agents, whereas I study a model with two state variables and atomic players. However, in both cases, computing equilibrium involves solving a coupled HJB system and the numerical algorithms we use are conceptually similar.

E.2 Model Variant with Single Spatial State Variable

Here I describe a slight reformulation of the duopoly model that reduces the number of state variables: instead of tracking the quality levels of both platforms separately, it suffices for each platform to track only the share of attention received by platform 1. In the process, I remark on the close connection between the duopoly model and that of Budd et al. (1993). As discussed in Section 7, this connection (together with the comment in Footnote 33) motivates the conjecture that a nontrivial amount of noise is sufficient for equilibrium existence in my setting. This is because Budd et al. 1993 show in their similar model that any nontrivial amount of noise ensures equilibrium existence (provided some standard and permissive conditions on other parameters are met).

Suppose that the effect of investment on quality scaled with quality. That is, suppose instead of (25) we have

$$dq_{kt} = (\ell_{kt}^\varphi - \delta) q_{kt} dt + \eta q_{kt} dB_{kt}.$$

Then one can show that the share of attention x_{kt} that platform k receives

is a Markov process. Namely, using Itô's formula,

$$\begin{aligned} dx_{1t} = & x_{1t}(1 - x_{1t}) \left[(\epsilon - 1)(\ell_{1t}^\varphi - \ell_{2t}^\varphi) + (\epsilon - 1)^2 \eta^2 (1 - 2x_{1t}) \right] dt \\ & + (\epsilon - 1) \eta x_{1t}(1 - x_{1t}) (dB_{1t} - dB_{2t}). \end{aligned}$$

where recall that $x_{1t} = q_{1t}^{\epsilon-1} / (q_{1t}^{\epsilon-1} + q_{2t}^{\epsilon-1})$.

Therefore we can express

$$\begin{aligned} dx_{1t} = & x_{1t}(1 - x_{1t}) \left[(\epsilon - 1)(\ell_{1t}^\varphi - \ell_{2t}^\varphi) + (\epsilon - 1)^2 \eta^2 (1 - 2x_{1t}) \right] dt \\ & + (\epsilon - 1) \eta x_{1t}(1 - x_{1t}) \sqrt{2} dB_t. \end{aligned} \tag{47}$$

where $B \equiv (1/\sqrt{2})(B_1 - B_2)$ is a standard Brownian motion.

Thus, in this setup it is logical to look for a steady state equilibrium that is Markov with only a single state variable: x_{1t} . That is, the two quality state variables collapse onto a single state variable.

The model in Budd et al. (1993) consists of a duopoly setting where profits depend on market share x_{kt} as in my model through some exogenously given function $\pi(\cdot)$. They do not focus on a specific microfoundation for market share and profits in terms of quality as I do but rather assume x_{kt} is directly controlled by the investment of the two players via the law of motion

$$dx_{kt} = (\ell_{kt} - \ell_{-kt}) dt + \sigma dB_t$$

where σ is a positive constant in the interior of the state space $[0, 1]$ and zero at the boundaries. Budd et al. (1993) also assume that there is some given flow cost $c(\cdot)$ of investment for both players.

Thus our models are qualitatively similar except for the fact that in (47) there is an extra factor $x_{1t}(1 - x_{1t})$ that multiplies the drift and volatility terms and there is also the second term in brackets in the drift term.⁴¹

Note that at the boundaries of the state space, the volatility disappears in both of our models.

F Supplemental Material for Section 8

I present a characterization of the steady state equilibrium and a sketch of its derivation.

⁴¹Other differences are merely cosmetic such as the fact that investment is raised to the power φ in my model but not in Budd et al. (1993)—these differences disappear by appropriately relabeling variables.

F.1 Steady State Equilibrium Characterization

The following Theorem 3 summarizes the steady state equilibrium properties.

Theorem 3. *Suppose that $\epsilon - 1 < 1/\varphi$ and that A is the unique solution of $\max_a a\nu(a)^{\epsilon-1}$. In any steady state equilibrium with increasing bidding functions the following hold:*

1. *Consumer i 's demands for products are as in (27) and her demands for platforms are as in (15).*
2. *Firm j sets prices as in (1).*
3. *Platform k displays ads at rate A as in (16).*
4. *The size of consideration sets is $M = A/\lambda_f$.*
5. *Firm j 's expected flow profits from sales are as in (2).*
6. *Bidding functions $\mathbf{B} = (B_1, B_2)$ for the two groups are the fixed point of the operator $\mathbf{\Lambda}$ defined by (51) which is a contraction map whenever values are bounded.*
7. *The attention shares received by the two groups are given by (53).*
8. *The rates of investment by the two groups are given by (52).*

Proof. I now sketch the proof, i.e., the procedure to solve for the steady state equilibrium. Much of the analysis of the baseline model ports over to this extended setup. Demands, prices, ad rates, and the size of consideration sets will be as in equations (27), (15), (16), and (29). However, we now must keep track of the joint distribution of the two signals inside and outside of consideration sets.

Step 1. Law of Motion of H_t and H_t^c : Let H_t^c denote the joint cdf of signals outside of consideration sets at time t . Let H_t denote the joint cdf of signals inside of consideration sets at time t . Let h_t^c and h_t be their corresponding pdfs. Let $H_t^c(\zeta_1, \infty)$ denote $\lim_{\zeta \rightarrow \infty} H_t^c(\zeta_1, \zeta)$ and let $H_t^c(\infty, \zeta_2)$ be defined analogously. Suppose that the winner in an auction on a platform in group z is the firm with the highest group z signal. Then, assuming M_t is

at its steady state level, $M = A/\lambda_f$, the law of motion of h_t must satisfy

$$\begin{aligned} M \frac{d}{dt} [h_t(\zeta)] \\ = A \left[x_{1t} N H_t^c(\zeta_1, \infty)^{N-1} h_t^c(\zeta) + x_{2t} N H_t^c(\infty, \zeta_2)^{N-1} h_t^c(\zeta) - h_t(\zeta) \right]. \end{aligned} \quad (48)$$

In (48), with abuse of notation, x_{zt} denotes the total share of attention devoted to group z platforms. The first two terms in the brackets represents the inflow coming from the winners in the auctions on the two platform groups. The third term in the bracket represents the outflow as the consumer forgets about products. Note that since M is in steady state, both the aggregate rate of inflow and the aggregate rate of outflow must be equal to A .

To derive the steady state h , first fix an initial guess of x_1 , the steady state level of x_{1t} . Then set $x_{1t} = x_1$, $x_{2t} = 1 - x_1$, and use the accounting identity $M_t h_t + (J - M_t) h_t^c = Jg$ to iterate (48) forward to convergence at each point ζ in a fine grid on a region that contains almost all of G 's mass.

Step 2. Calculate Bidding Strategies: Given M and h , we next compute equilibrium bidding strategies. To do so let

$$\mu_H = \int_{\mathbb{R}^2} \mathbb{E}[v_{ij}|\zeta] h(\zeta) d\zeta,$$

$$\pi_J = \frac{I}{\sigma M \mu_H},$$

$$O_1(\cdot) = H^c(\cdot, \infty)^{N-1},$$

and

$$O_2(\cdot) = H^c(\infty, \cdot)^{N-1}.$$

Above, μ_H is the average value of the firms in consideration sets, π_J is the coefficient of firms' flow profits, O_1 determines the probability that a firm wins an auction if it takes place on a platform in group 1, and O_2 determines the probability that a firm wins an auction if it takes place on a platform in group 2.

In a steady state, bidding strategies correspond to a pair of functions $\mathbf{B} = (B_1, B_2)$. Here, $B_z : \mathbb{R} \mapsto [0, \infty)$ maps firm j 's group z signal ζ_{zij} to its bid $B_z(\zeta_{zij})$ in a group z auction for consumer i . To derive $\mathbf{B}$, let $\zeta_{ij} = (\zeta_{1ij}, \zeta_{2ij})$ and let $V(\zeta_{ij})$ be firm j 's continuation value from selling to consumer i at the

time of auction entry if it knows ζ_{ij} but does not know which platform hosts the auction.

More precisely, V satisfies the recursive equation

$$V(\zeta_{ij}) = \sum_{z=1}^2 \left(x_z O_z(\zeta_{zij}) \left[\frac{\pi_{\mathbb{J}}}{\lambda_f + \rho} \mathbb{E}[v_{ij} | \zeta_{ij}] + \frac{\lambda_f}{\lambda_f + \rho} \frac{\lambda_a}{\lambda_a + \rho} V(\zeta_{ij}) \right] + x_z \left[- \int_{-\infty}^{\zeta_{zij}} B_z(s) dO_z(s) + [1 - O_z(\zeta_{zij})] \frac{\lambda_a}{\lambda_a + \rho} V(\zeta_{ij}) \right] \right) \quad (49)$$

The first term in the bracket is the discounted expected flow profit that firm j earns from entering Ω_{it} . It exits at rate λ_f and subsequently enters another auction at rate λ_a which corresponds to the second term in the bracket. On the second line, the first term in brackets is the expected payment in a group z auction. The last term is the continuation value in the event that firm j loses the auction, weighted by the probability that this happens.

Since auctions are second-price, in each auction, firm j simply bids the gain in its continuation value from winning the auction. Then,

$$\begin{aligned} B_z(\zeta_{zij}) &= \mathbb{E} \left[\frac{\pi_{\mathbb{J}}}{\lambda_f + \rho} v_{ij} + \frac{\lambda_f}{\lambda_f + \rho} \frac{\lambda_a}{\lambda_a + \rho} V(\zeta_{ij}) \middle| \zeta_{zij}, j \in \Omega_{it}^c \right] \\ &\quad - \mathbb{E} \left[\frac{\lambda_a}{\lambda_a + \rho} V(\zeta_{ij}) \middle| \zeta_{zij}, j \in \Omega_{it}^c \right] \\ &= \mathbb{E} \left[\frac{\pi_{\mathbb{J}}}{\lambda_f + \rho} v_{ij} - \frac{\rho}{\lambda_f + \rho} \frac{\lambda_a}{\lambda_a + \rho} V(\zeta_{ij}) \middle| \zeta_{zij}, j \in \Omega_{it}^c \right]. \end{aligned} \quad (50)$$

Above, $\lambda_a = NA/(J - M)$ is the rate of auction entry. The expectation is conditional on only the group z signal and the fact that the firm is outside the consideration set since this is all that the firm knows when it bids.

Using (50) and (49), we can show that $\mathbf{B}$ is the fixed point of an operator $\mathbf{\Lambda} := (\Lambda_1, \Lambda_2)$ where $\Lambda_z : C^+(\mathbb{R})^2 \rightarrow C^+(\mathbb{R})$ takes in a pair of functions $\mathbf{f} = (f_1, f_2)$ and outputs another function⁴²

$$\Lambda_z(\mathbf{f})(\cdot) = \mathbb{E} \left[\frac{\pi_{\mathbb{J}} v_{ij} + \lambda_a \sum_{l=1}^2 x_l \int_{-\infty}^{\zeta_{lij}} f_l(s) dO_l(s)}{\lambda_f + \rho + \lambda_a \sum_{l=1}^2 x_l O_l(\zeta_{lij})} \middle| \zeta_{zij} = \cdot, j \in \Omega_{it}^c \right]. \quad (51)$$

The map $\mathbf{\Lambda}$ is a contraction with modulus $\lambda_a/(\lambda_a + \lambda_f + \rho)$ with respect to the sup-norm whenever values v_{ij} are bounded above by some level $\bar{v}$. For

⁴² $C^+(\mathbb{R})$ denotes the set of nonnegative continuous functions on $\mathbb{R}$.

numerical purposes, this will always be the case. Even without this bounded support assumption, we see that $\mathbf{\Lambda}$ is increasing. Thus, starting from an initial $\mathbf{B}$ such that $B_z > \Lambda_z(\mathbf{B})$ for each $z \in \{1, 2\}$ it follows that $\{\mathbf{\Lambda}^n(\mathbf{B})\}_{n=1}^\infty$ is a decreasing sequence which converges to a fixed point. Thus we can compute the equilibrium bidding strategies by iterating on (51).

Step 3. Calculate Investment: In a steady state equilibrium, a platform in group $z \in \{1, 2\}$ invests a constant level $\ell_{\mathbb{K}z}$ to maintain quality level $q_z = \ell_{\mathbb{K}z}^\varphi / \delta$.

Let platform k belong to group z . The Hamiltonian for platform k 's optimization problem is

$$\mathcal{H}(t, q_{kt}, \lambda_t, \ell_{kt}) = \pi_{\mathbb{K}z} A \frac{q_{kt}^{\epsilon-1}}{m_z q_z^{\epsilon-1} + m_{-z} q_{-z}^{\epsilon-1}} - \ell_{kt} + \lambda_t (\ell_{kt}^\varphi - \delta q_{kt})$$

where λ_t , the costate variable, evolves according to

$$\rho \lambda_t - \dot{\lambda}_t = \pi_{\mathbb{K}z} A (\epsilon - 1) \frac{q_{kt}^{\epsilon-2}}{m_z q_z^{\epsilon-1} + m_{-z} q_{-z}^{\epsilon-1}} - \lambda_t \delta.$$

By the Maximum Principle, a necessary condition for optimality is that the control ℓ_{kt} maximizes the Hamiltonian along the optimal trajectory:

$$\lambda_t \varphi \ell_{kt}^{\varphi-1} = 1.$$

Under the conjectured stationary strategy then

$$\lambda_t \varphi \ell_{\mathbb{K}z}^{\varphi-1} = 1.$$

This implies that λ_t must be a constant λ . By the costate evolution equation,

$$\lambda = \frac{\pi_{\mathbb{K}z} A (\epsilon - 1)}{\rho + \delta} \frac{q_z^{\epsilon-2}}{m_z q_z^{\epsilon-1} + m_{-z} q_{-z}^{\epsilon-1}}.$$

Substituting, we have

$$\frac{\pi_{\mathbb{K}z} A (\epsilon - 1)}{\rho + \delta} \varphi \ell_{\mathbb{K}z}^{\varphi-1} = m_z q_z + m_{-z} \left(\frac{q_{-z}}{q_z} \right)^{\epsilon-2} q_{-z}.$$

This implies that

$$\frac{\pi_{\mathbb{K}z} A (\epsilon - 1)}{\rho + \delta} \varphi \ell_{\mathbb{K}z}^{\varphi-1} = m_z \frac{\ell_{\mathbb{K}z}^\varphi}{\delta} + m_{-z} \left(\frac{\ell_{\mathbb{K}-z}}{\ell_{\mathbb{K}z}} \right)^{\varphi(\epsilon-2)} \frac{\ell_{\mathbb{K}-z}^\varphi}{\delta}.$$

Dividing both sides by $\ell_{\mathbb{K}z}^\varphi/\delta$ we arrive at

$$\frac{\delta\pi_{\mathbb{K}z}A(\epsilon-1)}{\rho+\delta}\varphi\ell_{\mathbb{K}z}^{-1} = m_z + m_{-z} \left(\frac{\ell_{\mathbb{K}-z}}{\ell_{\mathbb{K}z}} \right)^{\varphi(\epsilon-1)}.$$

By symmetry by considering the problem of a platform k in group $-z$,

$$\frac{\delta\pi_{\mathbb{K}-z}A(\epsilon-1)}{\rho+\delta}\varphi\ell_{\mathbb{K}-z}^{-1} = m_{-z} + m_z \left(\frac{\ell_{\mathbb{K}z}}{\ell_{\mathbb{K}-z}} \right)^{\varphi(\epsilon-1)}.$$

Let $y := \ell_{\mathbb{K}z}/\ell_{\mathbb{K}-z}$. Using the above two equations, I derive

$$\frac{\pi_{\mathbb{K}z}}{\pi_{\mathbb{K}-z}} \frac{1}{y} = \frac{m_z + m_{-z}y^{-\varphi(\epsilon-1)}}{m_{-z} + m_z y^{\varphi(\epsilon-1)}}.$$

Equivalently,

$$y = \left(\frac{\pi_{\mathbb{K}z}}{\pi_{\mathbb{K}-z}} \right)^{\frac{1}{1-\varphi(\epsilon-1)}}.$$

Thus,

$$\ell_{\mathbb{K}z} = \frac{\varphi\delta\pi_{\mathbb{K}z}A(\epsilon-1)}{\rho+\delta} \frac{1}{m_z + m_{-z} \left(\frac{\pi_{\mathbb{K}-z}}{\pi_{\mathbb{K}z}} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}}} \quad (52)$$

Note that if $\epsilon \leq 2$ then the Hamiltonian is jointly concave in the state and control and so I have identified the optimal control. If $\epsilon - 1 < \frac{1}{\varphi}$ one can verify using the same approach outline in Step 5 for the proof of Theorem 1.

Step 4. Calculate Quality Levels and Attention Shares: From (52) we also have the quality level

$$q_z = \frac{\ell_{\mathbb{K}z}^\varphi}{\delta}$$

and attention share

$$x_z = \frac{m_z q_z^{\epsilon-1}}{m_z q_z^{\epsilon-1} + m_{-z} q_{-z}^{\epsilon-1}} = \frac{m_z \left(\frac{\pi_{\mathbb{K}z}}{\pi_{\mathbb{K}-z}} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}}}{m_z \left(\frac{\pi_{\mathbb{K}z}}{\pi_{\mathbb{K}-z}} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} + m_{-z}} \quad (53)$$

of group z platforms.

Step 5. Calculate Surpluses Consumer surplus is simply $u(C, X)/\rho$ where

$$C = \frac{\sigma - 1}{\sigma} I(M\mu_H)^{\frac{1}{\sigma-1}}$$

and

$$X = \nu(A) \left(\sum_{z=1}^2 m_z q_z^{\epsilon-1} \right)^{\frac{1}{\epsilon-1}}$$

are product and platform consumption respectively. $\square$

F.2 Summary of the Algorithm

For ease of viewing, I summarize the algorithm once more here in the appendix.

1. Guess a value of x_1 .
2. Iterate (48) forward to compute h .
3. Iterate (51) to compute per-unit income bid functions and average ad prices.
4. Check whether the guess of x_1 aligns with (53).
5. If yes, done. If not, repeat with a revised guess.

All other equilibrium objects are characterized in closed form in terms of the output of this algorithm and primitives. Though inefficient, one can simply run steps 2-4 for each guess of x_1 in a fine grid on $[0, 1]$. This is relatively fast and allows one to solve for *all* steady state equilibria and in particular, check uniqueness.

G Discussion of Dynamics

In this appendix, I describe some methodological advantages of a dynamic analysis.

Consider an alternative one period model in which firms participate in multiple auctions for a given consumer. In such a model, there will not exist a bidding equilibrium in symmetric strategies (and it is unclear if there are other asymmetric equilibria). It is easiest to show this when values are supported on a $[0, \bar{v}]$ where $\bar{v} < \infty$. Suppose that there was a symmetric bidding equilibrium with increasing bidding strategies. Then a firm with value $\bar{v}$ would

have a profitable deviation to bidding a zero amount in one of the auctions it participates on. This is because the firm is guaranteed to win all of the auctions it participates on if it follows the equilibrium strategy. But, the firm has only unit demand for displaying an ad. By deviating in this way, the firm can reduce its cost while still displaying an ad. By spreading the auction competition out over time I am able to avoid this issue. Thus, dynamics allows us to have an auction analysis in which consumers multi-home and there is interplatform competition in the sale of ads.

Of course, we could consider a one period model where there is no interplatform competition in the sale of ads and each firm participates in one auction each. There would be some matching of firms to auctions which we would have to take a stance on. If N or A are sufficiently large, then regardless of the matching some firms *must* participate on multiple auctions—there aren’t enough distinct bidders to be allocated to fill up the auctions. In other words, comparative statics on N or A would have to be limited to a certain range where this is not the case. This is unattractive for the model especially since A is endogenous. You could of course assume N and A are sufficiently small so that not all auctions are filled. But, these additional modeling assumptions, in my opinion, seem unnatural. An attempt to explain them would probably appeal to unmodeled frictions such as the fact that it takes time for firms to locate auctions for a given consumer. The dynamic modeling simply makes this intuition formal.

Consider an alternative one period formulation with competitive pricing in the ad market rather than auctions. The baseline model can be solved in a competitive pricing environment. However, in that model, ads would be sold consumer by consumer and it would be as though the platforms inform firms about the *identity* i of the consumer when they make their purchases. In reality, firms only see signals and they do not know if they correspond to the same individual just as they do in the baseline model of Section 3. Of course, a disadvantage of switching to competitive pricing is that future work can not explore the model using ad auction level data. Moreover, its not clear how to solve the extended version of the model where platforms may have different data with competitive pricing. In this model, we can not let platforms sell ads consumer by consumer as if the platforms know the consumers’ identities because then the firms could combine the data they receive from the different platforms. Thus, suppose each firm sees only signals of the consumers’ valuations but not their identities when choosing which ads to purchase. Firms would have to do some inference about the likelihood they will also purchase an ad for the same consumer on the other platforms. This inference effect leads to complicated purchasing strategies that are nontrivial to characterize.

H Extension: Network Effects

I extend the baseline model to allow for network effects.

H.1 Setup

I redefine the CES aggregate for platform consumption to be

$$X_{it} = \left[\int_{\mathbb{K}} [\eta(x_{kt}) \nu(a_{kt}) q_{kt} x_{ikt}]^{\frac{\epsilon-1}{\epsilon}} dk \right]^{\frac{\epsilon}{\epsilon-1}}$$

where $\eta(x) = x^\zeta$ where $\zeta > 0$. I retain all other aspects of the baseline model of Section 3.

H.2 Equilibrium Characterization

Theorem 4. *Suppose that $\hat{A}$ is the unique solution of $\max_a a \nu(a)^{\frac{\epsilon-1}{1-\zeta(\epsilon-1)}}$. If $\hat{A}/\lambda_f < J$ and $0 < (\epsilon - 1)/[1 - \zeta(\epsilon - 1)] < 1/\varphi$, then there exists a unique equilibrium in which each platform $k \in \mathbb{K}$ receives a positive amount of attention $x_{kt} > 0$ at all times t for any feasible initial conditions M_0 , H_0 , and q_0 . The equilibrium converges to a steady state and has the following properties:*

1. *Consumer i 's demands for products are as in (27) and her demands for platforms are as in (56).*
2. *Firm j sets prices as in (1).*
3. *Platform k displays ads at rate $\hat{A}$.*
4. *The size of consideration sets is given by (29) and the cdfs of the expected values of firms inside and outside of them are characterized by (30) and (4) with $\hat{A}$ in place of A .*
5. *Firm j 's expected flow profits from sales are as in (2) and the rates at which firm j matches with consumers are as in (11).*
6. *Firm j bids according to (31).*
7. *Platform k 's quality and investment solve the boundary-value problem (33) except with $(\epsilon - 1)/[1 - \zeta(\epsilon - 1)]$ in place of $\epsilon - 1$.*

Moreover the sufficient conditions are almost necessary: if either $\hat{A}/\lambda_f \geq J$ or $(\epsilon - 1)/[1 - \zeta(\epsilon - 1)] > 1/\varphi$ then there does not exist an equilibrium.

Proof. As discussed in Section 9, the attention received by platform k solves

$$x_{kt} = \frac{x_{kt}^{\zeta(\epsilon-1)} [\nu(a_{kt}) q_{kt}]^{\epsilon-1}}{Y} \quad (54)$$

where

$$Y = \int_{\mathbb{K}} x_{kt}^{\zeta(\epsilon-1)} [\nu(a_{kt}) q_{kt}]^{\epsilon-1} dk.$$

Solving (54) for x_{kt} yields two possibilities:

$$x_{kt} = \frac{[\nu(a_{kt}) q_{kt}]^{\frac{\epsilon-1}{1-\zeta(\epsilon-1)}}}{Y^{\frac{1}{1-\zeta(\epsilon-1)}}} \quad (55)$$

or $x_{kt} = 0$. Under the equilibrium refinement, all platforms must receive positive attention share (55). Integrating both sides of (55) over $\mathbb{K}$ yields

$$Y^{\frac{1}{1-\zeta(\epsilon-1)}} = \int_{\mathbb{K}} [\nu(a_{kt}) q_{kt}]^{\frac{\epsilon-1}{1-\zeta(\epsilon-1)}} dk.$$

Then substituting into (55) gives

$$x_{kt} = \frac{[\nu(a_{kt}) q_{kt}]^{\frac{\epsilon-1}{1-\zeta(\epsilon-1)}}}{\int_{\mathbb{K}} [\nu(a_{kt}) q_{kt}]^{\frac{\epsilon-1}{1-\zeta(\epsilon-1)}} dk}. \quad (56)$$

Thus, the only change relative to the baseline model is that the elasticity of attention with respect to platform quality is now higher. The rest of the equilibrium derivation follows the same steps as in Section 4. $\square$

I Extension: Heterogeneous Platform Productivity

I solve an extension of the baseline model in which platforms may differ in the productivity of their investments.

I.1 Setup

Platform k now solves

$$\max_{\{\ell_{kt}\}} \int_0^\infty e^{-\rho t} \left(\pi_{\mathbb{K}t} A \frac{q_{kt}^{\epsilon-1}}{\int_{\mathbb{K}} q_{zt}^{\epsilon-1} dz} - \alpha_k \ell_{kt} \right) dt$$

where

$$\dot{q}_{kt} = \ell_{kt}^\varphi - \delta q_{kt}.$$

The only difference relative to the baseline model is the parameter $\alpha_k > 0$ which controls the productivity of platform k . Let P denote the frequency distribution of $\alpha_k, k \in \mathbb{K}$. I retain all other aspects of the baseline model.

I.2 Equilibrium Characterization

Theorem 5. *Suppose that A is the unique solution to*

$$\max_a a\nu(a)^{\epsilon-1},$$

that $\epsilon - 1 < 1/\varphi$, and that

$$\int \left(\frac{1}{\alpha}\right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha) < \infty.$$

Suppose further that initial qualities satisfy

$$q_{k0} = \left(\frac{1}{\alpha_k}\right)^{\frac{\varphi}{1-\varphi(\epsilon-1)}} q_0.$$

Then there exists a unique equilibrium in which

$$\ell_{kt} = \left(\frac{1}{\alpha_k}\right)^{\frac{1}{1-\varphi(\epsilon-1)}} \ell_t \quad \text{and} \quad q_{kt} = \left(\frac{1}{\alpha_k}\right)^{\frac{\varphi}{1-\varphi(\epsilon-1)}} q_t,$$

where ℓ_t and q_t solve

$$\begin{aligned} (\rho + \delta) \frac{1}{\varphi} \ell_t - \frac{1 - \varphi}{\varphi} \dot{\ell}_t &= \frac{\pi_{\mathbb{K}t} A(\epsilon - 1)}{\int \left(\frac{1}{\alpha}\right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha)} \frac{\ell_t^\varphi}{q_t}, \\ \dot{q}_t &= \ell_t^\varphi - \delta q_t, \end{aligned}$$

with initial condition q_0 and boundary condition

$$\lim_{t \rightarrow \infty} \ell_t = \frac{\delta \pi_{\mathbb{K}}^{ss} A(\epsilon - 1) \varphi}{\rho + \delta} \left[\int \left(\frac{1}{\alpha}\right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha) \right]^{-1},$$

where $\pi_{\mathbb{K}}^{ss} \equiv \lim_{t \rightarrow \infty} \pi_{\mathbb{K}t}$.

Proof. All equilibrium objects besides investment are derived as in the proof of Theorem 1.

The Hamiltonian for platform k 's problem is

$$\mathcal{H}(t, q_{kt}, \ell_{kt}, \lambda_{kt}) = \pi_{\mathbb{K}t} A \frac{q_{kt}^{\epsilon-1}}{\int_{\mathbb{K}} q_{zt}^{\epsilon-1} dz} - \alpha_k \ell_{kt} + \lambda_{kt} (\ell_{kt}^{\varphi} - \delta q_{kt}),$$

where the costate variable satisfies

$$\rho \lambda_{kt} - \dot{\lambda}_{kt} = \pi_{\mathbb{K}t} A (\epsilon - 1) \frac{q_{kt}^{\epsilon-2}}{\int_{\mathbb{K}} q_{zt}^{\epsilon-1} dz} - \delta \lambda_{kt}.$$

The first-order condition for maximizing the Hamiltonian yields

$$\lambda_{kt} = \frac{\alpha_k}{\varphi} \ell_{kt}^{1-\varphi}.$$

Differentiating with respect to time,

$$\dot{\lambda}_{kt} = \alpha_k \frac{1-\varphi}{\varphi} \ell_{kt}^{-\varphi} \dot{\ell}_{kt}.$$

Substituting into the costate equation gives

$$(\rho + \delta) \frac{\alpha_k}{\varphi} \ell_{kt}^{1-\varphi} - \alpha_k \frac{1-\varphi}{\varphi} \ell_{kt}^{-\varphi} \dot{\ell}_{kt} = \pi_{\mathbb{K}t} A (\epsilon - 1) \frac{q_{kt}^{\epsilon-2}}{Q_t}, \quad (57)$$

where

$$Q_t = \int_{\mathbb{K}} q_{zt}^{\epsilon-1} dz.$$

Step 1. Steady State. I first derive the steady-state equilibrium before solving for the transition path. In steady state,

$$(\rho + \delta) \frac{\alpha_k}{\varphi} \ell_k^{1-\varphi} = \pi_{\mathbb{K}} A (\epsilon - 1) \frac{q_k^{\epsilon-2}}{Q},$$

and

$$q_k = \frac{\ell_k^{\varphi}}{\delta}.$$

Hence

$$(\rho + \delta) \frac{\alpha_k}{\varphi} \ell_k^{1-\varphi} = \pi_{\mathbb{K}} A (\epsilon - 1) \frac{\ell_k^{(\epsilon-2)\varphi}}{\delta^{\epsilon-2} Q},$$

which implies

$$\ell_k = \left(\frac{\pi_{\mathbb{K}} A(\epsilon - 1)}{\delta^{\epsilon-2} Q(\rho + \delta)} \frac{\varphi}{\alpha_k} \right)^{\frac{1}{1-\varphi(\epsilon-1)}}.$$

Therefore,

$$q_k = \frac{1}{\delta} \left(\frac{\pi_{\mathbb{K}} A(\epsilon - 1)}{\delta^{\epsilon-2} Q(\rho + \delta)} \frac{\varphi}{\alpha_k} \right)^{\frac{\varphi}{1-\varphi(\epsilon-1)}}.$$

Using the definition of Q ,

$$Q = \frac{1}{\delta^{\epsilon-1}} \int \left(\frac{\pi_{\mathbb{K}} A(\epsilon - 1)}{\delta^{\epsilon-2} Q(\rho + \delta)} \frac{\varphi}{\alpha} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha),$$

which implies

$$Q = \left[\frac{1}{\delta^{\epsilon-1}} \int \left(\frac{\pi_{\mathbb{K}} A(\epsilon - 1)}{\delta^{\epsilon-2}(\rho + \delta)} \frac{\varphi}{\alpha} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha) \right]^{1-\varphi(\epsilon-1)}.$$

Substituting back yields

$$\ell_k = \left(\frac{1}{\alpha_k} \right)^{\frac{1}{1-\varphi(\epsilon-1)}} \frac{\delta \pi_{\mathbb{K}} A(\epsilon - 1) \varphi}{\rho + \delta} \left[\int \left(\frac{1}{\alpha} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha) \right]^{-1}.$$

Step 2. Transition Path. Let ℓ_t and q_t denote the investment and quality of a platform with $\alpha_k = 1$. I conjecture that

$$\ell_{kt} = \left(\frac{1}{\alpha_k} \right)^{\frac{1}{1-\varphi(\epsilon-1)}} \ell_t$$

and

$$q_{kt} = \left(\frac{1}{\alpha_k} \right)^{\frac{\varphi}{1-\varphi(\epsilon-1)}} q_t.$$

The second relation is consistent with the law of motion because

$$\dot{q}_{kt} = \left(\frac{1}{\alpha_k} \right)^{\frac{\varphi}{1-\varphi(\epsilon-1)}} \dot{q}_t = \ell_{kt}^{\varphi} - \delta q_{kt},$$

provided that the initial conditions satisfy

$$q_{k0} = \left(\frac{1}{\alpha_k} \right)^{\frac{\varphi}{1-\varphi(\epsilon-1)}} q_0.$$

Substituting the conjecture into (57), the left-hand side becomes

$$\alpha_k \left(\frac{1}{\alpha_k} \right)^{\frac{1-\varphi}{1-\varphi(\epsilon-1)}} \left[(\rho + \delta) \frac{1}{\varphi} \ell_t^{1-\varphi} - \frac{1-\varphi}{\varphi} \ell_t^{-\varphi} \dot{\ell}_t \right].$$

Moreover,

$$q_{kt}^{\epsilon-2} = \left(\frac{1}{\alpha_k} \right)^{\frac{\varphi(\epsilon-2)}{1-\varphi(\epsilon-1)}} q_t^{\epsilon-2}.$$

The powers of α_k coincide because

$$1 - \frac{1-\varphi}{1-\varphi(\epsilon-1)} = -\frac{\varphi(\epsilon-2)}{1-\varphi(\epsilon-1)}.$$

Hence (57) reduces to

$$(\rho + \delta) \frac{1}{\varphi} \ell_t^{1-\varphi} - \frac{1-\varphi}{\varphi} \ell_t^{-\varphi} \dot{\ell}_t = \pi_{\mathbb{K}t} A(\epsilon-1) \frac{q_t^{\epsilon-2}}{Q_t}.$$

Under the conjectured quality profile,

$$Q_t = q_t^{\epsilon-1} \int \left(\frac{1}{\alpha} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha).$$

Therefore,

$$(\rho + \delta) \frac{1}{\varphi} \ell_t^{1-\varphi} - \frac{1-\varphi}{\varphi} \ell_t^{-\varphi} \dot{\ell}_t = \frac{\pi_{\mathbb{K}t} A(\epsilon-1)}{\int \left(\frac{1}{\alpha} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha)} \frac{1}{q_t}.$$

Multiplying both sides by ℓ_t^φ gives

$$(\rho + \delta) \frac{1}{\varphi} \ell_t - \frac{1-\varphi}{\varphi} \dot{\ell}_t = \frac{\pi_{\mathbb{K}t} A(\epsilon-1)}{\int \left(\frac{1}{\alpha} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha)} \frac{\ell_t^\varphi}{q_t}.$$

Together with

$$\dot{q}_t = \ell_t^\varphi - \delta q_t,$$

this is exactly the reduced ODE system stated in the theorem.

Finally, Step 1 implies

$$\lim_{t \rightarrow \infty} \ell_t = \frac{\delta \pi_{\mathbb{K}} A(\epsilon-1) \varphi}{\rho + \delta} \left[\int \left(\frac{1}{\alpha} \right)^{\frac{\varphi(\epsilon-1)}{1-\varphi(\epsilon-1)}} dP(\alpha) \right]^{-1}$$

for any investment path that converges to steady state. Existence and uniqueness of the solution to the reduced ODE system follow by the analogous arguments as in the proof of Theorem 1. $\square$

J Extension: Zero Prices

In this section, I give an informal argument that zero prices can arise, for some parameter conditions, endogenously in a variant of the baseline model where platforms can charge nonnegative prices. I rule out negative prices on the grounds that it is too difficult for platforms to verify human usage as opposed to usage by bots. Under this premise, charging a negative price is unsustainable for a platform.

It is easiest to make the point when there are an integer number K of atomic platforms and when consumers have Cobb-Douglas utility as in Section 6. The core of the argument is simply that if consumers enjoy product consumption much more than platform consumption, then the attention spent on a platform will decrease quickly in the price set by the platform. This is because, if the consumer spends more attention on the platform, the consumer can spend less income on consuming products. When the elasticity of attention with respect to price is sufficiently high, it is better for a platform to rely solely on advertising to earn revenue.

Suppose that all platforms but platform 1 charge a zero price and have quality level q . Let q_1 denote platform 1's quality level. Let $p \geq 0$ denote the price charged by platform 1. Consumer i chooses how much attention x_1 to allocate to platform 1 to maximize flow utility which amounts to maximizing

$$(I - px_1)^{1-\tau} (M\mu_H)^{\frac{1-\tau}{\sigma-1}} \left[(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\tau\epsilon}{\epsilon-1}} \nu(A)^\tau.$$

The first two terms comprise product consumption and the second two terms comprise platform consumption. Above, I have used the fact that the consumer will want to allocate attention evenly across the $K-1$ remaining platforms. After spending px_1 units of income on consuming platform 1, the consumer has only $I - px_1$ left to spend on products.

The first order condition for consumer i 's problem is

$$\begin{aligned} p(1-\tau)(I - px_1)^{-\tau} \left[(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\tau\epsilon}{\epsilon-1}} = \\ (I - px_1)^{1-\tau} \tau \left[(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\tau\epsilon}{\epsilon-1}-1} \\ \times \left[q_1 (x_1 q_1)^{\frac{\epsilon-1}{\epsilon}-1} - q \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}-1} \right]. \end{aligned}$$

Canceling out some terms and rearranging yields

$$p \left[(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}} \right] = \frac{\tau}{1-\tau} \left[q_1 (x_1 q_1)^{\frac{\epsilon-1}{\epsilon}-1} - q \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}-1} \right] (I - p x_1)$$

which is linear in the price p .

Solving for p yields the inverse demand curve:

$$p(x_1) = \frac{I \left[q_1 (x_1 q_1)^{\frac{\epsilon-1}{\epsilon}-1} - q \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}-1} \right]}{\frac{1-\tau}{\tau} \left[(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}} \right] + \left[q_1 (x_1 q_1)^{\frac{\epsilon-1}{\epsilon}-1} - q \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}-1} \right] x_1}.$$

Since $I - p x_1$ must be positive and p must be nonnegative, the domain of the inverse demand curve is the set of demands x_1 such that the numerator is nonnegative. That is, the domain is

$$x_1 \in \left[0, \frac{q_1^{\epsilon-1}}{q_1^{\epsilon-1} + (K-1)q^{\epsilon-1}} \right].$$

This is intuitive: the domain consists of attention levels that are less than that which would arise if platform 1 also charged a price of zero.

We can now formulate platform 1's pricing problem which is to choose x_1 in this domain to maximize flow profit:

$$p(x_1)x_1 + \pi_{\mathbb{K}} A x_1.$$

We are interested in parameter conditions for when

$$x_1 = \frac{q_1^{\epsilon-1}}{q_1^{\epsilon-1} + (K-1)q^{\epsilon-1}}$$

is optimal which corresponds to setting a zero price. This amounts to looking for parameter conditions such that

$$\frac{d[p(x_1)x_1]}{dx_1} + \pi_{\mathbb{K}} A = p'(x_1)x_1 + p(x_1) + \pi_{\mathbb{K}} A > 0 \quad (58)$$

for all x_1 in the domain. This will happen if $p(\cdot)$ does not decrease too fast. Intuitively this will be the case when τ is close to zero so that the consumer

cares little about platform use and so attention will be very sensitive to the price set by platform 1.

Indeed, by inspection, if τ is close to 0,

$$p(x_1) \approx I \frac{\tau}{1-\tau} \frac{q_1(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}-1} - q \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}-1}}{(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}}} \approx 0.$$

I show later that we can bound

$$\frac{d[p(x_1)x_1]}{dx_1}$$

from *below* for all x_1 in the domain by an amount that can be made arbitrarily close to 0 by making τ sufficiently close to 0. Thus, when τ is sufficiently close to 0, it follows that (58) holds for all x_1 in the domain and a price of zero is optimal. I will take this fact as given now and show it formally later.

I have therefore shown that platform 1 does not have a profitable deviation to charging a positive price when τ is close to 0. In principle the platform could deviate both in its investment strategy and in its pricing strategy. But, so far, our analysis has fixed an arbitrary quality level for q_1 . For some value of τ , say $\tau(q_1)$, which depends on q_1 we have shown it is not profitable to charge a positive price. Let $\bar{q}$ be relatively large and $\underline{q}$ be relatively small and consider

$$\tau^* = \inf_{q_1 \in [\underline{q}, \bar{q}]} \tau(q_1) < 1.$$

Then for parameter $\tau = \tau^*$ it is never optimal for a platform to deviate to any quality $q \in [0, \bar{q}]$. By setting $\bar{q}$ sufficiently high, investment costs are sufficiently high that it is obviously not optimal to deviate in terms of investment to end up at any quality $q \geq \bar{q}$. Similarly if $q \leq \underline{q}$ for some $\underline{q}$ sufficiently low deviating by cutting back on investment is not profitable because profits are too low at any positive price that the platform can set.

I will now show that we can bound the derivative of

$$p(x_1)x_1 = \frac{I \left[q_1(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}-1} - q \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}-1} \right] x_1}{\frac{1-\tau}{\tau} \left[(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}} \right] + \left[q_1(x_1 q_1)^{\frac{\epsilon-1}{\epsilon}-1} - q \left(\frac{1-x_1}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}-1} \right] x_1}$$

from below by an amount arbitrarily close to 0 as claimed.

Define f such that

$$p(x_1)x_1 = I \frac{f'(x_1)x_1}{\frac{1-\tau}{\tau} \frac{\epsilon-1}{\epsilon} f(x_1) + f'(x_1)x_1}.$$

Specifically,

$$\frac{\epsilon-1}{\epsilon} f(x) = (xq_1)^{\frac{\epsilon-1}{\epsilon}} + (K-1) \left(\frac{1-x}{K-1} q \right)^{\frac{\epsilon-1}{\epsilon}}, \quad f'(x) = q_1(xq_1)^{-\frac{1}{\epsilon}} - q \left(\frac{1-x}{K-1} q \right)^{-\frac{1}{\epsilon}}.$$

Then

$$\begin{aligned} \frac{1}{I} \frac{d[p(x_1)x_1]}{dx_1} &= \frac{f''(x_1)x_1 + f'(x_1)}{\frac{1-\tau}{\tau} \frac{\epsilon-1}{\epsilon} f(x_1) + f'(x_1)x_1} \\ &\quad - \frac{f'(x_1)x_1}{\left[\frac{1-\tau}{\tau} \frac{\epsilon-1}{\epsilon} f(x_1) + f'(x_1)x_1 \right]^2} \left[\left(\frac{1-\tau}{\tau} \frac{\epsilon-1}{\epsilon} + 1 \right) f'(x_1) + f''(x_1)x_1 \right]. \end{aligned}$$

Can we bound this from below? Consider taking the limit as τ tends to 0. Then the above, for a fixed $x_1 > 0$, converges to 0. However, we need to make sure that the above over the entire range of $x_1 \in \left[0, \frac{q_1^{\epsilon-1}}{q_1^{\epsilon-1} + (K-1)q^{\epsilon-1}} \right]$ converges to 0. This will necessarily be the case if we can bound $f'(x_1)$, $f''(x_1)$, $f(x_1)$. The concern is about points x_1 near 0 since some of these terms explode there. Suppose we know that for any τ close to 0 that it is not optimal to set $x_1 < \tilde{\epsilon}$ for some $\tilde{\epsilon} > 0$, then we are done since $f'(x_1)$, $f''(x_1)$, and $f(x_1)$ are bounded on $\left[\tilde{\epsilon}, \frac{q_1^{\epsilon-1}}{q_1^{\epsilon-1} + (K-1)q^{\epsilon-1}} \right]$.

To show this, note that

$$p(x)x + \pi_{\mathbb{K}}Ax \leq p(x)x + \pi_{\mathbb{K}}A\tilde{\epsilon}$$

whenever $x \leq \tilde{\epsilon}$. But

$$p(x)x \leq I \frac{f'(x)x}{\frac{\epsilon-1}{\epsilon} f(x) + f'(x)x}$$

for all $\tau \leq 1/2$. Then

$$p(x)x + \pi_{\mathbb{K}}Ax \leq \sup_{x \in [0, \tilde{\epsilon}]} I \frac{f'(x)x}{\frac{\epsilon-1}{\epsilon} f(x) + f'(x)x} + \pi_{\mathbb{K}}A\tilde{\epsilon}$$

and the upper bound holds uniformly over all $\tau \leq \frac{1}{2}$.

Note that the sup on the right-hand side goes to zero as $\tilde{\epsilon}$ goes to zero. Thus, for some $\tilde{\epsilon} > 0$ the right hand side is always less than

$$\pi_{\mathbb{K}} A \frac{q_1^{\epsilon-1}}{q_1^{\epsilon-1} + (K-1)q^{\epsilon-1}}.$$

Thus, there is some $\tilde{\epsilon} > 0$ lower bound such that its not optimal to set $x_1 < \tilde{\epsilon}$ for any parameter $\tau \leq 1/2$.

K Extension: Firm and Platform Entry

I extend the baseline model to allow for entry of firms and platforms.

K.1 Setup

To enter the market, a firm must pay a cost $e_{\mathbb{J}} > 0$ and a platform must pay a cost $e_{\mathbb{K}} > 0$. I retain all other aspects of the baseline model of Section 3 except that in equilibrium, the measure J of firms and the measure K of platforms are such that firms and platforms earn zero profits net of entry costs.

K.2 Steady State

I focus on steady-state equilibria. Initial conditions (q_0, M_0, H_0) are such that q_t , M_t , and H_t remain constant over time. Firms are initially distributed across consumers' consideration sets according to a symmetric stationary distribution consistent with (M_0, H_0) , implying constant flow profits for firms. Platforms are initialized at the steady-state quality level. The equilibrium masses of firms and platforms are then determined by the free-entry conditions.

Theorem 6. *Suppose that A is the unique solution of $\max_a a\nu(a)^{\epsilon-1}$ and $\epsilon - 1 < 1/\varphi$. Then if a steady-state equilibrium exists, it has the same steady-state properties as in Theorem 1 for a given J and K which satisfy*

$$K = \frac{1}{\rho} \frac{\pi_{\mathbb{K}} A}{e_{\mathbb{K}}} \left(1 - \frac{\varphi \delta (\epsilon - 1)}{\rho + \delta} \right), \quad (59)$$

and

$$J = \frac{1}{\rho} \left(\frac{\frac{I}{\sigma} - \pi_{\mathbb{K}} A}{e_{\mathbb{J}}} \right). \quad (60)$$

Under Condition 1, the steady-state equilibrium is unique.

Proof. Equations 59 and (60) are zero profit conditions. Note that in (60), $\pi_{\mathbb{K}}$ depends on J but not K . Thus, to prove uniqueness under Condition 1, I must prove there is a unique solution for J in (60). This follows by Lemma 14 which shows that $\pi_{\mathbb{K}}$ is increasing in J and so, by monotonicity only one J can satisfy (60). The other parts of the theorem follows the same roadmap as in the proof of Theorem 1. $\square$

Remark 1. *An alternative interpretation is that $e_{\mathbb{J}}$ and $e_{\mathbb{K}}$ are fixed flow costs incurred each period to maintain operation rather than one-time entry costs. Under this interpretation, the same steady-state characterization applies, except that the factor $1/\rho$ is omitted from the zero-profit conditions. Comparative statics might then be interpreted as changes in the steady-state masses of firms and platforms arising through entry and exit.*

One can extend most of the comparative statics for steady state in Appendix C to this setting.

Lemma 18. *Under Condition 1, in steady state, banning data use in the sense of Proposition 2 leads to a decrease in J and an increase in K .*

Proof. Suppose for contradiction J increases. Holding fixed J , a ban on data use raises ad revenue, as show in Proposition 2 of the main text. An increase in J further increases ad revenue as shown by Lemma 14. Thus ad revenue must increase. But then by (60), J must decrease, a contradiction. Thus, J must decrease.

Because ad revenue must increase, inspecting (59), reveals K must increase. $\square$

Lemma 19. *Under Condition 1, in steady state, an increase in platform substitutability ϵ leads to a decrease in J and, if the induced decrease in the ad display rate A is sufficiently small, leads to a decrease in K .*

Proof. An increase in ϵ increases ad revenue by Proposition 6. Suppose, toward a contradiction, that J increases. By Lemma 14, the increase in J further increases ad revenue. Hence, ad revenue must increase. Equation (60) then implies that J decreases, a contradiction. Thus, J decreases.

If the decrease in A induced by the increase in ϵ is sufficiently small, then (59) implies that K decreases. $\square$