# Online Appendix

Estimating Models of Supply and Demand: Instruments and Covariance Restrictions

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March 11, 2024

## Additional Appendices

*Note: Appendices A, B, and C are included with the main text.*

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## D Proofs

### D.1 A Consistent and Unbiased Quasi-Estimator of $\Delta\xi$

Our proofs make use of the following lemma, which identifies a consistent and unbiased quasi-estimator for the unobserved term in a linear regression when one of the covariates is endogenous. We refer to it as a *quasi-estimator* because it depends on unobservables and cannot be constructed from the data. It turns out to be a useful input in our proofs. Though demonstrated in the context of semi-linear demand, the proof also applies for any endogenous covariate, including when (transformed) quantity depends on a known transformation of price, as no supply-side assumptions are required. For example, we may replace $p$ with $\ln p$ everywhere and obtain the same results.

For convenience, in this section, we omit the market-period subscripts $jt$ on scalar variables such as $p$, $h$, and $\xi$ and the $K \times 1$ (row) vector $x$.

**Lemma 1.** *A consistent and unbiased quasi-estimator of $\Delta\xi$ is given by $\hat{\Delta\xi}_1 = \hat{\Delta\xi}^{OLS} + \left(\hat{\alpha}^{OLS} - \alpha\right) p^*$*

For some intuition, note that we can construct both the true demand shock and OLS residuals (at the probability limit) as:

$$\Delta\xi = h - \alpha p - x\beta$$
$$\Delta\xi^{OLS} = h - \alpha^{OLS} p - x\beta^{OLS}$$

where this holds even in small samples. Recall that $E[\Delta\xi_{jt}] = 0$. The true demand shock is given by $\Delta\xi_0 = \Delta\xi^{OLS} + (\alpha^{OLS} - \alpha)p + x(\beta^{OLS} - \beta)$.

We desire to show that the quasi-estimator of the demand shock, $\hat{\Delta\xi}_1 = \hat{\Delta\xi}^{OLS} + \left(\hat{\alpha}^{OLS} - \alpha\right) p^*$, is consistent and unbiased. This eliminates the need to estimate the true $\beta$ parameters. It suffices to show that $(\hat{\alpha}^{OLS} - \alpha)p^* = (\hat{\alpha}^{OLS} - \alpha)p + x(\hat{\beta}^{OLS} - \beta) + \Upsilon$, where $\Upsilon$ is such that $E[\Upsilon = 0]$ and $\Upsilon \to 0$ as $T$ gets large. It is straightforward to show this using the projection matrices for $p$ and $x$.[1]

### D.2 Proof of Proposition 1 (Set Identification)

From equation (9), we have $\hat{\alpha}^{OLS} \xrightarrow{p} \alpha + \frac{Cov(p^*, \Delta\xi)}{Var(p^*)}$. The general form for a firm's first-order condition is $p = mc + \mu$, where $mc$ is the marginal cost and $\mu$ is the markup. We can write $p = p^* + \hat{p}$, where $\hat{p}$ is the projection of $p$ on the exogenous variables $\tilde{x}$ that include product and market fixed effects. If we substitute the first-order condition $p^* = mc + \mu - \hat{p}$ into the bias

[1] Please contact the authors if interested in the full proof.

term from the OLS regression, we obtain

$$\alpha^{OLS} - \alpha = \frac{Cov(p^*, \Delta\xi)}{Var(p^*)} = \frac{Cov(\Delta\xi, mc + \mu - \hat{p})}{Var(p^*)}$$
$$= \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} + \frac{Cov(\Delta\xi, \mu)}{Var(p^*)} \tag{1}$$

where the second line follows from the exogeneity assumption that $E[\Delta\xi|\tilde{\boldsymbol{x}}] = 0$ and that, by assumption, $mc = x\gamma + \eta$. The exogeneity assumption implies that $\Delta\xi$ is orthogonal to the product-specific and time-specific terms in $mc$, as these are included in $\tilde{x}$ as fixed effects.

From Lemma 1, we can construct a consistent estimate of the unobserved demand shock as $\Delta\xi = \Delta\xi^{OLS} + \left(\alpha^{OLS} - \alpha\right) p^*$. We substitute this expression into $\frac{Cov(\Delta\xi,\mu)}{Var(p^*)}$, along with the above expression for $\left(\alpha^{OLS} - \alpha\right)$ to obtain

$$\frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)} = \frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \left(\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} + \frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)}\right)\frac{Cov(p^*, \mu)}{Var(p^*)}$$
$$\frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)}\left(1 - \frac{Cov(p^*, \mu)}{Var(p^*)}\right) = \frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\frac{Cov(p^*, \mu)}{Var(p^*)}$$
$$\frac{Cov\left(\Delta\xi, \mu\right)}{Var(p^*)} = \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} +$$
$$\frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\frac{Cov(p^*, \mu)}{Var(p^*)}$$

Plugging this into equation (1) yields

$$\alpha^{OLS} = \alpha + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \frac{\frac{Cov(p^*,\mu)}{Var(p^*)}}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}$$
$$\alpha^{OLS} = \alpha + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov\left(\Delta\xi^{OLS}, \mu\right)}{Var(p^*)} + \frac{1}{1 - \frac{Cov(p^*,\mu)}{Var(p^*)}}\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}$$

Thus, we obtain an expression for the plim of the OLS estimator in terms of the OLS residuals, the residualized prices, the markup, and the correlation between unobserved demand and cost shocks.

If the markup can be parameterized as a function of observable data, and if the correlation in unobserved shocks can be calibrated, we have a method to estimate $\alpha$ from the OLS regression. Under our supply and demand assumptions, $\mu = -\frac{1}{\alpha}\lambda$, and plugging in obtains the first equation of the proposition:

$$\alpha^{OLS} = \alpha - \frac{1}{\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)}}\frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)} + \alpha\frac{1}{\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)}}\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}.$$

The second equation in the proposition is obtained by rearranging terms. QED.

### D.3 Proof of Proposition 2 (Point Identification)

**Part (1)**. We first prove the sufficient condition, i.e., that under assumptions 1 and 2, $\alpha$ is the lower root of equation (11) if the following condition holds:

$$0 \leq \alpha^{OLS} \frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)} \tag{2}$$

Consider a generic quadratic, $ax^2 + bx + c$. The roots of the quadratic are $\frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$. Thus, if $4ac < 0$ and $a > 0$ then the upper root is positive and the lower root is negative. In equation (11), $a = 1$, and $4ac < 0$ if and only if equation (2) holds. Because the upper root is positive, $\alpha < 0$ must be the lower root, and point identification is achieved given knowledge of $Cov(\Delta\xi, \Delta\eta)$. QED.

**Part (2)**. In order to prove the necessary and sufficient condition for point identification, we first state and prove a lemma:

**Lemma 2.** *The roots of equation* (11) *are* $\alpha$ *and* $\frac{Cov(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}$.

**Proof of Lemma 2**. We first provide equation (11) for reference:

$$\begin{aligned} 0 &= \alpha^2 \\ &+ \left(\frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} - \alpha^{OLS}\right)\alpha \\ &+ \left(-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}\right) \end{aligned}$$

To find the roots, begin by applying the quadratic formula

$$\begin{aligned} (r_1, r_2) &= \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4AC}\right) \\ &= \frac{1}{2}\left(\alpha^{OLS} - \frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right) \\ &\pm \frac{1}{2}\sqrt{\left(\alpha^{OLS} + \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)^2 + 4\frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)} + \left(\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\right)^2 - 2\frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)}\left(\alpha^{OLS} - \frac{Cov(p^*, \lambda)}{Var(p^*)}\right)} \end{aligned} \tag{3}$$

Looking inside the radical, consider the first part: $\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\Delta\xi^{OLS},\lambda\right)}{Var(p^*)}$

$$\begin{aligned}
&\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\Delta\xi^{OLS},\lambda\right)}{Var(p^*)}\\
&=\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\Delta\xi-p^*(\alpha^{OLS}-\alpha),\lambda\right)}{Var(p^*)}\\
&=\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\Delta\xi,\lambda\right)}{Var(p^*)}-4\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
&=\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\Delta\xi,\lambda\right)}{Var(p^*)}-4\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}+\frac{Cov(\Delta\xi,-\frac{1}{\alpha}\lambda)}{Var(p^*)}\right)\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
&=\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\Delta\xi,\lambda\right)}{Var(p^*)}\left(1+\frac{1}{\alpha}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)-4\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} \qquad (4)
\end{aligned}$$

To simplify this expression, it is helpful to use the general decomposition of a firm's first-order condition, $p=mc+\mu$, where $mc$ is the marginal cost and $\mu$ is the markup. We can write $p=p^*+\hat{p}$, where $\hat{p}$ is the projection of $p$ onto the exogenous demand variables, $\tilde{x}$. By assumption, $c=x\gamma+\eta$. It follows that

$$\begin{aligned}
p^* &= x\gamma+\eta+\mu-\hat{p}\\
&= x\gamma+\eta-\frac{1}{\alpha}\lambda-\hat{p}
\end{aligned}$$

Therefore

$$Cov(p^*,\Delta\xi)=Cov(\Delta\xi,\Delta\eta)-\frac{1}{\alpha}Cov(\Delta\xi,\lambda)$$

and

$$\begin{aligned}
Cov(\Delta\xi,\lambda) &= -\alpha\left(Cov(p^*,\Delta\xi)-Cov(\Delta\xi,\Delta\eta)\right)\\
\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)} &= -\alpha\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}-\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right) \qquad (5)
\end{aligned}$$

Returning to equation (4), we can substitute using equation (5) and simplify:

$$
\begin{aligned}
&\left(\alpha^{OLS}+\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)^2+4\frac{Cov\left(\Delta\xi,\lambda\right)}{Var(p^*)}\left(1+\frac{1}{\alpha}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)-4\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
=&\left(\alpha^{OLS}\right)^2+\left(\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)^2+2\alpha^{OLS}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}-4\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
&+4\frac{Cov\left(\Delta\xi,\lambda\right)}{Var(p^*)}+4\frac{1}{\alpha}\frac{Cov\left(\Delta\xi,\lambda\right)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
=&\left(\alpha+\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)^2+\left(\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)^2+2\left(\alpha+\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}-4\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
&-4\alpha\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}-\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)-4\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}-\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
=&\left(\alpha+\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)^2+\left(\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)^2+2\left(\alpha+\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
&-4\alpha\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}\right)-4\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}\right)\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\\
=&\alpha^2+\left(\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)^2+\left(\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)^2+2\alpha\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\\
&-2\alpha\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}-2\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\\
=&\left(\left(\alpha+\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)-\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)^2+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}
\end{aligned}
$$

Now, consider the second part inside of the radical in equation (3):

$$
\begin{aligned}
&\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\left(\alpha^{OLS}-\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)\\
=&\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\left(\alpha+\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-\frac{1}{\alpha}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)}-\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)\\
=&\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-2\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2+2\frac{1}{\alpha}\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&-\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\left(\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}-\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\alpha-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}
\end{aligned}
$$

Combining yields a simpler expression for the terms inside the radical of equation (3):

$$
\begin{aligned}
&\left(\left(\alpha+\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)-\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)^2+4\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\\
&+\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\alpha-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&\left(\left(\alpha+\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}\right)-\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}\right)^2+\left(\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2\\
&+2\alpha\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}-2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\Delta\xi)}{Var(p^*)}+2\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\frac{Cov(p^*,\lambda)}{Var(p^*)}\\
=&\left(\alpha+\frac{Cov\left(p^*,\lambda\right)}{Var(p^*)}-\frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)}+\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2
\end{aligned}
$$

Plugging this back into equation (3), we have:

$$\begin{aligned}(r_1, r_2) &= \frac{1}{2}\left(\alpha^{OLS} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right.\\ &\left.\pm\sqrt{\left(\alpha + \frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2}\right)\\ &= \frac{1}{2}\left(\alpha + \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right.\\ &\left.\pm\sqrt{\left(\alpha + \frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2}\right)\end{aligned}$$

The roots are given by

$$\frac{1}{2}\left(\alpha + \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right) + \frac{1}{2}\left(\alpha + \frac{Cov\,(p^*,\lambda)}{Var(p^*)} - \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right) = \alpha$$

and

$$\frac{1}{2}\left(\alpha + \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right) + \frac{1}{2}\left(-\alpha - \frac{Cov\,(p^*,\lambda)}{Var(p^*)} + \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right) = \frac{Cov\,(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}$$

which completes the proof of the intermediate result. QED.

**Part (3)**. Consider the roots of equation (11), $\alpha$ and $\frac{Cov(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}$. The price parameter $\alpha$ may or may not be the lower root.[2] However, $\alpha$ is the lower root iff

$$\begin{aligned}
\alpha &< \frac{Cov(p^*,\Delta\xi)}{Var(p^*)} - \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
\alpha &< -\alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} + \alpha\frac{Cov(p^*,-\frac{1}{\alpha}\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
\alpha &< -\alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} + \alpha\frac{Cov(p^*,p^*-c)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
\alpha &< \alpha\frac{Var(p^*)}{Var(p^*)} - \alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} - \alpha\frac{Cov(p^*,\Delta\eta)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
0 &< -\alpha\frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} - \alpha\frac{Cov(p^*,\Delta\eta)}{Var(p^*)} - \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \\
0 &< \frac{Cov(p^*,-\frac{1}{\alpha}\Delta\xi)}{Var(p^*)} + \frac{Cov(p^*,\Delta\eta)}{Var(p^*)} + \frac{1}{\alpha}\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}
\end{aligned}$$

The third line relies on the expression for the markup, $p-c = -\frac{1}{\alpha}\lambda$. The final line holds because $\alpha < 0$ so $-\alpha > 0$. It follows that $\alpha$ is the lower root of equation (11) iff

$$-\frac{1}{\alpha}\frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} \leq \frac{Cov\left(p^*,-\frac{1}{\alpha}\Delta\xi\right)}{Var(p^*)} + \frac{Cov\left(p^*,\Delta\eta\right)}{Var(p^*)}$$

in which case $\alpha$ is point identified given knowledge of $Cov(\Delta\xi,\Delta\eta)$. QED.

## D.4 Proof of Proposition 3 (Approximation)

The demand and supply equations are given by:

$$\begin{aligned}
h &= \alpha p + x\beta + \xi \\
p &= x\gamma - \frac{1}{\alpha}\frac{dh}{dq}q + \eta
\end{aligned}$$

[2]Consider that the first root is the upper root if

$$\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} > 0$$

because, in that case,

$$\sqrt{\left(\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2} = \alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}$$

When $\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)} < 0$, then $\sqrt{\left(\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,\Delta\xi)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)^2} = -\left(\alpha + \frac{Cov\left(p^*,\lambda\right)}{Var(p^*)} - \frac{Cov\left(p^*,\Delta\xi\right)}{Var(p^*)} + \frac{Cov(\Delta\xi,\Delta\eta)}{Var(p^*)}\right)$, and the first root is then the lower root (i.e., minus the negative value).

where $\frac{dh}{dq}q = \lambda$ for single-product firms. For ease of exposition, here we slightly abuse notation and assume that $\xi$ and $\eta$ are exogenous (and $x$ includes dummy variables to absorb fixed effects). Using an first-order expansion of $h$ about $q$, $h \approx \overline{h} + \overline{\frac{dh}{dq}}(q - \overline{q})$, we can solve for a reduced-form for $p$ and $h$. It follows that

$$\begin{aligned}\overline{h} + \overline{\frac{dh}{dq}}(q - \overline{q}) &\approx \alpha p + x\beta + \xi \\ \overline{\frac{dh}{dq}}q &\approx \alpha p + x\beta + \xi - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\end{aligned}$$

Letting $\frac{dh}{dq}q = \frac{\tilde{dh}}{dq}q + \overline{\frac{dh}{dq}}q$, we have

$$\begin{aligned}p &\approx x\gamma - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q - \frac{1}{\alpha}\left(\alpha p + x\beta + \xi - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\right) + \eta \\ 2p &\approx x\gamma + \frac{1}{\alpha}x\beta - \frac{1}{\alpha}\overline{h} + \frac{1}{\alpha}\overline{\frac{dh}{dq}}\overline{q} - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q + \eta + \frac{1}{\alpha}\xi \\ p &\approx \frac{1}{2}\left(x\gamma + \frac{1}{\alpha}x\beta - \frac{1}{\alpha}\overline{h} + \frac{1}{\alpha}\overline{\frac{dh}{dq}}\overline{q} - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q + \eta + \frac{1}{\alpha}\xi\right).\end{aligned}$$

Let $H^*$ denote the residual from a regression of $\frac{\tilde{dh}}{dq}q$ on $\boldsymbol{x}$. Then $p^*$, the residual from a regression of $p$ on $x$, is

$$p^* \approx \frac{1}{2}\left(\eta + \frac{1}{\alpha}\xi - \frac{1}{\alpha}H^*\right). \tag{6}$$

Likewise, as $h - \overline{h} + \overline{\frac{dh}{dq}}\overline{q} \approx \overline{\frac{dh}{dq}}q$,

$$p \approx x\gamma - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q - \frac{1}{\alpha}\overline{\frac{dh}{dq}}q + \eta$$

$$\begin{aligned}h &\approx \alpha\left(x\gamma - \frac{1}{\alpha}\frac{\tilde{dh}}{dq}q - \frac{1}{\alpha}\overline{\frac{dh}{dq}}q + \eta\right) + x\beta + \xi \\ h &\approx \alpha x\gamma + x\beta - \frac{\tilde{dh}}{dq}q - \left(h - \overline{h} + \overline{\frac{dh}{dq}}\overline{q}\right) + \alpha\eta + \xi \\ 2h &\approx \alpha x\gamma + x\beta - \frac{\tilde{dh}}{dq}q + \overline{h} - \overline{\frac{dh}{dq}}\overline{q} + \alpha\eta + \xi.\end{aligned}$$

Similarly, the residual from a regression of $h$ on $x$ is:

$$h^* \approx \frac{1}{2}\left(\alpha\eta + \xi - H^*\right). \tag{7}$$

Equations (6) and (7) provide an approximation for $\alpha$.

$$
\begin{aligned}
-\sqrt{\frac{Var(h^*)}{Var(p^*)}} &\approx -\sqrt{\frac{\frac{1}{4}Var\left(\alpha\eta+\xi-H^*\right)}{\frac{1}{4}Var\left(\eta+\frac{1}{\alpha}\xi-\frac{1}{\alpha}H^*\right)}} \\
&\approx -\sqrt{\frac{\alpha^2 Var\left(\eta+\frac{1}{\alpha}\xi-\frac{1}{\alpha}H^*\right)}{Var\left(\eta+\frac{1}{\alpha}\xi-\frac{1}{\alpha}H^*\right)}} \\
&\approx \alpha
\end{aligned}
$$

QED.

## D.5 Proof of Lemma 1 (Monotonicity in $Cov(\Delta\xi, \Delta\eta)$)

We return to the quadratic formula for the proof. The lower root of a quadratic $ax^2 + bx + c$ is $L \equiv \frac{1}{2}\left(-b - \sqrt{b^2 - 4ac}\right)$. In our case, $a = 1$.

We wish to show that $\frac{\partial L}{\partial \gamma} < 0$, where $\gamma = Cov(\Delta\xi, \Delta\eta)$. We evaluate the derivative to obtain

$$
\frac{\partial L}{\partial \gamma} = -\frac{1}{2}\left(1 + \frac{b}{(b^2-4c)^{\frac{1}{2}}}\right)\frac{\partial b}{\partial \gamma}.
$$

We observe that, in our setting, $\frac{\partial b}{\partial \gamma} = \frac{1}{Var(p^*)}$ is always positive. Therefore, it suffices to show that

$$
1 + \frac{b}{(b^2-4c)^{\frac{1}{2}}} > 0. \tag{8}
$$

We have two cases. First, when $c < 0$, we know that $\left|\frac{b}{(b^2-4c)^{\frac{1}{2}}}\right| < 1$, which satisfies equation (8). Second, when $c > 0$, it must be the case that $b > 0$ also. Otherwise, both roots are positive, invalidating the model. When $b > 0$, it is evident that the left-hand side of equation (8) is positive. This demonstrates monotonicity.

Finally, we obtain the range of values for $L$ by examining the limits as $\gamma \to \infty$ and $\gamma \to -\infty$. From the expression for $L$ and the result that $\frac{\partial b}{\partial \gamma}$ is a constant, we obtain

$$
\begin{aligned}
\lim_{\gamma \to -\infty} L &= 0 \\
\lim_{\gamma \to \infty} L &= -\infty
\end{aligned}
$$

When $c < 0$, the domain of the quadratic function is $(-\infty, \infty)$, which, along with monotonicity, implies the range for $L$ of $(0, -\infty)$. When $c > 0$, the domain is not defined on the interval $(-2\sqrt{c}, 2\sqrt{c})$, but $L$ is equal in value at the boundaries of the domain. QED.

Additionally, we note that the upper root, $U \equiv \frac{1}{2}\left(-b + \sqrt{b^2 - 4ac}\right)$ is increasing in $\gamma$. When the upper root is a valid solution (i.e., negative), it must be the case that $c > 0$ and $b > 0$, and it is straightforward to follow the above arguments to show that $\frac{\partial U}{\partial \gamma} > 0$ and that the range of the upper root is $[-\frac{1}{2}b, 0)$.

### D.6 Proof of Proposition 4 (Covariance Bound)

The proof involves an application of the quadratic formula. Any generic quadratic, $ax^2 + bx + c$, with roots $\frac{1}{2}\left(-b \pm \sqrt{b^2 - 4ac}\right)$, admits a real solution if and only if $b^2 \geq 4ac$. Given the formulation of equation (11), real solutions satisfy the condition:

$$\left(\frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} - \alpha^{OLS}\right)^2 \geq 4\left(-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}\right).$$

As $a = 1$, a solution is always possible if $c < 0$. This is the sufficient condition for point identification from the text. If $c \geq 0$, it must be the case that $b \geq 0$; otherwise, both roots are positive. Therefore, a real solution is obtained if and only if $b \geq 2\sqrt{c}$, that is

$$\left(\frac{Cov(p^*, \lambda)}{Var(p^*)} + \frac{Cov(\Delta\xi, \Delta\eta)}{Var(p^*)} - \alpha^{OLS}\right) \geq 2\sqrt{-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}}.$$

Solving for $Cov(\Delta\xi, \Delta\eta)$, we obtain the model-based bound,

$$Cov(\Delta\xi, \Delta\eta) \geq Var(p^*)\alpha^{OLS} - Cov(p^*, \lambda) + 2Var(p^*)\sqrt{-\alpha^{OLS}\frac{Cov(p^*, \lambda)}{Var(p^*)} - \frac{Cov\left(\Delta\xi^{OLS}, \lambda\right)}{Var(p^*)}}.$$

This bound exists if the expression inside the radical is positive, which is the case if and only if the sufficient condition for point identification from Proposition 2 fails. QED.

### D.7 Proof of Corollary 1 (Marginal Cost Functions)

Under the semi-linear marginal cost schedule of equation (26) and the assumption that $Cov(\Delta\xi, \Delta\eta) = 0$, the plim of the OLS estimator is equal to

$$\text{plim } \hat{\alpha}^{OLS} = \alpha + \frac{Cov(\Delta\xi, g)}{Var(p^*)} - \frac{1}{\alpha}\frac{Cov\left(\Delta\xi, \lambda\right)}{Var(p^*)}.$$

This is obtain directly by plugging in the first–order condition for $p$: $Cov(p^*, \Delta\xi) = Cov(g(q;\tau) + \eta - \frac{1}{\alpha}\lambda - \hat{p}, \Delta\xi) = Cov(\Delta\xi, g) - \frac{1}{\alpha}Cov(\Delta\xi, \lambda)$ under the assumptions. Next, we re-express the terms, including the unobserved demand shocks, in terms of OLS residuals. As shown by Lemma 1, the estimated residuals are given by $\Delta\xi^{OLS} = \Delta\xi + \left(\alpha - \alpha^{OLS}\right)p^*$. As $\alpha - \alpha^{OLS} =$

$\frac{1}{\alpha}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,g)}{Var(p^*)}$, we obtain $\Delta\xi^{OLS} = \Delta\xi + \left(\frac{1}{\alpha}\frac{Cov(\Delta\xi,\lambda)}{Var(p^*)} - \frac{Cov(\Delta\xi,g)}{Var(p^*)}\right)p^*$. This implies

$$Cov\left(\Delta\xi^{OLS},\lambda\right) = \left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\Delta\xi,\lambda) - \frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\Delta\xi,g)$$
$$Cov\left(\Delta\xi^{OLS},g(q;\tau)\right) = \frac{1}{\alpha}\frac{Cov(p^*,g)}{Var(p^*)}Cov\left(\Delta\xi,\lambda\right) + \left(1-\frac{Cov(p^*,g)}{Var(p^*)}\right)Cov(\Delta\xi,g)$$

We write the system of equations in matrix form and invert to solve for the covariance terms that include the unobserved demand shock:

$$\begin{bmatrix} Cov(\Delta\xi,\lambda) \\ Cov(\Delta\xi,g) \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} & -\frac{Cov(p^*,\lambda)}{Var(p^*)} \\ \frac{1}{\alpha}\frac{Cov(p^*,g)}{Var(p^*)} & 1-\frac{Cov(p^*,g)}{Var(p^*)} \end{bmatrix}^{-1} \begin{bmatrix} Cov(\Delta\xi^{OLS},\lambda) \\ Cov(\Delta\xi^{OLS},g) \end{bmatrix}$$

where

$$\begin{bmatrix} 1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} & -\frac{Cov(p^*,\lambda)}{Var(p^*)} \\ \frac{1}{\alpha}\frac{Cov(p^*,g)}{Var(p^*)} & 1-\frac{Cov(p^*,g)}{Var(p^*)} \end{bmatrix}^{-1} =$$
$$\frac{1}{1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g)}{Var(p^*)}} \begin{bmatrix} 1-\frac{Cov(p^*,g)}{Var(p^*)} & \frac{Cov(p^*,\lambda)}{Var(p^*)} \\ -\frac{1}{\alpha}\frac{Cov(p^*,g)}{Var(p^*)} & 1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} \end{bmatrix}.$$

Therefore, we obtain the relations

$$Cov(\Delta\xi,\lambda) = \frac{\left(1-\frac{Cov(p^*,g)}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},\lambda) + \frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\Delta\xi^{OLS},g)}{1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g)}{Var(p^*)}}$$
$$Cov(\Delta\xi,g) = \frac{-\frac{1}{\alpha}\frac{Cov(p^*,g)}{Var(p^*)}Cov(\Delta\xi^{OLS},\lambda) + \left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},g)}{1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g)}{Var(p^*)}}.$$

In terms of observables, we can substitute in for $Cov(\Delta\xi,g) - \frac{1}{\alpha}Cov\left(\Delta\xi,\lambda\right)$ in the plim of the OLS estimator and simplify:

$$\begin{aligned}
&\left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)} - \frac{Cov(p^*,g)}{Var(p^*)}\right)\left(Cov(\Delta\xi,g) - \frac{1}{\alpha}Cov\left(\Delta\xi,\lambda\right)\right) \\
&= -\frac{1}{\alpha}\frac{Cov(p^*,g)}{Var(p^*)}Cov(\Delta\xi^{OLS},\lambda) + \left(1+\frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},g) \\
&\quad - \frac{1}{\alpha}\left(1-\frac{Cov(p^*,g)}{Var(p^*)}\right)Cov(\Delta\xi^{OLS},\lambda) - \frac{1}{\alpha}\frac{Cov(p^*,\lambda)}{Var(p^*)}Cov(\Delta\xi^{OLS},g) \\
&= Cov(\Delta\xi^{OLS},g) - \frac{1}{\alpha}Cov(\Delta\xi^{OLS},\lambda).
\end{aligned}$$

Thus, we obtain an expression for the probability limit of the OLS estimator,

$$\text{plim}\hat{\alpha}^{OLS} = \alpha - \frac{\frac{Cov(\Delta\xi^{OLS},\lambda)}{Var(p^*)} - \alpha\frac{Cov(\Delta\xi^{OLS},g)}{Var(p^*)}}{\alpha + \frac{Cov(p^*,\lambda)}{Var(p^*)} - \alpha\frac{Cov(p^*,g)}{Var(p^*)}},$$

and the following quadratic $\alpha$.

$$\begin{aligned} 0 = & \left(1 - \frac{Cov(p^*, g)}{Var(p^*)}\right)\alpha^2 \\ & + \left(\frac{Cov(p^*, \lambda)}{Var(p^*)} - \hat{\alpha}^{OLS} + \frac{Cov(p^*, g)}{Var(p^*)}\hat{\alpha}^{OLS} + \frac{Cov(\Delta\xi^{OLS}, g)}{Var(p^*)}\right)\alpha \\ & + \left(-\frac{Cov(p^*, \lambda)}{Var(p^*)}\hat{\alpha}^{OLS} - \frac{Cov(\Delta\xi^{OLS}, \lambda)}{Var(p^*)}\right). \end{aligned}$$

QED.