WHEN MONEY DIES: THE DYNAMICS OF SPECULATIVE HYPERINFLATIONS

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SUPPLEMENTARY APPENDICES

Appendix: Proofs of propositions and corollaries

Proof of Proposition 2. The value of money obeys $\dot{y}_t = f(y_t)$ where

$$f(y_t) \equiv [\rho + \alpha\sigma(1-M)]\, y_t - \alpha\sigma(1-M)u(y_t).$$

Steady state. A steady-state monetary equilibrium (SSME) is a $y^s > 0$ solution to $f(y^s) = 0$, i.e.,

$$\frac{u(y^s)}{y^s} = 1 + \frac{\rho}{\alpha\sigma(1-M)}.$$

From the strict concavity of u and the assumption $u(0) = 0$, the left side is strictly decreasing in y^s. Hence, if a SSME exists, it is unique. From the assumption, $\lim_{y\to+\infty} u'(y) < 1$, there exists a SSME if and only if

$$\lim_{y\to 0} \frac{u(y)}{y} = u'(0) > 1 + \frac{\rho}{\alpha\sigma(1-M)}.$$

Speculative hyperinflations. If $u'(0) > 1 + \rho/\left[\alpha\sigma(1-M)\right]$, so that a SSME exists, then $f'(0) < 0$. Given that $f(0) = 0$, it follows that for all $y \in (0, y^s)$, $f(y) < 0$, i.e., $\dot{y}_t < 0$. Moreover, f, which is continuously differentiable, is locally Lipschitz continuous. By the Cauchy-Lipschitz theorem, for all $y_0 \in (0, y^s)$, there is a unique solution to the ODE. It is such that $\dot{y}_t < 0$ as long as $y_t > 0$. Hence, $\lim_{t\to+\infty} y_t = 0$.

For all $y_t > 0$, the ODE can be rewritten as:

$$\frac{\dot{y}_t}{f(y_t)} = 1.$$

For given $y_0 \in (0, y^s)$, let $T \in (0, +\infty]$ denote the time at which y_t reaches 0. Integrate both sides from 0 to T to obtain:

$$\int_0^T \frac{\dot{y}_t}{f(y_t)} dt = T.$$

Use the change of variable $y = y_t$, and hence $dy = \dot{y}_t dt$, to rewrite the equation as:

$$\int_{y_0}^0 \frac{1}{f(y)} dy = T,$$

where, by the definition of T, $y_T = 0$. Substitute $f(y)$ by its expression to obtain (13).

A necessary condition for $T < +\infty$. A necessary condition for $T < +\infty$ is $u'(0) = +\infty$. To see this, suppose $u'(0) < +\infty$. Since $u(y)/y$ is strictly decreasing, $u(y)/y < \lim_{y \to 0} u(y)/y = u'(0)$ for all $y > 0$. It follows that:

$$\alpha\sigma(1-M)u(y) - [\rho + \alpha\sigma(1-M)]\, y < y \left\{ \alpha\sigma(1-M)u'(0) - [\rho + \alpha\sigma(1-M)] \right\},$$

for all $y > 0$. Take the reciprocal on both sides,

$$\frac{1}{\alpha\sigma(1-M)u(y) - [\rho + \alpha\sigma(1-M)]\, y} > \frac{1}{y \left\{ \alpha\sigma(1-M)u'(0) - [\rho + \alpha\sigma(1-M)] \right\}},$$

for all $y > 0$. Since $u'(0) > 1 + \rho / [\alpha\sigma(1-M)]$, so that a speculative hyperinflation equilibrium exists, and $\int_0^{y_0} (1/y) dy = +\infty$, the integral of the right side from 0 to $y_0 > 0$ is $+\infty$. Hence, $T = +\infty$.

Necessary and sufficient conditions for $T < +\infty$. From (13) $T < +\infty$ if and only if

$$\int_0^{y_0} \frac{1}{\alpha\sigma(1-M)u(y) - [\rho + \alpha\sigma(1-M)]\, y} dy < +\infty.$$

I simplify this condition by establishing that $T < +\infty$ if and only if $1/u(y)$ is integrable over $(0, y_0)$.

Necessity ($\Longrightarrow$). From (13), if $T < +\infty$ for $y_0 \in (0, y^s)$, then $-1/f(y)$ is integrable over $(0, y_0)$. Since $\alpha\sigma(1-M)u(y) > -f(y)$ for all $y > 0$, $1/[\alpha\sigma(1-M)u(y)] < -1/f(y)$ for all $y \in (0, y^s)$. Since the right side is integrable, it follows that $1/u(y)$ is integrable over $(0, y_0)$.

Sufficiency ($\Longleftarrow$). Next, suppose $1/u(y)$ is integrable over $(0, y_0)$ for some $y_0 \in (0, y^s)$. Then, it is integrable for any $\tilde{y}_0 \in (0, y^s)$ since u is continuous on $(0, y^s)$. As shown above, a necessary condition is $u'(0) = +\infty$. I now establish that there is a $\beta > 0$ and a $\tilde{y}_0 > 0$ such that

$$\alpha\sigma(1-M)u(y) - [\rho + \alpha\sigma(1-M)]\, y > \beta u(y) \quad \forall y \in (0, \tilde{y}_0).$$

To see this, rearrange the inequality as:

$$[\alpha\sigma(1-M) - \beta] \frac{u(y)}{y} > \rho + \alpha\sigma(1-M) \quad \forall y \in (0, \tilde{y}_0).$$

As y goes to 0, $u(y)/y$ tends to $u'(0) = +\infty$. Hence, there is a $\beta < \alpha\sigma(1-M)$ and a $\tilde{y}_0$ small enough so that the inequality holds. Take the reciprocal on both sides,

$$0 < \frac{1}{\alpha\sigma(1-M)u(y) - [\rho + \alpha\sigma(1-M)]\, y} < \frac{1}{\beta u(y)} \quad \forall y \in (0, \tilde{y}_0).$$

Since the right side is integrable, so is the middle term. Also, recall that if it is integrable over $(0, \tilde{y}_0)$, it is integrable over $(0, y_0)$ for all $y_0 \in (0, y^s)$. By the definition of T in (13) it follows that $T < +\infty$. ■

Proof of Corollary 2. I show that if $u'(0) = +\infty$ and $\lim_{y \to 0} \eta(y) > 0$, then $1/u(y)$ is integrable over $(0, y_0)$ for all $y_0 \in (0, y^s)$. (Since $u(y)$ is continuous, if $1/u(y)$ is integrable over $(0, y_0)$ for some $y_0 \in (0, y^s)$ then it is integrable for all $y_0 \in (0, y^s)$.) Hence, by Proposition 2, $T < +\infty$ for all $y_0 \in (0, y^s)$. The proof is based on the following claim:

Claim: There is a $a \in (0, 1)$, a $b > 0$, and a $y_0 > 0$ such that

$$u(y) > by^{1-a} \quad \forall y \in (0, y_0).$$

Let $\ell \equiv \lim_{y \to 0} u(y) y^{a-1}$ for some $a \in (0, 1)$ and $\eta_0 \equiv \lim_{y \to 0} \eta(y)$. Since $u(0) = 0$, $u' > 0$ and $u'' < 0$, $u(y) > u'(y)y$ for all $y > 0$. Equivalently, $\eta(y) \in (0, 1)$ for all $y > 0$. Hence, $\eta_0 \in [0, 1]$. From L'Hôpital's rule,

$$\begin{aligned} \ell &\equiv \lim_{y \to 0} \frac{u(y) y^a}{y} = \lim_{y \to 0} \left[u'(y) y^a + a u(y) y^{a-1} \right] \\ &= \lim_{y \to 0} u(y) y^{a-1} \left[\frac{u'(y) y}{u(y)} + a \right]. \end{aligned}$$

Using the definition of $\eta_0 \equiv \lim_{y \to 0} [1 - yu'(y)/u(y)]$, it follows that

$$\ell = \ell \left(1 - \eta_0 + a\right).$$

Since $\eta_0 \in [0, 1]$, $1 - \eta_0 \in [0, 1]$, and $1 - \eta_0 + a > 0$. If $a \neq \eta_0$, the solution ℓ to the equation above is either $\ell = 0$ or $\ell = +\infty$. In order to distinguish the two cases, I compute the limit of the slope of $u(y) y^{a-1}$ as y tends to 0:

$$\begin{aligned} \lim_{y \to 0} \frac{\partial \left[u(y) y^{a-1} \right]}{\partial y} &= \lim_{y \to 0} \left[u'(y) y^{a-1} + (a-1) u(y) y^{a-2} \right] \\ &= \lim_{y \to 0} \frac{u(y)}{y^{2-a}} \left[\frac{u'(y) y}{u(y)} + a - 1 \right] \\ &= (a - \eta_0) \lim_{y \to 0} \frac{u(y)}{y^{2-a}}. \end{aligned}$$

In order to obtain the third equality, I used that $\eta_0 \equiv \lim_{y \to 0} [1 - yu'(y)/u(y)]$. The limit on the right side can be rewritten as

$$\lim_{y \to 0} \frac{u(y)}{y^{2-a}} = \lim_{y \to 0} \left[\frac{u(y)}{y} \frac{1}{y^{1-a}} \right] = +\infty \text{ for all } a \in (0, 1).$$

It follows that if $a < \eta_0$, which requires $\eta_0 > 0$, then

$$\lim_{y \to 0} \frac{\partial \left[u(y) y^{a-1} \right]}{\partial y} = -\infty.$$

In that case, ℓ cannot be 0. Indeed, if $u(y)y^{a-1}$ is 0 at $y = 0$ and its derivative is $-\infty$, then $u(y)y^{a-1} < 0$ for y close to 0, which contradicts $u(y)y^{a-1} > 0$ for all $y > 0$. Hence, $\ell = +\infty$.

I am now in position to prove the claim above, which can be reexpressed as follows: there is a $a \in (0,1)$, a $b > 0$, and a $y_0 > 0$ such that

$$u(y)y^{a-1} > b \quad \forall y \in (0, y_0).$$

Assume $a \in (0, \eta_0)$. Using that the left side of the inequality is continuous and approaches $+\infty$ as y tends to 0, then for any $b > 0$, there is a $y_0 > 0$ such that $u(y)y^{a-1} > b$ for all $y \in (0, y_0)$. It follows that $1/u(y) < 1/(by^{1-a})$ for all $y \in (0, y_0)$. Since $1/y^{1-a}$ is integrable when $a \in (0,1)$, i.e., the primitive is y^a/a which is finite at $y = 0$, so is $1/u(y)$. From Proposition 2, if $1/u(y)$ is integrable then $T < +\infty$. ■

Proof of Lemma 1. Suppose first that buyers' preferences are of the CRRA type, $u(y) = y^{1-\eta}/(1-\eta)$, while sellers' preferences are linear, $w(y) = y$. The pricing function is such that $p(y) = w(y) = y$. It follows that $p'(y) = w'(y) = 1$. If the liquidity constraint, $p \leq m$, binds then $y = m$. Hence, $u'[y(m)] = m^{-\eta}$. Otherwise, $y = y^* = 1$, $p = 1$, and $u'[y(m)] = 1$. By definition,

$$L(m) = \frac{u'[y(m)]}{p'[y(m)]} - 1.$$

Hence,

$$\begin{aligned} L(m) &= m^{-\eta} - 1 \text{ if } m \leq 1 \\ &= 0 \text{ otherwise.} \end{aligned}$$

The second case is when buyers' preferences are linear, $u(y) = u'y$, while seller's disutility is strictly convex, $w(y) = y^{1+\gamma}/(1+\gamma)$, where $\gamma = \eta/(1-\eta)$ and $u' = (1-\eta)^{-\eta}$. The quantity y^* solves $u' = y^\gamma$, i.e.,

$$y^* = (1-\eta)^{-(1-\eta)}.$$

The pricing function is $p(y) = w(y)$. It follows that $m^* = p(y^*) = 1$. Hence, if $m \leq m^* = 1$, then $p \leq m$ holds at equality and y solves $w(y) = m$, which gives

$$y(m) = [(1+\gamma)\, m]^{\frac{1}{1+\gamma}}.$$

Using that $p'(y) = w'(y) = y^\gamma$, it follows that

$$p'[y(m)] = [(1+\gamma)\, m]^{\frac{\gamma}{1+\gamma}}.$$

Using that $\gamma = \eta/(1-\eta)$,

$$p'[y(m)] = \frac{1}{(1-\eta)^\eta} m^\eta.$$

Using that $u' = (1-\eta)^{-\eta}$,

$$L(m) = \frac{u'}{p'[y(m)]} - 1 = m^{-\eta} - 1 \quad \text{if } m \leq 1.$$

If $m > 1$, $y = y^*$ and $L(m) = 0$. ■

Proof of Proposition 3. I consider equilibria such that $m_t \in (0,1)$ for all $t \in [0, T)$. From (22), $L(m_t) = m_t^{-\eta} - 1$ for all $t \in [0, T)$. Hence, m_t is a solution to (23). I operate the change of variable, $x_t = m_t^\eta$. By differentiating x_t with respect to t, I obtain $\dot{x}_t = \eta m_t^{\eta-1} \dot{m}_t$. Substitute $\dot{m}_t = \dot{x}_t m_t^{1-\eta}/\eta$ into (23) and rearrange to obtain the following nonautonomous, linear differential equation:

$$\dot{x}_t = (\alpha\sigma + \rho + \pi_t)\eta x_t - \alpha\sigma\eta \quad \text{for all } t \in (0, T).$$

This ODE is solved using the method of the integrating factor. Multiply both sides of the ODE by $e^{-\eta(\alpha\sigma+\rho)t - \eta\Pi(t)}$, where $\Pi(t) = \int_0^t \pi_s ds$, to obtain:

$$e^{-\eta[(\alpha\sigma+\rho)t+\Pi(t)]}\left[\dot{x}_t - (\alpha\sigma + \rho + \pi_t)\eta x_t\right] = -e^{-\eta[(\alpha\sigma+\rho)t+\Pi(t)]}\alpha\sigma\eta,$$

for all $t \in (0, T)$. A primitive of the left side is $e^{-\eta[(\alpha\sigma+\rho)t+\Pi(t)]}x_t$. Integrate from t to T:

$$e^{-\eta[(\alpha\sigma+\rho)T+\Pi(T)]}x_T - e^{-\eta[(\alpha\sigma+\rho)t+\Pi(t)]}x_t = -\int_t^T e^{-\eta[(\alpha\sigma+\rho)s+\Pi(s)]}\alpha\sigma\eta ds.$$

Rearrange the terms to obtain:

$$e^{-\eta[(\alpha\sigma+\rho)t+\Pi(t)]}x_t = \int_t^T e^{-\eta[(\alpha\sigma+\rho)s+\Pi(s)]}\alpha\sigma\eta ds + e^{-\eta[(\alpha\sigma+\rho)T+\Pi(T)]}x_T.$$

Finally, multiply both sides by $e^{\eta[(\alpha\sigma+\rho)t+\Pi(t)]}$ to solve for x_t:

$$x_t = \int_t^T e^{-\eta[(\alpha\sigma+\rho)(s-t)+\Pi(s)-\Pi(t)]}\alpha\sigma\eta ds + e^{-\eta[(\alpha\sigma+\rho)(T-t)+\Pi(T)-\Pi(t)]}x_T. \tag{65}$$

I define the *nonspeculative* solution as the solution obtained by taking the limit as T goes to $+\infty$. From the restriction to time-paths such that $x_t \in (0,1)$ and the assumption $\pi_t + \rho > 0$ for all t, the limit of the second term on the right side of (65) is

$$\lim_{T\to+\infty} e^{-\eta[(\alpha\sigma+\rho)(T-t)+\Pi(T)-\Pi(t)]}x_T = \lim_{T\to+\infty} e^{-\eta\left[\alpha\sigma(T-t)+\int_t^T(\pi_s+\rho)ds\right]}x_T = 0.$$

Hence, from (65),

$$\bar{x}_t \equiv \lim_{T\to+\infty} x_t = \int_t^{+\infty} e^{-\eta[(\alpha\sigma+\rho)(s-t)+\Pi(s)-\Pi(t)]}\alpha\sigma\eta ds.$$

By assumption, $\Pi(s) - \Pi(t) > -\rho(s-t)$. Hence,

$$e^{-\eta[(\alpha\sigma+\rho)(s-t)+\Pi(s)-\Pi(t)]} < e^{-\eta\alpha\sigma(s-t)} \text{ for all } s > t.$$

Integrating both sides from $s = t$ to $s = +\infty$, $\bar{x}_t$ is bounded above by

$$\bar{x}_t < \int_t^{+\infty} e^{-\eta\alpha\sigma(s-t)}\alpha\sigma\eta ds = 1, \quad \forall t > 0.$$

Moreover, by assumption, $\Pi(s) - \Pi(t) \leq (s-t)\bar{\pi}$, which implies

$$\bar{x}_t \geq \int_t^{+\infty} e^{-\eta(\alpha\sigma+\rho+\bar{\pi})(s-t)}\alpha\sigma\eta ds = \frac{\alpha\sigma}{\alpha\sigma+\rho+\bar{\pi}} > 0, \quad \forall t > 0.$$

So, $\bar{x}_t \in (0,1)$ for all $t > 0$, and hence $\bar{m}_t = (\bar{x}_t)^{\frac{1}{\eta}} \in (0,1)$ for all $t > 0$. So the solution satisfies the initial restriction. Using that $\bar{m}_t = (\bar{x}_t)^{\frac{1}{\eta}}$, I obtain (26).

The speculative hyperinflation equilibria correspond to the continuum of solutions, indexed by $T \in (0,+\infty)$, where money becomes valueless at time $T < +\infty$, $x_T = 0$. From (65),

$$x_t = \alpha\sigma\eta \int_t^T e^{-\eta\{(\alpha\sigma+\rho)(s-t)+[\Pi(s)-\Pi(t)]\}}ds \leq \bar{x}_t \text{ for all } t \leq T.$$

Using that $m_t = (x_t)^{\frac{1}{\eta}}$, I obtain (25). At $t = T$, $m_t = 0$ and, from (23), $\dot{m}_t = 0$. For all $t \geq T$, $m_t = 0$. So, the time-path is continuous and differentiable at T and hence it is an equilibrium. ■

Proof of Corollary 3.

From (25):

$$(m_t)^\eta = e^{\eta[(\alpha\sigma+\rho)t+\Pi(t)]}\alpha\sigma\eta\int_t^T e^{-\eta[(\alpha\sigma+\rho)s+\Pi(s)]}ds, \quad \forall t\in[0,T].$$

From (26):

$$(\bar{m}_t)^\eta = e^{\eta[(\alpha\sigma+\rho)t+\Pi(t)]}\alpha\sigma\eta\int_t^{+\infty} e^{-\eta[(\alpha\sigma+\rho)s+\Pi(s)]}ds, \quad \forall t\geq 0.$$

By taking the difference between these two equations:

$$(\bar{m}_t)^\eta-(m_t)^\eta = e^{\eta[(\alpha\sigma+\rho)t+\Pi(t)]}\alpha\sigma\eta\int_T^{+\infty} e^{-\eta[(\alpha\sigma+\rho)s+\Pi(s)]}ds, \quad \forall t\in[0,T].$$

At $t=0$,

$$(\bar{m}_0)^\eta-(m_0)^\eta = \alpha\sigma\eta\int_T^{+\infty} e^{-\eta[(\alpha\sigma+\rho)s+\Pi(s)]}ds.$$

Hence, from the last two equations:

$$(\bar{m}_t)^\eta-(m_t)^\eta = e^{\eta[(\alpha\sigma+\rho)t+\Pi(t)]}\left[(\bar{m}_0)^\eta-(m_0)^\eta\right], \quad \forall t\in[0,T].$$

It can be rearranged to give (27). The time T at which $m_T=0$ solves

$$(\bar{m}_T)^\eta = e^{\eta[(\alpha\sigma+\rho)T+\Pi(T)]}\left[(\bar{m}_0)^\eta-(m_0)^\eta\right].$$

Divide both sides by $(\bar{m}_T)^\eta$ and take the log to obtain (28). ■

Proof of Proposition 4. Existence of speculative hyperinflation equilibria. An equilibrium is a differentiable function, m_t, solution to $\dot{m}_t = f(m_t)$ where

$$f(m)\equiv(\rho+\pi)m-\alpha\sigma mL(m).$$

A positive steady state is a $m^s>0$ solution to

$$f(m^s)=0 \Longleftrightarrow \rho+\pi=\alpha\sigma L(m^s).$$

Since $L'(m)<0$ for all $m<p(y^*)$, the right side is decreasing in m^s, it is equal to $\alpha\sigma L(0)$ when $m^s=0$, and it is equal to zero for $m^s=p(y^*)<+\infty$. The left side is constant and greater than zero. Hence, a solution exists provided that the right side evaluated at

$m^s = 0$ is greater than the left side, i.e., $\alpha\sigma L(0) > \rho + \pi$. If this condition holds and $\lim_{m\to 0^+} L(m)m = 0$, speculative hyperinflation equilibria exist by the following logic. As shown in the phase diagram in Figure 3, any solution to $\dot{m}_t = f(m_t)$ with $m_0 \in (0, m^s)$ is such that $m_t \in [0, m^s]$ for all $t \in \mathbb{R}$. Consider an open interval $\Omega \subset (0, m^s)$. By assumption, f is continuously differentiable for all $m > 0$. Hence, $f : \Omega \to \mathbb{R}$ is locally Lipschitz continuous for all $x \in \Omega$. By the Cauchy-Lipschitz theorem, for all $m_0 \in (0, m^s)$, $\dot{m}_t = f(m_t)$ has a unique solution in the neighborhood of m_0. Since $\dot{m}_t < 0$ for all $m_t \in (0, m^s)$, this solution is decreasing and it approaches 0 at $t \to +\infty$.

If $\rho + \pi > \alpha\sigma L(0)$, there is no positive steady state and $f(m) > 0$ for all $m > 0$. Hence, one cannot construct time-paths where m_t decreases over time. If $\lim_{m\to 0^+} L(m)m > 0$ then any time-path such that $m_t > 0$ for all $t < T$ and $m_t = 0$ for all $t \geq T$ is not differentiable at $t = T$ since

$$\dot{m}_{T^-} = -\alpha\sigma \lim_{m\to 0^+} L(m)m < \dot{m}_{T^+} = 0.$$

Hence, it does not satisfy the definition of an equilibrium.

Characterization of T_δ. Assume $m_0 \in (0, m^s)$ and let $T_\delta > 0$ be the time at which m_t reaches δm_0, i.e., $m_{T_\delta} = \delta m_0$, for some $\delta \in (0, 1]$. For all $t \in (0, T_\delta)$, $\dot{m}_t / f(m_t) = 1$. Integrate both sides from 0 to T_δ to get:

$$T_\delta = \int_0^{T_\delta} \frac{\dot{m}_t}{f(m_t)} dt.$$

Use the change of variable $m = m_t$ to obtain:

$$T_\delta = \int_{m_0}^{\delta m_0} \frac{1}{f(m)} dm.$$

Using that $f(m)$ is continuous and such that $f(m) \in (0, +\infty)$ for all $m \in (0, m^s)$, $1/f(m)$ is integrable and $T_\delta \in (0, +\infty)$ for all $\delta \in (0, 1]$. Substitute $f(m)$ by its expression,

$$T_\delta = \int_{\delta m_0}^{m_0} \frac{1}{\alpha\sigma m L(m) - (\rho + \pi)\, m} dm.$$

I now adopt the change of variable $x = m/m_0$ to rewrite T_δ as:

$$T_\delta = \int_\delta^1 \frac{1}{\alpha\sigma x L(x m_0) - (\rho + \pi)\, x} dx.$$

Conditions for money to die in finite time. The time it takes for money to lose all of its value is

$$T \equiv T_0 = \int_0^{m_0} \frac{1}{\alpha\sigma m L(m) - (\rho + \pi)\, m} dm.$$

A necessary condition for $T < +\infty$ is $L(0) = +\infty$. By contradiction, suppose $L(0) < +\infty$. Using that $L(m)$ is decreasing for all $m \in (0, p(y^*))$, then

$$\frac{1}{\alpha\sigma m L(m) - (\rho+\pi) m} > \frac{1}{m\left[\alpha\sigma L(0) - (\rho+\pi)\right]} \text{ for all } m > 0.$$

From the observation that

$$\int_0^{m_0} \frac{1}{m\left[\alpha\sigma L(0) - (\rho+\pi)\right]} dm = +\infty,$$

where I used that $\alpha\sigma L(0) > \rho + \pi$, it follows that $T = +\infty$. By the logic of the proof of Proposition 2, where $u(y)$ is replaced with $mL(m)$, a necessary and sufficient condition for $T < +\infty$ is $\int_0^{m_0} 1/\left[mL(m)\right] dm < +\infty$.

Approximate solutions. Suppose $L(0) < +\infty$ and L is differentiable at $m = 0$. I linearize the ODE in the neighborhood of $m_t = 0$ to obtain:

$$\dot{m}_t = \left[\rho + \pi - \alpha\sigma L(0)\right] m_t,$$

where I used that $L(m)$ is differentiable at $m = 0$ so that $L'(0) \in (-\infty, 0)$ and $\lim_{m\to 0} mL'(m) = 0$. The solution is (35). Under the condition for the existence of a positive steady state, i.e., $\alpha\sigma L(0) > \rho + \pi$, the term on the right side between squared brackets is negative, $\partial \dot{m}_t / \partial m_t \in (-\infty, 0)$. Hence, m_t converges to 0 but only at the limit as $t \to +\infty$.

The linearization of the ODE in the neighborhood of m^s gives

$$\dot{m}_t = \left[\rho + \pi - \alpha\sigma L(m^s) - \alpha\sigma m^s L'(m^s)\right](m_t - m^s).$$

From (31), $\rho + \pi = \alpha\sigma L(m^s)$, and hence

$$\dot{m}_t = \alpha\sigma\left[1 + L(m^s)\right] \frac{-m^s L'(m^s)}{1 + L(m^s)} (m_t - m^s).$$

Using the notation $\eta(m^s) \equiv -m^s L'(m^s)/\left[1 + L(m^s)\right]$ and $\alpha\sigma\left[1 + L(m^s)\right] = \rho + \pi + \alpha\sigma$, one obtains

$$\dot{m}_t = (\rho + \pi + \alpha\sigma)\, \eta(m^s)\, (m_t - m^s).$$

The solution to this linear differential equation is (34). Note that m_t close to m^s requires that m_0 is also close to m^s. ■

Proof of Corollary 4. Preliminary result: If a speculative hyperinflation with finite duration exists, then $\eta_0 \equiv \lim_{m\to 0} -L'(m)m/\left[1+L(m)\right] \in [0,1]$.

The proof of this result is as follows. From Proposition 4, a necessary condition for $T < +\infty$ is $L(0) = +\infty$. Hence, $\eta_0 = \lim_{m\to 0} -L'(m)m/L(m)$. In order for a speculative hyperinflation equilibrium to exist, $\lim_{m\to 0} L(m)m = 0$. Since $L(m)m \geq 0$ for all $m > 0$, it follows that $\lim_{m\to 0} \partial\left[L(m)m\right]/\partial m \geq 0$, i.e., $\lim_{m\to 0} \left\{L(m)\left[1-\eta(m)\right]\right\} \geq 0$. Hence, $\eta_0 \in [0,1]$.

I now turn to the main part of the proof. From Proposition 4, $T < +\infty$ if and only if $1/\left[L(m)m\right]$ is integrable. In order to establish that $1/\left[L(m)m\right]$ is integrable if $\eta_0 > 0$, I show that there exists an $a \in (0,1)$, a $b > 0$, and a $m_0 > 0$ such that

$$L(m)m > bm^{1-a} \text{ for all } m \in (0, m_0).$$

This inequality can be reexpressed as

$$L(m)m^a > b \text{ for all } m \in (0, m_0).$$

Consider the limit of the left side as m tends to 0.

$$\begin{aligned} \ell &\equiv \lim_{m\to 0} L(m)m^a \\ &= \lim_{m\to 0} \frac{L(m)m^{1+a}}{m} \\ &= \lim_{m\to 0} \left\{L'(m)m^{1+a} + (1+a)L(m)m^a\right\} \\ &= \lim_{m\to 0} L(m)m^a \left\{L'(m)m/L(m) + 1 + a\right\} \\ &= \ell\left(1+a-\eta_0\right). \end{aligned}$$

To go from the second line to the third line, I apply L'Hôpital's rule. To go from the fourth line to the last line I use the definitions of ℓ and η_0. From the preliminary result above, under the assumptions $\lim_{m\to 0} L(m)m = 0$ and $L(0) = +\infty$, $\eta_0 \in [0,1]$, and hence $1 + a - \eta_0 > 0$. If $a \neq \eta_0$, then the solution to the $\ell = \ell\left(1+a-\eta_0\right)$ is either $\ell = 0$ or $\ell = +\infty$.

In order to determine whether $\ell = 0$ or $\ell = +\infty$ is the solution, I compute the limit of the slope of $L(m)m^a$ as m tends to 0:

$$\begin{aligned}\lim_{m\to 0}\left\{\frac{\partial L(m)m^a}{\partial m}\right\} &= \lim_{m\to 0}\left\{L'(m)m^a + aL(m)m^{a-1}\right\} \\ &= \lim_{m\to 0}\frac{L(m)}{m^{1-a}}\left[\frac{L'(m)m}{L(m)}+a\right] \\ &= (a-\eta_0)\lim_{m\to 0}\frac{L(m)}{m^{1-a}}.\end{aligned}$$

If $a < \eta_0$, this limit is $-\infty$. It implies $\ell = +\infty$. To see this, suppose, a contrario, that $\ell \equiv \lim_{m\to 0} L(m)m^a = 0$. From $\partial\left[L(m)m^a\right]/\partial m = -\infty$ at $m = 0$, it follows that $L(m)m^a < 0$ for small m, which is inconsistent with $L(m)m^a \geq 0$ for all $m > 0$.

Using the result that $\ell \equiv \lim_{m\to 0} L(m)m^a = +\infty$, it follows by continuity that for any $b > 0$, there is a $m_0 > 0$ such that $L(m)m^a > b$ for all $m \in (0, m_0)$. Equivalently,

$$\frac{1}{L(m)m} < \frac{1}{bm^{1-a}} \quad \text{for all } m \in (0, m_0)\,.$$

Since the right side is integrable, i.e., $\int_0^{m_0} m^{a-1}dm = (m_0)^a/a$, so is the left side. Hence, $T < +\infty$. ■

Proof of Corollary 5.

Part 1. Under the generalized CRRA preferences, for every element of the sequence $\{b_n\}_{n=0}^{+\infty}$, the ODE, (21), can be rewritten as

(66) $$\dot{m}_{n,t} = (\rho + \pi + \alpha\sigma)\, m_{n,t} - \alpha\sigma m_{n,t}(m_{n,t} + b_n)^{-\eta}.$$

The positive steady state, denoted m_n^s, solves $\dot{m}_{n,t} = 0$, i.e., $\alpha\sigma(m_n^s + b_n)^{-\eta} = \rho + \pi + \alpha\sigma$. Solving for m_n^s in closed form gives (37). Since $b_n \in \left(0, \left[\alpha\sigma/(\alpha\sigma + \rho + \pi)\right]^{\frac{1}{\eta}}\right)$, $m_n^s > 0$. Using that $\{b_n\}_{n=0}^{+\infty}$ is decreasing, $\{m_n^s\}_{n=0}^{+\infty}$ is increasing.

Part 2. The uniqueness of the solution to the ODE, (66), follows from Proposition 4 and the fact that the right side of (66) is continuously differentiable for all $m_{n,t} > -b_n$. Since $\{m_n^s\}_{n=0}^{+\infty}$ is increasing, the condition $m_0 < m_0^s$ guarantees that $m_{n,0} = m_0 < m_n^s$ for all n. From Proposition 4, for all $m_{n,0} \in (0, m_n^s)$, $\dot{m}_{n,t} < 0$ and $m_{n,t} \to 0$ as $t \to +\infty$. Moving backward in time, $m_{n,t} \to m_n^s$ as $t \to -\infty$.

Part 3. Under generalized CRRA preferences, $L(xm_0) = (xm_0 + b)^{-\eta} - 1$. Hence, from (32),

$$T_{n,\delta} = \int_\delta^1 \frac{1}{\alpha\sigma x(xm_0 + b_n)^{-\eta} - (\rho + \pi + \alpha\sigma)x} dx.$$

Hence, $T_{n,\delta}$ increases in b_n or, equivalently, decreases in n. As b_n tends to 0, it approaches

$$T_{\infty,\delta} = \int_\delta^1 \frac{1}{\alpha\sigma(m_0)^{-\eta}x^{1-\eta} - (\rho + \pi + \alpha\sigma)x} dx.$$

I adopt the change of variable $u = x^\eta$. Then, $dx = x^{1-\eta}du/\eta$ and

$$T_{\infty,\delta} = \frac{1}{\eta}\int_\delta^1 \frac{1}{\alpha\sigma(m_0)^{-\eta} - (\rho + \pi + \alpha\sigma)u} du.$$

A primitive of the integrand is

$$\frac{-\ln\left[\alpha\sigma(m_0)^{-\eta} - (\rho + \pi + \alpha\sigma)u\right]}{\rho + \pi + \alpha\sigma}.$$

Hence,

$$\begin{aligned} T_{\infty,\delta} &= \frac{1}{\eta(\rho + \pi + \alpha\sigma)} \ln\left[\frac{\alpha\sigma(m_0)^{-\eta} - \delta(\rho + \pi + \alpha\sigma)}{\alpha\sigma(m_0)^{-\eta} - (\rho + \pi + \alpha\sigma)}\right] \\ &= \frac{1}{\eta(\rho + \pi + \alpha\sigma)} \ln\left[1 + (1-\delta)\frac{(\rho + \pi + \alpha\sigma)}{\alpha\sigma(m_0)^{-\eta} - (\rho + \pi + \alpha\sigma)}\right] \\ &= \frac{1}{\eta(\rho + \pi + \alpha\sigma)} \ln\left[1 + (1-\delta)\frac{1}{(m_\infty^s/m_0)^\eta - 1}\right], \end{aligned}$$

where to obtain the second equality I rearranged the terms between squared brackets and to obtain the third equality I used that $m_\infty^s = \left[\alpha\sigma/\left(\rho + \pi + \alpha\sigma\right)\right]^{\frac{1}{\eta}}$. ■

Proof of Proposition 5. Under quadratic preferences, the ODE, m_t obeys

(67) $$\dot{m}_t = \left[\rho + \pi + \alpha\sigma(1 - A)\right] m_t + \alpha\sigma\varepsilon\left(m_t\right)^2.$$

The positive steady state solves $\dot{m}_t = 0$ and $m_t > 0$, which gives (41). In order for $m^s > 0$, $\alpha\sigma A > \alpha\sigma + \rho + \pi$, which can be rewritten as (40). The ODE, (67), is a Bernoulli equation. Assuming $m_t > 0$, I adopt the change of variable $x_t = m_t^{-1}$. Then $\dot{x}_t = -\dot{m}_t/(m_t)^2$. Substitute $\dot{m}_t = -\dot{x}_t(m_t)^2$ into (67) to obtain

$$\dot{x}_t = -\left[\rho + \pi + \alpha\sigma(1 - A)\right] x_t - \alpha\sigma\varepsilon.$$

The solution to this linear ODE is

$$x_t = x^s + (x_0 - x^s)e^{[\alpha\sigma(A-1)-\rho-\pi]t},$$

where $x^s = \alpha\sigma\varepsilon/\left[\alpha\sigma(A-1)-(\rho+\pi)\right] = 1/m^s$. Using that $m_t = 1/x_t$, I obtain (42). It is easy to check that for all $m_0 \in (0, m^s)$, $m_t > 0$ for all $t > 0$, as conjectured above.

From (32), using that $L(m^s) = (\rho+\pi)/(\alpha\sigma)$, the time it takes for m_t to reach δm_0 starting from m_0 is

$$T_\delta = \int_\delta^1 \frac{1}{\alpha\sigma x\left[L(xm_0) - L(m^s)\right]}dx.$$

Under quadratic preferences, $L(m) = A - \varepsilon m$ so that

$$T_\delta = \frac{1}{\alpha\sigma\varepsilon}\int_\delta^1 \frac{1}{x\left(m^s - xm_0\right)}dx.$$

The integrand can be reexpressed as:

$$\frac{1}{x\left(m^s - xm_0\right)} = \frac{1/m^s}{x} + \frac{m^0/m^s}{m^s - xm_0}.$$

A primitive is

$$\frac{1}{m^s}\ln x - \frac{1}{m^s}\ln\left(m^s - xm_0\right) = \frac{1}{m^s}\ln\left(\frac{x}{m^s - xm_0}\right).$$

It follows that T_δ can be rewritten as in (43). ■

Proof of Proposition 6. From Proposition 4, a necessary condition for $T < +\infty$ is $L(0) = +\infty$. Hence, in order to establish that $T = +\infty$, it suffices to show that $L(0) < +\infty$. From (45),

$$L(m) = \frac{u'\left[y(m)\right]}{p'\left[y(m)\right]} - 1 = \frac{\theta\left\{u'\left[y(m)\right] - w'\left[y(m)\right]\right\}}{\theta w'\left[y(m)\right] + (1-\theta)u'\left[y(m)\right]},$$

where $y(m)$ is the solution to $p(y) = \min\{p(y^*), m\}$ with $p(y) \equiv \theta w(y) + (1-\theta)u(y)$. Divide the numerator and the denominator by $w'\left[y(m)\right]$:

$$\begin{aligned} L(m) &= \frac{\theta\left\{u'\left[y(m)\right]/w'\left[y(m)\right] - 1\right\}}{\theta + (1-\theta)u'\left[y(m)\right]/w'\left[y(m)\right]} \\ &\leq \frac{\theta u'\left[y(m)\right]/w'\left[y(m)\right]}{\theta + (1-\theta)u'\left[y(m)\right]/w'\left[y(m)\right]}. \end{aligned}$$

The right side is increasing with the term u'/w'. So, an upper bound is obtained by taking the limit as this term goes to infinity. This gives:

$$L(m) \leq \lim_{x \to +\infty} \frac{\theta x}{\theta + (1-\theta)x} = \frac{\theta}{1-\theta}.$$

So, if $\theta < 1$, $L(0) < +\infty$. ■

Proof of Proposition 8. Part 1.

Steady states. From (55) with $b = 0$, $\dot{m}_t = f(m_t)$ where

$$f(m_t) \equiv (\rho + \alpha\sigma)\, m_t + g_t - \alpha\sigma(m_t)^{1-\eta}.$$

A steady-state equilibrium is a m^s solution to $f(m^s) = 0$, i.e.,

$$\alpha\sigma(m^s)^{1-\eta} - (\rho + \alpha\sigma)\, m^s = g. \tag{68}$$

Since $\eta \in (0,1)$, the left side of (68) is strictly concave, equal to 0 when either $m^s = 0$ or $m^s = [\alpha\sigma/\left(\rho + \alpha\sigma\right)]^{1/\eta}$. It reaches a maximum when $m^s = m_{\max} \equiv [\alpha\sigma\left(1-\eta\right)/\left(\rho + \alpha\sigma\right)]^{1/\eta}$. The right side is constant and equal to $g > 0$. Hence, if the left side when evaluated at $m^s = m_{\max}$ is greater than g, i.e.,

$$g < \left[\frac{\alpha\sigma\left(1-\eta\right)}{\rho + \alpha\sigma}\right]^{\frac{1}{\eta}} \frac{\eta\left(\rho + \alpha\sigma\right)}{\left(1-\eta\right)},$$

then there are two steady-state equilibria, $0 < m^s_\ell < m^s_h$. Because the left side of (68) is increasing in m^s for all $m^s < m_{\max}$, an increase in g raises $m^s_\ell < m_{\max}$. By a symmetric reasoning, an increase in g reduces m^s_h.

Speculative equilibria. From (55), $\partial\dot{m}_t/\partial m_t = f'(m_t) < 0$ for all $m_t < m_{\max}$. From the result that $m^s_\ell < m_{\max}$ and $\dot{m}_t = 0$ when $m_t = m^s_\ell$, it follows that $\dot{m}_t > 0$ for all $m_t < m^s_\ell$. Thus, there are no equilibria where the value of money converges to 0. By a similar reasoning, for all $m_0 \in (m^s_\ell, m^s_h)$, $\dot{m}_t < 0$. Hence, there are a continuum of equilibrium paths where m_t converges to m^s_ℓ. These dynamics are illustrated in the phase diagram in Figure 9.

Approximation. From (55),

$$\left.\frac{\partial \dot{m}_t}{\partial m_t}\right|_{m_t = m^s_\ell} = f'(m^s_\ell) = \rho + \alpha\sigma - (1-\eta)\alpha\sigma(m^s_\ell)^{-\eta}.$$

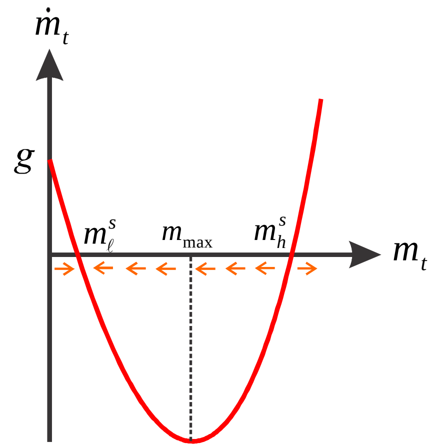

Figure 9: Phase diagram of ODE (55)

I use this expression to linearize (55) in the neighborhood of m_ℓ^s:

$$\dot{m}_t = \left[\rho + \alpha\sigma - (1-\eta)\alpha\sigma (m_\ell^s)^{-\eta}\right](m_t - m_\ell^s).$$

Using that $\partial \dot{m}_t / \partial m_t < 0$ at $m_t = m_\ell^s$, any solution to the linear ODE above converges to m_ℓ^s. The closed-form solution is given by (58). Similarly, I linearize (55) in the neighborhood of m_h^s to obtain (57). Since m_t diverges from m_h^s when $m_0 \neq m_h^s$, in order for m_t to be in the neighborhood of m_h^s, m_0 must also be in the neighborhood of m_h^s.

Part 2. If $u(y) = \ln(y+b) - \ln(b)$, with $b \in (0,1)$, then the ODE for m_t, (54), is rewritten as

$$\dot{m}_t = f(m_t; b) \equiv (\rho + \alpha\sigma)\, m_t + g - \frac{\alpha\sigma m_t}{m_t + b}. \tag{69}$$

Part 2a. A steady state, m^s, solves $f(m^s; b) = 0$, i.e.,

$$\overbrace{(\rho + \alpha\sigma)\, m^s + g}^{\text{LHS}} = \overbrace{\frac{\alpha\sigma m^s}{m^s + b}}^{\text{RHS}}.$$

The left-hand side (LHS) is linear increasing in m^s with a positive intercept. The right-hand side (RHS) is a strictly increasing and strictly concave function of m^s with a slope given by

$$\frac{\partial RHS}{\partial m^s} = \frac{\alpha\sigma b}{(m^s + b)^2}.$$

So, $\partial RHS / \partial m^s|_{m^s = 0} = \alpha\sigma / b$ and $\partial RHS / \partial m^s|_{m^s \to +\infty} = 0$. Moreover,

$$\frac{\partial RHS}{\partial m^s} = \frac{\partial LHS}{\partial m^s} \Longleftrightarrow \frac{\alpha\sigma b}{(m^s + b)^2} = \rho + \alpha\sigma.$$

The solution is $m^s = \hat{m}$ where

$$\hat{m} \equiv \sqrt{\frac{\alpha\sigma b}{\rho + \alpha\sigma}} - b.$$

Finally,

$$RHS|_{m^s=0} = 0 < LHS|_{m^s=0} = g$$

and

$$RHS|_{m^s=1-b} = \alpha\sigma(1-b) < LHS|_{m^s=1-b} = \alpha\sigma(1-b) + \rho(1-b) + g.$$

Hence, if a solution to $f(m^s; b) = 0$ exists, then there are two solutions, $m_\ell^s \in (0, \hat{m}]$ and $m_h^s \in [\hat{m}, 1-b)$. The determination of the steady states is represented graphically in Figure 10.

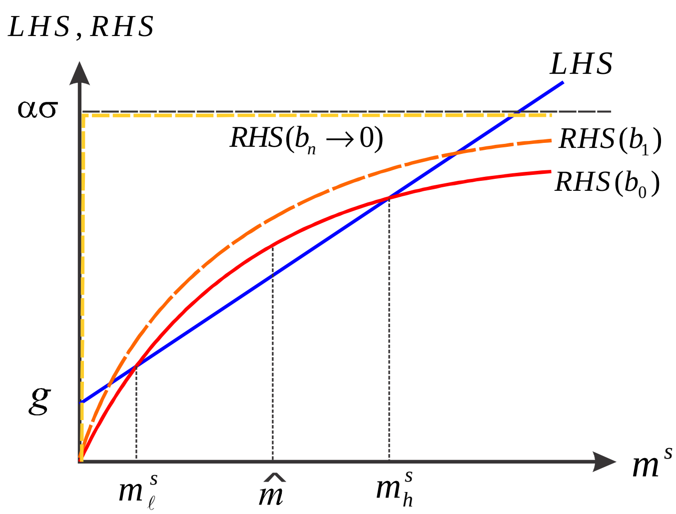

Figure 10: Determination of steady states under logarithmic preferences

Consider a decreasing sequence, $\{b_n\}_{n=0}^{+\infty}$, that converges to 0. As b_n approaches zero, RHS increases and approaches $\alpha\sigma$ for all $m^s > 0$, as shown in Figure 10. Hence, if $g < \alpha\sigma$, there is a $N \geq 0$ such that for all $n \geq N$, there are two steady states, $m_{\ell,n}^s$ and $m_{h,n}^s$. Since RHS is decreasing in b, it shifts upward as b decreases. Moreover, RHS intersects LHS by below at m_ℓ^s. Hence, $m_{\ell,n}^s$ is decreasing in n. By a similar logic, $m_{h,n}^s$ is increasing in n. Moreover, by the squeeze theorem, since $0 \leq m_{\ell,n}^s \leq \hat{m}(b_n)$ and $\lim_{b\to 0} \hat{m} = 0$, it follows that $\lim_{n\to+\infty} m_{\ell,n}^s = 0$. The high steady state converges to the solution to $(\rho + \alpha\sigma)\, m^s + g = \alpha\sigma$, i.e., (59).

Part 2b. In the following, I abstract from the index, n. Suppose $m_0 \in (m_\ell^s, m_h^s)$. Let T_δ denote the time at which m_t reaches $\delta m_0 + (1-\delta)m_\ell^s$, i.e.,

$$m_{T_\delta} = \delta m_0 + (1-\delta)m_\ell^s.$$

The ODE, $\dot{m}_t = f(m_t; b)$, can be rewritten as

$$\frac{\dot{m}_t}{f(m_t; b)} = 1.$$

Integrate both sides from $t = 0$ to $t = T_\delta$ to obtain

$$T_\delta = \int_0^{T_\delta} \frac{\dot{m}_t}{f(m_t; b)} dt.$$

Use that $f(m_t; b) \equiv (\rho + \alpha\sigma)\, m_t + g - \alpha\sigma m_t/(m_t + b)$ and adopt the change of variable $m = m_t$:

$$T_\delta = \int_{\delta m_0 + (1-\delta)m_\ell^s}^{m_0} \frac{m+b}{\alpha\sigma m - [(\rho+\alpha\sigma)\, m + g]\,(m+b)} dm.$$

The denominator of the integrand, which is quadratic, can be rewritten as:

$$\alpha\sigma m - [(\rho+\alpha\sigma)\, m + g]\,(m+b) = (\rho+\alpha\sigma)\,(m_h^s - m)(m - m_\ell^s),$$

where I used that $f(m_\ell^s; b) = f(m_h^s; b) = 0$ and the coefficient in front of the quadratic term is $-(\rho + \alpha\sigma)$. Hence, the integrand can be rewritten as

$$I(m) \equiv \frac{m+b}{\alpha\sigma m - [(\rho+\alpha\sigma)\, m + g]\,(m+b)} = \frac{m+b}{(\rho+\alpha\sigma)\,(m_h^s - m)(m - m_\ell^s)}.$$

It can be checked that

$$\frac{m+b}{(m_h^s - m)(m - m_\ell^s)} = \frac{1}{m_h^s - m_\ell^s}\left(\frac{b + m_h^s}{m_h^s - m} + \frac{m_\ell^s + b}{m - m_\ell^s}\right).$$

Multiplying both sides of the equality above by $(\rho + \alpha\sigma)^{-1}$, the integrand can be rewritten as:

$$I(m) = \frac{1}{(\rho+\alpha\sigma)\,(m_h^s - m_\ell^s)}\left(\frac{b + m_h^s}{m_h^s - m} + \frac{m_\ell^s + b}{m - m_\ell^s}\right).$$

A primitive of the right side is:

$$F(m) = \frac{-(b + m_h^s)\ln(m_h^s - m) + (m_\ell^s + b)\ln(m - m_\ell^s)}{(\rho+\alpha\sigma)\,(m_h^s - m_\ell^s)}.$$

It follows that:

$$\begin{aligned} T_\delta &= F(m_0) - F(\delta m_0 + (1-\delta) m_\ell^s) \\ &= \frac{1}{(\rho+\alpha\sigma)} \left\{ \left(\frac{m_h^s + b}{m_h^s - m_\ell^s} \right) \ln \left[1 + (1-\delta) \left(\frac{m_0 - m_\ell^s}{m_h^s - m_0} \right) \right] - \left(\frac{m_\ell^s + b}{m_h^s - m_\ell^s} \right) \ln(\delta) \right\}. \end{aligned}$$

The expression for $T_{\delta,n}$ given by (60) is obtained from T_δ above where $m_\ell^s = m_{\ell,n}^s$, $m_h^s = m_{h,n}^s$, and $b = b_n$.

Part 2c. Using that $m_{\ell,n}^s \searrow 0$ and $m_{h,n}^s \nearrow m^s$, for all $m_0 \in (0, m^s)$, there is a $\tilde{N} \geq 0$ such that for all $n \geq \tilde{N}$, $m_0 \in (m_{\ell,n}^s, m_{h,n}^s)$. From the expression for $T_{\delta,n}$ given by (60),

$$\tilde{T}_\delta \equiv \lim_{n \to +\infty} T_{\delta,n} = \frac{1}{(\rho+\alpha\sigma)} \ln \left[1 + (1-\delta) \left(\frac{m_0}{m^s - m_0} \right) \right].$$

Since $f \in C^1$ for all $m > -b$, it is locally Lipschitz continuous. By the theorem of Cauchy-Lipschitz, the ODE, $\dot{m}_t = f(m_t; b_n)$, admits a unique solution, $m_{n,t}$, given the initial condition, m_0, and it is such that $m_{n,t}$ converges to $m_{\ell,n}^s$ as $t \to +\infty$. Since f is continuously differentiable with respect to m and b for all (m, b) such that $m + b > 0$, by the theorem of continuous dependence (see, e.g., Grant, 2014, page 20), the solution to the ODE is continuous in b_n. As $b_n \to 0$, the ODE converges to

$$\dot{m}_t = (\rho + \alpha\sigma) m_t + g - \alpha\sigma, \text{ for all } m_t > 0.$$

The solution to this linear ODE is $\tilde{m}_t$ given by (62) where T is defined by $\tilde{m}_T = 0$ and corresponds to $\tilde{T}_0$. So, for all $t \in (0, T)$, $m_{n,t}$ converges pointwise to $\tilde{m}_t$ as $n \to +\infty$. ■

Proof of Proposition 9. The ODE (54) becomes

$$\dot{m}_t = \alpha\sigma\varepsilon(m_t)^2 - [\alpha\sigma(A-1) - \rho] m_t + g. \tag{70}$$

The steady-state solutions to (70) are given by

$$\begin{aligned} m_h^s &= \frac{\alpha\sigma(A-1) - \rho + \sqrt{[\alpha\sigma(A-1)-\rho]^2 - 4\alpha\sigma\varepsilon g}}{2\alpha\sigma\varepsilon} \\ m_\ell^s &= \frac{\alpha\sigma(A-1) - \rho}{\alpha\sigma\varepsilon} - m_h^s. \end{aligned}$$

Note that the right side of (70) is equal to $g > 0$ when $m_t = 0$. It implies that the two roots, if they exist, are both positive or both negative. Two real, positive, and distinct solutions exist if and only if $\alpha\sigma(A-1)-\rho>0$, so that $m_h^s+m_\ell^s>0$, and

$$g<\frac{[\alpha\sigma(A-1)-\rho]^2}{4\alpha\sigma\varepsilon},$$

so that the discriminant associated with the quadratic equation is positive and the roots are real and distinct.

The ODE (70) is a Ricatti equation that admits an explicit solution. Adopt the change of variable, $z_t=m_t-m_\ell^s$, to rewrite (70) as

$$\begin{aligned}\dot{z}_t &= \alpha\sigma\varepsilon(z_t+m_\ell^s)^2-[\alpha\sigma(A-1)-\rho]\,(z_t+m_\ell^s)+g\\ &= \alpha\sigma\varepsilon(z_t)^2+2\alpha\sigma\varepsilon m_\ell^s z_t-[\alpha\sigma(A-1)-\rho]\,z_t\\ &\quad +\alpha\sigma\varepsilon\left(m_\ell^s\right)^2-[\alpha\sigma(A-1)-\rho]\,m_\ell^s+g\end{aligned}$$

By definition of m_ℓ^s, the last three terms on the right side add up to 0. Hence,

$$\dot{z}_t=\alpha\sigma\varepsilon(z_t)^2+\{2\alpha\sigma\varepsilon m_\ell^s-[\alpha\sigma(A-1)-\rho]\}\,z_t.$$

The ODE in z_t is a Bernoulli equation. Using the change of variable $x_t=1/z_t$, which implies $\dot{x}_t=-\dot{z}_t/(z_t)^2$, it becomes

$$-\dot{x}_t(z_t)^2=\alpha\sigma\varepsilon(z_t)^2+\{2\alpha\sigma\varepsilon m_\ell^s-[\alpha\sigma(A-1)-\rho]\}\,z_t.$$

Suppose $z_t>0$. I divide both sides by $-(z_t)^2$ and I use that $x_t=1/z_t$ to obtain

$$\dot{x}_t=-\alpha\sigma\varepsilon-\{2\alpha\sigma\varepsilon m_\ell^s-[\alpha\sigma(A-1)-\rho]\}\,x_t.$$

Using that $\alpha\sigma\varepsilon m_\ell^s+\alpha\sigma\varepsilon m_h^s=\alpha\sigma(A-1)-\rho$, the term between brackets can be rewritten as

$$2\alpha\sigma\varepsilon m_\ell^s-[\alpha\sigma(A-1)-\rho]=\alpha\sigma\varepsilon\left(m_\ell^s-m_h^s\right).$$

Moreover, the steady state is

$$x^s=\frac{1}{m_h^s-m_\ell^s}.$$

Hence, the solution is

$$x_t=\frac{1}{m_h^s-m_\ell^s}+\left(x_0-\frac{1}{m_h^s-m_\ell^s}\right)e^{\alpha\sigma\varepsilon\left(m_h^s-m_\ell^s\right)t}.$$

For all $x_0\in(0,x^s)$, the solution is such that $x_t>0$ for all t, and hence $z_t>0$ for all t. Using that $x_0=1/(m_0-m_\ell^s)$ and $m_t=1/x_t+m_\ell^s$, the solution to (70) is (64). ■

Proposition 10 ***(Speculative dollarization.)*** *The steady-state monetary equilibrium is such that* $m^s = \min\{m_0^s, m_1^s\}$ *and* $a^s + m^s = \max\{m_0^s, \omega_1^s\}$ *where*

$$m_0^s \equiv \left(\frac{\alpha\sigma}{\alpha\sigma + \rho + \pi}\right)^{\frac{1}{\eta}} \tag{71}$$

$$m_1^s \equiv \left(\frac{\alpha\sigma\chi_m}{\alpha\sigma\chi_m + r_a + \pi}\right)^{\frac{1}{\eta}} \tag{72}$$

$$\omega_1^s \equiv \left(\frac{\alpha\sigma\chi_2}{\rho - r_a + \alpha\sigma\chi_2}\right)^{\frac{1}{\eta}}. \tag{73}$$

There exists a continuum of speculative hyperinflation equilibria indexed by $m_0 \in (0, m^s)$. *The time at which the economy starts dollarizing,* $T_0 \equiv \inf\{t \in \mathbb{R}_+ : a_t > 0\}$, *is*

$$\begin{aligned} T_0 &= \frac{1}{\eta(\alpha\sigma + \rho + \pi)} \ln\left[\frac{(m_0^s)^\eta - (\omega_1^s)^\eta}{(m_0^s)^\eta - (m_0)^\eta}\right] \quad \textit{if } \omega_1^s < m_0 < m_0^s \\ &= 0 \textit{ otherwise.} \end{aligned} \tag{74}$$

The time at which the economy is fully dollarized is $T_0 + T_1 \equiv \sup\{t \in \mathbb{R}_+ : m_t > 0\}$ *where*

$$T_1 = \frac{\ln\left[1 - (m_{T_0}/m_1^s)^\eta\right]^{-1}}{(\alpha\sigma\chi_m + r_a + \pi)\,\eta}, \tag{75}$$

and $m_{T_0} = \min\{m_0, \omega_1^s\}$. *The time-path for real balances is*

$$\begin{aligned} m_t = &\left\{(m_0^s)^\eta - e^{\eta(\alpha\sigma+\rho+\pi)t}\left[(m_0^s)^\eta - (m_0)^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T_0]}(t) \\ &+ \left\{(m_1^s)^\eta - e^{\eta(\alpha\sigma\chi_m + r_a + \pi)(t-T_0)}\left[(m_1^s)^\eta - (m_{T_0})^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[T_0,T_0+T_1]}(t). \end{aligned}$$

The time-path for real holdings of dollars is

$$a_t = (\omega_1^s - m_t)\, \mathbb{I}_{[T_0,+\infty)}(t). \tag{76}$$

Proof of Proposition 10. Using that $r_t = \dot{m}_t/m_t - \pi$, (47) can be rewritten as:

$$\frac{\dot{m}_t}{m_t} = \rho + \pi - \alpha\sigma\chi_m\upsilon'(m_t) - \alpha\sigma\chi_2\upsilon'(m_t + a_t).$$

From (48),

$$\alpha\sigma\chi_2\upsilon'(m_t + a_t) = \min\left\{\alpha\sigma\chi_2\upsilon'(m_t), \rho - r_a\right\}.$$

Assuming $u(y) = y^{1-\eta}/(1-\eta)$ and $w(y) = p(y) = y$, $\upsilon'(x) = x^{-\eta} - 1$ if $x \leq 1$. Substitute $\upsilon'(m_t + a_t)$ by its expression into the ODE for m_t, the equilibrium condition is

$$\frac{\dot{m}_t}{m_t} = \rho + \pi + \alpha\sigma\chi_m - \alpha\sigma\chi_m (m_t)^{-\eta} - \min\left\{\alpha\sigma\chi_2\left[(m_t)^{-\eta} - 1\right], \rho - r_a\right\}. \tag{77}$$

From the last term on the right side, if the domestic currency is the only means of payment, i.e., $a_t = 0$, then its liquidity value in type-2 meetings cannot be greater than the holding cost of dollars, $\rho - r_a$. An equilibrium is a time-path, (m_t, a_t), where m_t solves (77) and, given m_t, a_t solves (48).

The ODE (77) can be rewritten as

$$\frac{\dot{m}_t}{m_t} = \max\left\{\Gamma_0(m_t), \Gamma_1(m_t)\right\},$$

where

$$\begin{aligned}\Gamma_0(m) &\equiv (\alpha\sigma + \rho + \pi) - \alpha\sigma m^{-\eta} \\ \Gamma_1(m) &\equiv (\alpha\sigma\chi_m + r_a + \pi) - \alpha\sigma\chi_m m^{-\eta}.\end{aligned}$$

From (71), m_0^s is the unique solution to $\Gamma_0(m_0^s) = 0$. From (72), m_1^s is the unique solution to $\Gamma_1(m_1^s) = 0$. Both $\Gamma_0(m_t)$ and $\Gamma_1(m_t)$ are increasing functions that intersect once at

$$m = \omega_1^s = \left(\frac{\alpha\sigma\chi_2}{\alpha\sigma\chi_2 + \rho - r_a}\right)^{\frac{1}{\eta}}.$$

For all $m < \omega_1^s$, $\Gamma_0(m) < \Gamma_1(m)$ and for all $m > \omega_1^s$, $\Gamma_0(m) > \Gamma_1(m)$. Hence,

$$\frac{\dot{m}_t}{m_t} = \Gamma_0(m_t)\mathbb{I}_{\{m_t \geq \omega_1^s\}} + \Gamma_1(m_t)\mathbb{I}_{\{m_t < \omega_1^s\}}.$$

The right side of the ODE is increasing in m_t with

$$\lim_{m_t \searrow 0} \frac{\dot{m}_t}{m_t} = -\infty \text{ and } \left.\frac{\dot{m}_t}{m_t}\right|_{m_t = 1} = \rho + \pi > 0.$$

Hence, there exists a unique positive steady state, $m^s \in (0, 1)$. The two candidates are m_0^s and m_1^s. Since $\max\left\{\Gamma_0(m^s), \Gamma_1(m^s)\right\} = 0$, $m^s = \min\{m_0^s, m_1^s\}$. To see this, suppose $m^s = m_0^s$. Then, $\Gamma_0(m^s) = 0 \geq \Gamma_1(m^s)$. Using that Γ_1 is increasing and $\Gamma_1(m_1^s) = 0$, $m^s \leq m_1^s$. I illustrate these arguments graphically in Figure 11 where I represent the functions $\Gamma_0(m_t)$ and $\Gamma_1(m_t)$ and the determination of m_0^s, m_1^s, and m^s.

Characterization of speculative hyperinflations. For all $m_t < m^s$, $\dot{m}_t < 0$. Hence, there are a continuum of speculative hyperinflation equilibria indexed by $m_0 \in (0, m^s)$ and such that m_t decreases over time with $\lim_{t \to +\infty} m_t = 0$. From (48),

$$a_t \begin{cases} = \\ > \end{cases} 0 \text{ if } \alpha\sigma\chi_2 \left[(m_t)^{-\eta} - 1\right] \begin{cases} < \\ > \end{cases} \rho - r_a.$$

In words, it is optimal to hold a positive amount of dollars if the liquidity premium in type-2 matches when $a = 0$ is greater than the holding cost of dollars. Equivalently, using the definition of ω_1^s in (73), i.e., $\alpha\sigma\chi_2 \left[(\omega_1^s)^{-\eta} - 1\right] = \rho - r_a$,

$$a_t \begin{cases} = \\ > \end{cases} 0 \text{ if } m_t \begin{cases} > \\ < \end{cases} \omega_1^s.$$

Since m_t is decreasing over time and approaches zero asymptotically, if $m_0 > \omega_1^s$ then there is a $T_0 > 0$ such that for all $t < T_0$, $a_t = 0$ and for all $t > T_0$, $a_t > 0$. If $m_0 \leq \omega_1^s$ then $T_0 = 0$. It follows that the ODE (77) can be rewritten as

$$\begin{aligned} \frac{\dot{m}_t}{m_t} &= \Gamma_0(m_t) \text{ for all } t \in (0, T_0) \\ &= \Gamma_1(m_t) \text{ for all } t > T_0. \end{aligned}$$

This ODE is solved by backward induction. For all $t > T_0$, the ODE $\dot{m}_t / m_t = \Gamma_1(m_t)$ is identical to (23) where $\alpha\sigma$ has been replaced with $\alpha\sigma\chi_m$ and ρ has been replaced with r_a. The initial condition is $m_{T_0} = \min\{m_0, \omega_1^s\} < m_1^s$. Hence, from Corollary 3, the solution is

$$m_t = \left\{(m_1^s)^\eta - e^{\eta(\alpha\sigma\chi_m + r_a + \pi)(t - T_0)} \left[(m_1^s)^\eta - (m_{T_0})^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[T_0, T_0 + T_1]}(t),$$

where, from (30) by replacing $\alpha\sigma$ with $\alpha\sigma\chi_m$ and ρ with r_a,

$$T_1 = \frac{\ln\left[1 - (m_{T_0}/m_1^s)^\eta\right]^{-1}}{(\alpha\sigma\chi_m + r_a + \pi)\,\eta}.$$

From (48) at equality, $a_t + m_t = \omega_1^s$ for all $t > T_0$.

For all $t < T_0$, the ODE $\dot{m}_t / m_t = \Gamma_0(m_t)$ is identical to (23). Hence, from Corollary 3, the solution is

$$m_t = \left\{(m_0^s)^\eta - e^{\eta(\alpha\sigma + \rho + \pi)t} \left[(m_0^s)^\eta - (m_0)^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0, T_0]}(t) \text{ for all } t < T_0. \tag{78}$$

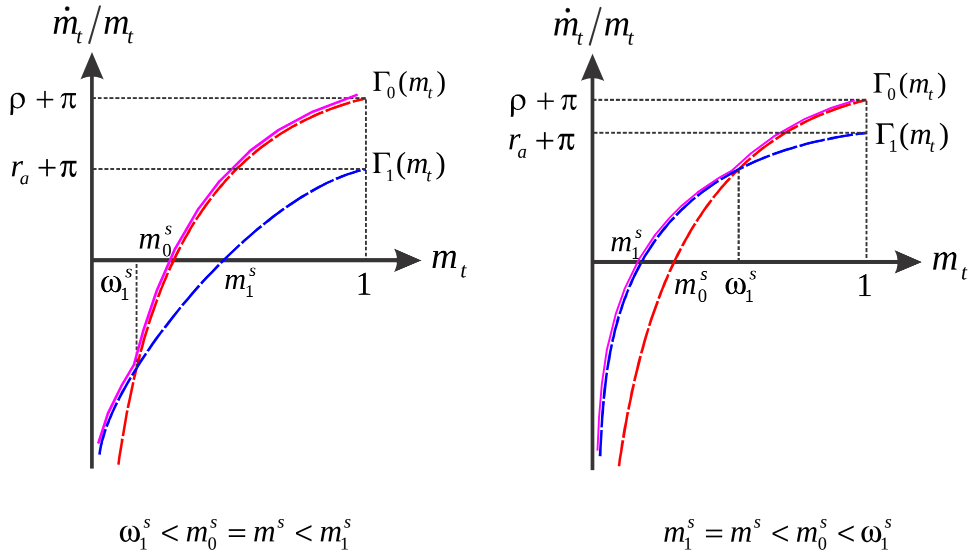

Figure 11: Representation of the functions Γ_0 and Γ_1

$\underline{T_0 = 0 \text{ versus } T_0 > 0}$. As shown in the right panel of Figure 11, if $m_1^s < m_0^s$, then $m^s = m_1^s < \omega_1^s$. The condition $m_1^s < \omega_1^s$ is equivalent to $r_a > \chi_m \rho - \chi_2 \pi$. From (48), $a^s + m^s = \omega_1^s$ and the steady-state holdings of dollars are

(79) $$a_1^s = \left(\frac{\alpha\sigma\chi_2}{\rho - r_a + \alpha\sigma\chi_2}\right)^{\frac{1}{\eta}} - \left(\frac{\alpha\sigma\chi_m}{\alpha\sigma\chi_m + r_a + \pi}\right)^{\frac{1}{\eta}}.$$

For all $m_0 < m^s$, $m_0 < \omega_1^s$. Hence, $a_t > 0$ for all t, i.e., $T_0 = 0$.

As shown in the left panel of Figure 11, if $m_0^s < m_1^s$, then $m^s = m_0^s > \omega_1^s$ and $a^s = 0$. The condition $m_0^s > \omega_1^s$ is equivalent to $r_a < \chi_m \rho - \chi_2 \pi$. For all $m_0 \in (\omega_1^s, m^s)$, $T_0 > 0$. If $m_0 \leq \omega_1^s$, then $T_0 = 0$.

$\underline{\text{Determination of } T_0}$. Suppose $m^s = m_0^s > \omega_1^s$ and $m_0 \in (\omega_1^s, m^s)$ so that $T_0 > 0$. From (77), T_0 is the solution to

$$\alpha\sigma\chi_2 \left[(m_{T_0})^{-\eta} - 1\right] = \rho - r_a,$$

or, equivalently,

$$m_{T_0} = \omega_1^s = \left(\frac{\alpha\sigma\chi_2}{\alpha\sigma\chi_2 + \rho - r_a}\right)^{\frac{1}{\eta}}.$$

Using the expression for m_t given by (78), T_0 is the solution to

$$\left\{(m_0^s)^\eta - e^{\eta(\alpha\sigma+\rho+\pi)T_0}\left[(m_0^s)^\eta - (m_0)^\eta\right]\right\}^{\frac{1}{\eta}} = \omega_1^s.$$

Solving for T_0,

$$T_0 = \frac{1}{\eta(\alpha\sigma + \rho + \pi)} \ln \left[\frac{(m_0^s)^\eta - (\omega_1^s)^\eta}{(m_0^s)^\eta - (m_0)^\eta} \right],$$

which corresponds to (74). ■

Appendix S1: The role of continuous time

The observation that the breakdown of backward induction hinges on time being uncountable raises the question of the robustness of continuous-time monetary equilibria. In order to address this concern, I will show that equilibria in continuous time are the limits of equilibria in discrete time, for which backward induction holds, when the length of the period approaches zero. It means that for any $T \in (0, +\infty)$ and any small $\varepsilon > 0$, one can find a period length small enough in the discrete-time model so that there exist equilibria where the value of money is less than ε at time T.

The Shi-Trejos-Wright model

In discrete time, the model can be rewritten as:

$$\begin{aligned} V_{1,n\Delta} &= \beta_\Delta \left\{ \alpha_\Delta \sigma (1-M) \left[u(y_{(n+1)\Delta}) + V_{0,(n+1)\Delta} - V_{1,(n+1)\Delta} \right] + V_{1,(n+1)\Delta} \right\} \\ V_{0,n\Delta} &= \beta_\Delta \left\{ \alpha_\Delta \sigma M \left(-y_{(n+1)\Delta} + V_{1,(n+1)\Delta} - V_{0,(n+1)\Delta} \right) + V_{0,(n+1)\Delta} \right\}, \end{aligned}$$

where $n \in \mathbb{N}_0$ and $\Delta \in \mathbb{R}_+$ is the length of time period. The take-it-or-leave-it offer by sellers gives

$$y_{(n+1)\Delta} = V_{1,(n+1)\Delta} - V_{0,(n+1)\Delta}.$$

Hence, an equilibrium of the STW model in discrete time as a sequence, $\{y_{n\Delta}\}_{n=0}^{+\infty}$, solution to

$$y_{n\Delta} = \beta_\Delta \left\{ \alpha_\Delta \sigma (1-M) \left[u(y_{(n+1)\Delta}) - y_{(n+1)\Delta} \right] + y_{(n+1)\Delta} \right\}. \tag{80}$$

The length of a period, Δ, affects both the discount factor, $\beta_\Delta = e^{-\rho\Delta}$, and the arrival rate of meetings, $\alpha_\Delta = 1 - e^{-\alpha\Delta}$. It can be checked that the difference equation (80) converges to the differential equation (4) as Δ approaches zero. To see this, denote $t = n\Delta$, and rearrange (80) as follows:

$$\left(\frac{1-\beta_\Delta}{\beta_\Delta} \right) y_t = \alpha_\Delta \sigma (1-M) \left[u(y_{t+\Delta}) - y_{t+\Delta} \right] + y_{t+\Delta} - y_t.$$

Divide by Δ and take the limit as $\Delta \to 0$ to obtain (4).

The Lagos-Wright model

Suppose that competitive markets (CMs) open infrequently at times $n\Delta$ for $n \in \mathbb{N}_0$. So, Δ is the length of time between two consecutive CMs. Time can be continuous but the times at which the CMs open are countable. Between CMs, each agent meets a trading partner with probability α_Δ. This description corresponds to the discrete-time model of Lagos and Wright (2003). Under a constant money supply ($\pi = 0$) and CRRA preferences, the time-path for real balances, $\{m_{n\Delta}\}_{n=0}^{+\infty}$, solves

$$m_{n\Delta} = \beta_\Delta m_{(n+1)\Delta} \overbrace{\left[1 + \alpha_\Delta \sigma \left(m_{(n+1)\Delta}^{-\eta} - 1\right)\right]}^{\text{liquidity factor}}, \quad \forall n \in \mathbb{N}_0, \tag{81}$$

where the discount factor, $\beta_\Delta = e^{-\rho\Delta}$, and the probability of a meeting between two consecutive CMs, $\alpha_\Delta = 1 - e^{-\alpha\Delta}$, depend on Δ. The value of money at time $n\Delta$ is equal to the discounted value of money at time $(n+1)\Delta$, multiplied by a liquidity factor.

The difference equation (81) converges to the differential equation (23) as Δ approaches zero. To see this, denote $t = n\Delta$, and rearrange (81) as follows:

$$m_t \left(\frac{1 - \beta_\Delta}{\beta_\Delta}\right) = \alpha_\Delta \sigma m_{t+\Delta} \left(m_{t+\Delta}^{-\eta} - 1\right) + m_{t+\Delta} - m_t. \tag{82}$$

Divide by Δ and take the limit as $\Delta \to 0$ and use that $\alpha_\Delta/\Delta \to 0$, $(1 - \beta_\Delta)/\Delta \to \rho$ and $\beta_\Delta \to 1$ to obtain (23). It suggests that the dynamics of the model in discrete time when CMs open at high frequency approach the dynamics in continuous time.

I confirm this point in Figure 12. In the top panel, I plot the solution to (81) for different values of Δ.[29] The time-paths in discrete time converge to the equilibrium time-path in continuous time as Δ goes to zero. In the bottom panel, I plot real balances at some arbitrary date, $t = 10$, for different frequencies of CM openings, $n = 10/\Delta$. As n increases, Δ decreases in order to keep $t = n\Delta$ constant. The value of money decreases toward 0 as n becomes large, i.e., $\lim_{\Delta \to 0, \Delta n \to 10} m_{10} = 0$. Even though money does not die, for any arbitrary $\varepsilon > 0$, one can find a $\Delta_\varepsilon > 0$ such that if $\Delta < \Delta_\varepsilon$ then the value of money at $t = 10$ is less than ε.

[29] The value, m_t, is computed as $m_{n\Delta}$ where n is the integer part of t/Δ. The parameter values are $\eta = 0.5$, $\alpha = 5$, $\sigma = 0.2$, $\rho = 0.01$, and the initial value is $m_0 = 0.85$.

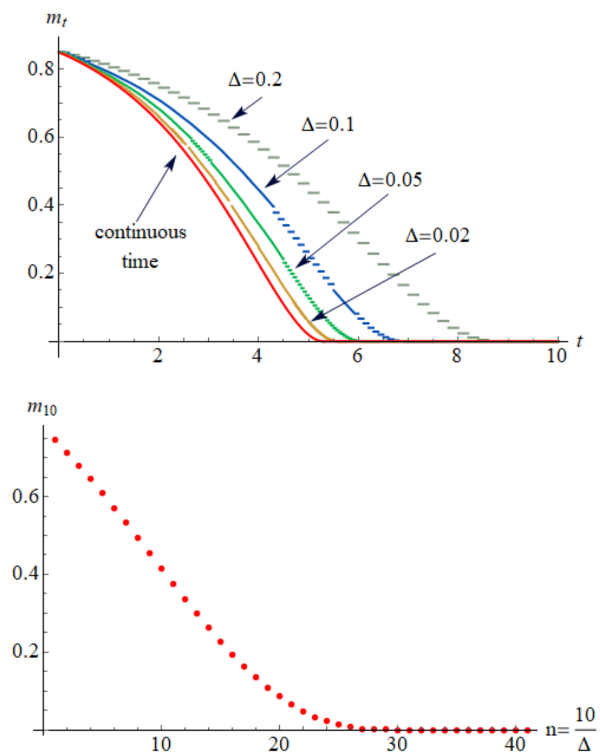

Figure 12: Top panel: Time-paths for real balances in the discrete-time model for different Δ and in the continuous-time model. Bottom panel: Value of real money balances at time $t = n\Delta = 10$ as Δ vanishes.

Appendix S2: When credit dies

So far I studied speculative hyperinflation in economies with fiat monies. As the use of fiat money in developed economies recedes, do speculative hyperinflations become less relevant? I now describe a phenomenon similar to speculative hyperinflations in a pure credit economy with limited commitment, as in Kehoe and Levine (1993).[30]

I maintain the assumption from the pure currency economy that there is no technology to enforce the repayment of debts, i.e., repayment has to be self-enforcing. In order for credit arrangements to be incentive feasible, there is a record-keeping technology that keeps track of transactions and repayments. In equilibrium, buyers will have incentives to repay their debts in order to maintain their access to credit. I assume that the record-keeping technology is imperfect in the following sense. A default event is recorded with probability $\lambda \in [0,1]$. Hence, there is a probability $1-\lambda$ that an agent who defaulted is not reported publicly and hence is not excluded from future credit.[31]

Credit transactions take place as follows. At times T_n, $n \in \mathbb{N}$, a buyer receives opportunities to consume by being matched bilaterally with sellers. While she cannot produce in the match, she can promise to repay her debt as soon as the meeting is over, at time T_n^+. The maximum amount a buyer can promise to repay at time t expressed in the numéraire, called *debt limit*, is denoted d_t. This real payment capacity, which is analogous to m_t in the monetary economy, is endogenous.

The expected lifetime utility of a buyer solves the following HJB equation:

$$\rho V_t^b = \alpha\sigma\upsilon(d_t) + \dot{V}_t^b. \tag{83}$$

The interpretation is similar to (16) where m_t is replaced with d_t. The debt limit is defined as the highest amount of debt that a buyer would repay willingly knowing that if she defaults she will be excluded from future transactions with probability λ, in which case her lifetime utility is 0 (under the assumption that $u(0)=0$). It solves

$$d_t = \lambda V_t^b. \tag{84}$$

[30] Such equilibria have been described in a discrete-time competitive economy by Bloise, Reichlin, and Tirelli (2013).

[31] This model is a version of Gu, Mattesini, Monnet, and Wright (2013). A characterization of the perfect Bayesian equilibria of such a pure credit economy is provided in Bethune, Hu, and Rocheteau (2018).

By defaulting, the buyer saves the cost of repayment, d_t, but incurs the cost of being excluded from future credit, V_t^b, with probability λ. Substitute d_t by its expression given by (84) into (83) to obtain

$$\rho d_t = \alpha\sigma\lambda\upsilon(d_t) + \dot{d}_t. \tag{85}$$

An equilibrium is a time-path, d_t, solution to (85).

In order to solve for equilibria in closed form, I make the following assumptions. Buyers make take-it-or-leave-it offers to sellers, i.e., $p(y) = y$, and preferences are of the type $u(y) = y^{1-\eta}$ with $\eta \in (0,1)$. In order to keep the analysis succinct, I focus on the region in the parameter space where the debt limit at the steady state, d^s, is less than y^*, which requires $\rho y^* > \alpha\sigma\lambda\left[u(y^*) - y^*\right]$. This condition holds is agents are sufficiently impatient. The ODE (85) can be rewritten as:

$$\dot{d}_t = (\rho + \alpha\sigma\lambda)\, d_t - \alpha\sigma\lambda(d_t)^{1-\eta}. \tag{86}$$

This ODE is formally identical to the ODE in the STW model, (6), where $1 - M$ has been replaced with λ. By the same logic as in Proposition 1, I obtain the following proposition.

Proposition 11 ***(Speculative debt limits.)*** *Consider a pure credit economy under limited commitment. The positive steady state is*

$$d^s = \left(\frac{\alpha\sigma\lambda}{\rho + \alpha\sigma\lambda}\right)^{\frac{1}{\eta}}. \tag{87}$$

In addition, there are a continuum of nonstationary equilibria, indexed by $d_0 \in (0, d^s)$, such that

$$d_t = \left\{(d^s)^\eta - \left[(d^s)^\eta - (d_0)^\eta\right] e^{\eta(\rho+\alpha\sigma\lambda)t}\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t) \tag{88}$$

where

$$T = \frac{-\ln\left[1 - (d_0/d^s)^\eta\right]}{\eta\,(\rho + \alpha\sigma\lambda)}. \tag{89}$$

Along those speculative equilibria, the real borrowing capacity of buyers goes to zero in finite time. The rate at which d_t diverges from its steady-state value, $\eta\,(\rho + \alpha\sigma\lambda)$, increases with the reliability of the record-keeping technology.

Appendix S3: Time since take-off in the Solow growth model

The representation of the equilibrium of a macroeconomic model by a Bernoulli equation is not unique to the STW model. Indeed, the equilibrium of the textbook model of economic growth from Solow (1956) can also be represented by a Bernoulli equation. In the following, I show the analogy between the two equilibrium conditions and I establish a little-known result for the Solow growth model that mirrors the result from the STW model according to which money loses its value in finite time.

Under a Cobb-Douglas production function, the capital stock per worker obeys the following differential equation,

$$\dot{k}_t = s k_t^a - \delta k_t, \tag{90}$$

where $s \in (0,1)$ is the savings rate, $\delta > 0$ is the rate of depreciation. (Without loss, I omit technological progress and population growth.) The phase line is the mirror image of the phase line of the STW model relative to the horizontal axis. There is an active and an inactive steady state, the active steady state is dynamically stable. By the same logic as above, using the change of variable $x = k^{1-a}$, the closed form solution is

$$k_t = \left[\left(k_0^{1-a} - \frac{s}{\delta}\right) e^{-\delta(1-a)t} + \frac{s}{\delta}\right]^{\frac{1}{1-a}}.$$

As time goes to infinity, k_t approaches its steady state, $(s/\delta)^{1/(1-a)}$, asymptotically.[32] Given any $k_0 < (s/\delta)^{1/(1-a)}$, we can compute the finite time at which the economy started to grow:

$$T = \frac{-1}{\delta(1-a)} \ln\left(\frac{s}{s - \delta k_0^{1-a}}\right) > -\infty. \tag{91}$$

So, just like the steady state nonmonetary equilibrium is reached in finite time in the STW model, the inactive steady state is reached in finite "backward time" in the Solow model.

Corollary 6 *Consider the Solow growth model and suppose $k_0 = 0$. There exists a continuum of equilibria indexed by $T \in \mathbb{R}_+$ such that the economy takes off at time T and reaches the active steady state asymptotically.*

[32] Sato (1963) is the first to identify the ODE of the Solow growth model under Cobb-Douglas production function as a Bernoulli equation and to provide a closed-form solution.

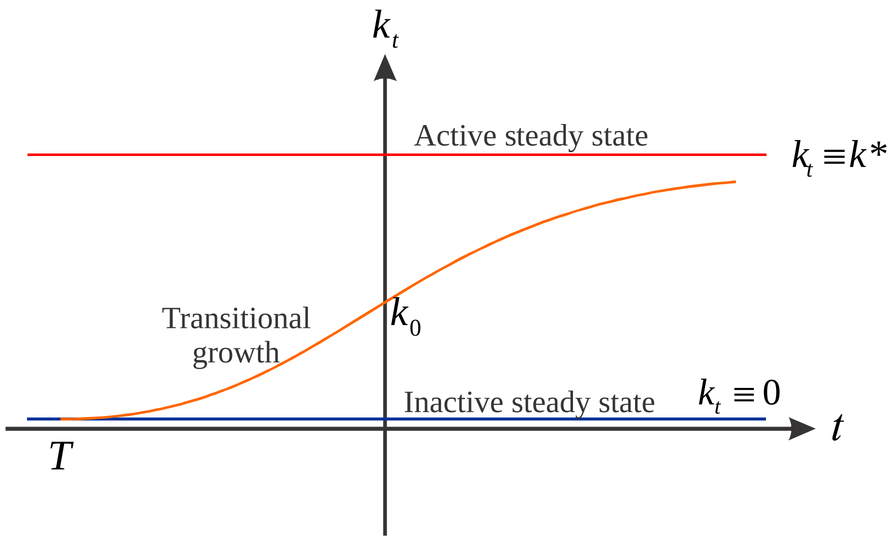

Figure 13: Solutions to the Solow growth model

The solution to the equilibrium ODE of the Solow growth model is not unique when $k_0 = 0$. There is a solution where the capital stock remains at zero forever. But there are also a continuum of solutions indexed by the time at which the economy takes off.[33]

How can the economy take off without capital? In the discrete-time Solow growth model, if $k_0 > 0$, then $k_{n\Delta} > 0$ for all $n \in \mathbb{Z}$, where Δ represents the length of a period. By the same reasoning as the one in Appendix S1, $k_{n\Delta}$ approaches 0 as Δ tends to 0 and $n\Delta \to t < T$, where T is defined by (91). So, along a sequence of discrete-time economies indexed by Δ and such that Δ goes to zero, capital stocks before the take-off date are positive but converge to 0. The infinitesimal capital before the take-off combined with an infinite marginal product allows the economy to take off at time T.[34] By viewing the continuous-time economy as the limit of discrete-time economies, one can see that it would be misleading to interpret the result from the corollary above as suggesting that the economy can take off spontaneously.

[33] A version of this corollary was proven in Hakenes and Irmen (2008).

[34] There is a parallel with the Friedmann equations in physical cosmological physics that govern the expansion of space. These equations show that the universe is of finite age, and had its origin in a mathematical singularity. See https://ned.ipac.caltech.edu/level5/Peacock/Peacock3_2.html

Appendix S4: Convertible money

I have been focusing in this paper on fiat monies defined as intrinsically useless, inconvertible objects that serve as media of exchange. In this appendix, I explore a model variant where money is temporarily convertible. Until time $T_C > 0$, each unit of money can be exchanged on demand from the government for $k > 0$ units of numéraire. After T_C, money loses its convertibility and becomes a pure fiat money.

The government prints money at a constant rate, $\pi > 0$, and it balances its budget at each point in time with lump-sum transfers or taxes to buyers. Consequently, the quantity of money issued up to time t is $A_t = A_0 e^{\pi t}$. Under the assumption that money is convertible, the quantity of money in circulation, $M_t \leq A_t$, is endogenous. I examine equilibria where M_t is continuously differentiable almost everywhere (allowing for a countable number of discontinuities). I assume CRRA preferences, $u(y) = y^{1-\eta}/(1-\eta)$, with $\eta \in (0,1)$, $w(y) = y$, and buyers make take-it-or-leave-it offers to sellers, $p(y) = y$.

Money is always convertible ($T_C = +\infty$).

I start by characterizing equilibria under the assumption that money is always convertible. At any point in time, an agent can acquire units of money at price ϕ_t and redeem them for k units of numéraire. If $\phi_t < k$, the demand for money driven by profit opportunities becomes unbounded so that the money market cannot clear. Therefore, in any equilibrium, the price of money has a lower bound, $\phi_t \geq k$, and speculative equilibria where money dies do not exist.

Proposition 12 ***(Convertible money.)*** *Suppose money is always convertible, $T_C = +\infty$. The time-path for real balances is*

$$(92) \qquad m_t = \left\{(m_\pi^s)^\eta + \left[(m_0^s)^\eta - (m_\pi^s)^\eta\right] e^{-\eta(\pi+\rho+\alpha\sigma)(\tau-t)}\right\}^{\frac{1}{\eta}} \quad \forall t < \tau$$
$$= m_0^s \quad \forall t \geq \tau,$$

where

$$m_\pi^s \equiv \left(\frac{\alpha\sigma}{\alpha\sigma+\rho+\pi}\right)^{\frac{1}{\eta}} < m_0^s \equiv \left(\frac{\alpha\sigma}{\alpha\sigma+\rho}\right)^{\frac{1}{\eta}},$$

and the time at which $\phi_t \geq k$ *binds is*

$$\tau = \frac{1}{\pi} \max \left\{ \frac{1}{\eta} \ln \left(\frac{\alpha\sigma}{\alpha\sigma + \rho} \right) - \ln (A_0 k), 0 \right\}. \tag{93}$$

Along an equilibrium path with convertible money, ϕ_t decreases over time until it reaches its lower bound, k, at time τ. For all $t > \tau$, the price of money is equal to k and its real rate of return is equal to $r_t = 0$. According to (92), aggregate real balances increase until they reach the value consistent with a constant money supply, m_0^s. At that point, agents convert the units of money issued by the government, πA_t, in order to keep the money in circulation constant. The equilibrium is analogous to the nonspeculative equilibrium of a fiat money economy where the government commits at time 0 to reduce the money growth rate at time τ from $\pi > 0$ to 0.

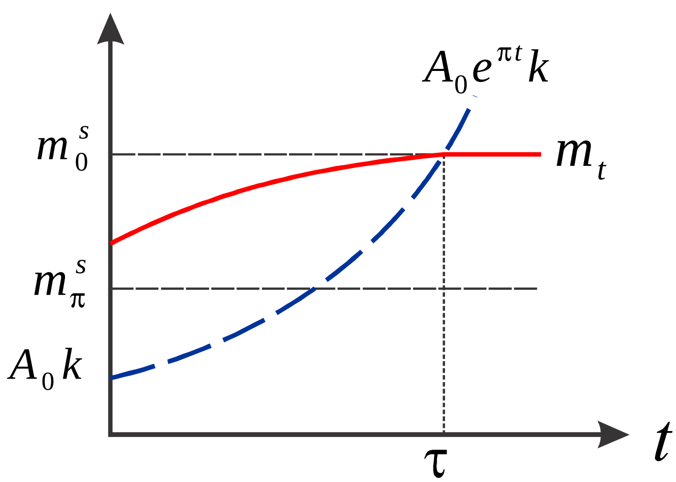

Figure 14: Time-path of real money balances, m_t, when money is convertible.

Figure 14 plots the time-path of real money balances, m_t. From $t = 0$ to $\tau = \tau$, real money balances, which are between m_π^s and m_0^s, increase until they reach m_0^s. From τ onwards, m_t is constant. The time at which agents start exercising the convertibility option, τ, is determined at the intersection of the supply of money, A_t, valued at the convertibility price, k, and the demand for money when its rate of return is $r = 0$, m_0^s. From (93), $\tau = 0$ if $m_0^s < A_0 k$, in which case $m_t = m_0^s$ for all t. Under this condition, agents convert a discrete amount of the initial money supply so that $M_0 k = m_0^s$.

Money is temporarily convertible $(T_C < +\infty)$**.**

I now consider the case where the convertibility of money is transitory.

Proposition 13 ***(Temporary convertibility)*** *Suppose money is convertible until time $T_C < +\infty$ and ceases to be convertible afterwards. There are three types of equilibria.*

1. *Convertibility is not binding. If $A_0 k < e^{-\pi T_C} m^s_\pi$, then there exists a continuum of equilibria indexed by $T \in (T_0, +\infty)$, where $T_0 \in (T_C, +\infty)$, such that: $\phi_t > k$ for all $t \leq T_C$;*

$$m_t = m^s_\pi \left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T-t)}\right]^{\frac{1}{\eta}} \quad \forall t \in (0, T); \tag{94}$$

   $m_t = 0$ for all $t > T$; and $M_t = A_t$ for all $t \geq 0$.

2. *Convertibility is binding. If $A_0 k > e^{-\pi T_C} m^s_0$, then there exists a continuum of equilibria indexed by $m_{T_C^+} \in [0, m^s_\pi]$ where: $\phi_t = k$ for all $t \in (\tau, T_c)$ where $\tau < T_C$ is given by (93);*

$$m_t = \begin{cases} \left\{(m^s_\pi)^\eta + \left[(m^s_0)^\eta - (m^s_\pi)^\eta\right] e^{-\eta(\pi+\rho+\alpha\sigma)(\tau-t)}\right\}^{\frac{1}{\eta}} \\ m^s_0 \\ m^s_\pi \left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T-t)}\right]^{\frac{1}{\eta}} \\ 0 \end{cases} \quad \forall t \in \begin{cases} (0, \tau) \\ [\tau, T_C] \\ (T_C, T) \\ [T, +\infty) \end{cases} \tag{95}$$

   where $M_{T_C^+} k = m_{T_C^+}$ and T solves

$$T = T_C - \frac{1}{\eta(\alpha\sigma+\rho+\pi)} \ln\left[1 - \left(\frac{m_{T_C^+}}{m^s_\pi}\right)^\eta\right]. \tag{96}$$

3. *Convertibility only binds at T_C. If $A_0 k \leq e^{-\pi T_C} m^s_0$, there exists a continuum of equilibria indexed by $m_{T_C^+} \leq \min\{m^s_\pi, e^{\pi T_C} A_0 k\}$ such that: $\phi_t > k$ for all $t < T_C$ and $\phi_{T_C} = k$;*

$$m_t = \begin{cases} \left\{(m^s_\pi)^\eta \left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T_C-t)}\right] + e^{-\eta(\rho+\alpha\sigma+\pi)(T_C-t)} e^{\pi T_C} (A_0 k)^\eta\right\}^{\frac{1}{\eta}} \\ m^s_\pi \left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T-t)}\right]^{\frac{1}{\eta}} \\ 0 \end{cases} \quad \forall t \in \begin{cases} [0, T_C] \\ (T_C, T) \\ [T, +\infty), \end{cases} \tag{97}$$

   where $M_{T_C^+} k = m_{T_C^+}$ and T solves (96).

Proposition 13 distinguishes three types of equilibria. There are equilibria where the convertibility constraint is never binding. They correspond to the speculative equilibria studied in the main text. Along such equilibria, the value of money is decreasing over time and real money balances are nonincreasing. This requires the convertibility value, k, to be small and the period of convertibility, T_C, to be short.

There is a second type of equilibrium where agents exercise the convertibility option before T_C. In that case, the equilibrium path from $t = 0$ to $t = T_C$ is analogous to the one described in Proposition 12. Real money balances increase initially until they reach m_0^s at time τ. Over the time interval, (τ, T_C), m_t remains constant as agents convert the units of money they receive from the government. At time T_C, agents convert a discrete quantity of money so that the real balances at time T_C^+, $M_{T_{C+}}k$, correspond to the initial condition of an equilibrium of a pure currency economy. While real money balances can be discontinuous at time T_C, the price of money is continuous.

Finally, there is a third type of equilibrium where agents only exercise the option to convert some of their units of money at time T_C. If $m_\pi^s < e^{\pi T^C} A_0 k$, then real money balances increase until they reach $e^{\pi T^C} A_0 k$ at time T_C^-. If $m_\pi^s > e^{\pi T^C} A_0 k$, then real money balances decrease until they reach $e^{\pi T^C} A_0 k$ at time T_C^-. Agents convert some amount of money at time T_C. From T_C^+ onward, the equilibrium time-path corresponds to an equilibrium of the economy with fiat money.

The different equilibria are represented in Figure 15. The top left panel plots an equilibrium where the convertibility option is never exercised. The top right panel corresponds to an equilibrium where the convertibility option is exercised over the time interval, (τ, T_C). The bottom panels correspond to equilibria when the convertibility option is only exercised at time $t = T_C$. In the left panel, m_π^s is smaller than $A_{T_C}k$ and hence m_t increases over time. In the right panel, m_π^s is larger than $A_{T_C}k$ and m_t decreases over time.

Proofs of propositions

Proof of Proposition 12. First, I rule out equilibria where M_t is discontinuous at any time $t > 0$. Suppose a contrario that agents convert a discrete amount of money at some time, $\tau > 0$. If it is the case, $\phi_\tau = k$ and $M_{\tau^-} > M_{\tau^+}$. Using that $\phi_t \geq k$ for all t, $\dot{\phi}_{\tau^-} \leq 0$

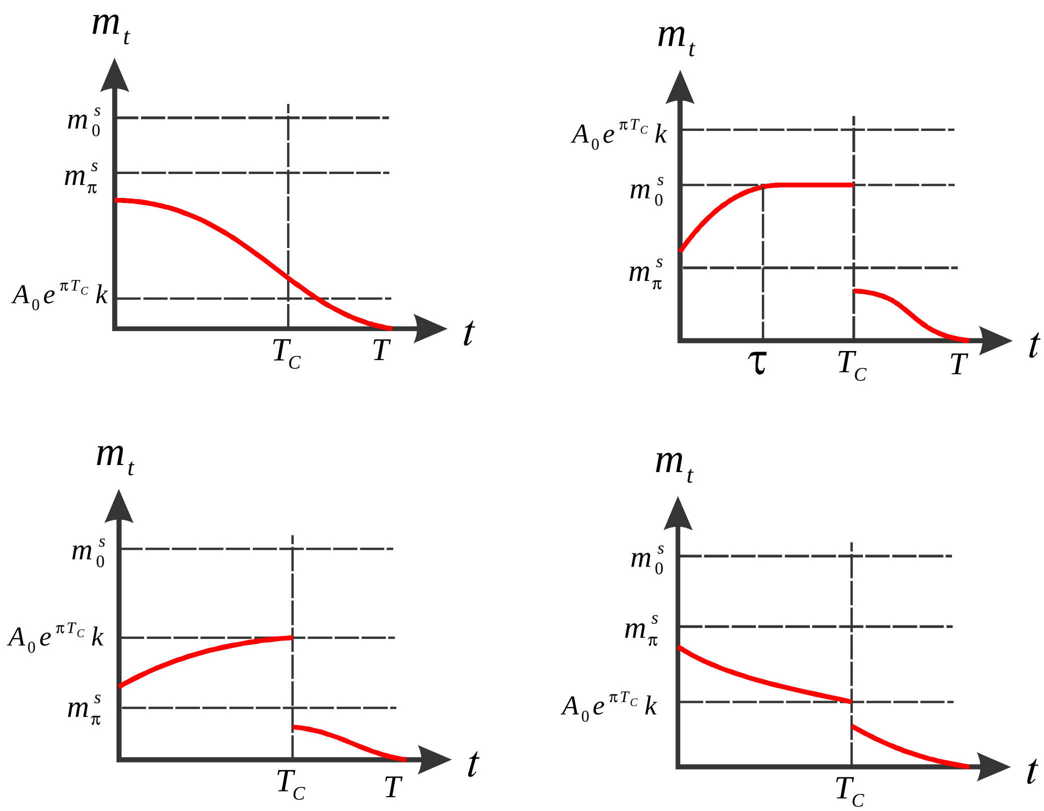

Figure 15: Time-paths for real money balances. Top left panel: no conversion along the equilibrium path. Top right panel: conversion occurs during time interval $[\tau, T_C]$. Bottom panels: conversion occurs at time T_C.

and $\dot{\phi}_{\tau^+} \geq 0$. The value of money must be nonincreasing the instant before it reaches its lower bound, k, and nondecreasing the instant after. Equivalently, $r_{\tau^-} \leq 0 \leq r_{\tau^+}$. From the first-order condition, (20), $m_{\tau^-} = \phi_\tau M_{\tau^-} \leq m_{\tau^+} = \phi_\tau M_{\tau^+}$. A contradiction.

Second, suppose the constraint $\phi_t \geq k$ binds over some nonempty time interval, $(\tau, \tau') \subseteq \mathbb{R}_+$, i.e., $\phi_t = k$ for all $t \in (\tau, \tau')$. It follows that $r_t = \dot{\phi}_t/\phi_t = 0$ for all $t \in (\tau, \tau')$. From the first-order condition, (20), $u'(m_t) = 1 + \rho/\alpha\sigma$, i.e., $m_t = m_0^s \equiv [\alpha\sigma/(\alpha\sigma + \rho)]^{\frac{1}{\eta}}$ for all $t \in (\tau, \tau')$. Thus, both m_t and ϕ_t are constant. Since $m_t = M_t\phi_t$, the money in circulation is constant, $M_t = m_0^s/k$ for all $t \in (\tau, \tau')$. Equivalently, $\hat{\pi}_t = 0$ for all $t \in (\tau, \tau')$, where $\hat{\pi}_t$ is the growth rate of M_t.

Third, I establish that in any equilibrium where $\phi_t = k$ over some nonempty time interval, (τ, τ'), then $\tau' = +\infty$. I showed above that $\hat{\pi}_t = 0$ if $\phi_t = k$ and $\hat{\pi}_t = \pi > 0$ if $\phi_t > k$.

Hence, for all $s > t$, $\hat{\Pi}(s) - \hat{\Pi}(t) \geq 0$, where $\hat{\Pi}(t) = \int_0^t \hat{\pi}_s ds$ is the cumulative growth of M_t. From (26), aggregate real balances are given by the nonspeculative solution,

$$m_t = \left[\alpha\sigma\eta \int_t^{+\infty} e^{-\eta\alpha\sigma(s-t)} e^{-\eta\left[\rho(s-t)+\hat{\Pi}(s)-\hat{\Pi}(t)\right]} ds\right]^{\frac{1}{\eta}}, \tag{98}$$

Thus, $m_t \leq m_0^s$. Divide both sides by M_t to rewrite the inequality as $\phi_t \leq m_0^s/M_t$. From the constraint, $\phi_t \geq k$, it follows that $M_t \leq m_0^s/k$. As shown above this upper bound for M_t is achieved for all $t \in (\tau, \tau')$. Using that M_t is nondecreasing, it must be that $\tau' = +\infty$ so that $\hat{\pi}_t = 0$ and $\phi_t = k$ for all $t > \tau$.

Fourth, I compute the entire path for m_t from (98) and the result that $\hat{\pi}_t = \pi\mathbb{I}_{\{t<\tau\}}$. For all $t \geq \tau$, $\hat{\pi}_t = 0$. Hence, $m_t = m_0^s$. For all $t < \tau$:

$$m_t = \left\{\frac{\alpha\sigma}{\alpha\sigma+\rho+\pi}\left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(\tau-t)}\right] + \frac{\alpha\sigma}{\rho+\alpha\sigma} e^{-\eta(\alpha\sigma+\rho+\pi)(\tau-t)}\right\}^{\frac{1}{\eta}}.$$

The terms between brackets can be rearranged to obtain (92). Assuming $\tau > 0$, τ solves $\phi_\tau = k$ where $\phi_\tau = m_\tau/A_\tau$. Since there is no conversion before τ, the quantity of money in circulation at the time where $\phi_t \geq k$ starts binding is equal to the total amount of money issued by the government, A_τ. Using that $m_\tau = m_0^s$ and $A_\tau = A_0 e^{\pi t}$, the condition can be rewritten as $A_0 e^{\pi\tau} k = m_0^s$. Solving for τ, one obtains

$$\tau = \frac{1}{\pi}\left[\frac{1}{\eta}\ln\left(\frac{\alpha\sigma}{\alpha\sigma+\rho}\right) - \ln(A_0 k)\right].$$

If the right side is negative, then $\tau = 0$. Hence, (93).

Finally, if $\tau = 0$, then $m_t = m_0^s$ for all $t \geq 0$. The price of money is $\phi_t = k$ and the initial supply of money in circulation is $M_0 \leq A_0$ so that $M_0 k = m_0^s$. ■

Proof of Proposition 13. I consider equilibria with the following properties: $\phi_t = k$ for all $t \in (\tau, T_C)$ and $\phi_t > k$ for all $t < \tau$. The interval (τ, T_C) can be empty. For all $t \in (\tau, T_C)$, $r_t = 0$ and $m_t = m_0^s = A_\tau k$. The money in circulation, M_t, is constant over (τ, T_C). I allow M_t to be discontinuous at times 0 and T_C. In the following, I distinguish three types of equilibria.

<u>Type-1 equilibria: $\phi_t > k$ for all $t \leq T_C$.</u> The option to convert units of money into the numéraire is never exercised on the equilibrium path, i.e., $M_t = A_t$ for all t. The time-path

for m_t is given by the solution to the ODE,

$$\dot{m}_t = (\alpha\sigma + \rho + \pi)m_t - \alpha\sigma m_t^{1-\eta}.$$

Use the change of variable, $x_t = m_t^\eta$, to rewrite this ODE as

$$\dot{x}_t = \eta(\alpha\sigma + \rho + \pi)x_t - \eta\alpha\sigma.$$

The solution such that $x_T = 0$ is

$$x_t = \frac{\alpha\sigma}{\alpha\sigma + \rho + \pi}\left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T-t)}\right],$$

which, from $m_t = (x_t)^{\frac{1}{\eta}}$, gives (94). This time-path constitutes an equilibrium if $\phi_t > k$ for all $t \leq T_C$. Since $\dot{m}_t < 0$ and $\dot{M}_t > 0$, $\phi_t = m_t/M_t$ decreases over time. So, a necessary and sufficient condition is $\phi_{T_C} > k$ or, equivalently, $m_{T_C} > M_{T_C}k = A_{T_C}k$. Using the closed-form solution for m_t given by (94), this condition can be rewritten as

$$k < \frac{1}{e^{\pi T_C}A_0}\left(\frac{\alpha\sigma}{\alpha\sigma + \rho + \pi}\right)^{\frac{1}{\eta}}\left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T-T_C)}\right]^{\frac{1}{\eta}}. \tag{99}$$

The right side is equal to 0 when $T = T_C$ and it is increasing in T. Hence, there exist an interval of values for $T > T_C$ such that (99) holds if the right side evaluated at $T = +\infty$ is greater than the left side, i.e., $k < e^{-\pi T_C}m_\pi^s/A_0$. Under that condition, the minimum value for T consistent with (99) is T_0 given by

$$T_0 = T_C - \frac{1}{\eta(\alpha\sigma + \rho + \pi)}\ln\left[1 - \frac{\alpha\sigma + \rho + \pi}{\alpha\sigma}\left(e^{\pi T_C}A_0k\right)^\eta\right]. \tag{100}$$

Type-2 equilibria: $\phi_t = k$ for all $t \in (\tau, T_C) \neq \varnothing$. The equilibrium path from $t = 0$ to $t = T_C^-$ is as characterized in Proposition 12: $m_\tau = m_0^s$, and for all $t < T_C$, m_t is given by (92). The expression for τ is given by (93). Thus, the condition, $\tau < T_C$, can be rewritten as

$$\frac{1}{\eta}\ln\left(\frac{\alpha\sigma}{\alpha\sigma + \rho}\right) - \ln(A_0k) < \pi T_C.$$

It can be rearranged as $k > e^{-\pi T_C}m_0^s/A_0$. The money in circulation at time T_C can jump downward, $M_{T_C^-} \geq M_{T_C^+}$. From T_C^+ onwards, the equilibrium path corresponds to the one of a fiat-money economy. Thus, $\phi_{T_C^+}M_{T_C^+} = M_{T_C^+}k$ is no greater than m_π^s, i.e., $m_{T_C^+} = M_{T_C^+}k \leq$

m_π^s. Note that $\phi_{T_C^+} = 0$ is also an equilibrium, in which case agents convert all their money at time T_C. The time at which money dies, T, solves

$$m_\pi^s \left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T-T_C)}\right]^{\frac{1}{\eta}} = m_{T_C^+}.$$

This equation can be solved for T.

Type-3 equilibria: $\phi_t > k$ for all $t < T_C$ and $\phi_{T_C} = k$. Since $\phi_t > k$ for all $t < T_C$, agents do not exercise the option to convert units of money before T_C and $M_t = A_t$ for all $t < T_C$. It follows that $m_{T_C^-} = A_{T_C} k$. Real balances solve the ODE

$$\dot{m}_t = (\alpha\sigma + \rho + \pi)m_t - \alpha\sigma m_t^{1-\eta}.$$

By the same logic as above,

$$m_t = \left\{ \frac{\alpha\sigma}{\alpha\sigma + \rho + \pi} \left[1 - e^{-\eta(\alpha\sigma+\rho+\pi)(T_C-t)}\right] + e^{-\eta(\rho+\alpha\sigma+\pi)(T_C-t)} \left(A_{T_C} k\right)^\eta \right\}^{\frac{1}{\eta}}.$$

Since $\phi_t > k$ for all $t < T_C$, $\dot{\phi}_{T_C^-} \leq 0$ and $r_{T_C^-} \leq 0$. The condition $r_{T_C^-} \leq 0$ is equivalent to $m_{T_C^-} \leq m_0^s$, which can be rewritten as $k \leq e^{-\pi T_C} m_0^s / A_0$. At time T_C^+, real balances must satisfy $M_{T_C^+} k \leq m_\pi^s$ in order to be part of an equilibrium where money is no longer convertible. It follows that $M_{T_C^+} k \leq \min\{m_\pi^s, M_{T_C^-} k\} = \min\{m_\pi^s, e^{\pi T_C} A_0 k\}$. ■

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