

Growth and Redistribution: The Hedging Perspective

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teaching materials

1 Kelly's Betting

2 Macroeconomic Reinterpretation

3 Analysis

Kelly '56

assets $i \in I$, discrete time t

stationary portfolio $p(i) \in \Delta(I)$

- share $p(i)$ of wealth invested in each asset i each morning

each asset i has a gross return $r(i, \omega_t)$ in each period t

- iid states $\omega_t \sim q^0(\omega) \in \Delta(\Omega)$

investor maximizes long-run growth rate of her wealth:

$$\max_{p(i)} E_{q^0(\omega)} \ln \left(\sum_i p(i) r(i, \omega) \right)$$

Example

$$\Omega = I$$

$$\omega \sim q^0(\omega)$$

exogenous return function: $r(i, \omega) = \text{const.} \times \mathbb{1}_{\omega=i}$

optimal allocation: $p^*(i) = q^0(i)$

in contrast, maximization of expected return leads to a.s. bankruptcy

Outcome Distribution

def: conditional outcome distribution

share of wealth in asset i at the end of a period with $\omega_t = \omega$:

$$o_p(i | \omega) := \frac{p(i)r(i, \omega)}{\sum_j p(j)r(j, \omega)}.$$

def: outcome distribution

share of wealth on i at the end of a random period:

$$o_p(i) := E_{q^0(\omega)} o(i | \omega).$$

Kelly's Optimality Condition

First Order Condition

Optimal portfolio eliminates systematic redistribution of investments:

$$p^*(i) = o_{p^*}(i).$$

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Macroeconomic Reinterpretation

individuals $i \in I$ with random gross returns $r(i, \omega_t)$

social planner maximizes long-run growth rate of aggregate wealth

we add two novel features:

- a constraint on the endowment distribution $p(i)$
- constrained control over the return function $r(i, \omega)$

the set \mathcal{P} of feasible policies $(p(i), r(i, \omega))$ needs not be a product set

- e.g. chosen endowment distribution may affect returns via incentives

Main Result

Proposition

Growth-maximizing endowment distribution p^* minimizes KL-divergence from the induced outcome distribution

$$p^*(i) \in \arg \min_{p(i) \in \mathcal{A}^*} \text{KL} (o_{p^*, r^*}(i) \parallel p(i)),$$

where \mathcal{A}^* is the set of feasible endowment distributions given optimized returns $r^*(i, \omega)$.

recall: $\text{KL} (o(i) \parallel p(i)) = E_{o(i)} \ln \frac{o(i)}{p(i)}$ is a pseudo-distance

note the **myopicity**: the outcome distribution is fixed to o_{p^*, r^*}

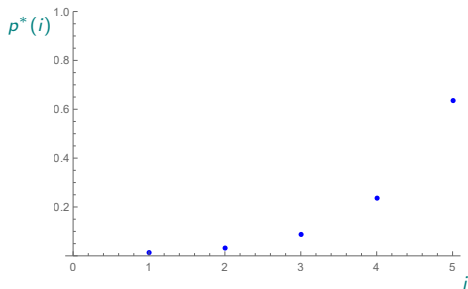
Example

no uncertainty

individual $i = 1, \dots, 5$ has a return i

inequality constraint $H(p(i)) \geq 1$

growth-optimal endowment distribution



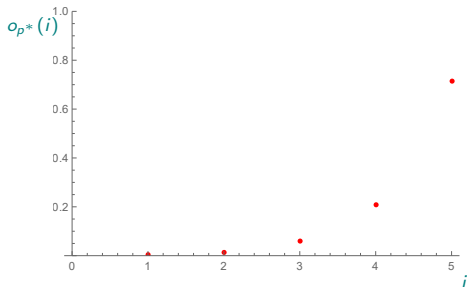
Example

no uncertainty

individual $i = 1, \dots, 5$ has a return i

inequality constraint $H(p(i)) \geq 1$

induced outcome distribution



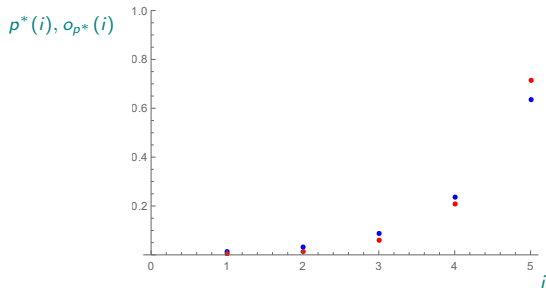
Example

no uncertainty

individual $i = 1, \dots, 5$ has a return i

inequality constraint $H(p(i)) \geq 1$

the 'closest' feasible endowment distribution to σ_p^* coincides with p^*



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Change of the Notation

relabel the return function $r(i, \omega)$ as $p(\omega | i)$

- generalized (non-normalized) conditional distribution

define (generalized) joint distribution

$$p(i, \omega) = p(i)p(\omega | i)$$

it is a (sufficient statistics for) the policy

- so we refer to $p(i, \omega)$ as **policy**

Auxiliary Problem

known as Variational Autoencoder in machine learning

let $\tilde{\mathcal{P}}$ be the set of feasible policies $p(i, \omega)$

Proposition

Policy $p^*(i, \omega)$ maximizes the growth rate if and only if solves

$$\min_{q(i, \omega), p(i, \omega)} \text{KL} \left(q(i, \omega) \parallel p(i, \omega) \right)$$

$$\text{s.t. } q(i, \omega) \in \Delta(I \times \Omega)$$

$$q(\omega) = q^0(\omega)$$

$$p(i, \omega) \in \tilde{\mathcal{P}},$$

together with some $q^*(i, \omega)$.

studied in machine learning in the context of cognition/perception

Dynasties of Dollars

interpretation of $q(i, \omega)$

a dynasty originates in \$1 at $t = 1$

it moves from one individual to another

it multiplies by the return of its current owner

define **pattern** (of circulation) $q(i, \omega)$ as:

- the empirical frequency of a dynasty being in hands of i and state ω

all patterns $q(i, \omega)$ s.t: $q(\omega) = q^0(\omega)$ are followed by some dynasties

Wealth of a Pattern

informal proof

fix a policy p

define wealth of pattern q as the total wealth of all dynasties following it

wealth of pattern q grows at the rate:

$$E_{q(i,\omega)} \ln p(\omega | i) - E_{q^0(\omega)} \text{KL}(q(i | \omega) || p(i))$$

- growth rate of each dynasty circulating according the patter q
- rate of decline in the measure of such dynasties (Sanov's Theorem)

the fastest growing pattern q_p^* dominates

Wealth of a Pattern

informal proof

fix a policy p

define wealth of pattern q as the total wealth of all dynasties following it

wealth of pattern q grows at the rate:

$$- \text{KL} (q(i, \omega) \parallel p(i, \omega)) + \text{const.}$$

- growth rate of each dynasty circulating according the patter q
- rate of decline in the measure of such dynasties (Sanov's Theorem)

the fastest growing pattern q_p^* dominates

Optimal Path

define **joint** outcome distribution as $o_p(i, \omega) = q^0(\omega) o_p(i | \omega)$

- prob. that a random end-of-period dollar is in hands of i in state ω

lemma

Joint outcome distribution equals the optimal pattern:

$$o_p(i, \omega) = q_p^*(i, \omega).$$

intuition:

- almost all wealth circulates according to $q_p^*(i, \omega)$.

Proof of The Main Result

growth-maximizing policy solves

$$\min_{p(i,\omega)} \text{KL} \left(q^*(i,\omega) \parallel p(i,\omega) \right)$$

$$\text{s.t. } p(i,\omega) \in \tilde{\mathcal{P}}.$$

Proof of The Main Result

growth-maximizing policy solves

$$\min_{p(i)} \left\{ \text{KL}(q^*(i) \parallel p(i)) + \sum_i q^*(i) \text{KL}(q^*(\omega | i) \parallel p(\omega | i)) \right\}$$

s.t. $p(i) \in \mathcal{A}^*$.

Proof of The Main Result

growth-maximizing policy solves

$$\min_{p(i)} \left\{ \text{KL}(o_{p^*}(i) \parallel p(i)) + \sum_i q^*(i) \text{KL}(q^*(\omega \mid i) \parallel p(\omega \mid i)) \right\}$$

$$\text{s.t. } p(i) \in \mathcal{A}^*.$$