Robust Monopoly Regulation
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Online Appendix

D Derivation of q_α, r_α, and s_α

Derivation of q_α and r_α. Let $T^u = q(1-f_\alpha)\bar{v} - qp\log q$ and $T^o = q(f_\alpha\bar{v} - p)$ denote the two terms in the definition of r_α:

$$r_\alpha = \max_{(q,p)\in[0,1]\times[0,f_\alpha\bar{v}]} \min\left\{q(1-f_\alpha)\bar{v} - qp\log q,\ q(f_\alpha\bar{v}-p)\right\}$$

It is readily verified that T^u increases in price p whereas T^o decreases in p. Moreover, $T^u \leqslant T^o$ when $p=0$ and $T^u \geqslant T^o$ when $p = f_\alpha\bar{v}$. Hence, for any fixed q, $\min\{T^u, T^o\}$ is achieved by the value of p which satisfies $T^u = T^o$, that is:

$$p = \frac{\alpha f_\alpha \bar{v}}{1-\log q}.$$

Substituting this value of p into T^u and T^o, we have

$$T^u = T^o = qf_\alpha\bar{v}\left(1 - \frac{\alpha}{1-\log q}\right). \tag{18}$$

This term (18) is concave in q and increases in q at $q=0$. Moreover, if $\alpha \leqslant 1/2$, this term (18) increases in q also at $q=1$. In this case, the maximum is achieved at $q=1$ so:

$$q_\alpha = 1,\ \text{and}\ r_\alpha = \frac{1-\alpha}{2-\alpha}\bar{v} = (1-f_\alpha)\bar{v}.$$

If $\alpha > 1/2$, the term (18) decreases in q at $q=1$ so the maximum is achieved at an interior q. In this case, q_α is given by setting the derivative of (18) with respect to q to zero, so:

$$q_\alpha = e^{1-\frac{\alpha+\sqrt{\alpha(\alpha+4)}}{2}},\ \text{and}\ r_\alpha = \frac{\left(2+\alpha-\sqrt{\alpha(\alpha+4)}\right)e^{1-\frac{\alpha+\sqrt{\alpha(\alpha+4)}}{2}}}{2(2-\alpha)}\bar{v}.$$

Derivation of s_α. The value of s_α is given by:

$$\begin{aligned} s_\alpha &= \left(\sup\{q(f_\alpha \bar{v} - p) : q(1-f_\alpha)\bar{v} - qp\log q > r_\alpha, (q,p) \in [0,1]\times[0, f_\alpha \bar{v}]\}\right)^+ \\ &= \left(\sup\{T^o : T^u > r_\alpha, (q,p) \in [0,1]\times[0, f_\alpha \bar{v}]\}\right)^+ . \end{aligned}$$

We first explain that the value of s_α is at most r_α. Given the definition of r_α, $\min\{T^u, T^o\} \leqslant r_\alpha$ for any $(q,p) \in [0,1]\times[0, f_\alpha \bar{v}]$. Hence, for any $(q,p) \in [0,1]\times[0, f_\alpha \bar{v}]$ such that $T^o > r_\alpha$, it holds that $T^u \leqslant r_\alpha$. The value of s_α is the supremum of such T^u, so it is at most r_α.

We next argue that for $\alpha > 1/2$, the value of s_α equals r_α. Consider the quantity-price pair $\left(q_\alpha, \frac{\alpha f_\alpha \bar{v}}{1-\log q_\alpha} + \varepsilon\right)$, which is in $[0,1]\times[0, f_\alpha \bar{v}]$ for small enough $\varepsilon > 0$. The value of T^u under this pair is strictly above r_α, because (i) T^u equals r_α under the pair $\left(q_\alpha, \frac{\alpha f_\alpha \bar{v}}{1-\log q_\alpha}\right)$, and (ii) T^u is strictly increasing in p for any $q \in (0,1)$. As ε goes to zero, the value of T^o under the pair $\left(q_\alpha, \frac{\alpha f_\alpha \bar{v}}{1-\log q_\alpha} + \varepsilon\right)$ goes to r_α.

We next consider the case in which $\alpha \leqslant 1/2$. The condition $T^u > r_\alpha$ is satisfied if and only if $q \in (0,1)$ and

$$p > \frac{\frac{(\alpha-1)\bar{v}}{\alpha-2} - \frac{r_\alpha}{q}}{\log q} = (1-f_\alpha)\bar{v}\frac{1-q}{-q\log q}. \tag{19}$$

The lower bound in (19) decreases in q, so it is at least $(1-f_\alpha)\bar{v}$. Since $(1-f_\alpha)\bar{v} = f_\alpha \bar{v}$ for $\alpha = 0$, it follows that there exists no $(q,p) \in [0,1]\times[0, f_\alpha \bar{v}]$ such that $T^u > r_\alpha$. Hence, for $\alpha = 0$, s_α equals zero. For $\alpha \in (0, 1/2]$, since T^o decreases in price p, the supremum of T^o is achieved when p approaches the lower bound in (19). Substituting this lower bound into T^o, we have:

$$T^o = \frac{\bar{v}((1-\alpha)(1-q) + q\log q)}{(2-\alpha)\log q}, \text{ for } q \in (0,1).$$

This term is convex in q and equals zero when $q = 0$, so the supremum of T^o is achieved when q approaches 1, and is equal to:

$$\frac{\alpha}{2-\alpha}\bar{v} = \alpha f_\alpha \bar{v}, \text{ for } \alpha \in (0, 1/2].$$

E An example that illustrates the role of $(-d(q))$

Suppose that $\overline{P} \equiv 1$ and that $\underline{P}(z) = 1$ if $z \leqslant b$ and $\underline{P}(z) = 0$ if $z > b$ for some parameter $b \in (0, 1)$. Suppose that $\alpha = 0$, so $f_0 = 1/2$. Then, $\underline{d}(q) = q/2$ for $q \leqslant b$ and $\underline{d}(q) = b/2$ for $q > b$. There is an optimal policy with s being zero. Substituting $s = 0$ and $\underline{d}(q)$ into the optimal policy (9), we reduce the policy to:

$$\rho(q,p) = \begin{cases} \frac{q}{2}, & \text{if } q \leqslant b, \\ \frac{b}{2} + \min\left\{p, \frac{1}{2}\right\}(q-b), & \text{if } q > b. \end{cases}$$

According to this policy, for the first b units the firm produces, its average revenue is $1/2$, which is a fraction f_0 of the value to a consumer. For the remaining units, about which the regulator does not know the value to a consumer, the firm gets the market price p per unit, capped by a fraction f_0 of the highest possible value to a consumer. If the firm chooses $(q, p) = (1, 0)$, the total consumer value $\Theta(1, 0)$ that the firm proves it has created is b. However, the regulator only gives the firm $b/2$ instead of $\min\left\{f_0\overline{V}(1), \Theta(1, 0)\right\} = \min\{1/2, b\}$.