

SUPPLEMENTAL APPENDIX

Spatial Externalities, Inefficiency, and Sufficient Statistics

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Proof of Proposition 1: The proof uses the implicit function theorem around the current equilibrium. We can write the equilibrium conditions 1 and 2 as a multivariate function $G(c, \pi) = \pi - F(u^0 + c + e(\pi))$. The Jacobian with respect to π evaluated at the initial equilibrium (c^0, π^0) is given by

$$\frac{d}{d\pi}G(c^0, \pi^0) = I - DE'$$

Since eigenvalues of DE' are same as eigenvalues of $E'D$, the Jacobian is non-singular because of Assumption 1. The implicit function theorem implies that there is an open neighborhood of c^0 given by $B(c^0)$ and a unique function $\pi(c)$ defined on $B(c^0)$ such that $G(c, \pi(c)) = 0$ for all $c \in B(c^0)$. Furthermore, the π function is continuously differentiable.

Since $(c, \pi(c))$ satisfy equilibrium conditions 1 and 2, we have

$$u(c) = u^0 + c + e(\pi(c)) \quad \text{and} \quad \pi(c) = F(u(c)),$$

Defining $K = d\pi/dc$ and computing derivatives using chain rule gives

$$\begin{aligned} du &= dc + (de(\pi)/d\pi)d\pi = dc + E'd\pi = (1 + E'K)dc \\ d\pi &= (dF/du)(du/dc)dc = D(1 + E'K)dc \end{aligned}$$

We then have $K = D(1 + E'K) \implies K = (1 - DE')^{-1}D$. It is easy to show that K also equals $D(I - E'D)^{-1}$.

We turn to proving the last part of Proposition 1, namely the welfare effect of small charges. Using the envelope theorem and substituting from the above results, we get

$$\begin{aligned} dW/dc &= \frac{d}{dc} \left(\mathbb{E} \max_k [u_k + \epsilon_k(\omega)] - c'\pi \right) \\ &= \pi'(du/dc) - c'(d\pi/dc) - \pi' \\ &= \pi'(I + E'K) - c'K - \pi' \\ &= (E\pi - c)'K. \end{aligned}$$

This concludes the proof of Proposition 1.

Proposition 3. *Assume that the following conditions are satisfied*

- *An unpriced equilibrium $(c = 0, u = u^0, \pi = \pi^0)$ exists.*
- *Externalities $e(\pi)$ are bounded and differentiable functions of $\pi = F(u)$.*
- *Assumption 1 holds true globally: for any $u, \pi = F(u)$, assumption 1 is true with $E' = de(\pi)/d\pi$ and $K = d\pi/du$.*

Then, given any charges c , a unique equilibrium exists.

Equilibrium can be considered as a pair of charges and utilities (c, u) satisfying conditions 1 and 2. E', D are functions of u since $\pi = F(u)$ and $D = d\pi/du, E' = e(\pi)/e\pi$.

Proof:

For a vector x , define $G(x) = x - e(F(u^0 + x)) + e(F(u^0))$. Then the equilibrium conditions 1, 2 can be summarized as $c - c^0 = G(u - u^0)$. It is sufficient to prove that G is a bijection. Note that G satisfies the following properties

- $G(0) = 0$
- The Jacobian determinant of G is non-zero everywhere as $J_G = \frac{dG}{dx} = 1 - E'D$ where matrices E, D are evaluated at $u = u^0 + x$
- G is a *proper* map. To see this, we show that for any compact set $K \subset \mathbb{R}^N$, the preimage $G^{-1}(K)$ is also compact. Since K is compact, it is bounded and closed. The set $G^{-1}(K)$ is given by

$$G^{-1}(K) = \{x : x - e(F(u^0 + x)) + e(F(u^0)) \in K\}$$

Since $-e(F(u^0 + x)) + e(F(u^0))$ is bounded, $G^{-1}(K)$ is also bounded. Continuity of G implies that $G^{-1}(K)$ is closed. Hence $G^{-1}(K)$ is compact.

With these properties, the Hadamard-Caccioppoli theorem implies that G is a bijective map (Krantz and Parks (2003), p. 125). Hence, for any charge c there exists a unique $u = u(c)$ and $\pi = F(u(c))$ satisfying equilibrium conditions.

This concludes the proof.

Proof of Proposition 2: With D and E constant,

$$\begin{aligned} dW &= (E\pi - c)'Kdc \\ &= (E\pi(0) + (EK - I)c)'Kdc \end{aligned}$$

We integrate this over $\lambda \in [0, 1]$ to get

$$\begin{aligned} W(c) - W(0) &= \int_0^1 dW(\lambda c) d(\lambda c) \\ &= \int_0^1 (E\pi(0) + (EK - I)\lambda c)'Kc d\lambda \\ &= (E\pi(0))'Kc + (c'(EK - I)'Kc) \int_0^1 \lambda d\lambda \\ &= (E\pi(0))'Kc + \frac{1}{2}c'(EK - I)'Kc \end{aligned}$$

$c^{so} = (I - EK)^{-1}E\pi(0)$ is a stationary point of $W(c)$. When $R = (EK - I)'K$ is negative semidefinite, welfare is maximized at $c = c^{so}$ and the deadweight loss is given by

$$DWL \equiv W(c^{so}) - W(0) = -\frac{1}{2}(c^{so})'Rc^{so}$$

Assumption 3 ensures that R is negative semidefinite. We can write R explicitly

$$R = -(I - DE)^{-1}(I - DE - DE')D(I - E'D)^{-1}(c^{so}).$$

D is positive semi-definite by definition and eigenvalues of $ED + E'D$ are same as eigenvalues of the symmetric matrix $\sqrt{D}(E + E')\sqrt{D}$. Therefore, eigenvalues of $\sqrt{D}(E + E')\sqrt{D}$ are real and less than 1 by assumption 3. Hence the inner matrix $(I - DE - DE')D = \sqrt{D}(1 - \sqrt{D}(E + E')\sqrt{D})\sqrt{D}$ is positive semi-definite and R is negative semi-definite.

This concludes the proof of proposition 2.

A crucial approximation in the proposition is that E and D are constant in a big-enough neighborhood around $c = 0$. This can be expressed formally as the following assumption.

Assumption 4. *There exists a ball B_r of radius r centered at $c = 0$ such that for any $c_1, c_2 \in B_r$, $\frac{dE^{c_1}}{d\pi} \pi^{c_1} \ll E^0$ and $\frac{dK^{c_1}}{dc} c_2 \ll K^0$. B_r is large enough such that $E\pi^0, (1 - EK)^{-1}E\pi^0 \in B_r$.*

We also report a more general result without assuming constant E and D :

Proposition 4. *Exact deadweight loss:* *Adding charges c to an unpriced initial equilibrium modifies welfare by (up to second order in charges)⁸*

$$W(c) - W(0) = (E^0 \pi^0)' K^0 c + \frac{1}{2} c' R^0 c + o(c^3)$$

$$\text{where } R^0 \equiv \underbrace{\frac{dK^0}{dc}}_{\text{tensor}} + \left[\underbrace{\frac{dE^0}{d\pi}}_{\text{tensor}} \pi^0 K^0 + (E^0 K^0 - I)' \right] K^0$$

where ⁰ superscripts indicate terms evaluated at the unpriced equilibrium. When R^0 is positive semi-definite, welfare is maximized by socially optimal charges c^{so} satisfying $(K^0)' E^0 \pi^0 + R^0 c^{so} \approx 0$. The deadweight loss up to the second order in c^{so} is

$$DWL \equiv W(c^{so}) - W(0) \approx -\frac{1}{2} (c^{so})' R^0 c^{so}$$

This result can be obtained by directly applying Taylor's series. Setting the tensor terms to zero gives Proposition 2.

A1. Example: Two Locations and Global Externality

We express a textbook model with externality in our setting. Consider two locations (or goods), with externality matrix

$$E = \begin{pmatrix} 0 & e \\ 0 & e \end{pmatrix},$$

for some $e < 0$. This means that location 2 imposes a externality on all agents (irrespective of their location). We begin by assuming perfectly elastic supply. The Slutsky matrix is

$$D = \begin{pmatrix} -\kappa_D & \kappa_D \\ \kappa_D & -\kappa_D \end{pmatrix},$$

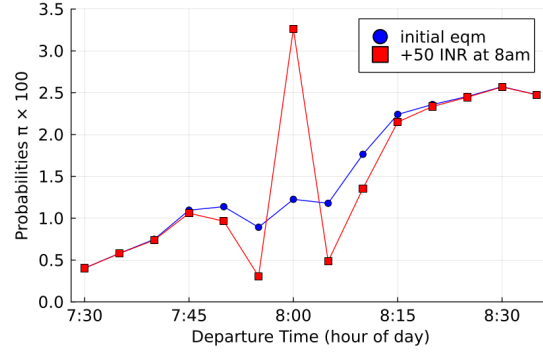
where $\kappa_D < 0$ is the slope of the demand curve. Externalities do not affect the incentive of where to locate, so $E'D = 0$, hence $R = K = D$. Optimal charges and deadweight loss are

$$c^{so} \approx \begin{pmatrix} 0 \\ e \end{pmatrix} \quad \text{and} \quad DWL \approx -\frac{1}{2} e^2 \kappa_D = \frac{1}{2} e \Delta \pi$$

(When adding elastic supply, deadweight loss becomes $-\frac{1}{2} e^2 K_{22}$ where the equilibrium elasticity $K_{22} = \frac{1}{\frac{1}{\kappa_S} - \frac{1}{\kappa_D}}$ and κ_S is the slope of the supply curve.)

⁸ $\frac{dK^0}{dc}$ and $\frac{dE^0}{d\pi}$ are tensors such that $[\frac{dK^0}{dc}]_{ij} = \sum_k \frac{dK_{ik}^0}{dc_j} b_k$

Figure A1. : Local Demand Substitution Patterns



Highly localized demand substitution in the estimated model from Kreindler (2024) due to an additional 50 INR transfer for agents starting trips between 08:00 and 08:05.

A2. Example 2: Peak-Hour Congestion

We report here the results of our approach for a setting where a scalar measure of unobserved heterogeneity (called ζ) interacts with externalities. Equilibrium conditions are

$$u(\zeta) = u^0(\zeta) + e(\pi, \zeta) + c, \forall \zeta$$

$$\pi = \int F(u(\zeta), \zeta) dG(\zeta)$$

By integrating over ζ we obtain

$$DWL \approx (c^{so})'(I - M_{ED} - M_{EDE'}K)'Kc^{so}$$

where

$$c^{so} = (I - M_{ED} - M_{EDE'}K)^{-1}M_{E\pi^0}$$

$$K = (I - M'_{ED})^{-1}M_D$$

$$M_D = \int D(\zeta)dG(\zeta)$$

$$M_{E\pi^0} = \int E(\zeta)\pi^0(\zeta)dG(\zeta)$$

$$M_{ED} = \int E(\zeta)D(\zeta)dG(\zeta)$$

$$M_{EDE'} = \int E(\zeta)D(\zeta)E(\zeta)'dG(\zeta).$$

These expressions exactly mirror those in the main paper, except that, whenever necessary, we integrate over ζ .

REFERENCES

Krantz, Steven G., and Harold R. Parks. 2003. *The Implicit Function Theorem: History, Theory, and Applications*. Boston, MA, Birkhäuser.