Sustainability in a Risky World

John Y. Campbell Ian W. R. Martin

Appendix

Details regarding the derivation of equation (3). It follows by applying Itô's formula for semimartingales to (2) that

$$\text{(A1)} \qquad d\log C = \left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta \right) dt + \alpha\sigma\, dZ + \log\left(1 - \alpha L\right) dN.$$

See, for example, Proposition 8.19 of Cont and Tankov (2004). Heuristically, we can derive (A1) by writing

$$d\log C = \frac{1}{C}dC - \frac{1}{2!}\frac{1}{C^2}\left(dC\right)^2 + \frac{1}{3!}\frac{2}{C^3}\left(dC\right)^3 - \frac{1}{4!}\frac{6}{C^4}\left(dC\right)^4 + \cdots$$

and using the relationships $dt\, dN = dZ\, dN = 0$ and $dN^k = dN$ for all $k > 0$, in addition to the standard properties of dZ and the fact that $\log\left(1+x\right) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$ if $|x| < 1$, which holds when $x = -\alpha L$ because the agent will never risk bankruptcy.

Integrating forwards, exponentiating, using $C_0 = \theta W_0$, and raising to the power $1-\gamma$, we have

$$\text{(A2)} \quad C_t^{1-\gamma} = W_0^{1-\gamma}\theta^{1-\gamma}\exp\left\{(1-\gamma)\left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta\right)t + \alpha(1-\gamma)\sigma Z_t\right\}\prod_{i=1}^{N_t}\left(1-\alpha L_i\right)^{1-\gamma}.$$

Writing L for a representative of the i.i.d. L_i, we have

(A3)

$$\mathrm{E}\, C_t^{1-\gamma} = W_0^{1-\gamma}\theta^{1-\gamma}\exp\left\{(1-\gamma)\left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 - \theta\right)t + \omega\mathrm{E}\left[\left(1-\alpha L\right)^{1-\gamma} - 1\right]t\right\}.$$

This holds because N_t, Z_t, and L_i are independent, and using (i) the law of iterated expectations, (ii) the fact that N_t is a Poisson random variable with parameter ωt, (iii) the i.i.d. nature of the L_i, and (iv) the series definition of the exponential function to calculate

$$\begin{aligned}
\mathrm{E}\prod_{i=1}^{N_t}(1-\alpha L_i)^{1-\gamma} &\overset{(i)}{=} \mathrm{E}\left[\mathrm{E}\left(\prod_{i=1}^{N_t}(1-\alpha L_i)^{1-\gamma} \;\middle|\; N_t\right)\right] \\
&\overset{(ii)}{=} \sum_{n=0}^{\infty} e^{-\omega t}\frac{(\omega t)^n}{n!}\mathrm{E}\prod_{i=1}^{n}(1-\alpha L_i)^{1-\gamma} \\
&\overset{(iii)}{=} \sum_{n=0}^{\infty} e^{-\omega t}\frac{(\omega t)^n}{n!}\left(\mathrm{E}\left[(1-\alpha L)^{1-\gamma}\right]\right)^n \\
&\overset{(iv)}{=} \exp\left\{\omega \mathrm{E}\left[(1-\alpha L)^{1-\gamma}-1\right]t\right\}.
\end{aligned}$$

Details regarding the derivation of equation (12). As above, equation (12) follows directly from Itô's lemma, but we can understand the evolution of the rescaled variable $X = W^{1-\gamma}$ heuristically by writing

$$dX = (1-\gamma)W^{-\gamma}\,dW + \frac{\gamma(\gamma-1)}{2}W^{-\gamma-1}\,dW^2 - \frac{\gamma(\gamma-1)(\gamma+1)}{6}W^{-\gamma-2}\,dW^3 + \cdots$$

Rearranging, we have

$$\begin{aligned}
\frac{dX}{X} &= (1-\gamma)\frac{dW}{W} + \frac{\gamma(\gamma-1)}{2}\left(\frac{dW}{W}\right)^2 - \frac{\gamma(\gamma-1)(\gamma+1)}{6}\left(\frac{dW}{W}\right)^3 + \cdots \\
&= (1-\gamma)\left(r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2\right)dt + (1-\gamma)\alpha\sigma dZ + \\
&\qquad + \left[(\gamma-1)\alpha L + \frac{\gamma(\gamma-1)}{2}\alpha^2 L^2 + \frac{\gamma(\gamma-1)(\gamma+1)}{6}\alpha^3 L^3 + \cdots\right] dN \\
&= (1-\gamma)\left(r_f + \alpha\widehat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2\right)dt + (1-\gamma)\alpha\sigma dZ + \left[(1-\alpha L)^{1-\gamma} - 1\right] dN\,,
\end{aligned}$$

as required, where we use the binomial expansion of $(1-\alpha L)^{1-\gamma}$ in the last line.

PROOF OF RESULT 3:

Define $\boldsymbol{m}(x) = \mathrm{E}\left[(1-\alpha L)^{-x}\right]$. This is the moment-generating function of the random variable $J = -\log(1-\alpha L)$, so $\boldsymbol{m}(x)$ is a convex function of x (indeed, the cumulant-generating function $\log \boldsymbol{m}(x)$ is convex). By Result 1 we can write the maximum sustainable consumption-wealth ratio either as

$$\theta_{\text{con}} = r_f + \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega\left[\boldsymbol{m}(\gamma) - \boldsymbol{m}(\gamma-1) - \frac{\boldsymbol{m}(\gamma-1) - \boldsymbol{m}(0)}{\gamma-1}\right]$$

or as

$$\theta_{\text{con}} = r_f + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2 - \omega\left[\frac{\boldsymbol{m}(\gamma-1)-\boldsymbol{m}(0)}{\gamma-1} - (\boldsymbol{m}(0)-\boldsymbol{m}(-1))\right].$$

The upper and lower bounds follow by noting that the expressions in the square brackets are (collectively) positive in each equation, by virtue of the convexity of $\boldsymbol{m}(x)$.

Adding the two together and dividing by two, it follows that

$$\theta_{\text{con}} = \frac{r_f + r_f + \alpha\mu}{2} + \frac{\omega}{2}\left[\boldsymbol{m}(\gamma) - \boldsymbol{m}(\gamma-1) - 2\frac{\boldsymbol{m}(\gamma-1)-\boldsymbol{m}(0)}{\gamma-1} + (\boldsymbol{m}(0)-\boldsymbol{m}(-1))\right].$$

The condition for $\theta_{\text{con}} \geq \frac{r_f+r_f+\alpha\mu}{2}$ is therefore that

$$\text{(A4)} \qquad \boldsymbol{m}(\gamma) - \boldsymbol{m}(\gamma-1) - 2\frac{\boldsymbol{m}(\gamma-1)-\boldsymbol{m}(0)}{\gamma-1} + \boldsymbol{m}(0) - \boldsymbol{m}(-1) \geq 0.$$

This may or may not hold for different size distributions L. Suppose, however, that L is deterministic and $L \geq 0$. Then $\boldsymbol{m}(x) = e^{Jx}$ where $J = -\log(1-\alpha L) \geq 0$. In this case, condition (A4) becomes

$$e^{J\gamma} - e^{J(\gamma-1)} - 2\frac{e^{J(\gamma-1)}-1}{\gamma-1} + 1 - e^{-J} \geq 0.$$

We fix $\gamma > 1$ and view the left-hand side of this inequality as a function of J. Defining

$$h(J) = e^{J\gamma} - e^{J(\gamma-1)} - 2\frac{e^{J(\gamma-1)}-1}{\gamma-1} + 1 - e^{-J},$$

we must show that $h(J) \geq 0$ for arbitrary $J \geq 0$. As $h(0) = 0$, it is enough to show that $h'(J) \geq 0$ for $J \geq 0$. By direct calculation, $h'(J) = \gamma e^{J\gamma} - (\gamma+1)e^{J(\gamma-1)} + e^{-J}$. As $h'(0) = 0$, it remains to show that $h''(J) \geq 0$ for $J \geq 0$, as this will establish that $h'(J) \geq 0$ for arbitrary $J \geq 0$, and hence that $h(J) \geq 0$ for arbitrary $J \geq 0$. But this holds:

$$h''(J) = \gamma^2\left(e^{J\gamma} - e^{J(\gamma-1)}\right) + e^{-J}\left(e^{J\gamma} - 1\right) \geq 0.$$

If $L \leq 0$ and hence $J \leq 0$, the same logic applies but the inequality is reversed.

PROOF OF RESULT 4:

As

$$\text{(A5)} \qquad \frac{dW}{W} = \left\{\frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega}{\gamma-1}\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1\right]\right\} dt + \alpha\sigma\, dZ - \alpha L\, dN,$$

the drift of wealth is

$$\text{(A6)} \qquad \mathrm{E}\frac{dW}{W} = \left(\frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega}{\gamma-1}\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1 + \alpha L(1-\gamma)\right]\right) dt,$$

and both terms in the brackets are positive. (To see that the second term is positive, note that Bernoulli's inequality states that $(1+x)^r \geq 1 + rx$ if $1+x$ is positive and $r \leq 0$ or $r \geq 1$. Under our maintained assumption that $\gamma > 1$, it follows that $(1-\alpha L)^{1-\gamma} \geq 1 + (\gamma-1)\alpha L$.)

Similarly,

(A7)

$$\mathrm{E}\, d\log W = \left(\frac{1}{2}(\gamma-1)\alpha^2\sigma^2 + \frac{\omega}{\gamma-1}\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1 - (1-\gamma)\log(1-\alpha L)\right]\right) dt\,.$$

Again both terms in the brackets are positive. This is obvious for the first term; for the second, write

$$\mathrm{E}\left[(1-\alpha L)^{1-\gamma} - 1 - (1-\gamma)\log(1-\alpha L)\right] = \mathrm{E}\left[e^{(\gamma-1)J} - 1 - (\gamma-1)J\right],$$

where $J = -\log(1-\alpha L)$, and use the fact that $e^y \geq 1 + y$ for all $y \in \mathbb{R}$.

Sustainability with population growth. The analysis in the body of the paper imposes sustainability on a social welfare function defined over aggregate consumption. This is equivalent to sustainability of individual utility only if the population is constant. We now show how to modify our analysis to make individual utility sustainable given constant exogenous population growth at rate g.

If there is population growth, then wealth at time t is shared between more people. Normalizing the initial population size to 1, the wealth of an average individual at time t is $W_t e^{-gt}$, where $g > 0$ is the population growth rate. To ensure that such an average individual's expected utility is nondecreasing, we require[1] that $\widetilde{X}_t$ has nonpositive drift, where $\widetilde{X}_t = e^{g(\gamma-1)t}X_t$.

As $d\widetilde{X}/\widetilde{X} = g(\gamma-1)\,dt + dX/X$, the sustainability constraint (11) becomes

$$\text{(A8)} \qquad \theta \leq r_{\mathrm{CE}} - g.$$

The right-hand side of (A8) subtracts the population growth rate g from the previous formula for the sustainable consumption-wealth ratio. Sustainability of individual utility is a more demanding requirement in the presence of population growth. However, for realistic population growth rates the central message of the

[1]This condition also ensures nondecreasing expected utility for any class of individuals who have a constant share of the wealth of society. For example, a Blanchard (1985) model with population growth implies that a newborn person has lower wealth than the average currently living person, because more people are born than die at each instant; however, with a constant population growth rate the wealth share of newborn individuals is constant over time. Thus, the constraint that $\widetilde{X}_t$ has nonpositive drift ensures that the expected utility of newborn individuals does not decline over time.

paper remains unchanged: in the presence of risk, the sustainable consumption-wealth ratio exceeds the riskless interest rate r_f and substantially exceeds $r_f - g$, which would be its value in the riskless economy considered by Arrow et al. (2004).

The case of multiple assets. Our results generalize without modification if there are multiple assets whose returns are i.i.d. over time (but potentially correlated across assets), as for any portfolio of asset holdings consumption growth will continue to satisfy equation (2) for appropriate μ, σ, ω, and L.

The log utility case. With log utility, the investor's objective function is

$$U = \mathrm{E}\int_0^\infty e^{-\rho t}\log C_t\,dt\,, \quad \text{where } \rho > 0.$$

It follows from equation (A1) that

$$\log C_t = \log C_0 + \left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta\right)t + \alpha\sigma Z_t + \sum_{i=1}^{N_t}\log\left(1 - \alpha L_i\right),$$

and hence

$$\mathrm{E}\log C_t = \log C_0 + \left(r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta\right)t + \omega\mathrm{E}\left[\log\left(1 - \alpha L\right)\right]t.$$

Thus the objective function can be evaluated explicitly as

$$U = \frac{\log W_0 + \log\theta}{\rho} + \frac{r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 - \theta + \omega\mathrm{E}\left[\log\left(1 - \alpha L\right)\right]}{\rho^2}.$$

Maximizing with respect to θ and α we find the first-order conditions for an unconstrained optimum,

$$\theta = \rho \quad \text{and} \quad \widehat{\mu} - \alpha\sigma^2 = \omega\mathrm{E}\left[L\left(1 - \alpha L\right)^{-1}\right].$$

The objective function at time t is affine in $\log W_t$, so the sustainability condition requires that $d\log W_t$, or equivalently $d\log C_t$, is driftless, i.e. that

$$\theta \le r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 + \omega\mathrm{E}\left[\log\left(1 - \alpha L\right)\right].$$

We define the constrained solution as before, giving

$$\theta_{\text{con}} = r_f + \alpha\widehat{\mu} - \frac{1}{2}\alpha^2\sigma^2 + \omega\mathrm{E}\left[\log\left(1 - \alpha L\right)\right].$$

When the constraint binds, we have

$$U = \frac{\log W_0 + \log \theta}{\rho},$$

so α is chosen to maximize the constrained consumption-wealth ratio. We end up with the same first-order condition as in the unconstrained case. Thus the optimal investment choice is the same in the constrained and unconstrained cases, as before. Equations (14) and (15) also hold as before.