Hedging When Applying: Simultaneous Search with Correlation

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Ann is applying to a portfolio of schools, and is uncertain about which would admit her.

She faces idiosyncratic and correlated uncertainty:

- Centralized Admissions: Many school and university systems stipulate that students first apply to colleges and universities and then take a test whose results determine admissions.

  (China, Ghana, Kenya, Mexico, Turkey, and the United Kingdom)

- Decentralized Admissions: She does not know where she stands in the pool of applicants. Also application essay, extracurriculars, letters of recommendation affect admissions across schools.

Rejection from College i raises conditional odds of rejection from more selective college.

Ann may hedge by applying to reaches, matches, and safeties.

Diversification features in practice (and appears intuitive):

> *Before you start your applications, strengthen your list to include three reach colleges, two match colleges, and one safety college to ensure you apply to a balanced list of schools that match your academic abilities. – The College Board*

But not in theory: Leading analysis builds on **Simultaneous Search** by Chade and Smith'06.

- Stochastic independence: No correlation across colleges in admissions prospects.
- No Safety Schools: Ann foregoes colleges with high admissions probabilities.
- Upward Expansion: Optimal $(k+1)$-portfolio nests optimal k-portfolio and expands upwards.

By the same token, "safety schools" can only be understood if acceptances are not independent.…This remains a challenging but important research avenue.

– Chade, Eeckhout, and Smith (2017)

What We Do

Develop simple approach to simultaneous search with correlation: analysis is graphical and elementary.

Use that approach to study how optimal portfolio varies with beliefs and risk attitudes.

Main Findings:

- Applicant will apply to safety schools.
- Optimal $(k+1)$-portfolio includes both more and less selective colleges than the most and least selective colleges in the optimal k-portfolio.
- Optimal solution can be found generally using a time-efficient algorithm.
- Results qualitatively remain mixing the independent and common score approach.

Example

Portfolio Problem with A Common Score

Ann can apply to a subset of Colleges: Reach, Match, and Safety.

How admissions works:

- Ann first chooses her portfolio.
- Then there is a common score, s, which is uniformly distributed on $[0, 1]$.
- College i accepts an application from Ann if her score exceeds its threshold τ_i.
- If every school rejects Ann, she obtains her outside option, which offers a utility of 0.

Corresponds to centralized processes where s is a test score that determines admissions.

For decentralized admissions: score may reflect where Ann is viewed relative to the pool.

College Characteristics

	Reach	**Match**	**Safety**
Utility (u_i)	1	0.45	0.25
Score Threshold (τ_i)	0.78	0.5	0.125
Admissions Probability	0.22	0.5	0.875

Compare optimal single-college and two-college portfolios.

Optimal Single-College Portfolio

	Reach	Match	Safety
Utility (u_i)	1	0.45	0.25
Score Threshold (τ_i)	0.78	0.5	0.125
Admissions Probability (p_i)	0.22	0.5	0.875

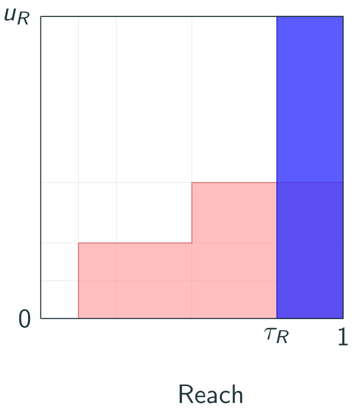
Optimal Single-College Portfolio:

- Reach: 1×0.22.
- Match: 0.45×0.5.
- Safety: 0.25×0.875.

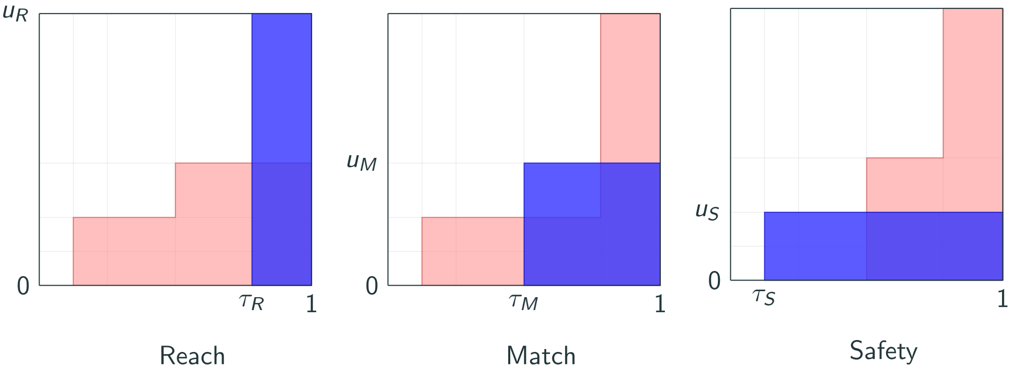
Optimal Portfolios = Coverage Problem

One may view a portfolio as a coverage problem:

- Consider a 1×1 square.
- A single-college portfolio covers a rectangle.
- Optimal portfolio = Largest rectangle



Optimal Single-College Portfolios



Optimal Two-College Portfolios

Suppose Ann can apply to two colleges:

(i) Match and Reach

(ii) Match and Safety

(iii) Reach and Safety

Will see that she picks (iii).

Approaches

Elementary reasoning + Pictures.

Bad-News Effect: *What is a good backup if I am rejected by my first choice?*

Risk-Loving Effect: *What is a good college to shoot for given that I have a backup?*

$\{$Reach,Safety$\}$ vs. $\{$Reach,Match$\}$

Score Thresholds

- Reach: 0.78
- Match: 0.5
- Safety: 0.125

For both portfolios: Ann enrolls in R if $s \geq 0.78$.

Decision is relevant only if $s < 0.78$.

Payoff from Portfolio {R,M}:

$$\Pr(\text{Match accepts} | s < 0.78) \times u_M$$

Payoff from Portfolio {R,S}:

$$\Pr(\text{Safety accepts} | s < 0.78) \times u_S$$

$\{$Reach,Safety$\}$ vs. $\{$Reach,Match$\}$

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- Reach: 0.78
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Decision is relevant only if $s < 0.78$.

Payoff from Portfolio $\{$R,M$\}$:

$$\Pr(\text{Match accepts} | s < 0.78) \times u_M \approx 0.16.$$

Payoff from Portfolio $\{$R,S$\}$:

$$\Pr(\text{Safety accepts} | s < 0.78) \times u_S \approx 0.21.$$

{Reach,Safety} vs. {Reach,Match}

Match dominates Safety ex ante:

$$u_S \times \Pr(\text{S accepts}) < u_M \times \Pr(\text{M accepts}).$$

But Safety is relatively better once Ann conditions on being rejected by Reach:

$$u_S \times \Pr(\text{S accepts} \mid \text{R rejects}) > u_M \times \Pr(\text{M accepts} \mid \text{R rejects}).$$

Key idea:
Rejection by the Reach is bad news for Ann's odds at other schools but worse news for the Match:

$$\frac{\Pr(\text{M accepts} \mid \text{R rejects})}{\Pr(\text{S accepts} \mid \text{R rejects})} < \frac{u_S}{u_M} < \frac{\Pr(\text{M accepts})}{\Pr(\text{S accepts})}.$$

The Bad-News Effect

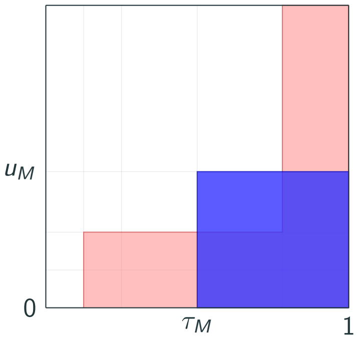
Key idea:

Rejection by Reach is bad news for Ann's odds at other schools but worse news for Match:

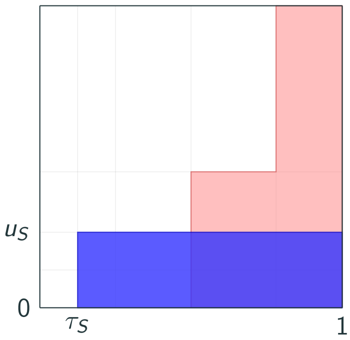
$$\frac{\Pr(\text{M accepts} \mid \text{R rejects})}{\Pr(\text{S accepts} \mid \text{R rejects})} < \frac{u_S}{u_M} < \frac{\Pr(\text{M accepts})}{\Pr(\text{S accepts})}.$$

Rejection from top choice $\Rightarrow$ Beliefs $\downarrow$ in LR-sense $\Rightarrow$ Less aggressive choice.

Using Coverage Problem to see why {R,S} beats {R,M}

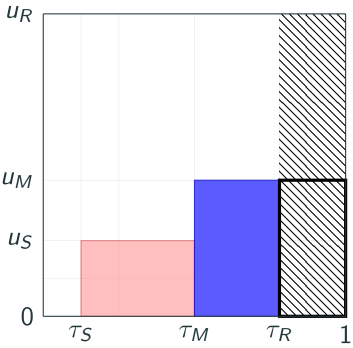


Match

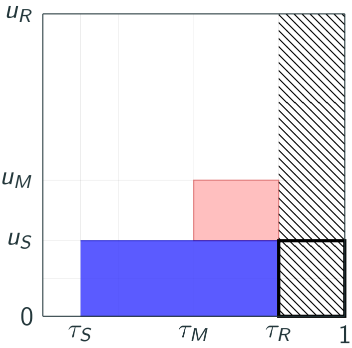


Safety

Using Coverage Problem to see why {R,S} beats {R,M}



Match, without overlap



Safety, without overlap

{Reach,Safety} vs. {Match,Safety}

Score Thresholds

- Reach: 0.78
- Match: 0.5
- Safety: 0.125

For both portfolios: $s < 0.125 \implies$ Ann is rejected by all colleges.

Decision is relevant only if $s \geq 0.125$.

Payoff from Portfolio {R,S}:

$$u_S + \Pr(s \geq 0.78 | s \geq 0.125) \times (u_R - u_S)$$

Payoff from Portfolio {M,S}:

$$u_S + \Pr(s \geq 0.5 | s \geq 0.125) \times (u_M - u_S)$$

{Reach,Safety} vs. {Match,Safety}

Score Thresholds

- Reach: 0.78
- Match: 0.5
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For both portfolios: $s < 0.125 \implies$ Ann is rejected by all colleges.

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Payoff from Portfolio {M,S}:

$$u_S + \Pr(s \geq 0.5 | s \geq 0.125) \times (u_M - u_S)$$

{Reach,Safety} vs. {Match,Safety}

If Ann applies to only one college, she obtains a payoff of 0 if rejected $\Longrightarrow$ R is too risky.

If Ann applies to two colleges, she has a backup if rejected $\Longrightarrow$ She is willing to gamble on R.

Better outside option

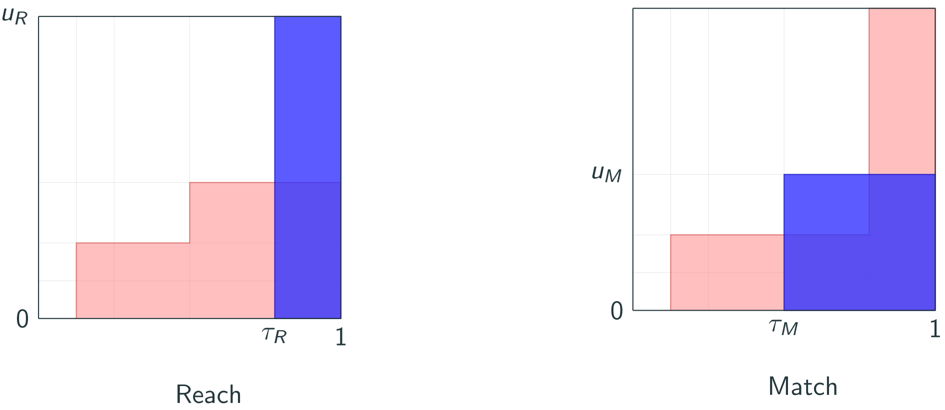
$\Downarrow$

More risk-loving in Arrow-Pratt sense

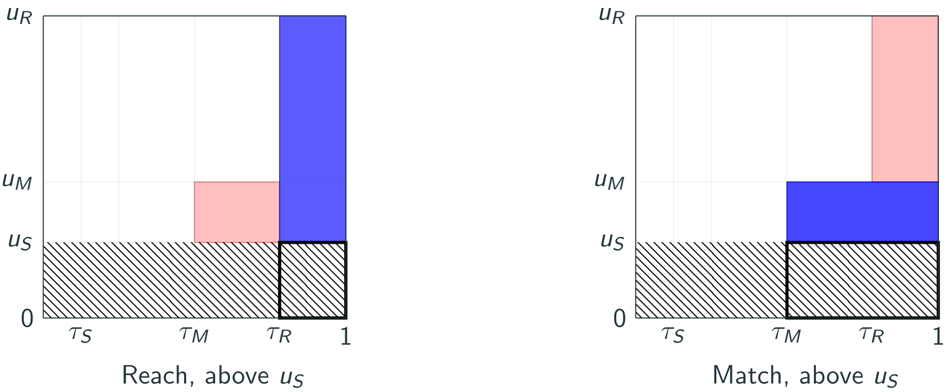
$\Downarrow$

More aggressive choices

Using Coverage Problem to see why {R,S} beats {M,S}



Using Coverage Problem to see why {R,S} beats {M,S}

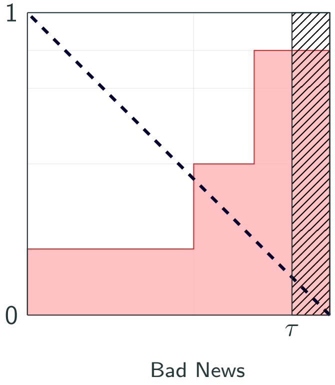


Duality between Beliefs and Risk Attitudes

Analysis uses two ideas:

- Pessimistic beliefs $\rightarrow$ less aggressive choices.
- Better outside option $\rightarrow$ more aggressive choices.

These are in fact the dual of each other.



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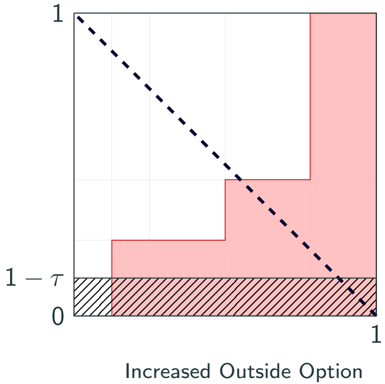


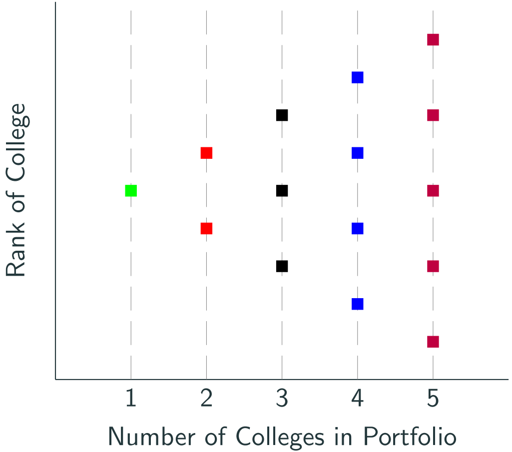
Diversification

Optimal portfolio of 2 colleges expands in both directions from optimal single-college portfolio.

Moreover, excludes optimal single-college portfolio.

Generally: if $k' > k$:

- Top k colleges in optimal-k' portfolio more aggressive than optimal k-portfolio.
- Bottom k colleges in optimal-k' portfolio less aggressive than optimal k-portfolio.



Comparison to Independent Success (Chade & Smith)

Standard model studies stochastically independent admissions:

$$\Pr(\text{Accepted by MIT} \mid \text{Rejected by Penn State}) = \Pr(\text{Accepted by MIT}).$$

In this setting, Chade & Smith show that applicants do not apply to safety schools.

What accounts for this difference?

- Independent admissions probabilities nullifies downward force:

  Rejection from top choice $\implies$ ~~Bad~~ No news on odds at backup schools.

- Risk-loving effect continues to push up: More backups $\implies$ Go for selective top choices.

Model

Primitives

Ann applies to a subset of colleges $C := \{1, \ldots, n\}$ ordered in her preference: $u_1 \geq u_2 \geq \ldots \geq u_n$.

Each college has its own threshold $\tau_i \in [0, 1]$.

Ann chooses a portfolio $P \subseteq C$; after she chooses a portfolio, a score s is determined.

Ann's application to College i accepted if $s \geq \tau_i$.

Ann observes which colleges accept her and enrolls in her favorite among them.

If she is rejected by all colleges, she obtains her outside option u_o.

Primitives

Definition

A utility assessment is $U := (u_o; u_1, \ldots, u_n)$.

Ann's prior on s is atomless cdf F with strictly positive density on its support.

Value of Portfolio

$$V(P, U, F) := \int_0^1 \max\{\max_{\{i \in P : s \geq \tau_i\}} u_i, u_o\} \, dF$$

Cost of portfolio is $\phi(|P|)$ where $\phi(0) = 0$ and ϕ is non-decreasing.

Simplifying Assumptions

No dominated colleges or replicas: $i < j \implies u_i > u_j$ and $\tau_i > \tau_j$.

Non-redundancy: Ann does not apply to colleges that she believes would reject her w.p. 1 or she would reject w.p. 1.

Genericity: Every pair of feasible portfolios has distinct values.

A (sub)-Optimization Problem

Much of our analysis leans on identifying the optimal portfolio of k colleges:

$$P^*(k, U, F) := \underset{\{|P| \leq k\}}{\arg\max}\, V(P, U, F).$$

Our analysis identifies how P^* varies with utility assessment U, beliefs F, and size of portfolio k.

Ordering Portfolios by Aggressiveness

Definition

Portfolio P is more aggressive than $\tilde{P}$, denoted $P \succcurlyeq_A \tilde{P}$, if any of the following is true:

- $|P| = |\tilde{P}|$ and the k^{th} ranked college in P is higher ranked than that for $\tilde{P}$.
- $|P| < |\tilde{P}|$ and more aggressive than the top $|P|$ items in $\tilde{P}$.
- $|P| > |\tilde{P}|$ and the bottom $|\tilde{P}|$ items of P are higher ranked than $\tilde{P}$.

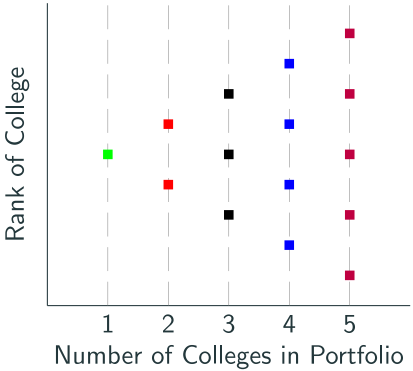
Ordering Portfolios by Dispersiveness

Definition

Portfolio P is more dispersed than $\tilde{P}$, denoted $P \succcurlyeq_D \tilde{P}$, if all of the following are true:

- Portfolio P has more colleges: $|P| \geq |\tilde{P}|$.
- The $|\tilde{P}|$ most selective colleges in P is more aggressive than $\tilde{P}$.
- The $|\tilde{P}|$ least selective colleges in P is less aggressive than $\tilde{P}$.

Ordering Portfolios by Dispersiveness



The Bad News Effect

As Ann applies to colleges, she realizes that her score will be lower than she initially anticipated.

How do her beliefs about her admissions chances influence her portfolio?

For a distribution G, let $\mu(i, G)$ denote the probability that i is the best college that accepts Ann:

$$\mu(i, G) = \underbrace{G(\tau_{i-1})}_{(i-1) \text{ rejects her}} - \underbrace{G(\tau_i)}_{i \text{ accepts her}} .$$

Definition

Distribution H has bad news relative to G if for college i and less selective college j:

$$\frac{\Pr^H(i \text{ is best college})}{\Pr^H(j \text{ is best college})} = \frac{\mu(i, H)}{\mu(j, H)} \leq \frac{\mu(i, G)}{\mu(j, G)} = \frac{\Pr^G(i \text{ is best college})}{\Pr^G(j \text{ is best college})}.$$

We then write $G \succcurlyeq_{LR} H$.

The Bad-News Effect

Definition

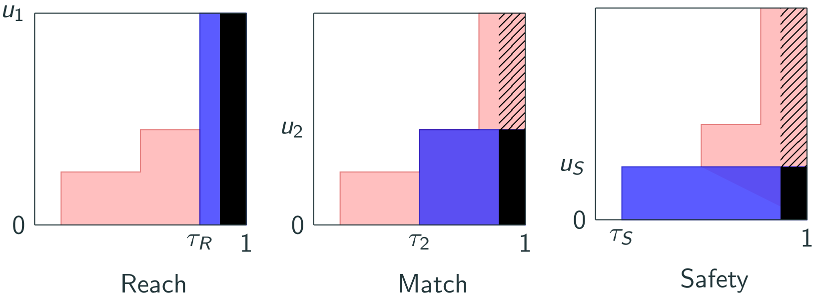
$G \succcurlyeq_{LR} H$ if for every pair of colleges i and j where $j > i$: $\dfrac{\mu(i,H)}{\mu(j,H)} \leq \dfrac{\mu(i,G)}{\mu(j,G)}$

Proposition 1

Bad news leads to a less aggressive portfolio:

$$G \succcurlyeq_{LR} H \Longrightarrow P^*(k,U,G) \succcurlyeq_A P^*(k,U,H).$$

Example exhibited version of this where H is a right truncation of G: $H(s) := G(s|s \leq \tau)$.



Truncation removes relatively more area from more selective colleges than from less selective colleges.

LR-dominance is not limited to removing slices from the right, but generally removes more mass from the right than from the left.

Proposition 1

Bad news leads to a less aggressive portfolio:

$$G \succcurlyeq_{LR} H \Longrightarrow P^*(k, U, G) \succcurlyeq_A P^*(k, U, H).$$

Proof approach:

- Show that highest ranked college (i) in $P^*(k, U, G)$ is more selective (j) than that in $P^*(k, U, H)$.
- Remainder then follows by induction:

$$\tau_i \geq \tau_j \Longrightarrow G(s|s < \tau_i) \succcurlyeq_{LR} H(s|s < \tau_j).$$

In other words, downward force propagates through the portfolio.

Risk Sensitivity

Risk-loving

As Ann applies to colleges, she realizes that not going to college isn't so bad after all.

How does a change in her risk sensitivity influence her portfolio?

Definition

Utility assessment $U' := ({u_o}'; u'_1, \ldots, u'_n)$ is more risk loving than $U := (u_o; u_1, \ldots, u_n)$ if there exists a convex non-decreasing transformation $v : \mathbb{R} \to \mathbb{R}$ such that for every college i,

$$\max\{u'_i, {u_o}'\} = v(\max\{u_i, u_o\}).$$

We denote $U' \succcurlyeq_{RL} U$.

Risk-loving

Definition

$U' \succcurlyeq_{RL} U$ if $\exists$ a convex non-decreasing transformation $v : \mathbb{R} \to \mathbb{R}$ such that for every college i,

$$\max\{u_i', u_o{}'\} = v(\max\{u_i, u_o\}).$$

Proposition 2

Being more risk loving leads to a more aggressive portfolio:

$$U' \succcurlyeq_{RL} U \implies P^*(k, U', G) \succcurlyeq_A P^*(k, U, G).$$

Risk-loving

Proposition 2

Being more risk loving leads to a more aggressive portfolio:

$$U' \succcurlyeq_{RL} U \Longrightarrow P^*(k, U', G) \succcurlyeq_A P^*(k, U, G).$$

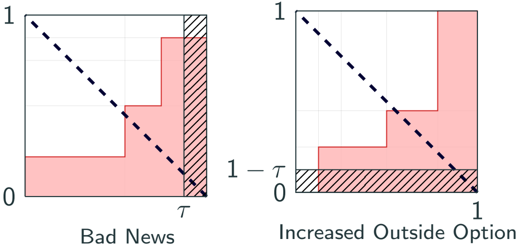
Example exhibited the special case of an increased outside option effect:

$$u_o{}' > u_o \Longrightarrow \max\{u_i, u_o{}'\} = \max\{\max\{u_i, u_o\}, u_o{}'\},$$

which is a convex non-decreasing transformation.

General argument exploits duality between beliefs and utility.

We transpose utilities and probabilities, and observe that these lead to isometric coverage problems.



Key observation: $U' \succcurlyeq_{RL} U \iff T[U'] \preccurlyeq_{LR} T[U]$.

Proposition 1 $\implies$ $T[U']$ leads to less aggressive portfolio $\implies$ U' leads to more aggressive portfolio.

Step 1: Normalize utility assessment $U(u_o; u_1, \ldots, u_n)$ to values in $[0, 1]$: $w_i = \frac{\max\{u_i - u_o, 0\}}{u_1 - u_o}$.

Step 2: Now we transpose:

- Original: utilities $(0; u_1, \ldots, u_n)$ & rejection prob $(G(\tau_1), \ldots, G(\tau_n))$ (where $G(\tau) = \Pr^G(s < \tau)$).
- Transposition: utilities $(0; 1 - G(\tau_n), \ldots, 1 - G(\tau_1))$ and rejection probabilities $(1 - u_n, \ldots, 1 - u_1)$. (Note that this flips order of colleges.)
- Isometric coverage problem.

Step 3: Equivalence between Arrow-Pratt and Likelihood-Ratio Dominance Order.

Optimal Diversification

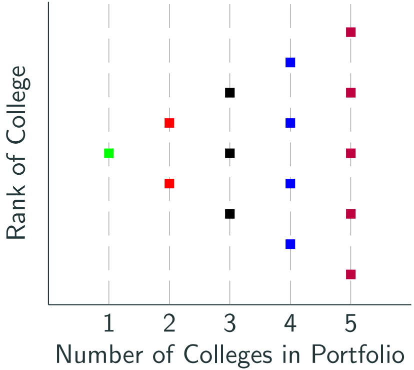
Taking stock, we have shown:

- Pessimism leads to less aggressive portfolio.
- Risk-loving leads to more aggressive portfolio.

Using these results, we establish the following claim:

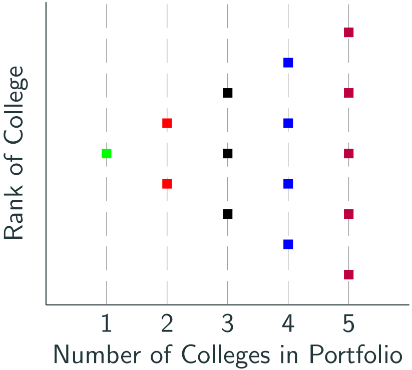
Theorem 1 (informally stated)

Larger optimal portfolios are more dispersed.



Theorem 1

$k' > k \Rightarrow P^*(k', U, F) \succcurlyeq_D P^*(k, U, F).$



Proof sketch

Suppose $k' = k + 1$:

$$\lceil P(k+1, U, F) \rceil^k \succcurlyeq_A P(k, U, F)$$

and

$$\lfloor P(k+1, U, F) \rfloor^k \preccurlyeq_A P(k, U, F)$$

Suffices to establish the above as theorem then follows by induction.

Suppose $k' = k + 1$:

$$\lfloor P(k+1, U, F) \rfloor^k \preccurlyeq_A P(k, U, F)$$

Argument: Choosing the optimal $(k+1)$-portfolio is equivalent to the following two-stage process:

- Pick one's first choice, i.
- Pick the best k backup colleges if one is rejected by that first choice.

Backups matter only if rejected by first choice $\Longrightarrow$ they should be chosen conditioning on that event.

Conditioning on that rejection $\Longrightarrow$ posterior beliefs are $F(\cdot | s < \tau_i)$.

But $F(\cdot | s < \tau_i) \preccurlyeq_{LR} F(\cdot)$, and so conclusion follows from Proposition 1.

Suppose $k' = k + 1$:

$$\lceil P(k+1, U, F) \rceil^k \succcurlyeq_A P(k, U, F)$$

Argument: Choosing the optimal $(k+1)$-portfolio is equivalent to the following two-stage process:

- Pick one's last choice / safety school, i.
- Pick the best k improvements.

Improvements matter only if she is accepted at least by College $i \implies$ Ann conditions on that event.

Conditioning on that event $\implies$ she obtains at least u_i if she is rejected by these improvements.

As $u_i > u_o$, the conclusion follows from Proposition 2.

Reduction in Application Fees

Suppose an application fee schedule ϕ' has lower marginal cost than ϕ.

Corollary

The optimal portfolio for fee schedule ϕ' is more dispersed than that for ϕ.

Connection with Evidence

Applicant applies to safety school i.e., school less selective than optimal single-college portfolio.

More broadly, Pallais'15 and Ajayi'22 find evidence of greater diversification for lower application fees.

Other Results

Algorithm → Optimal Portfolio

We used elementary decision-theoretic arguments to partially characterize optimal solution.

Generally, optimal solution can be found using an efficient (polynomial-time) algorithm that mirrors dynamic programming:

- When constructing optimal k-portfolio, can proceed by backward induction to identify optimal "backup options" after every list of $k' < k$ colleges.
- Backup options depend only on identity of least selective college.
- Can use the bad-news lemma to speed up the algorithm.

Theoretically, the number of steps required is $O(n^2 \log n)$.

Fu, Guo, Smith, and Sorensen (2022) highlight that structural model *ought* to allow for correlation:

- Choice of 4 colleges among 80 results in over 1.5 million portfolios.
- Our approach reduces it to ≈ 2240 portfolios.
- Their approach (and that of others) presume independence, which would infer from applications to safety schools that applicants prefer those schools.

Idoux (2023) considers single-tie breaking for overdemanded seats, as done for public schools in NYC and Amsterdam, and presumes a behavioral heuristic.

- If applicant ranks 12 of 60 schools, there are more than 10^{20} lists.
- Our approach would reduce this number to ≈ 4320.

Algorithm already used by Ajayi and Sidibe (2024).

Bridging Common Score and Independent Success

Consider model where Ann has a school-specific score for each school:

$$s_i = \rho s + \left(\sqrt{1-\rho^2}\right)\varepsilon_i,$$

where $\rho \in [0,1]$, $s \sim N(\mu_s, \sigma_s^2)$ and $\varepsilon_i \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$.

Each college has threshold τ_i.

We show that if $\rho > 0$:

- Every optimal portfolio of two or more schools includes a safety and a reach, so long as the choice set is sufficiently fine.
- Large portfolios include many copies of both the best and worst schools (and potentially others).

Comparative statics: With two-college portfolios, increasing ρ never reduces dispersion.

Options as Projects

Suppose each option is a project, instead of a school application:

- Projects vary in difficulty: harder projects bring higher rewards.
- Success at a project hinges on researcher's type, which she is uncertain about.
- Simplification: If a researcher succeeds at project $i \Rightarrow$ she would succeed at all easier projects.
- Researcher can follow through with only one idea.

This could model R&D decisions (or choices of an early-stage researcher) to try several ideas simultaneously before investing in one.

Our results establish the value of a healthy mix of safe and risky projects.

In this context, we also compare simultaneous to sequential search.

Sequential Search

We consider a search-with-recall setting:

- The Agent can try up to k projects for free, but no more.
- The Agent observes whether an attempt succeeds before deciding what to try next.
- The Agent can “consume” only one project.

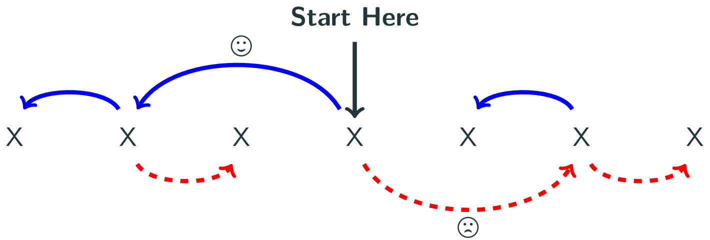
Theorem

The optimal k strategy achieves the same value as the optimal $(2^k - 1)$ portfolio.

Sequential Search

Theorem

The optimal k strategy achieves the same value as the optimal $(2^k - 1)$ portfolio.



Conclusion

Summary

Proposed and analyzed model of college portfolio choices that allows for correlation:

- Simple framework of a common score.
- Framework lends itself to a graphical approach with elementary decision-theoretic arguments.
- Predicts diversification and applying to safety schools.

Potential Directions

Endogenize school thresholds, where these are derived in equilibrium from college applications.

How Correlation Affects Welfare: Are applicants better or worse off with correlation?

- A standard intuition: Independence gives each applicant more draws.
- Countervailing force: given a fixed capacity, school thresholds adjust in equilibrium.
- Applicants may be better off with more correlation.

Directed Search: In this model, prize values are exogenous. But in labor markets, one might envision that firms post wages anticipating its effect on workers' application strategies.

- A literature does this assuming independence across firms.