ONLINE APPENDIX FOR "A THEORY OF DYNAMIC INFLATION TARGETS"

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October 2024

A Proofs

A.1 Proof of Proposition 1

Under full information, the objective function of the government is

$$\sup_{\pi_t} E_0 \sum_{t=0}^{\infty} \beta^t U_t \left(\pi_t, E_t \left[\pi_{t+1}|\theta_t\right], \theta_t\right).$$

Taking the FOC in π_t, we have

$$0 = \beta^{t-1} \frac{\partial U_{t-1}}{\partial \mathbb{E}_{t-1}\pi_t} \frac{\partial \mathbb{E}_{t-1}\pi_t}{\partial \pi_t(\theta^t)} f(\theta^{t-1}) + \beta^t \frac{\partial U_t}{\partial \pi_t} f(\theta^t)$$

From here, we have $\frac{\partial \mathbb{E}_{t-1}\pi_t}{\partial \pi_t(\theta^t)} = f(\theta_t|\theta_{t-1})$, so that we have

$$0 = \frac{\partial U_{t-1}}{\partial \mathbb{E}_{t-1}\pi_t} + \beta \frac{\partial U_t}{\partial \pi_t}$$

from which the result follows.

A.2 Proof of Proposition 3

The proof strategy is as follows. First, we derive the relevant envelope condition associated with local incentive compatibility, which defines necessary conditions on the value function associated with an incentive compatible mechanism (as in e.g., Farhi and Werning 2013, Pavan et al. 2014). We then show that the value function generated by our mechanism satisfies this envelope condition.

Envelope condition. Suppose that the central bank has a history θ^{t-1} of reports and has a current true type θ_t. Given a mechanism with transfer rule T_t and allocation rule π_t, the value function of a central bank that has truthfully reported in the past, assuming truthful reporting in the future, as a function of its current report is given by equation (6),

$$\mathcal{W}_t(\theta^{t-1}, \tilde{\theta}_t|\theta_t) = U_t\left(\pi_t(\theta^{t-1}, \tilde{\theta}_t), \pi^e_t(\theta^{t-1}, \tilde{\theta}_t), \theta_t\right) + T_t(\theta^{t-1}, \tilde{\theta}_t) + \beta \mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1}|\theta_{t+1})\Big|\theta_t\Big].$$

Recall that $\pi^e_t(\theta^{t-1}, \tilde{\theta}_t) = \mathbb{E}_t\left[\pi_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1})|\tilde{\theta}_t\right]$ is a function of the reported type, not the true type, at date t. Furthermore recall that $\mathcal{W}_{t+1}$ is also a function of the reported type $\tilde{\theta}_t$ but not the true type θ_t. As a result, the Envelope Condition, obtained by Envelope Theorem, in the true type θ_t, evaluated at truthful reporting $\tilde{\theta}_t = \theta_t$, is

$$\frac{\partial \mathcal{W}_t(\theta^t)}{\partial \theta_t} = \frac{\partial U_t\left(\pi_t, \pi_t^e, \theta_t\right)}{\partial \theta_t} + \beta \frac{\partial \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1})\middle|\theta_t\right]}{\partial \theta_t}\Bigg|_{\tilde{\theta}_t=\theta_t}$$

where we have

$$\begin{aligned}\frac{\partial \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1})\middle|\theta_t\right]}{\partial \theta_t} &= \frac{\partial}{\partial \theta_t}\int_{\underline{\theta}}^{\overline{\theta}} \mathcal{W}_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1}) f(\theta_{t+1}|\theta_t) d\theta_{t+1} \\ &= \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1}) \frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)}\middle|\theta_t\right]\end{aligned}$$

Substituting in and evaluating at truthful reporting, we obtain

$$\frac{\partial \mathcal{W}_t(\theta^t)}{\partial \theta_t} = \frac{\partial U_t\left(\pi_t, \pi_t^e, \theta_t\right)}{\partial \theta_t} + \beta \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1}) \frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)}\middle|\theta_t\right]$$

which provides a conventional envelope condition for incentive compatibility. For clarity, note that $\frac{\partial U_t(\pi_t, \pi_t^e, \theta_t)}{\partial \theta_t}$ is the derivative of U_t in the direct type θ_t, but *not* including the Phillips curve expectation, which is the derivative in the reported type.

Verifying the envelope condition. We now verify the value function under our mechanism satisfies the envelope condition. Our mechanism has a transfer rule $T_t(\theta^t) = -\nu_{t-1}(\theta^{t-1})(\pi_t(\theta^t) - \mathbb{E}_{t-1}[\pi_t|\theta_{t-1}])$ and an allocation rule given by the constrained efficient allocation of Proposition 1. The value function associated with this mechanism is

$$\mathcal{W}_t(\theta^t) = -\nu_{t-1}\Big(\pi_t - \mathbb{E}_{t-1}[\pi_t|\theta_{t-1}]\Big) + U_t\left(\pi_t, \mathbb{E}_t\left[\pi_{t+1}|\theta_t\right], \theta_t\right) + \beta \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1})\middle|\theta_t\right]$$

where $\nu_{t-1}, \pi_t, \mathbb{E}_{t-1}[\pi_t|\theta_{t-1}]$ are the constrained efficient values associated with Proposition 1, given the realized shock history. From here, recall that ν_{t-1} and $\mathbb{E}_{t-1}[\pi_t|\theta_{t-1}]$ are only functions of θ^{t-1}. Therefore, $\frac{\partial \nu_{t-1}}{\partial \theta_t} = \frac{\partial \mathbb{E}_{t-1}[\pi_t|\theta_{t-1}]}{\partial \theta_t} = 0$. Thus differentiating the value function in θ_t, we have

$$\begin{aligned}\frac{\partial \mathcal{W}_t(\theta^t)}{\partial \theta_t} =& \frac{\partial U_t}{\partial \theta_t} + \beta \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1}) \frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)}\middle|\theta_t\right] \\ &- \nu_{t-1}\frac{\partial \pi_t}{\partial \theta_t} + \frac{\partial U_t}{\partial \pi_t}\frac{\partial \pi_t}{\partial \theta_t} + \frac{\partial U_t}{\partial \mathbb{E}_t\left[\pi_{t+1}|\theta_t\right]}\frac{d\mathbb{E}_t\left[\pi_{t+1}|\theta_t\right]}{d\theta_t} + \beta \mathbb{E}_t\left[\frac{\partial \mathcal{W}_{t+1}(\theta^{t+1})}{\partial \theta_t}\middle|\theta_t\right]\end{aligned}$$

The first line on the RHS are the terms associated with the envelope condition. The second line are derivatives that arise because in equilibrium, the reported type equals the true type, and we have evaluated the value function given truthful reporting. It therefore remains to show that the second line sums to zero and hence our mechanism satisfies the required envelope condition.

It is helpful to write out the continuation value function $\mathcal{W}_{t+1}$ in sequence notation. Iterating forward, we obtain

$$\begin{aligned}\mathcal{W}_{t+1}(\theta^{t+1}) = & -\nu_t\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) \\ & - \mathbb{E}_{t+1}\left[\sum_{s=1}^{\infty}\beta^s \nu_{t+s}\Big(\pi_{t+1+s} - \mathbb{E}_{t+s}[\pi_{t+1+s}|\theta_{t+s}]\Big)\Big|\theta_{t+1}\right] \\ & + \mathbb{E}_{t+1}\left[\sum_{s=0}^{\infty}\beta^s U_{t+1+s}\left(\pi_{t+1+s}, \mathbb{E}_{t+1+s}\left[\pi_{t+2+s}|\theta_{t+1+s}\right], \theta_{t+1+s}\right)\Big|\theta_{t+1}\right]\end{aligned}$$

The first two lines on the RHS are total expected discounted value arising from transfers. The third line on the RHS is total expected discounted value arising from flow utility.

Notice from here that the second line is equal to zero. To see this, applying Law of Iterated Expectations, when $s \geq 1$ we have

$$\mathbb{E}_{t+1}\left[\nu_{t+s}\pi_{t+1+s}|\theta_{t+1}\right] = \mathbb{E}_{t+1}\left[\mathbb{E}_{t+s}\left[\nu_{t+s}\pi_{t+1+s}\Big|\theta_{t+s}\right]|\theta_{t+1}\right] = \mathbb{E}_{t+1}\left[\nu_{t+s}\mathbb{E}_{t+s}\left[\pi_{t+1+s}\Big|\theta_{t+s}\right]|\theta_{t+1}\right]$$

since ν_{t+s} is a function only of θ^{t+s}, and so is known at date $t+s$. As a result, the second line is zero, and we can write

$$\begin{aligned}\mathcal{W}_{t+1}(\theta^{t+1}) = & -\nu_t\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) \\ & + \mathbb{E}_{t+1}\left[\sum_{s=0}^{\infty}\beta^s U_{t+1+s}\left(\pi_{t+1+s}, \mathbb{E}_{t+1+s}\left[\pi_{t+2+s}|\theta_{t+1+s}\right], \theta_{t+1+s}\right)\Big|\theta_{t+1}\right]\end{aligned}$$

Observe that this is an *augmented Lagrangian* at date $t+1$: it is the date $t+1$ lifetime value (second line), plus an augmented penalty on date $t+1$ inflation. The Ramsey solution is a critical point of the augmented Lagrangian, which leads to a simple derivative. Formally from the Ramsey solution of Proposition 1, we know that

$$\frac{dU_{t+1+s}}{\partial \mathbb{E}_{t+1+s}\pi_{t+2+s}} + \beta\frac{\partial U_{t+2+s}}{\partial \pi_{t+2+s}} = 0, \quad s \geq 0$$

history by history. Therefore, we have

$$\begin{aligned}\frac{\partial \mathcal{W}_{t+1}(\theta^{t+1})}{\partial \theta_t} &= -\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) - \nu_t\Big(\frac{\partial \pi_{t+1}}{\partial \theta_t} - \frac{d\mathbb{E}_t[\pi_{t+1}|\theta_t]}{d\theta_t}\Big) + \frac{\partial U_{t+1}}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial \theta_t} \\ &= -\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) + \nu_t\frac{d\mathbb{E}_t[\pi_{t+1}|\theta_t]}{d\theta_t}\end{aligned}$$

where the second line follows since $\nu_t = \frac{\partial U_{t+1}}{\partial \pi_{t+1}}$ (Proposition 1).

Now substituting back into the expression for $\frac{\partial \mathcal{W}_t}{\partial \theta_t}$, we have

$$\begin{aligned}\frac{\partial \mathcal{W}_t(\theta^t)}{\partial \theta_t} =& \frac{\partial U_t}{\partial \theta_t} + \beta\mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t+1})\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\Big|\theta_t\Big] \\ & - \nu_{t-1}\frac{\partial \pi_t}{\partial \theta_t} + \frac{\partial U_t}{\partial \pi_t}\frac{\partial \pi_t}{\partial \theta_t} + \frac{\partial U_t}{\partial \mathbb{E}_t\left[\pi_{t+1}|\theta_t\right]}\frac{d\mathbb{E}_t\left[\pi_{t+1}|\theta_t\right]}{d\theta_t} \\ & + \beta\mathbb{E}_t\Big[-\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) + \nu_t\frac{d\mathbb{E}_t[\pi_{t+1}|\theta_t]}{d\theta_t}\Big|\theta_t\Big]\end{aligned}$$

The first term on the third line is zero, since

$$\mathbb{E}_t\Big[-\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big)\Big|\theta_t\Big] = -\frac{\partial \nu_t}{\partial \theta_t}\mathbb{E}_t\Big[\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big|\theta_t\Big] = 0.$$

From here, we can rearrange terms to get

$$\begin{aligned}\frac{\partial \mathcal{W}_t(\theta^t)}{\partial \theta_t} =& \frac{\partial U_t}{\partial \theta_t} + \beta\mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t+1})\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\Big|\theta_t\Big] \\ & + \Big[-\nu_{t-1} + \frac{\partial U_t}{\partial \pi_t}\Big]\frac{\partial \pi_t}{\partial \theta_t} + \Big[\frac{\partial U_t}{\partial \mathbb{E}_t\left[\pi_{t+1}|\theta_t\right]} + \beta\nu_t\Big]\frac{d\mathbb{E}_t\left[\pi_{t+1}|\theta_t\right]}{d\theta_t}\end{aligned}$$

By Proposition 1, we have $-\nu_{t-1} + \frac{\partial U_t}{\partial \pi_t} = 0$ and $\frac{\partial U_t}{\partial \mathbb{E}_t[\pi_{t+1}|\theta_t]} + \beta\nu_t = 0$.[56] Thus, the entire second line is zero, and we are left with

$$\frac{\partial \mathcal{W}_t(\theta^t)}{\partial \theta_t} = \frac{\partial U_t}{\partial \theta_t} + \beta\mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t+1})\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\Big|\theta_t\Big]$$

which is the required envelope condition. This concludes the proof.

[56] For completeness, note that when considering the date 0 value function, we have $\nu_{-1} = 0$ and have $\frac{\partial U_t}{\partial \pi_t} = 0$ by Proposition 1.

A.3 Proof of Lemma 4

Global incentive compatibility implies equation (6) holds. Under a dynamic inflation target, the transfer rule is

$$T(\vartheta_t^{t+s}) = -\nu_{t+s-1}(\vartheta_t^{t+s})\Big(\pi_{t+s}(\vartheta_t^{t+s}) - \pi^e_{t+s-1}(\vartheta_t^{t+s})\Big).$$

Therefore, we have

$$\begin{aligned}\mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t-1},\tilde\theta_t,\theta_{t+1}|\theta_{t+1})\Big|\theta_t\Big] =&\mathbb{E}_t\Big[-\nu_t(\vartheta_t^t)\Big(\pi_{t+1}(\vartheta^{t+1}) - \pi_t^e(\vartheta_t^t)\Big) + U_{t+1}(\pi_{t+1}(\vartheta_t^{t+1}),\pi^e_{t+1}(\vartheta_t^{t+1}),\theta_{t+1})\Big|\theta_t\Big]\\ &+\mathbb{E}_t\Big[\sum_{s=1}^{\infty}\beta^s\mathbb{E}_{t+1}\Big[-\nu_{t+s}(\vartheta_t^{t+s})\Big(\pi_{t+s+1}(\vartheta_t^{t+s+1}) - \pi_{t+s}(\vartheta_t^{t+s})\Big)\\ &+U_{t+s+1}(\pi_{t+s+1}(\vartheta_t^{t+s+1}),\pi^e_{t+s+1}(\vartheta_t^{t+s+1}),\theta_{t+s+1})\Big|\theta_{t+1}\Big]\Big|\theta_t\Big]\end{aligned}$$

and using that along a one-shot deviation we have $\pi^e_{t+s}(\vartheta_t^{t+s}) = \mathbb{E}_{t+s}[\pi_{t+s+1}(\vartheta_t^{t+s+1})|\theta_{t+s}]$ for $s \geq 1$, we obtain

$$\begin{aligned}\mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t-1},\tilde\theta_t,\theta_{t+1}|\theta_{t+1})\Big|\theta_t\Big] =&-\beta\nu_t(\vartheta_t^t)\Big(\mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\theta_t] - \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\tilde\theta_t]\Big)\\ &+\mathbb{E}_t\Big[\sum_{s=1}^{\infty}\beta^s U_{t+s}(\pi_{t+s}(\vartheta_t^{t+s}),\mathbb{E}_{t+s}[\pi_{t+s+1}(\vartheta_t^{t+s+1})|\theta_{t+s}]\theta_{t+s})\Big|\theta_t\Big]\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathcal{W}_t(\theta^{t-1},\tilde\theta_t|\theta_t) =&\nu_{t-1}(\theta^{t-1})\tau_{t-1}(\theta^{t-1}) + \mathcal{L}_t(\vartheta_t^t|\theta_t)\\ &+U_t\Big(\pi_t(\theta^{t-1},\tilde\theta_t),\mathbb{E}_t\Big[\pi_{t+1}(\vartheta_t^{t+1})\Big|\tilde\theta_t\Big],\theta_t\Big) - U_t\Big(\pi_t(\theta^{t-1},\tilde\theta_t),\mathbb{E}_t\Big[\pi_{t+1}(\vartheta_t^{t+1})\Big|\theta_t\Big],\theta_t\Big)\\ &+\beta\nu_t(\vartheta_t^t)\Big(\mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\tilde\theta_t] - \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\theta_t]\Big)\end{aligned}$$

Thus substituting into global IC obtains the result.

A.4 Proof of Proposition 7

We begin by describing the Ramsey allocation. Using $\nu_{t-1} = \frac{\partial U_t}{\partial \pi_t}$ and $-\beta\nu_t = \frac{\partial U_t}{\partial \pi_t^e}$, we obtain

$$\nu_{t-1} = \sum_{n=1}^{N}\frac{\partial \mathcal{U}_{tn}(x_{tn},\theta_t)}{\partial x_{tn}}c_{tn}$$

$$\nu_t = -\sum_{n=1}^{N} \frac{\partial \mathcal{U}_{tn}(x_{tn}, \theta_t)}{\partial x_{tn}} d_{tn}$$

A.4.1 A Tractable Representation of Augmented Lagrangian

Becuase $\mathcal{U}_{tn}$ is linear-quadratic in x_{tn}, we can do an *exact* second order Taylor series expansion of $\mathcal{U}_{tn}$ around $x_{tn}(\theta^t)$ to obtain for an alternate policy $\tilde{x}_{tn}$

$$\mathcal{U}_{tn}(\tilde{x}_{tn}, \theta_t) = \mathcal{U}_{tn}(x_{tn}(\theta^t), \theta_t) + \frac{\partial \mathcal{U}_{tn}(x_{tn}(\theta^t), \theta_t)}{\partial x_{tn}(\theta^t)}(\tilde{x}_{tn} - x_{tn}(\theta^t)) + \frac{1}{2}\frac{\partial^2 \mathcal{U}_{tn}(x_{tn}(\theta^t), \theta_t)}{\partial x_{tn}(\theta^t)^2}(\tilde{x}_{tn} - x_{tn}(\theta^t))^2$$

Observing that $\frac{\partial^2 U_{tn}(x_{tn}(\theta^t), \theta_t)}{\partial x_{tn}(\theta^t)^2} = -a_{tn}(\theta_t)$, then we can write

$$U_t(x_t(\theta^t), \theta_t) - \mathcal{U}_{tn}(\tilde{x}_t, \theta_t) = -\sum_{n=1}^{N} \frac{\partial U_{tn}(x_{tn}(\theta^t), \theta_t)}{\partial x_{tn}(\theta^t)}(\tilde{x}_{tn} - x_{tn}(\theta^t)) + \sum_{n=1}^{N} \frac{1}{2} a_{tn}(\theta_t)(\tilde{x}_{tn} - x_{tn}(\theta^t))^2$$

Thus, we can write the augmented Lagrangian gap as

$$\mathcal{L}(\theta^t|\theta_t) - \mathcal{L}(\tilde{x}|\theta_t) = -\nu_{t-1}\left[\pi_t(\theta^t) - \tilde{\pi}_t\right] + \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\sum_{n=1}^{N} \frac{\partial \mathcal{U}_{t+s,n}(x_{t+s,n}(\theta^{t+s}), \theta_{t+s})}{\partial x_{t+s,n}(\theta^{t+s})}(x_{t+s,n}(\theta^{t+s}) - \tilde{x}_{t+s,n})\right]$$
$$+ \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{n=1}^{N} \frac{1}{2} a_{t+s,n}(\theta_{t+s})(\tilde{x}_{t+s,n} - x_{t+s,n}(\theta^{t+s}))^2$$

The key observation is that the first line sums to zero for any one shot deviation $\tilde{\theta}_t$ in reporting strategy. This follows from the fact that the Ramsey policy is a critical point of the augmented Lagrangian (see also the proof of Proposition 3). Formally, observe that

$$x_{t+s,n}(\theta^{t+s}) - \tilde{x}_{t+s,n} = c_{t+s,n}(\pi_{t+s}(\theta^{t+s}) - \tilde{\pi}_{t+s}) + \beta d_{t+s,n}(\pi^e_{t+s}(\theta^{t+s}) - \tilde{\pi}^e_{t+s}),$$

which obtains a telescoping series. Therefore, we are left with the simple form of the augmented Lagrangian,

$$\mathcal{L}(\theta^t|\theta_t) - \mathcal{L}(\theta^{t-1}, \tilde{\theta}_t|\theta_t) = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{n=1}^{N} \frac{1}{2} a_{t+s,n}(\theta_{t+s})(x_{t+s,n}(\vartheta_t^{t+s}) - x_{t+s,n}(\theta^{t+s}))^2$$

Given the assumption $a_{tn}(\theta_t) \geq 0$, then this is weakly positive. This gives rise to the following result.

Corollary 19. In the linear-quadratic model, if shocks are independent over time then the dynamic inflation target is globally incentive compatible.

Proof. The result follows from the fact that $\mathcal{L}(\theta^t|\theta_t) - \mathcal{L}(\theta^{t-1}, \tilde{\theta}_t|\theta_t) \geq 0$ and that the RHS of equation (9) is zero under independent shocks. ■

A.4.2 Right hand side of global IC

We define $s_t \equiv \tilde{\theta}_t$ to be the reported type, for notational clarity in the analysis which follows.

Next, consider the right hand side of global IC, given by

$$RHS = U_t(\tilde{\pi}_t, \mathbb{E}_t[\tilde{\pi}_{t+1}|s_t], \theta_t) - U_t(\tilde{\pi}_t, \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t], \theta_t] + \beta\nu_t(\vartheta^t)\Big[\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t]\Big]$$

Observe that the gap between x_{tn} for these two allocations is given by

$$\Delta x_{tn} \equiv \beta d_{tn}\Big[\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t]\Big]$$

Therefore using our usual Taylor series expansion, we can write

$$U_t(\tilde{\pi}_t, \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t], \theta_t) = U_t(\tilde{\pi}_t, \mathbb{E}_t[\tilde{\pi}_{t+1}|s_t], \theta_t) - \sum_{n=1}^{N} \frac{\partial U_{tn}(\tilde{\pi}_t, \mathbb{E}_t[\tilde{\pi}_{t+1}|s_t], \theta_t)}{\partial x_{tn}}\Delta x_{tn} - \sum_{n=1}^{N}\frac{1}{2}a_{tn}(\theta_t)\Delta x_{tn}^2$$

Thus substituting in above, we have

$$RHS = \sum_{n=1}^{N} \frac{\partial U_{tn}(\tilde{\pi}_t, \mathbb{E}_t[\tilde{\pi}_{t+1}|s_t], \theta_t)}{\partial x_{tn}}\Delta x_{tn} + \sum_{n=1}^{N}\frac{1}{2}a_{tn}(\theta_t)\Delta x_{tn}^2 + \beta\nu_t(\vartheta^t)\Big[\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t]\Big]$$

The key derivative is

$$\frac{\partial U_{tn}}{\partial x_{tn}} = -a_{tn}(\theta_t)\Big[c_{tn}\pi_t(\vartheta^t) + \beta d_{tn}\mathbb{E}_t\Big[\pi_{t+1}(\vartheta^{t+1})|s_t\Big]\Big] + b_{tn}(\theta_t)$$

Using Assumption 6,

$$\begin{aligned}
\frac{\partial U_{tn}}{\partial x_{tn}} &= -a_{tn}\Big[c_{tn}\pi_t(\vartheta^t) + \beta d_{tn}\mathbb{E}_t\Big[\pi_{t+1}(\vartheta^{t+1})|s_t\Big]\Big] + b_{tn}(\theta_t) \\
&= -a_{tn}\Big[c_{tn}\pi_t(\vartheta^t) + \beta d_{tn}\mathbb{E}_t\Big[\pi_{t+1}(\vartheta^{t+1})|s_t\Big]\Big] + b_{tn}(s_t) + b_{tn}(\theta_t) - b_{tn}(s_t) \\
&= \frac{\partial U_{tn}(\vartheta^t)}{\partial x_{tn}} + b_{tn}(\theta_t) - b_{tn}(s_t)
\end{aligned}$$

Thus substituting in above, we have

$$RHS = \sum_{n=1}^{N} \left[\frac{\partial U_{tn}(\vartheta^t)}{\partial x_{tn}} + b_{tn}(\theta_t) - b_{tn}(s_t) \right] \beta d_{tn} \left[\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right]$$
$$+ \sum_{n=1}^{N} \frac{1}{2} a_{tn}(\theta_t) \Delta x_{tn}^2 + \beta \nu_t(\vartheta^t) \left[\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right]$$

Recall from here that

$$\nu_t(\vartheta^t) = -\sum_{n=1}^{N} \frac{\partial U_{tn}(\vartheta^t)}{\partial x_{tn}} d_{tn}$$

and therefore, we get

$$RHS = \beta \sum_{n=1}^{N} \left[b_{tn}(\theta_t) - b_{tn}(s_t) \right] d_{tn} \left[\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right] + \sum_{n=1}^{N} \frac{1}{2} a_{tn}(\theta_t) \left(\beta d_{tn} \left[\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right] \right)^2$$

or rearranging,

$$RHS = \beta \left(\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right) \sum_{n=1}^{N} \left[b_{tn}(\theta_t) - b_{tn}(s_t) \right] d_{tn} + \beta^2 \left(\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right)^2 \sum_{n=1}^{N} \frac{1}{2} a_{tn}(\theta_t) d_{tn}^2$$

A.4.3 Putting it together

Global IC therefore requires $LHS \geq RHS$, or in other words

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{n=1}^{N} \frac{1}{2} a_{t+s,n}(\theta_{t+s}) (\tilde{x}_{t+s,n} - x_{t+s,n}(\theta^{t+s}))^2$$
$$\geq \beta \left(\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right) \sum_{n=1}^{N} \left[b_{tn}(\theta_t) - b_{tn}(s_t) \right] d_{tn} + \beta^2 \left(\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] \right)^2 \sum_{n=1}^{N} \frac{1}{2} a_{tn}(\theta_t) d_{tn}^2$$

Now, Assumption 6 along with time-invariant coefficients comes in, and we can $b_{tn} = b_{n0} + b_{n1}\theta_t$ and $a_{tn}(\theta_t) = a_n$. We can use this to also show that the Ramsey solution is linear. In particular, the Ramsey solution has

$$\nu_{t-1} = \sum_{n=1}^{N} \left[-a_n x_{tn} + b_{n0} + b_{n1}\theta_t \right] c_n$$
$$-\nu_t = \sum_{n=1}^{N} \left[-a_n x_{tn} + b_{n0} + b_{n1}\theta_t \right] d_n$$

Thus using $x_{tn} = c_n \pi_t + \beta d_n \pi_t^e$, we can write

$$\nu_{t-1} + \pi_t \sum_{n=1}^{N} a_n c_n^2 + \pi_t^e \sum_{n=1}^{N} \beta a_n c_n d_n = \sum_{n=1}^{N} b_{n0} c_n + \theta_t \sum_{n=1}^{N} b_{n1} c_n$$

$$-\nu_t + \pi_t \sum_{n=1}^{N} a_n c_n d_n + \pi_t^e \sum_{n=1}^{N} \beta a_n d_n^2 = \sum_{n=1}^{N} b_{n0} d_n + \theta_t \sum_{n=1}^{N} b_{n1} d_n$$

We therefore obtain linear solutions,

$$\pi_t = \gamma_0 + \gamma_1 \nu_{t-1} + \gamma_2 \theta_t$$

$$\nu_t = \delta_0 + \delta_1 \nu_{t-1} + \delta_2 \theta_t$$

where the coefficients are obtained by coefficient matching in the above equations.

A key observation is that given this linear system, we can write

$$\pi_t(\vartheta_t^t) = \pi_t(\theta^t) + \gamma_2(s_t - \theta_t)$$

More generally at date $t+s$, the two policies differ only by the misreport at date t, which filters through target flexibility. Thus more generally, we have

$$\pi_{t+s}(\theta^{t+s}) - \pi_{t+s}(\vartheta_t^{t+s}) = \begin{cases} \gamma_1 \delta_1^{s-1} \delta_2 (\theta_t - s_t), & s \geq 1 \\ \gamma_2(\theta_t - s_t), & s = 0 \end{cases}$$

Therefore, we have

$$\mathbb{E}_{t+s}\left[\pi_{t+s+1}(\theta^{t+s+1}) - \pi_{t+s+1}(\vartheta_t^{t+s+1}) \middle| \theta_{t+s}\right] = \gamma_1 \delta_1^s \delta_2 (\theta_t - s_t)$$

From here, can can evaluate $x_{t+s,n}(\theta^{t+s}) - \tilde{x}_{t+s,n}$ for $\tilde{x}_{t+s,n} = x_{t+s,n}(\vartheta_t^{t+s})$. Substituting into the LHS of global IC, we have

$$\begin{aligned}
&\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{n=1}^{N} \frac{1}{2} a_n (\tilde{x}_{t+s,n} - x_{t+s,n}(\theta^{t+s}))^2 \\
&= \sum_{n=1}^{N} \frac{1}{2} a_n \left[(c_n \gamma_2 + \beta d_n \gamma_1 \delta_2)^2 (\theta_t - s_t)^2 + \sum_{s=1}^{\infty} \beta^s \delta_1^{2s} \left(c_n \gamma_1 \delta_1^{-1} \delta_2 + \beta d_n \gamma_1 \delta_2 \right)^2 (\theta_t - s_t)^2 \right] \\
&= \sum_{n=1}^{N} \frac{1}{2} a_n \left[(c_n \gamma_2 + \beta d_n \gamma_1 \delta_2)^2 + \frac{\beta \delta_1^2}{1 - \beta \delta_1^2} \left(c_n \gamma_1 \delta_1^{-1} \delta_2 + \beta d_n \gamma_1 \delta_2 \right)^2 \right] (\theta_t - s_t)^2
\end{aligned}$$

Thus, the left hand side is a constant multiplied by $(\theta_t - s_t)^2$.

Conducting the parallel decomposition for the right hand side and noting that $\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] -$

$\mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t] = \gamma_2\rho(s_t - \theta_t)$, we have

$$\beta\Big(\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t]\Big)\sum_{n=1}^{N}\Big[b_n(\theta_t) - b_n(s_t)\Big]d_n + \beta^2\Big(\mathbb{E}_t[\tilde{\pi}_{t+1}|s_t] - \mathbb{E}_t[\tilde{\pi}_{t+1}|\theta_t]\Big)^2\sum_{n=1}^{N}\frac{1}{2}a_n d_n^2$$

$$= \sum_{n=1}^{N}\Big[-\beta\gamma_2\rho b_{1n}d_n + \beta^2\gamma_2^2\rho^2\frac{1}{2}a_n d_n^2\Big](\theta_t - s_t)^2$$

Thus, the RHS also scales in $(\theta_t - s_t)^2$. Thus substituting into global IC, it reduces down to a condition on parameters of the model, given by

$$\sum_{n=1}^{N}\frac{1}{2}a_n\Big[(c_n\gamma_2 + \beta d_n\gamma_1\delta_2)^2 + \frac{\beta\delta_1^2}{1-\beta\delta_1^2}\Big(c_n\gamma_1\delta_1^{-1}\delta_2 + \beta d_n\gamma_1\delta_2\Big)^2\Big] \geq \sum_{n=1}^{N}\Big[-\beta\gamma_2\rho b_{1n}d_n + \beta^2\gamma_2^2\rho^2\frac{1}{2}a_n d_n^2\Big]$$

This equation defines our function Γ. Moreover, observe that the LHS is positive whereas the RHS is zero at $\rho = 0$. Therefore, we obtain a threshold ρ^*, concluding the proof.

A.4.4 Cost Push Shock Example

In the cost push shock model, suitable reduction in the above equation yields the condition

$$\rho - \frac{1}{2}\beta\gamma_1\rho^2 \leq \frac{1}{2}\frac{\gamma_1}{\alpha\beta}\Big[1 + \Big(1 + \alpha\Big[1 - \beta\gamma_1\Big]^2\Big)\frac{\beta(1-\gamma_1)^2}{1-\beta\gamma_1^2} + \alpha\Big[1 - \beta(\gamma_1 - 1)\Big]^2\Big]$$

where the right hand side is invariant to ρ. We can therefore define $\rho^*(\alpha,\beta)$ as the lower root of the quadratic equation $\rho - \frac{1}{2}\beta\gamma_1\rho^2 - \frac{1}{2}\frac{\gamma_1}{\alpha\beta}\Big[1 + \Big(1 + \alpha\Big[1 - \beta\gamma_1\Big]^2\Big)\frac{\beta(1-\gamma_1)^2}{1-\beta\gamma_1^2} + \alpha\Big[1 - \beta(\gamma_1 - 1)\Big]^2\Big] = 0$, and by convention set $\rho^*(\alpha,\beta) = 1$ if this lower root lies above 1.

A.5 Proof of Proposition 8

Consider reduced-form preferences,

$$U_t(\pi_t, \mathbb{E}_t\pi_{t+1}, \theta_t) = -\frac{1}{2}\pi_t^2 - \frac{1}{2}\alpha\Big(\pi_t - \beta\mathbb{E}_t\pi_{t+1}\Big)^2 + v(\mathbb{E}_t\pi_{t+1} + \theta_t)$$

where for notational convenience we use α in place of $\hat{\alpha} = \frac{\alpha}{\kappa^2}$ in the derivations. Thus, we have derivatives

$$\frac{\partial U_t}{\partial \pi_t} = -\pi_t - \alpha\Big(\pi_t - \beta\mathbb{E}_t\pi_{t+1}\Big)$$

$$\frac{\partial U_t}{\partial \mathbb{E}_t\pi_{t+1}} = \alpha\beta\Big(\pi_t - \beta\mathbb{E}_t\pi_t\Big) + v'(i_t^*)$$

Under the usual definitions of ν_t, we then have

$$\nu_{t-1} = -\pi_t - \alpha\left(\pi_t - \beta\mathbb{E}_t\pi_{t+1}\right) \tag{23}$$

$$\nu_t = -\alpha\left(\pi_t - \beta\mathbb{E}_t\pi_{t+1}\right) - v_1 + v_2\mathbb{E}_t\pi_{t+1} + v_2\theta_t \tag{24}$$

where we have used $v'(i_t) = \beta v_1 - \beta v_2 i_t$ and $i_t^* = \mathbb{E}_t\pi_{t+1} + \theta_t$.

We now guess and verify a linear solution of the form

$$\nu_t = \gamma_0 + \gamma_1\nu_{t-1} + \gamma_2\theta_t.$$

Rearranging equation (23), we get

$$\beta\mathbb{E}_t\pi_{t+1} = \frac{1}{\alpha}\nu_{t-1} + \frac{1+\alpha}{\alpha}\pi_t, \tag{25}$$

and substituting into equation (24) we get

$$\nu_t = -v_1 + \frac{(\alpha\beta + v_2)(1+\alpha) - \alpha^2\beta}{\alpha\beta}\pi_t + \frac{\alpha\beta + v_2}{\alpha\beta}\nu_{t-1} + v_2\theta_t.$$

From here, we denote $\frac{1}{\zeta} \equiv \frac{(\alpha\beta+v_2)(1+\alpha)-\alpha^2\beta}{\alpha\beta} > 0$. Thus rearranging the above equation, we have

$$\frac{1}{\zeta}\pi_t = \nu_t + v_1 - \frac{\alpha\beta + v_2}{\alpha\beta}\nu_{t-1} - v_2\theta_t \tag{26}$$

We now lead this equation forward one period and take expectations,

$$\frac{1}{\zeta}\mathbb{E}_t\pi_{t+1} = \mathbb{E}_t\nu_{t+1} + v_1 - \frac{\alpha\beta + v_2}{\alpha\beta}\nu_t - v_2\mathbb{E}_t\theta_{t+1}$$

and now, we can use the guess for ν_t along with the property $\mathbb{E}_t\theta_{t+1} = \rho\theta_t$ to obtain

$$\frac{1}{\zeta}\mathbb{E}_t\pi_{t+1} = \gamma_0 + v_1 + \left(\gamma_1 - \frac{\alpha\beta + v_2}{\alpha\beta}\right)\nu_t + (\gamma_2 - v_2)\rho\theta_t.$$

Now, equations (25) and (26) jointly imply

$$\frac{1}{\zeta}\mathbb{E}_t\pi_{t+1} = \frac{1}{\zeta}\frac{1}{\alpha\beta}\nu_{t-1} + \frac{1+\alpha}{\alpha\beta}\left(\nu_t + v_1 - \frac{\alpha\beta + v_2}{\alpha\beta}\nu_{t-1} - v_2\theta_t\right)$$

and so substituting in, we obtain

$$\gamma_0 + v_1 + \left(\gamma_1 - \frac{\alpha\beta + v_2}{\alpha\beta}\right)\nu_t + (\gamma_2 - v_2)\rho\theta_t = \frac{1}{\zeta}\frac{1}{\alpha\beta}\nu_{t-1} + \frac{1+\alpha}{\alpha\beta}\left(\nu_t + v_1 - \frac{\alpha\beta + v_2}{\alpha\beta}\nu_{t-1} - v_2\theta_t\right)$$

which rearranges and simplifies to

$$\left(\gamma_1 - \frac{1+\alpha+\alpha\beta+v_2}{\alpha\beta}\right)\nu_t = \left(\frac{1+\alpha-\alpha\beta}{\alpha\beta}v_1 - \gamma_0\right) - \frac{1}{\beta}\nu_{t-1} - \left(\frac{1+\alpha-\alpha\beta\rho}{\alpha\beta}v_2 + \gamma_2\rho\right)\theta_t.$$

The LHS is linear, so using our guess $\nu_t = \gamma_0 + \gamma_1\nu_{t-1} + \gamma_2\theta_t$ and coefficient matching, we have the system

$$\gamma_0 = \frac{\frac{1+\alpha(1-\beta)}{\alpha\beta}v_1 - \gamma_0}{\gamma_1 - \frac{1+\alpha+\alpha\beta+v_2}{\alpha\beta}}$$

$$\gamma_1 = -\frac{1}{\beta}\frac{1}{\gamma_1 - \frac{1+\alpha+\alpha\beta+v_2}{\alpha\beta}}$$

$$\gamma_2 = \frac{-\left(\frac{1+\alpha(1-\beta\rho)}{\alpha\beta}v_2 + \gamma_2\rho\right)}{\gamma_1 - \frac{1+\alpha+\alpha\beta+v_2}{\alpha\beta}}$$

The second equation rearranges to a quadratic $\beta\gamma_1^2 - \frac{1+\alpha+\alpha\beta+v_2}{\alpha}\gamma_1 + 1 = 0$ in γ_1. We choose the non-explosive lower root to maintain consistency with the transversality condition, which yields

$$\gamma_1 = \frac{1+\alpha(1+\beta)+v_2 - \sqrt{\left(1+\alpha(1+\beta)+v_2\right)^2 - 4\alpha^2\beta}}{2\alpha\beta}$$

From here, the equation for γ_0 can be rewritten as $\gamma_0 = -\beta\gamma_1\left(\frac{1+\alpha(1-\beta)}{\alpha\beta}v_1 - \gamma_0\right)$, and rearranging yields

$$\gamma_0 = -\gamma_1\frac{1+\alpha(1-\beta)}{\alpha(1-\beta\gamma_1)}v_1$$

Similarly, the equation for γ_2 is rewritten as $\gamma_2 = \beta\gamma_1\left(\frac{1+\alpha(1-\beta\rho)}{\alpha\beta}v_2 + \gamma_2\rho\right)$, which rearranges to

$$\gamma_2 = \frac{1}{\alpha}\frac{1+\alpha(1-\beta\rho)}{1-\beta\gamma_1\rho}\gamma_1 v_2$$

Thus, we have our solution for ν_t. Now recalling that $b_t = -\nu_t$, then we have

$$b_t = -\gamma_0 + \gamma_1 b_{t-1} - \gamma_2\theta_t.$$

Recall that $\gamma_0 < 0$, $\gamma_1 > 0$, and $\gamma_2 > 0$, then we can define

$$b_t = \delta_0 + \delta_1 b_{t-1} - \delta_2 \theta_t$$

where $\delta_0 = -\gamma_0$, $\delta_1 = \gamma_1$, and $\delta_2 = \gamma_2$ are all nonnegative.

From the derivations above, inflation is given by

$$\frac{1}{\zeta}\pi_t = \nu_t - \frac{\alpha\beta + v_2}{\alpha\beta}\nu_{t-1} + v_1 - v_2\theta_t.$$

Now, recall from above that we have $\mathbb{E}_t\pi_{t+1} = \frac{1}{\alpha\beta}\nu_{t-1} + \frac{1+\alpha}{\alpha\beta}\pi_t$. Thus we can substitute in for inflation and substitute in the rule for ν_t to obtain

$$\begin{aligned}
\tau_t &= \frac{1}{\alpha\beta}\nu_{t-1} + \zeta\frac{1+\alpha}{\alpha\beta}\left[\gamma_0 + \gamma_1\nu_{t-1} + \gamma_2\theta_t - \frac{\alpha\beta + v_2}{\alpha\beta}\nu_{t-1} + v_1 - v_2\theta_t\right] \\
&= \frac{1}{\alpha\beta}\nu_{t-1} + \zeta\frac{1+\alpha}{\alpha\beta}\Big(\gamma_0 + v_1\Big) + \zeta\frac{1+\alpha}{\alpha\beta}\left(\gamma_1 - \frac{\alpha\beta + v_2}{\alpha\beta}\right)\nu_{t-1} + \zeta\frac{1+\alpha}{\alpha\beta}(\gamma_2 - v_2)\theta_t \\
&= \zeta\frac{1+\alpha}{\alpha\beta}\Big(\gamma_0 + v_1\Big) + \zeta\frac{1+\alpha}{\alpha\beta}\left(\gamma_1 + \frac{1}{\zeta(1+\alpha)} - \frac{\alpha\beta + v_2}{\alpha\beta}\right)\nu_{t-1} + \zeta\frac{1+\alpha}{\alpha\beta}(\gamma_2 - v_2)\theta_t
\end{aligned}$$

where is readily re-expressed as $\tau_t = \chi_0 - \chi_1\nu_{t-1} - \chi_2\theta_t$. To show that $\chi_2 > 0$, we need only show that $\gamma_2 < v_2$. Substituting in the definition of γ_2, this is equivalent to

$$\frac{1}{\alpha}\frac{1+\alpha(1-\beta\rho)}{(1-\beta\gamma_1\rho)}\gamma_1 v_2 < v_2$$

$$\gamma_1 + \alpha\gamma_1 - \alpha\beta\rho\gamma_1 < \alpha - \alpha\beta\gamma_1\rho$$

$$\gamma_1 < \frac{\alpha}{1+\alpha}.$$

Substituting in the definition of γ_1 and rearranging, we have

$$\frac{\alpha\beta + v_2 + 1 + \alpha}{\alpha} - 2\frac{\alpha\beta}{1+\alpha} < \sqrt{\left(\frac{\alpha\beta + v_2 + 1 + \alpha}{\alpha}\right)^2 - 4\beta}$$

Squaring both sides (since if the LHS is negative we are already done), we get

$$\left(\frac{\alpha\beta + v_2 + 1 + \alpha}{\alpha}\right)^2 + 4\left(\frac{\alpha\beta}{1+\alpha}\right)^2 - 4\frac{\alpha\beta + v_2 + 1 + \alpha}{\alpha}\frac{\alpha\beta}{1+\alpha} < \left(\frac{\alpha\beta + v_2 + 1 + \alpha}{\alpha}\right)^2 - 4\beta$$

$$\frac{\alpha}{1+\alpha} < 1 + \frac{1}{\alpha\beta}v_2$$

which necessarily holds. Therefore, we have $\chi_2 > 0$.

We can next show that $\chi_0 > 0$, which follows since we have

$$\gamma_0 + v_1 = \gamma_0 + v_1 = \frac{\alpha - (1+\alpha)\gamma_1}{\alpha(1-\beta\gamma_1)} v_1 > 0$$

since we just showed that $\gamma_1 < \frac{\alpha}{1+\alpha}$.

Finally for χ_1, using the definition of ζ we have

$$\begin{aligned}
\chi_1 &= -\zeta \frac{1+\alpha}{\alpha\beta}\left[\gamma_1 + \frac{1}{\zeta(1+\alpha)} - \frac{\alpha\beta + v_2}{\alpha\beta}\right] \\
&= -\zeta \frac{1+\alpha}{\alpha\beta}\left[\gamma_1 - \frac{\alpha}{1+\alpha}\right] \\
&> 0
\end{aligned}$$

which follows again since $\gamma_1 < \frac{\alpha}{1+\alpha}$. Lastly substitute in $b_{t-1} = -\nu_{t-1}$ to get

$$\tau_t = \chi_0 + \chi_1 b_{t-1} - \chi_2 \theta_t$$

concluding the proof.

Parameters v_0, v_1, v_2. Finally, we briefly derive the parameters of v. Given $v(i_t^*) = -\int_{i_t^*}^{\bar{\epsilon}} [\lambda_0 - \lambda_1(i_t^* - \epsilon)] \frac{1}{\bar{\epsilon} - \underline{\epsilon}} d\epsilon$, then we have

$$\begin{aligned}
v(i_t^*) &= -\frac{1}{\bar{\epsilon} - \underline{\epsilon}}\left[(\lambda_0 - \lambda_1 i_t^*)(\bar{\epsilon} - i_t^*) + \frac{1}{2}\lambda_1(\bar{\epsilon}^2 - i_t^{*2})\right] \\
&= -\frac{1}{\bar{\epsilon} - \underline{\epsilon}}\left(\lambda_0\bar{\epsilon} + \frac{1}{2}\lambda_1\bar{\epsilon}^2\right) + \frac{(\lambda_0 + \lambda_1\bar{\epsilon})}{\bar{\epsilon} - \underline{\epsilon}} i_t^* - \frac{1}{2}\frac{\lambda_1}{\bar{\epsilon} - \underline{\epsilon}} i_t^{*2}
\end{aligned}$$

so that we have $v_0 = \frac{1}{\bar{\epsilon} - \underline{\epsilon}}\left(\lambda_0\bar{\epsilon} + \frac{1}{2}\lambda_1\bar{\epsilon}^2\right)$, $v_1 = \frac{1}{\beta}\frac{(\lambda_0 + \lambda_1\bar{\epsilon})}{\bar{\epsilon} - \underline{\epsilon}}$, and $v_2 = \frac{1}{\beta}\frac{\lambda_1}{\bar{\epsilon} - \underline{\epsilon}}$.

A.6 Proof of Proposition 9

Given reduced form preferences $U_t = -\frac{1}{2}\pi_t^2 + \theta_t \frac{\pi_t - \beta \mathbb{E}_t \pi_{t+1}}{\kappa}$, then we have

$$\frac{\partial U_t}{\partial \pi_t} = -\pi_t + \frac{1}{\kappa}\theta_t$$

$$\frac{\partial U_{t-1}}{\partial \mathbb{E}_{t-1}\pi_t} = -\frac{\beta}{\kappa}\theta_{t-1}$$

Thus substituting in the definitions,

$$\nu_{t-1} = -\pi_t + \frac{1}{\kappa/\theta_t}$$

$$\nu_{t-1} = \frac{1}{\kappa/\theta_{t-1}}$$

Thus putting them together, we get $\pi_t = \frac{1}{\kappa/\theta_t} - \frac{1}{\kappa/\theta_{t-1}}$. Finally, using $\mathbb{E}_t\theta_{t+1} = 1 - \rho + \rho\theta_t$ we get

$$\mathbb{E}_t\pi_{t+1} = \frac{\mathbb{E}_t\theta_{t+1} - \theta_t}{\kappa} = (1-\rho)\frac{1}{\kappa} - (1-\rho)\frac{\theta_t}{\kappa}$$

which gives the result.

A.7 Proof of Proposition 10

Consider the Ramsey problem,

$$\max_{\pi} \sum_{t=0}^{\infty} \beta^t U_t(\pi_t, \mathbb{E}_t[\pi_{t+1}|\theta_t], ..., \mathbb{E}_t[\pi_{t+K}|\theta_t], \theta_t)$$

It is expositionally helpful to extend the sum to include $U_{-1}, ..., U_{-K} = 0$. Under this extended sum, differentiating in $\pi_t(\theta^t)$ for $t \geq 0$, we have

$$0 = \sum_{s=t-K}^{t-1} \beta^s \frac{\partial U_s}{\partial \mathbb{E}_s[\pi_t|\theta_s]} \frac{\partial \mathbb{E}_s[\pi_t|\theta_s]}{\partial \pi_t(\theta^t)} f(\theta^s) + \beta^t \frac{\partial U_t}{\partial \pi_t} f(\theta^t).$$

From here, note that we have

$$\frac{\partial \mathbb{E}_s[\pi_t|\theta_s]}{\partial \pi_t(\theta^t)} f(\theta^s) = f(\theta^t|\theta^s) f(\theta^s) = f(\theta^t)$$

Thus rearranging and dividing through, we have

$$\frac{\partial U_t}{\partial \pi_t} = -\sum_{s=t-K}^{t-1} \beta^{s-t} \frac{\partial U_s}{\partial \mathbb{E}_s[\pi_t|\theta_s]}.$$

Substituting in the definition of $\nu_{t-k,t}$ gives the result.

A.8 Proof of Proposition 12

The proof strategy parallels that of Proposition 3. Defining $\pi^e_{t,t+k}(\theta^{t-1},\tilde{\theta}_t) \equiv \mathbb{E}_t[\pi_{t+k}(\theta^{t-1},\tilde{\theta}_t,\theta_{t+1},\dots,\theta_{t+k}|\tilde{\theta}_t]$, then we have

$$\mathcal{W}_t(\theta^{t-1},\tilde{\theta}_t|\theta_t) = U_t\left(\pi_t(\theta^{t-1},\tilde{\theta}_t),\pi^e_t(\theta^{t-1},\tilde{\theta}_t),\dots,\pi^e_{t,t+k}(\theta^{t-1},\tilde{\theta}_t),\theta_t\right) + T_t(\theta^{t-1},\tilde{\theta}_t)$$
$$+\beta\mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t-1},\tilde{\theta}_t,\theta_{t+1}|\theta_{t+1})\Big|\theta_t\right].$$

Global incentive compatibility is given by $\mathcal{W}_t(\theta^t|\theta_t) \geq \mathcal{W}_t(\theta^{t-1},\tilde{\theta}_t|\theta_t)$ for all $t,\theta^t,\tilde{\theta}_t$. By Envelope Theorem and the same steps as in the proof of Proposition 3, we obtain the Envelope Condition

$$\frac{\partial\mathcal{W}_t(\theta^t)}{\partial\theta_t} = \frac{\partial U_t\left(\pi_t,\mathbb{E}_t\left[\pi_{t+1}|\theta_t\right],...,\mathbb{E}_t[\pi_{t+K}|\theta_t],\theta_t\right)}{\partial\theta_t} + \beta\mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1})\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\Bigg|\theta_t\right]$$

What now remains is the verify the Envelope condition holds for the K-horizon dynamic inflation target.

Verifying the Envelope Condition. Our mechanism has a transfer rule

$$T_t = -\sum_{k=1}^{K}\nu_{t-k,t}(\pi_t - \mathbb{E}_{t-k}\pi_t)$$

and an allocation rule given by the constrained efficient allocation of Proposition 10. Recall the definition $\bar{v}_{t-1} = \sum_{k=1}^{K}\nu_{t-k,t}$. The value function evaluated at truthtelling and the Ramsey allocation is

$$\mathcal{W}_t(\theta^t) = -\sum_{k=1}^{K}\nu_{t-k,t}(\pi_t - \mathbb{E}_{t-k}\pi_t) + U_t\left(\pi_t,\mathbb{E}_t\pi_{t+1},...,\mathbb{E}_t\pi_{t+K},\theta_t\right) + \beta\mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1})\Big|\theta_t\right]$$

Differentiating in θ_t, we have

$$\frac{\partial\mathcal{W}_t(\theta^t)}{\partial\theta_t} = \frac{\partial U_t}{\partial\theta_t} + \beta\mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1})\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\Bigg|\theta_t\right]$$
$$-\bar{v}_{t-1}\frac{\partial\pi_t}{\partial\theta_t} + \frac{\partial U_t}{\partial\pi_t}\frac{\partial\pi_t}{\partial\theta_t} + \sum_{k=1}^{K}\frac{\partial U_t}{\partial\mathbb{E}_t\pi_{t+k}}\frac{d\mathbb{E}_t\pi_{t+k}}{d\theta_t} + \beta\mathbb{E}_t\left[\frac{\partial\mathcal{W}_{t+1}(\theta^{t+1})}{\partial\theta_t}\Bigg|\theta_t\right]$$

First from Proposition 10, we have $-\bar{v}_{t-1} + \frac{\partial U_t}{\partial\pi_t} = 0$, leaving the second line with only the latter two terms.

Expanding out the continuation value $\mathcal{W}_{t+1}(\theta^{t+1})$, we have

$$\mathcal{W}_{t+1}(\theta^{t+1}) = \mathbb{E}_{t+1}\sum_{s=0}^{\infty}\beta^s\Big[-\sum_{k=1}^{K}\nu_{t+1+s-k,t+1+s}(\pi_{t+1+s}-\mathbb{E}_{t+1+s-k}[\pi_{t+1+s}|\theta_{t+1+s-k}])+U_{t+1+s}\Big]$$

Observe that by Law of Iterated Expectations for $s \geq k$,

$$\begin{aligned}&\mathbb{E}_{t+1}\Big[\nu_{t+1+s-k,t+1+s}\Big(\pi_{t+1+s}-\mathbb{E}_{t+1+s-k}[\pi_{t+1+s}|\theta_{t+1+s-k}]\Big)\Big]\\ &=\mathbb{E}_{t+1}\Big[\nu_{t+1+s-k,t+1+s}\mathbb{E}_{t+1+s-k}\Big[\pi_{t+1+s}-\pi_{t+1+s}\Big|\theta_{t+1+s-k}\Big]\Big|\theta_{t+1}\Big]=0\end{aligned}$$

So we are left with

$$\mathcal{W}_{t+1}(\theta^{t+1}) = -\mathbb{E}_{t+1}\sum_{s=0}^{K-1}\beta^s\Big[\sum_{s<k\leq K}\nu_{t+1+s-k,t+1+s}\Big(\pi_{t+1+s}-\mathbb{E}_{t+1+s-k}[\pi_{t+1+s}|\theta_{t+1+s-k}]\Big)\Big]+\mathbb{E}_{t+1}\sum_{s=0}^{\infty}\beta^s U_{t+1+s}$$

Observe that, as in the proof of Proposition 3, this is also an augmented Lagrangian. For $s \geq K$, we have history by history

$$\sum_{k=1}^{K}\beta^{s-k}\frac{\partial U_{t+1+s-k}}{\partial \pi^e_{t+1+s-k,t+1+s}}+\beta^s\frac{\partial U_{t+1+s}}{\partial \pi_{t+1+s}}=0$$

which follows from Proposition 10. Likewise for $0 \leq s < K$, we have

$$-\beta^s\sum_{s<k\leq K}\nu_{t+1+s-k,t+1+s}+\sum_{k=1}^{s}\beta^k\frac{\partial U_{t+1+s-k}}{\partial \pi^e_{t+1+s-k,t+1+s}}+\beta^s\frac{\partial U_{t+1+s}}{\partial \pi_{t+1+s}}=0$$

which follows from Proposition 10 and the definitions of ν. Thus we obtain

$$\begin{aligned}\frac{\partial \mathcal{W}_{t+1}(\theta^{t+1})}{\partial \theta_t} &= -\mathbb{E}_{t+1}\sum_{s=0}^{K-1}\beta^s\Big[\sum_{s<k\leq K}\frac{\partial \nu_{t+1+s-k,t+1+s}}{\partial\theta_t}\Big(\pi_{t+1+s}-\mathbb{E}_{t+1+s-k}[\pi_{t+1+s}|\theta_{t+1+s-k}]\Big)\Big]\\ &\quad+\mathbb{E}_{t+1}\sum_{s=0}^{K-1}\beta^s\Big[\sum_{s<k\leq K}\nu_{t+1+s-k,t+1+s}\frac{d\mathbb{E}_{t+1+s-k}[\pi_{t+1+s}|\theta_{t+1+s-k}]}{d\theta_t}\Big]\end{aligned}$$

Lastly observe that $\nu_{t+1+s-k,t+1+s}$ is a date $t+1+s-k$ adapted constant and so, for $s<k$, depends only on θ_t when $k=s+1$. Thus we have

$$\frac{\partial \mathcal{W}_{t+1}}{\partial\theta_t} = -\mathbb{E}_{t+1}\sum_{s=1}^{K}\beta^{s-1}\frac{\partial \nu_{t,t+s}}{\partial\theta_t}\Big(\pi_{t+s}-\mathbb{E}_t[\pi_{t+s}|\theta_t]\Big)+\mathbb{E}_{t+1}\sum_{s=1}^{K}\beta^{s-1}\nu_{t,t+s}\frac{d\mathbb{E}_t[\pi_{t+s}|\theta_t]}{d\theta_t},$$

which reorders the indexation for clarity. By Law of Iterated Expectations,

$$\mathbb{E}_t\mathbb{E}_{t+1}\sum_{s=1}^{K}\beta^{s-1}\frac{\partial\nu_{t,t+s}}{\partial\theta_t}\Big(\pi_{t+s}-\mathbb{E}_t[\pi_{t+s}|\theta_t]\Big)=\sum_{s=1}^{K}\beta^{s-1}\frac{\partial\nu_{t,t+s}}{\partial\theta_t}\mathbb{E}_t\Big(\pi_{t+s}-\mathbb{E}_t[\pi_{t+s}|\theta_t]\Big)=0$$

and so substituting back into the equation for $\partial\mathcal{W}_t/\partial\theta_t$ we obtain

$$\begin{aligned}\frac{\partial\mathcal{W}_t(\theta^t)}{\partial\theta_t}=&\frac{\partial U_t}{\partial\theta_t}+\beta\mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1})\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\bigg|\theta_t\right]\\&+\sum_{k=1}^{K}\frac{\partial U_t}{\partial\mathbb{E}_t\pi_{t+k}}\frac{d\mathbb{E}_t\pi_{t+k}}{d\theta_t}+\beta\mathbb{E}_t\left[\sum_{s=1}^{K}\beta^{s-1}\nu_{t,t+s}\frac{d\mathbb{E}_t[\pi_{t+s}|\theta_t]}{d\theta_t}\bigg|\theta_t\right]\end{aligned}$$

The second line is zero from the definitions of $\nu_{t,t+s}$ (Proposition 10), leaving only the first line remaining, which is the required envelope condition. This concludes the proof.

A.9 Proof of Proposition 14

Recall that we have

$$\pi_t=\kappa y_t+(\beta\gamma+\tilde{\beta})\mathbb{E}_t\pi_{t+1}+\tilde{\beta}\mathbb{E}_t\Big[\sum_{s=1}^{\infty}\tilde{\delta}^s\pi_{t+1+s}\Big].$$

From Proposition 10 for $k\geq 1$,

$$\nu_{t,t+k}=-\frac{1}{\beta^k}\frac{\partial\mathcal{U}_t}{\partial y_t}\frac{\partial y_t}{\partial\mathbb{E}_t\pi_{t+k}}.$$

Thus, we can write for $k>1$,

$$\nu_{t,t+k}=\frac{1}{\beta^{k-1}}\frac{\frac{\partial y_t}{\partial\mathbb{E}_t\pi_{t+k}}}{\frac{\partial y_t}{\partial\mathbb{E}_t\pi_{t+1}}}\nu_{t,t+1}=\frac{1}{\beta^{k-1}}\frac{\tilde{\beta}\tilde{\delta}^{k-1}}{\beta\gamma+\tilde{\beta}}\nu_{t,t+1}=\beta^*\delta^{*(k-1)}\nu_{t,t+1}$$

where $\delta^*=\frac{\tilde{\delta}}{\beta}$ and $\beta^*=\frac{\tilde{\beta}}{\beta\gamma+\tilde{\beta}}$, completing the proof.

Now, consider the final part of the proposition. First, we have

$$\frac{\partial\delta^*}{\partial\gamma}=\frac{1}{\beta}\zeta\beta(\varepsilon-1)\gamma^{\varepsilon-2}>0$$

Next, we have

$$\frac{\partial\beta^*}{\partial\gamma}=\frac{\frac{\partial\tilde{\beta}}{\partial\gamma}(\tilde{\beta}+\beta\gamma)-(\frac{\partial\tilde{\beta}}{\partial\gamma}+\beta)\tilde{\beta}}{(\tilde{\beta}+\beta\gamma)^2}=\frac{\frac{\partial\tilde{\beta}}{\partial\gamma}\gamma-\tilde{\beta}}{(\tilde{\beta}+\beta\gamma)^2}\beta$$

From the definition of $\tilde{\beta}$, we have

$$\frac{\partial\tilde{\beta}}{\partial\gamma}=\beta(1-\zeta\gamma^{\varepsilon-1})(\varepsilon-1)-(\gamma-1)\beta\zeta(\varepsilon-1)\gamma^{\varepsilon-2}(\varepsilon-1)=\tilde{\beta}\left[\frac{1}{(\gamma-1)}-\frac{(\varepsilon-1)\zeta\gamma^{\varepsilon-2}}{(1-\zeta\gamma^{\varepsilon-1})}\right]$$

and therefore substituting above,

$$\frac{\partial\beta^*}{\partial\gamma} = \frac{\left[\frac{1}{\gamma-1} - \frac{(\varepsilon-1)\zeta\gamma^{\varepsilon-2}}{(1-\zeta\gamma^{\varepsilon-1})}\right]\gamma - 1}{(\tilde{\beta}+\beta\gamma)^2}\tilde{\beta}\beta = \frac{\frac{1}{\gamma-1} - \frac{(\varepsilon-1)\zeta\gamma^{\varepsilon-1}}{1-\zeta\gamma^{\varepsilon-1}}}{(\tilde{\beta}+\beta\gamma)^2}\tilde{\beta}\beta$$

The first step of the commitment curve is $\beta^*\delta^*$, so differentiating,

$$\begin{aligned}
\frac{\partial(\beta^*\delta^*)}{\partial\gamma} &= \frac{\partial\beta^*}{\partial\gamma}\delta^* + \beta^*\frac{\partial\delta^*}{\partial\gamma} \\
&= \left[\frac{\frac{1}{\gamma-1} - \frac{(\varepsilon-1)\zeta\gamma^{\varepsilon-1}}{1-\zeta\gamma^{\varepsilon-1}}}{\tilde{\beta}+\beta\gamma}\beta + \frac{(\varepsilon-1)}{\gamma}\right]\beta^*\delta^* \\
&= \left[\left(\frac{1}{\gamma-1} - \frac{(\varepsilon-1)\zeta\gamma^{\varepsilon-1}}{1-\zeta\gamma^{\varepsilon-1}}\right)\frac{\beta\gamma}{\tilde{\beta}+\beta\gamma} + (\varepsilon-1)\right]\frac{1}{\gamma}\beta^*\delta^*
\end{aligned}$$

which is positive for γ not too large, giving the result.

A.10 Proof of Proposition 15

Lemma 29 in Appendix E.1 proves a counterpart of Lemma 4: the K-horizon dynamic inflation target is globally incentive compatible if

$$\begin{aligned}
\mathcal{L}_t(\theta^t|\theta_t) - \mathcal{L}_t(\vartheta^t|\theta_t) \geq & U_t(\pi_t(\vartheta^t), \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\tilde{\theta}_t], \ldots, \mathbb{E}_t[\pi_{t+K}(\vartheta_t^{t+K})|\tilde{\theta}_t], \theta_t) \\
& - U_t(\pi_t(\vartheta^t), \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\theta_t], \ldots, \mathbb{E}_t[\pi_{t+K}(\vartheta_t^{t+K})|\theta_t], \theta_t) \\
& + \sum_{k=1}^{K}\beta^k\nu_{t,t+k}(\vartheta_t^t)\Big(\mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\tilde{\theta}_t] - \mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\theta_t]\Big)
\end{aligned}$$

where the augmented Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_t(\vartheta^t|\theta_t) = & -\mathbb{E}_t\left[\sum_{k=0}^{K-1}\beta^k \boldsymbol{V}_{t-1,t+k}\pi_{t+k}(\vartheta_t^{t+k})\middle|\theta_t\right] \\
& + \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s U_{t+s}(\pi_{t+s}(\vartheta^{t+s}), \mathbb{E}_{t+s}[\pi_{t+s+1}(\vartheta_t^{t+s+1})|\theta_{t+s}], \ldots, \mathbb{E}_{t+s}[\pi_{t+s+K}(\vartheta_t^{t+s+K})|\theta_{t+s}], \theta_{t+s})\middle|\theta_t\right]
\end{aligned}$$

The vector $\boldsymbol{V}_{t-1,t+k} \equiv \sum_{\ell\geq 0}\nu_{t-1-\ell,t+k}$ is cumulative historical commitments made at date $t-1$ and before to target flexibility at date $t+k$ (see also Appendix D).

A.10.1 Simplifying the LHS of Global IC (Augmented Lagrangian)

Observe that the Ramsey solution of Proposition 10 is a critical point of the augmented Lagrangian, which follows as in the proof of Proposition 7 but here with $\boldsymbol{V}_{t-1,t+k}$ encoding all prior commitments inherited for date $t+k$ (in the baseline model, we only had an inherited commitment for date t). Thus we can replicate the exact second order Taylor series expansion from the proof of Proposition 7, which relied on the allocation rule being the Ramsey solution, to obtain

$$\mathcal{L}(\theta^t|\theta_t) - \mathcal{L}(\vartheta_t^t|\theta_t) = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \sum_{n=1}^{N} \frac{1}{2} a_n (x_{t+s,n}(\vartheta_t^{t+s}) - x_{t+s,n}(\theta^{t+s}))^2.$$

We thus obtain a nonnegative left hand side of global incentive compatibility. This in turn allows us to replicate Corollary 19 in this setting (global incentive compatibility under iid shocks).

A.10.2 Simplifying the RHS of Global IC

Using Assumption 6, we can write

$$U_t(x_{t1}, \ldots, x_{tN}, \theta_t) = U_t(x_{t1}, \ldots, x_{tN}, s_t) + \sum_{n=1}^{N} b_{n1} x_{tn}(\theta_t - s_t)$$

when the policies x are held fixed. Therefore, we can write

$$\begin{aligned}
&U_t(\pi_t(\vartheta^t), \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\tilde{\theta}_t], \ldots, \mathbb{E}_t[\pi_{t+K}(\vartheta_t^{t+K})|\tilde{\theta}_t], \theta_t) - U_t(\pi_t(\vartheta^t), \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\theta_t], \ldots, \mathbb{E}_t[\pi_{t+K}(\vartheta_t^{t+K})|\theta_t], \theta_t) \\
&= U_t(\pi_t(\vartheta^t), \pi^e_{t,t+1}(\vartheta^t), \ldots, \pi^e_{t,t+K}(\vartheta^t), \tilde{\theta}_t) - U_t(\pi_t(\vartheta^t), \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\theta_t], \ldots, \mathbb{E}_t[\pi_{t+K}(\vartheta_t^{t+K})|\theta_t], \tilde{\theta}_t) \\
&\quad + \sum_{n=1}^{N} b_{n1}[x_{tn}(\vartheta^t) - x_{tn}(\vartheta^t|\theta_t)](\theta_t - \tilde{\theta}_t)
\end{aligned}$$

Observe that, as in the proof of Proposition 7, the exact second order Taylor series expansion of the second line has first order terms that cancel out with the second term on the RHS of global IC, $\sum_{k=1}^{K} \beta^k \nu_{t,t+k}(\vartheta_t^t) \Big(\mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\tilde{\theta}_t] - \mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\theta_t] \Big)$. Therefore we are left with

$$RHS = \frac{1}{2} \sum_{n=1}^{N} a_n \Big(x_{tn}(\vartheta^t) - x_{tn}(\vartheta^t|\theta_t) \Big)^2 + \sum_{n=1}^{N} b_{n1}[x_{tn}(\vartheta^t) - x_{tn}(\vartheta^t|\theta_t)](\theta_t - \tilde{\theta}_t)$$

A.10.3 Linear Solutions to the Ramsey Problem

It is easy to observe that given the linear-quadratic form, given the solution of Proposition 10, and given Assumption 6, we obtain linear solutions in $(\boldsymbol{V}_{t-1}, \theta_t)$, where $\boldsymbol{V}_{t-1} \in \mathbb{R}^K$ again encodes inherited commitments. It is therefore helpful to give a vector form representation to the system, that is $\pi_t = \gamma_0 + \gamma_1 \boldsymbol{V}_{t-1} + \gamma_2 \theta_t$ and $\boldsymbol{V}_t = \delta_0 + \delta_1 \boldsymbol{V}_{t-1} + \delta_2 \theta_t$, where $\gamma_0, \gamma_2 \in \mathbb{R}$, $\gamma_1, \delta_0, \delta_2 \in \mathbb{R}^K$, and δ_1 is a $K \times K$ matrix. Therefore, we can write

$$\pi_{t+s}(\theta^{t+s}) - \pi_{t+s}(\vartheta_t^{t+s}) = \begin{cases} \gamma_2(\theta_t - \tilde{\theta}_t), & s = 0 \\ \gamma_1 \delta_1^{s-1} \delta_2 (\theta_t - \tilde{\theta}_t), & s \geq 1 \end{cases}$$

where we note that $\gamma_1 \delta_1^{s-1} \delta_2$ is a scalar.

Therefore, for any $s \geq 0$ and any $k = 1, \ldots, K$

$$\mathbb{E}_{t+s}\left[\pi_{t+s+k}(\theta^{t+s+k}) - \pi_{t+s+k}(\vartheta_t^{t+s+k}) \middle| \theta_{t+s}\right] = \gamma_1 \delta_1^{s+k-1} \delta_2 (\theta_t - \tilde{\theta}_t)$$

Thus we have for $s \geq 1$

$$\begin{aligned} x_{t+s,n}(\theta^{t+s}) - x_{t+s,n}(\vartheta_t^{t+s}) &= c_n(\pi_{t+s}(\theta^{t+s})) - \pi_{t+s}(\vartheta_t^{t+s})) + \sum_{k=1}^{K} \beta^k d_{kn} \mathbb{E}_{t+s}\left[\pi_{t+s+k}(\theta^{t+s+k}) - \pi_{t+s+k}(\vartheta_t^{t+s+k}) \middle| \theta_{t+s}\right] \\ &= c_n \gamma_1 \delta_1^{s-1} \delta_2 (\theta_t - \tilde{\theta}_t) + \sum_{k=1}^{K} \beta^k d_{kn} \gamma_1 \delta_1^{s+k-1} \delta_2 (\theta_t - \tilde{\theta}_t) \\ &= \gamma_1 \left[c_n \delta_1^{s-1} + \sum_{k=1}^{K} \beta^k d_{kn} \delta_1^{s+k-1}\right] \delta_2 (\theta_t - \tilde{\theta}_t) \end{aligned}$$

Therefore we have

$$x_{t+s,n}(\theta^{t+s}) - x_{t+s,n}(\vartheta_t^{t+s}) = \begin{cases} \gamma_1 \left[c_n \delta_1^{s-1} + \sum_{k=1}^{K} \beta^k d_{kn} \delta_1^{s+k-1}\right] \delta_2 (\theta_t - \tilde{\theta}_t), & s \geq 1 \\ \left[c_n \gamma_2 + \gamma_1 \sum_{k=1}^{K} \beta^k d_{kn} \delta_2\right] (\theta_t - \tilde{\theta}_t), & s = 0 \end{cases}$$

We next construct $\mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\tilde{\theta}_t] - \mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\theta_t]$. For $k = 1$, we obtain

$$\mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\tilde{\theta}_t] - \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\theta_t] = \gamma_2 \rho (\tilde{\theta}_t - \theta_t)$$

For $k > 1$, we have

$$\begin{aligned} &\mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\tilde{\theta}_t] - \mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\theta_t] \\ &= \gamma_1 \left(\mathbb{E}_t\left[\boldsymbol{V}_{t+k-1}(\vartheta_t^{t+k-1}) \middle| \tilde{\theta}_t\right] - \mathbb{E}_t\left[\boldsymbol{V}_{t+k-1}(\vartheta_t^{t+k-1}) \middle| \theta_t\right]\right) + \gamma_2 (\mathbb{E}_t[\theta_{t+k}|\tilde{\theta}_t] - \mathbb{E}_t[\theta_{t+k}|\theta_t]) \\ &= \gamma_1 \left(\mathbb{E}_t\left[\boldsymbol{V}_{t+k-1}(\vartheta_t^{t+k-1}) \middle| \tilde{\theta}_t\right] - \mathbb{E}_t\left[\boldsymbol{V}_{t+k-1}(\vartheta_t^{t+k-1}) \middle| \theta_t\right]\right) + \gamma_2 \rho^k (\tilde{\theta}_t - \theta_t) \end{aligned}$$

From here, observe that we can write $\boldsymbol{V}_{t+1} = \delta_0 + \delta_1 \boldsymbol{V}_t + \delta_2 \theta_{t+1}$, or more generally

$$\boldsymbol{V}_{t+k} = \sum_{\ell=0}^{k-1} \delta_1^{\ell} \delta_0 + \delta_1^k \boldsymbol{V}_t + \sum_{\ell=0}^{k-1} \delta_1^{k-1-\ell} \delta_2 \theta_{t+1+\ell}$$

Therefore for any $k > 1$, we can write

$$\begin{aligned}
&\mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\tilde{\theta}_t] - \mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\theta_t] \\
&= \gamma_1 \sum_{\ell=0}^{k-2} \delta_1^{k-2-\ell} \delta_2 \rho^{1+\ell} (\tilde{\theta}_t - \theta_t) + \gamma_2 \rho^k (\tilde{\theta}_t - \theta_t) \\
&= \underbrace{\left[\gamma_1 \sum_{\ell=0}^{k-2} \delta_1^{k-2-\ell} \delta_2 \rho^{\ell} + \gamma_2 \rho^{k-1} \right]}_{\equiv \zeta_k} \rho (\tilde{\theta}_t - \theta_t)
\end{aligned}$$

Therefore, we can write

$$x_{tn}(\vartheta^t) - x_{tn}(\vartheta^t|\theta_t) = \sum_{k=1}^{K} \beta^k d_{kn} \left(\mathbb{E}_t \left[\pi_{t+k}(\vartheta_t^{t+k}) \middle| \tilde{\theta}_t \right] - \mathbb{E}_t \left[\pi_{t+k}(\vartheta_t^{t+k}) \middle| \theta_t \right] \right) = \left[\sum_{k=1}^{K} \beta^k d_{kn} \zeta_k \right] \rho (\tilde{\theta}_t - \theta_t)$$

A.10.4 Completing the Argument

Thus putting it all together, we have

$$LHS = \mathcal{L}(\theta^t|\theta_t) - \mathcal{L}(\vartheta_t^t|\theta_t) = \frac{1}{2}(\theta_t - \tilde{\theta}_t)^2 \Phi$$

where $\Phi = \sum_{n=1}^{N} a_n \left[\left(c_n \gamma_2 + \gamma_1 \sum_{k=1}^{K} \beta^k d_{kn} \delta_2 \right)^2 + \sum_{s=1}^{\infty} \beta^s \left(\gamma_1 \left[c_n \delta_1^{s-1} + \sum_{k=1}^{K} \beta^k d_{kn} \delta_1^{s+k-1} \right] \delta_2 \right)^2 \right]$ is a positive constant. Analogously, we can write

$$RHS = \frac{1}{2} \sum_{n=1}^{N} a_n \left(\left[\sum_{k=1}^{K} \beta^k d_{kn} \zeta_k \right] \right)^2 \rho^2 (\theta_t - \tilde{\theta}_t)^2 - \sum_{n=1}^{N} b_{n1} \left[\sum_{k=1}^{K} \beta^k d_{kn} \zeta_k \right] \rho (\theta_t - \tilde{\theta}_t)^2$$

Thus global IC requires $LHS \geq RHS$, or

$$\frac{1}{2}\Phi \geq \frac{1}{2} \sum_{n=1}^{N} a_n \left(\left[\sum_{k=1}^{K} \beta^k d_{kn} \zeta_k \right] \right)^2 \rho^2 - \sum_{n=1}^{N} b_{n1} \left[\sum_{k=1}^{K} \beta^k d_{kn} \zeta_k \right] \rho$$

We are thus left with a single condition on parameters of the model that needs to be checked. Moreover the RHS is positive whereas the LHS is zero at $\rho = 0$. Therefore, we obtain a threshold ρ^*. This concludes the proof.

A.11 Proof of Proposition 16

Given a penalty function $-\gamma_t P_t(\theta^t)$ augmenting the dynamic inflation target, we have a value function under truthtelling (given informed and uninformed firms have the same expectation) given by

$$\mathcal{W}_t(\theta^t) = U_t\left(\pi_t(\theta^t), \pi_t^e(\theta^t), \theta_t\right) - \nu_{t-1}\bigg(\pi_t(\theta^t) - \mathbb{E}_{t-1}[\pi_t|\theta_{t-1}]\bigg) - \gamma P_t(\theta^t) + \beta \mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t+1})\Big|\theta_t\Big].$$

Observe that this value function differs from the one in the proof of Proposition 3 only by the additional penalties. Thus from the proof of Proposition 3, we have

$$\begin{aligned}\frac{\partial \mathcal{W}_t(\theta^t)}{\partial \theta_t} =& \frac{\partial U_t}{\partial \theta_t} + \beta \mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t+1})\frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)}\Big|\theta_t\Big] \\ & - \gamma\frac{\partial P_t}{\partial \theta_t} - \gamma \mathbb{E}_t\left[\sum_{k=1}^{\infty} \beta^k \frac{\partial P_{t+k}}{\partial \theta_t}\middle|\theta_t\right]\end{aligned}$$

where the second line follows from the presence of the penalties. We will now construct penalties P_t so that the second line is exactly equal to the unaccounted for information rent, $-\gamma\omega_t$, from the Envelope Condition (equation 18). Thus we require

$$\frac{\partial P_t}{\partial \theta_t} + \mathbb{E}_t\left[\sum_{k=1}^{\infty} \beta^k \frac{\partial P_{t+k}}{\partial \theta_t}\middle|\theta_t\right] = \omega_t.$$

Totally differentiating the recursive formulation of $\overline{P}_t$, we have

$$\frac{\partial \overline{P}_t}{\partial \theta_t} = \frac{\partial P_t}{\partial \theta_t} + \beta \mathbb{E}_t[\frac{\partial \overline{P}_{t+1}}{\partial \theta_t}|\theta_t] + \beta \mathbb{E}_t[\overline{P}_{t+1}\frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)}|\theta_t].$$

Thus combining the two equations,

$$\frac{\partial \overline{P}_t}{\partial \theta_t} = \omega_t + \beta \mathbb{E}_t[\overline{P}_{t+1}\frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)}|\theta_t].$$

The final expression comes from integrating. Thus we have constructed the required penalty function to satisfy the envelope condition, completing the proof.

A.12 Proof of Proposition 17

Integrating the Envelope Condition (equation 7), we obtain integral incentive compatibility

$$\mathcal{W}_t(\theta^t) = \int_{\underline{\theta}}^{\theta_t} \frac{\partial U_t(\theta^{t-1}, s_t)}{\partial s_t} ds_t + \beta \int_{\underline{\theta}}^{\theta_t} \mathbb{E}_t\left[\mathcal{W}_{t+1}\frac{\partial f_t(\theta_{t+1}|s_t)/\partial s_t}{f_t(\theta_{t+1}|s_t)}|s_t\right] ds_t \tag{27}$$

where recall we have normalized the date 0 outside option to zero. From here, we can re-express the value function $\mathcal{W}_t$ as follows (see also Pavan et al. 2014).

Lemma 20. *The value function $\mathcal{W}_t$ can be represented as*

$$\mathcal{W}_t(\theta^t) = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s B_t^s(\theta^{t+s})\middle|\,\theta_t\right] \qquad \forall t,$$

where B_t^s is given by

$$B_t^s(\theta^{t+s}) = \prod_{k=0}^{s-1}\frac{1}{f_{t+k}(\theta_{t+k+1}|\theta_{t+k})}\times$$
$$\int_{s_t\le\theta_t,\ldots,s_{t+s}\le\theta_{t+s}}\frac{\partial U_{t+s}(\theta^{t-1},s_t,\ldots,s_{t+s})}{\partial s_{t+s}}\prod_{k=0}^{s-1}\frac{\partial f_{t+k}(\theta_{t+k+1}|s_{t+k})}{\partial s_{t+k}}ds_{t+s}\ldots ds_t.$$

Proof. Iterating the Envelope Condition forward one period,

$$\mathcal{W}_t(\theta^t) = \int_{\underline{\theta}}^{\theta_t}E_t\Bigg[\frac{\partial U_t(\theta^{t-1},s_t)}{\partial s_t}ds_t$$
$$+\frac{\partial f_t\left(\theta_{t+1}|s_t\right)/\partial s_t}{f_t\left(\theta_{t+1}|s_t\right)}\beta\left[\int_{\underline{\theta}}^{\theta_{t+1}}\frac{\partial U_t(\theta^{t-1},s_t,s_{t+1})}{\partial s_{t+1}}+E_{t+1}\left[\mathcal{W}_{t+2}\frac{f_{t+1}(\theta_{t+2}|s_{t+1})/\partial s_{t+1}}{f_{t+1}(\theta_{t+2}|s_{t+1})}|s_{t+1}\right]\right]\Bigg]ds_t$$

Define $\mathcal{B}_t^0(g,\theta) = \int_{\underline{\theta}}^{\theta}g ds_t$, with $g_t^0 = \frac{\partial U_t(\theta^{t-1},s_t)}{\partial s_t}$ yielding $\mathcal{B}_t^0(g_t^0,\theta)$ as the first term in the infinite series defining $\mathcal{W}_t$. We then define $\mathcal{B}_t^1\left(g,\theta\right) = \int_{\underline{\theta}}^{\theta}E_t\left[\frac{\partial f_t(\theta_{t+1}|s_t)/\partial s_t}{f_t(\theta_{t+1}|s_t)}g\middle|\,s_t\right]ds_t$, consider the function $g_t^1 = \int_{\underline{\theta}}^{\theta_{t+1}}\frac{\partial U_{t+1}(\theta^{t-1},s_t,s_{t+1})}{\partial s_{t+1}}ds_{t+1}$, and obtain $\beta\mathcal{B}_t^1\left(g_t^1,\theta_t\right)$ as the second term. Next consider a function g_t^s that is a date $t+s$ adapted function, and define $\mathcal{B}_t^2\left(g_t^2,\theta_t\right) = \mathcal{B}_t^1\left(\mathcal{B}_{t+1}^1\left(g_t^2,\theta_{t+1}\right),\theta_t\right)$. Thus we have

$$\mathcal{B}_t^2\left(g_t^2,\theta_t\right) = \int_{\underline{\theta}}^{\theta_t}E_t\left[\frac{\partial f_t(\theta_{t+1}|s_t)/\partial s_t}{f_t(\theta_{t+1}|s_t)}\int_{\underline{\theta}}^{\theta_{t+1}}E_{t+1}\left[\frac{\partial f_{t+1}(\theta_{t+2}|s_{t+1})/\partial s_{t+1}}{f_{t+1}(\theta_{t+2}|s_{t+1})}g_t^2\left(s_{t+1},\theta_{t+2}\right)\middle|\,s_{t+1}\right]ds_{t+1}\middle|\,s_t\right]ds_t$$

Using $g_t^2(s_t,s_{t+1},\theta_{t+2}) = \int_{\underline{\theta}}^{\theta_{t+2}}\frac{\partial U_{t+2}\left(\theta^{t-1},s_t,s_{t+1},s_{t+2}\right)}{\partial s_{t+2}}ds_{t+2}$ and multiplied by β^2 gives us the next term in the infinite series characterizing $\mathcal{W}_t$. Continuosly defining these recursive operators as such, and defining functions $g_t^s\left(s_t,\ldots,s_{t+s-1},\theta_{t+s}\right) = \int_{\underline{\theta}}^{\theta_{t+s}}\frac{\partial U_{t+s}\left(\theta^{t-1},s_t,\ldots,s_{t+s}\right)}{\partial s_{t+s}}ds_{t+s}$, we obtain the infinite series that characterizes $\mathcal{W}_t$.

To then simplify from here, for $\mathcal{B}_t^1(g,\theta_t)$ we have

$$\begin{aligned}
\mathcal{B}_t^1(g,\theta_t) &= \int_{\underline{\theta}}^{\theta_t} E_t\left[\frac{\partial f_t(\theta_{t+1}|s_t)/\partial s_t}{f_t(\theta_{t+1}|s_t)} g(s_t,\theta_{t+1})\middle| s_t\right] ds_t \\
&= \int_{\underline{\theta}}^{\theta_t}\int_{\theta_{t+1}} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t} g(s_t,\theta_{t+1}) d\theta_{t+1} ds_t \\
&= \int_{\theta_{t+1}}\left[\int_{\underline{\theta}}^{\theta_t} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t} g(s_t,\theta_{t+1}) ds_t\right] d\theta_{t+1} \\
&= \int_{\theta_{t+1}} \frac{\left[\int_{\underline{\theta}}^{\theta_t} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t} g(s_t,\theta_{t+1}) ds_t\right]}{f_t(\theta_{t+1}|\theta_t)} f_t(\theta_{t+1}|\theta_t) d\theta_{t+1} \\
&= E_t\left[\frac{1}{f_t(\theta_{t+1}|\theta_t)}\left[\int_{\underline{\theta}}^{\theta_t} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t} g(s_t,\theta_{t+1}) ds_t\right]\middle|\theta_t\right]
\end{aligned}$$

In particular, as applied to the function $g_t^1 = \int_{\underline{\theta}}^{\theta_{t+1}} \frac{\partial U_{t+1}(\theta^{t-1},s_t,s_{t+1})}{\partial s_{t+1}} ds_{t+1}$, we obtain:

$$\mathcal{B}_t^1(g,\theta_t) = E_t\left[\frac{1}{f_t(\theta_{t+1}|\theta_t)}\left[\int_{\underline{\theta}}^{\theta_t}\int_{\underline{\theta}}^{\theta_{t+1}} \frac{\partial U_{t+1}(\theta^{t-1},s_t,s_{t+1})}{\partial s_{t+1}} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t} ds_{t+1} ds_t\right]\middle|\theta_t\right]$$

Next considering $\mathcal{B}_t^2(g,\theta_t) = \mathcal{B}_t^1\left(\mathcal{B}_{t+1}^1(g,\theta_{t+1}),\theta_t\right)$, note we have along history (θ^{t-1},s_t)

$$\mathcal{B}_{t+1}^1(g,\theta_{t+1}) = E_{t+1}\left[\frac{1}{f_{t+1}(\theta_{t+2}|\theta_{t+1})}\left[\int_{\underline{\theta}}^{\theta_{t+1}} \frac{\partial f_{t+1}(\theta_{t+2}|s_{t+1})}{\partial s_{t+1}} g(s_t,s_{t+1},\theta_{t+2}) ds_{t+1}\right]\middle|\theta_{t+1}\right]$$

which then yields

$$\begin{aligned}
\mathcal{B}_t^2(g,\theta_t) &= E_t\left[\frac{1}{f_t(\theta_{t+1}|\theta_t)}\left[\int_{\underline{\theta}}^{\theta_t} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t} \mathcal{B}_{t+1}^1(g,\theta_{t+1}) ds_t\right]\middle|\theta_t\right] \\
&= E_t E_{t+1}\left[\frac{1}{f_t(\theta_{t+1}|\theta_t)}\left[\int_{\underline{\theta}}^{\theta_t} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t}\left[\frac{1}{f_{t+1}(\theta_{t+2}|\theta_{t+1})}\left[\int_{\underline{\theta}}^{\theta_{t+1}} \frac{\partial f_{t+1}(\theta_{t+2}|s_{t+1})}{\partial s_{t+1}} g(s_t,s_{t+1},\theta_{t+2}) ds_{t+1}\right]\middle|\theta_{t+1}\right] ds_t\right]\right| \\
&= E_t\left[\frac{1}{f_t(\theta_{t+1}|\theta_t)}\frac{1}{f_{t+1}(\theta_{t+2}|\theta_{t+1})}\left[\int_{\underline{\theta}}^{\theta_t}\int_{\underline{\theta}}^{\theta_{t+1}} \frac{\partial f_t(\theta_{t+1}|s_t)}{\partial s_t}\frac{\partial f_{t+1}(\theta_{t+2}|s_{t+1})}{\partial s_{t+1}} g(s_t,s_{t+1},\theta_{t+2}) ds_{t+1} ds_t\right]\middle|\theta_t\right]
\end{aligned}$$

and substituting in $g_t^2 = \int_{\underline{\theta}}^{\theta_{t+2}} \frac{\partial U_{t+2}\left(\theta^{t-1},s_t,s_{t+1},s_{t+2}\right)}{\partial s_{t+2}} ds_{t+2}$, we get the next expression from the Lemma. From here, the result follows from repeated iteration. ■

Thus given Lemma 20, we can construct the required transfer rule $T_t = \mathcal{W}_t - U_t - \beta\mathbb{E}_t[\mathcal{W}_{t+1}|\theta_t]$ to achieve that value function. This gives rise to the followed relaxed problem (i.e., requiring the envelope condition but not global incentive compatibility).

Lemma 21. *The relaxed problem is*

$$\max_{\{\pi_t(\theta^t)\}} \mathbb{E}_{-1}\left[\sum_{t=0}^{\infty}\beta^t\left[-\frac{\kappa}{1+\kappa}B_0^t + U_t\right]\right],$$

where B_0^t *is given as in Lemma 20.*

Proof. Since central bank welfare is $\mathcal{W}_0 = E_0\sum_{t=0}^{\infty}\left[\beta^t U_t + T_t\right]$, then

$$-E_0\sum_{t=0}^{\infty}T_t = E_0\sum_{t=0}^{\infty}\beta^t U_t - \mathcal{W}_0$$

Since government welfare is $E_{-1}\left[\sum_{t=0}^{\infty}\beta^t U_t - \kappa T_t\right]$, then substituting in

$$E_{-1}\left[-\kappa\mathcal{W}_0 + \sum_{t=0}^{\infty}\beta^t(1+\kappa)U_t\right],$$

The result obtains by substituting in $\mathcal{W}_0$ from Lemma 20. ■

We can now solve the relaxed problem of Lemma 21.[57] Denote the *realized value* of $\mathcal{B}_0^t$ by:

$$B_0^t(\theta^t) = \prod_{k=0}^{t-1}\frac{1}{f_k(\theta_{k+1}|\theta_k)}\int_{s_0\le\theta_0,\dots,s_t\le\theta_t}\frac{\partial U_t(s_0,\dots,s_t)}{\partial s_t}\prod_{k=0}^{t-1}\frac{\partial f_k(\theta_{k+1}|s_k)}{\partial s_k}ds_t\dots ds_0$$

so that $B_0^t(\theta^t)$ is a random variable derived from the history θ^t of shocks. From here the relaxed problem is

$$\max_{\{\pi_t\}} E_{-1}\left[\sum_{t=0}^{\infty}\beta^t\left[-\frac{\kappa}{1+\kappa}B_0^t(\pi_t,\pi_{t+1},\theta_t|\theta^{t-1}) + (1+\kappa)U_t(\pi_t,\pi_{t+1},\theta_t)\right]\right]$$

Consider the optimal choice of inflation $\pi_t(z^t)$, for a realized history $\theta^t = z^t$ of shocks. Note that the solution can be written in the form (for $t \ge 1$):

$$\frac{\partial U_{t-1}}{\partial\pi_t(z^t)}f(z^{t-1}) + \beta\frac{\partial U_t}{\partial\pi_t(z^t)}f(z^t) = \frac{\kappa}{1+\kappa}E_{-1}\sum_{s=t-1}^{t}\beta^{s-(t-1)}\frac{d}{d\pi_t(z^t)}B_0^s(\pi_s,\pi_{s+1},\theta_s|\theta^s)$$

so what remains is to characterize the derivatives of B_0^s with respect to $\pi_t(z^t)$. When $s = t$, we have:

$$\frac{d}{d\pi_t(z^t)}B_0^t(\theta^t) = \frac{d}{d\pi_t(z^t)}\left[\prod_{k=0}^{t-1}\frac{1}{f_k(\theta_{k+1}|\theta_k)}\int_{s_0\le\theta_0,\dots,s_t\le\theta_t}\frac{\partial U_t(s_0,\dots,s_t)}{\partial s_t}\prod_{k=0}^{t-1}\frac{\partial f_k(\theta_{k+1}|s_k)}{\partial s_k}ds_t\dots ds_0\right]$$

[57] We characterize the optimal allocation assuming that π_t is interior.

Note that $\pi_t(z^t)$ appears in $\frac{\partial U_t(s_0,\dots,s_t)}{\partial s_t}$ only along the path given by $s_0 = z_0, s_1 = z_1, \dots, s_t = z_t$, so we have

$$\frac{d}{d\pi_t(z^t)} B_0^t(\theta^t) = \mathbf{1}_{z_0 \le \theta_0, \dots, z_t \le \theta_t} \prod_{k=0}^{t-1} \frac{1}{f_k(\theta_{k+1}|\theta_k)} \frac{\partial^2 U_t}{\partial z_t \partial \pi_t(z^t)} \prod_{k=0}^{t-1} \frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}$$

Note the subtlety that the θ's are preserved, as the realization of the random history, whereas the s's are replaced by z's, as the path under the integrals that leads to the history z^t under the integrals. It is worth remembering then, when we substitute into the expectation, that θ_t is a random variable, and z^t is (fixed) the history being differentiated along, and so is not a random variable.

Note that by exactly the same logic, we obtain $\forall t \ge 2$

$$\frac{d}{d\pi_t(z^t)} B_0^{t-1}(\theta^{t-1}) = \mathbf{1}_{z_0 \le \theta_0, \dots, z_{t-1} \le \theta_{t-1}} \prod_{k=0}^{t-2} \frac{1}{f_k(\theta_{k+1}|\theta_k)} \frac{\partial^2 U_{t-1}}{\partial z_{t-1} \partial \pi_t(z^t)} \prod_{k=0}^{t-2} \frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}$$

As a result, the right-hand side of the first-order condition becomes $\forall t \ge 2$

$$\begin{aligned}
\frac{1+\kappa}{\kappa}\text{RHS} &= E_{-1} \sum_{s=t-1}^{t} \frac{d}{d\pi_t(z^t)} B_0^s(\pi_s, \pi_{s+1}, \theta_s|\theta^s) \\
&= E_{-1}\left[\mathbf{1}_{z_0 \le \theta_0, \dots, z_{t-1} \le \theta_{t-1}} \prod_{k=0}^{t-2} \frac{1}{f_k(\theta_{k+1}|\theta_k)} \frac{\partial^2 U_{t-1}}{\partial z_{t-1} \partial \pi_t(z^t)} \prod_{k=0}^{t-2} \frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}\right] \\
&\quad + \beta E_{-1}\left[\mathbf{1}_{z_0 \le \theta_0, \dots, z_t \le \theta_t} \prod_{k=0}^{t-1} \frac{1}{f_k(\theta_{k+1}|\theta_k)} \frac{\partial^2 U_t}{\partial z_t \partial \pi_t(z^t)} \prod_{k=0}^{t-1} \frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}\right] \\
&= \frac{\partial^2 U_{t-1}}{\partial z_{t-1} \partial \pi_t(z^t)} E_{-1}\left[\mathbf{1}_{z_0 \le \theta_0, \dots, z_{t-1} \le \theta_{t-1}} \prod_{k=0}^{t-2} \frac{1}{f_k(\theta_{k+1}|\theta_k)} \frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}\right] \\
&\quad + \frac{\partial^2 U_t}{\partial z_t \partial \pi_t(z^t)} \beta E_{-1}\left[\mathbf{1}_{z_0 \le \theta_0, \dots, z_t \le \theta_t} \prod_{k=0}^{t-1} \frac{1}{f_k(\theta_{k+1}|\theta_k)} \prod_{k=0}^{t-1} \frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}\right]
\end{aligned}$$

where recall that z^t is a specific history and so comes out of the expectation.

Now, consider these two expectations. Now, we define $\Omega_t(z^t)$ by:

$$\begin{aligned}
\Omega_t(z^t) &\equiv E_{-1}\left[\mathbf{1}_{z_0\leq\theta_0,\ldots,z_t\leq\theta_t}\prod_{k=0}^{t-1}\frac{1}{f_k(\theta_{k+1}|\theta_k)}\prod_{k=0}^{t-1}\frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}\right]\\
&= \int_{z_t}^{\overline{\theta}}\int_{z_{t-1}}^{\overline{\theta}}\ldots\int_{z_0}^{\overline{\theta}}\prod_{k=0}^{t-1}\frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}f(\theta_0)d\theta_t\ldots d\theta_0\\
&= \int_{z_t}^{\overline{\theta}}\frac{\partial f_k(\theta_t|z_{t-1})}{\partial z_k}\left[\int_{z_{t-1}}^{\overline{\theta}}\ldots\int_{z_0}^{\overline{\theta}}\prod_{k=0}^{t-2}\frac{\partial f_k(\theta_{k+1}|z_k)}{\partial z_k}f(\theta_0)d\theta_{t-1}\ldots d\theta_0\right]d\theta_t\\
&= \int_{z_t}^{\overline{\theta}}\frac{\partial f_k(\theta_t|z_{t-1})}{\partial z_{t-1}}\Omega_{t-1}(z^{t-1})d\theta_t\\
&= \Omega_{t-1}\left(z^{t-1}\right)\int_{z_t}^{\overline{\theta}}\frac{\partial f_k(\theta_t|z_{t-1})}{\partial z_{t-1}}d\theta_t
\end{aligned}$$

which is well-defined for all $t\geq 1$. However, it requires an initial condition $\Omega_0(z^0)$. It is helpful to define this initial condition in the date 1 FOC. Note that at date 1, we have:

$$\mathcal{B}_0^{t-1}(\theta^{t-1}) = \mathcal{B}_0^0(\theta^0) = \int_{\underline{\theta}}^{\theta_0}\frac{\partial U_0}{\partial s_0}ds_0$$

so that we have $\frac{d}{d\pi_t(z^t)}\mathcal{B}_0^{t-1}(\theta^{t-1}) = \mathbf{1}_{z_0\leq\theta_0}\frac{\partial U_0}{\partial\pi_1(z^1)}$. In particular then, the expectation is simply:

$$E_{-1}\left[\mathbf{1}_{z_0\leq\theta_0}\right] = \int_{z_0}^{\overline{\theta}}f(\theta_0)d\theta_0 = 1-F(z_0)$$

so the initial condition is $\Omega_0(z^0) = 1-F(z_0)$. This gives us a state space reduction property, where we can fully determine Ω_t from Ω_{t-1} and z_{t-1} by a recursive sequence, where the initial value is $\Omega_0(z^0) = 1-F(z_0)$.

From here, we can substitute back into the FOCs:

$$(1+\kappa)\left[\frac{\partial U_{t-1}}{\partial\pi_t(z^t)}f(z^{t-1})+\beta\frac{\partial U_t}{\partial\pi_t(z^t)}f(z^t)\right] = \kappa\left[\Omega_{t-1}(z^{t-1})\frac{\partial^2 U_{t-1}}{\partial z_{t-1}\partial\pi_t(z^t)}+\beta\Omega_t(z^t)\frac{\partial^2 U_t}{\partial z_t\partial\pi_t(z^t)}\right]$$

From here, it is helpful to divide through by $f(z^{t-1})$:

$$(1+\kappa)\left[\frac{\partial U_{t-1}}{\partial\pi_t(z^t)}+\beta\frac{\partial U_t}{\partial\pi_t(z^t)}f(z_t|z_{t-1})\right] = \kappa\left[\frac{\Omega_{t-1}(z^{t-1})}{f(z^{t-1})}\frac{\partial^2 U_{t-1}}{\partial z_{t-1}\partial\pi_t(z^t)}+\beta\frac{\Omega_t(z^t)}{f(z^t)}\frac{\partial^2 U_t}{\partial z_t\partial\pi_t(z^t)}f(z_t|z_{t-1})\right]$$

And from here, we define $\Gamma_t(z^t) = \frac{\Omega_t(z^t)}{f(z^t)}$. Note that we have:

$$\Gamma_t(z^t) = \frac{\Omega_t(z^t)}{f(z^t)} = \frac{\Omega_{t-1}(z^{t-1})}{f(z^{t-1})} \frac{\int_{z_t}^{\overline{\theta}} \frac{\partial f_k(\theta_t|z_{t-1})}{\partial z_k} d\theta_t}{f(z_t|z_{t-1})} = \Gamma_{t-1}(z^{t-1}) \frac{\int_{z_t}^{\overline{\theta}} \frac{\partial f_k(\theta_t|z_{t-1})}{\partial z_k} d\theta_t}{f(z_t|z_{t-1})}$$

which is itself a recursive sequence with initial condition $\Gamma_0 = \frac{1-F(z_0)}{f(z_0)}$. The characterization from the lemma follows from recalling that $\frac{\partial U_{t-1}}{\partial \pi_t(z^t)} = \frac{\partial U_{t-1}}{\partial \pi^e_{t-1}} f(z_t|z_{t-1})$.

Lastly, we can evaluate the FOC for π_0, which from the steps above yields

$$\frac{\partial U_0}{\partial \pi_0} = \frac{\kappa}{1+\kappa} \Gamma_0(z^0) \frac{\partial^2 U_0}{\partial z_0 \partial \pi_0}$$

This concludes the proof.

A.12.1 Second best with Average Transfers

In Section 5.2, we assumed the outside option was $\mathcal{W}_0(\theta^0) \geq 0$. We might alternatively have expressed this in the form

$$\int_{\theta_0} \mathcal{W}_0(\theta^0) f(\theta_0|\theta_{-1}) d\theta_0 \geq 0 \tag{28}$$

Intuitively, one can think of the former as a participation constraint when the central bank already knows θ_0, while the latter is a participation constraint when the central bank does not yet know θ_0. Under this structure, we can show a dynamic inflation target is optimal under costly transfers. Intuitively, the principal and agent have the same preferences (apart from transfers) and so agree that the Ramsey allocation maximizes total surplus. The average participation constraint allows the principal to extract full surplus without distorting the allocation rule.

Proposition 22. *Under an average participation constraint (28), the dynamic inflation target of Proposition 3 is an optimal mechanism.*

Proof. Lemma 20 still holds. Using $T_t(\theta^t) = \mathcal{W}_t - U_t - \beta \mathbb{E}_t [\mathcal{W}_{t+1}|\theta_t]$, we have from equation (28)

$$0 = E_{-1}\mathcal{W}_0 = \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t (U_t + T_t).$$

Thus substituting into the principal's problem, we have the relaxed problem

$$\max_{\{\pi_t\}} \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t (1+\kappa) U_t$$

so the principal's allocation rule is the Ramsey allocation, and hence is implemented by the dynamic inflation target (along with a date 0 lump sum transfer to achieve a binding participation constraint). ■

A.13 Proof of Corollary 18

The proof follows immediately from the definition of Γ_t, which is equal to zero if $\theta_t \in \{\underline{\theta}, \overline{\theta}\}$. When $\Gamma_t = 0$, the allocation rule is constrained efficient for all Γ_{t+k}, $k \geq 1$, so the optimal mechanism reverts to constrained efficiency, which is implemented by the dynamic inflation target.

B Applications Continued

This Appendix develops several additional applications as well as extensions of those presented in the main text. In Appendix B.1, we develop a canonical application of persistent cost-push shocks. In Appendix B.2, we characterize the dynamic inflation target response during lower bound spells. In Appendix B.3, we generalize the declining r^* application presented in Section 3.1 of the main text to the case where $\sigma > 0$. In Appendix B.4, we revisit our main applications allowing for costly mechanism transfers. Finally in Appendix B.5, we discuss how Rogoff (1985)'s classical conservative central banker relates to our dynamic inflation target mechanism.

B.1 Cost-Push Shocks

In this application, we study a persistent cost-push shock both with and without costly transfers. This revisits the related full-information environment of Svensson and Woodford (2004) and studies the properties of the dynamic inflation target. Social welfare is characterized by a New Keynesian loss function around a non-distorted steady state, $\mathcal{U}_t(\pi_t, y_t, \theta_t) = -\frac{1}{2}\pi_t^2 - \frac{1}{2}\alpha(y_t - \theta_t)^2$. For simplicity, we set the slope of the Phillips curve to be $\kappa = 1$. Internalizing the NKPC (11) into the loss function yields reduced-form preferences

$$U(\pi_t, \mathbb{E}_t\pi_{t+1}, \theta_t) = -\frac{1}{2}\pi_t^2 - \frac{1}{2}\alpha(\pi_t - \beta E_t\pi_{t+1} - \theta_t)^2. \tag{29}$$

Note that θ_t is a cost-push shock in the usual sense: higher θ_t means higher current inflation is needed in order to maintain the same output loss. We assume the cost-push shock satisfies $\mathbb{E}_t\theta_{t+1} = \rho\theta_t$, where $0 \leq \rho \leq 1$ is its persistence. The following result characterizes the dynamic inflation target.

Proposition 23. *The dynamic inflation target that implements the full-information Ramsey allocation is*

$$b_t = \gamma_1 b_{t-1} - \gamma_2\theta_t$$

$$\tau_t = (1-\gamma_1)\gamma_1 b_{t-1} + \gamma_2(\gamma_1 - 1 + \rho)\theta_t,$$

where $0 \leq \gamma_1 \leq 1$ *does not depend on* ρ*, and* $\gamma_2 \geq 0$ *increases in* ρ*. Optimal inflation sets* $\pi_t = -b_t + b_{t-1}$*.*

Proposition 23 specializes the dynamic inflation target of Proposition 3 to the cost-push shock application. In response to a positive and persistent innovation in the shock, i.e., a high θ_t realization, the central bank updates both parameters of the target for the next period. First, the target flexibility *decreases* in the sense that b_t falls. This happens because the cost-push shock leads to a larger output gap today, increasing the inflationary bias of the central bank.

Second, the response of the target level is ambiguous and depends on the shock persistence. When shocks are not persistent, a cost-push shock is followed by a *lower* target level. As shocks become more persistent, there is a critical level $\rho^* = 1 - \gamma_1$ after which the central bank raises the target level instead. This result reflects the common intuition of the cost-push shock model: The central bank would like to promise low future inflation to improve the contemporaneous inflation-output trade-off; as shocks become more persistent, however, it also wants to promise higher future inflation to mitigate future expected cost-push shocks.

The target also rises as the *previous* period's target flexibility parameter b_{t-1} rises. This reflects the history dependency: a high past inflationary bias leads to a desire for low inflation today, which in turn leads to a desire for low inflation tomorrow. This means that the decrease in b_t serves as a force for future deflationary pressures. Finally, contemporaneous inflation unambiguously rises in response to a positive cost-push shock. It is interesting to note that the target flexibility is *always* more responsive to a contemporaneous cost-push shock than its level, since we have $-1 < \gamma_1 - 1 + \rho < 1$.

B.1.1 Proof of Proposition 23

Given reduced from preferences are

$$U(\pi_t, \mathbb{E}_t\pi_{t+1}, \theta_t) = -\frac{1}{2}\pi_t^2 - \frac{1}{2}\alpha(\pi_t - \beta E_t\pi_{t+1} - \theta_t)^2$$

then we have

$$\frac{\partial U_t}{\partial \pi_t} = -\pi_t - \alpha(\pi_t - \beta E_t\pi_{t+1} - \theta_t)$$
$$\frac{\partial U_{t-1}}{\partial \mathbb{E}_{t-1}\pi_t} = \beta\alpha(\pi_{t-1} - \beta E_{t-1}\pi_t - \theta_{t-1}).$$

By definition, we have

$$\nu_{t-1} = -\frac{1}{\beta}\frac{\partial U_{t-1}}{\partial \mathbb{E}_{t-1}\pi_t} = -\alpha(\pi_{t-1} - \beta E_{t-1}\pi_t - \theta_{t-1}).$$

Therefore, we can write the FOC for the full-information Ramsey allocation, $\frac{\partial U_t}{\pi_t} = \nu_{t-1}$, equivalently as

$$-\pi_t - \nu_t = \nu_{t-1}$$

or in other words, $\pi_t = \nu_t - \nu_{t-1}$. Combined with the definition of ν_{t-1} and the initial condition $\nu_{-1} = 0$, this gives us a complete system.

Suppose that $\mathbb{E}_t\theta_{t+1} = \rho\theta_t$, where $\rho = 1$ corresponds to full persistence. We thus think of cost push shocks as reverting towards zero. We guess and verify a linear solution

$$\nu_t = \gamma_1 \nu_{t-1} + \gamma_2 \theta_t.$$

Given this conjecture, we know from the FOC that

$$\pi_t = (\gamma_1 - 1)\nu_{t-1} + \gamma_2 \theta_t.$$

Using the definition of ν_t,

$$\nu_t = -\alpha \pi_t + \alpha\beta \mathbb{E}_t \pi_{t+1} + \alpha \theta_t,$$

we substitute in the expression for π_t and our conjecture for ν_{t+1} to obtain

$$\nu_t = -\alpha \Big(\nu_t - \nu_{t-1} \Big) + \alpha\beta \Big((\gamma_1 - 1)\nu_t + \gamma_2 \mathbb{E}_t \theta_{t+1} \Big) + \alpha \theta_t.$$

Now using that $\mathbb{E}_t \theta_{t+1} = \rho \theta_t$ and rearranging, we get

$$\nu_t = \frac{\alpha}{1 + \alpha + (1 - \gamma_1)\alpha\beta} \nu_{t-1} + \frac{\alpha \Big(\beta \gamma_2 \rho + 1 \Big)}{1 + \alpha + (1 - \gamma_1)\alpha\beta} \theta_t$$

Thus coefficient matching, we have the system of equations

$$\frac{\alpha}{1 + \alpha + (1 - \gamma_1)\alpha\beta} = \gamma_1$$

$$\frac{\alpha \Big(\beta \gamma_2 \rho + 1 \Big)}{1 + \alpha + (1 - \gamma_1)\alpha\beta} = \gamma_2$$

The first equation is defined solely in terms of γ_1. Thus taking it and rearranging, we obtain the quadratic

$$\alpha\beta\gamma_1^2 - \gamma_1(1 + \alpha + \alpha\beta) + \alpha = 0.$$

This quadratic has two roots, with the upper root being explosive since $\beta < 1$ implies $\gamma_1^+ > 1$. Thus selecting the non-explosive root gives $0 \le \gamma_1 \le 1$, where

$$\gamma_1 = \frac{1 + \alpha + \alpha\beta - \sqrt{(1 + \alpha + \alpha\beta)^2 - 4\alpha^2\beta}}{2\alpha\beta}.$$

Note that to see why this root lies between 0 and 1, the quadratic above equals $\alpha > 0$ for $\gamma_1 = 0$ and equals $-1 < 0$ when $\gamma_1 = 1$.

Given that $0 \leq \gamma_1 \leq 1$, we can solve for γ_2 using the second equation, which gives

$$\gamma_2 = \frac{\gamma_1}{1 - \beta\rho\gamma_1},$$

which is positive since $\beta\rho\gamma_1 \leq 1$. Thus we have our solution. Given this solution, the parameters of the target are

$$\nu_t = \gamma_1 \nu_{t-1} + \gamma_2 \theta_t$$

and

$$\begin{aligned}\tau_t &= \mathbb{E}_t \pi_{t+1} \\ &= (\gamma_1 - 1)\nu_t + \gamma_2 \rho \theta_t \\ &= -(1 - \gamma_1)\gamma_1 \nu_{t-1} + \gamma_2(\gamma_1 - 1 + \rho)\theta_t\end{aligned}$$

B.2 Lower Bound Spells: Target Adjustments as Unconventional Policy

When the economy is at the effective (zero) lower bound, which we refer to as a "lower bound spell", the central bank loses its conventional policy instrument (short-term interest rates). Historically, central banks have then resorted to unconventional policy, focusing largely on forward guidance and asset purchases. Some commentators have explicitly raised the question whether changes in the targeting framework could and should be seen as a potential additional unconventional monetary policy instrument. Our theory provides a natural framework to ask this question.[58]

Zero lower bound spells are commonly represented by a constraint $i_t \geq 0$ (Eggertsson and Woodford, 2003; Werning, 2011). Consider a canonical loss function at a distorted steady state, $\mathcal{U}(\pi_t, y_t) = -\frac{1}{2}\pi_t^2 - \frac{1}{2}\alpha y_t^2 + \lambda y_t$. When explicitly accounting for the zero lower bound constraint, $i_t \geq 0$, social welfare can be associated with the Lagrangian $\mathbb{E}\sum_{t=0}^{\infty} \beta^t \left[\mathcal{U}(\pi_t, y_t) + \vartheta_t i_t\right]$. The Lagrange multiplier ϑ_t can be interpreted as the shadow value of being able to set negative nominal rates. In other words, when the economy falls into a liquidity trap, the shadow value on policies that push the economy away from the constraint rises—for example by raising inflation expectations, lowering current output, or raising future expected output.

In this application, we represent the mechanism design problem directly over the reduced-form loss function $\mathcal{U}_t(\pi_t, y_t) + \theta_t i_t$, which encodes $\theta_t i_t$ as a reduced form utility benefit/cost of the nominal interest rate. A positive innovation to θ_t qualitatively captures the same economics as an explicit lower bound spell ϑ_t: a higher θ_t increases the utility value of higher nominal interest rates, consistent with a lower bound spell. We associate a persistent lower bound spell with a persistently

[58] Crucially, we implicitly abstract from asset purchases: That is, we do not allow the central bank to use any other unconventional tool that would allow it to make the lower bound constraint slack again. We assume that instruments are incomplete to such an extent that the economy experiences a lower bound spell.

high value θ_t.

We assume that $\mathbb{E}_t\theta_{t+1} = \rho\theta_t$ for $0 \leq \rho \leq 1$. We associate $\rho = 0$ with a transitory liquidity trap, where the lower bound constraint is expected not to bind in the following period. In this application, we abstract from shocks to the slope of the Phillips curve, $\kappa_t = \kappa$, innovations in the natural rate, $r_t^* = r^*$, and demand shocks, $\epsilon_t = 0$. Substituting the NKPC (11) into the dynamic IS equation (12) then implies

$$i_t = \mathbb{E}_t\pi_{t+1} + r^* + \frac{\sigma}{\kappa}\Big[- \pi_t + (1+\beta)\mathbb{E}_t\pi_{t+1} - \beta\mathbb{E}_t\pi_{t+2}\Big]. \tag{30}$$

This means that, after substituting out for i_t and y_t in preferences $\mathcal{U}_t(\pi_t, y_t) + \theta_t i_t$, we can represent reduced-form preferences by $U_t(\pi_t, \mathbb{E}_t\pi_{t+1}, \mathbb{E}_t\pi_{t+2}, \theta_t)$. Since $\mathbb{E}_t\pi_{t+2}$ appears in this implementability condition, the resulting time consistency problem has a horizon of more than one period. We study longer-horizon time consistency problems in Section 4. In this application, we set $\sigma = 0$ so that the time consistency problem reverts to a single period. We can then rewrite the reduced-form utility function as

$$U_t(\pi_t, \mathbb{E}_t\pi_{t+1}, \theta_t) = -\frac{1}{2}\pi_t^2 - \frac{1}{2}\hat{\alpha}\Big(\pi_t - \beta\mathbb{E}_t\pi_{t+1}\Big)^2 + \hat{\lambda}\Big(\pi_t - \beta\mathbb{E}_t\pi_{t+1}\Big) + \theta_t\Big(\mathbb{E}_t\pi_{t+1} + r^*\Big)$$

where $\hat{\alpha} = \frac{\alpha}{\kappa^2}$ and $\hat{\lambda} = \frac{\lambda}{\kappa}$.[59] We now characterize the dynamic inflation target of Proposition 3 when the economy experiences a lower bound spell.

Proposition 24. *The dynamic inflation target that implements the full-information Ramsey allocation is*

$$b_t = -\gamma_0 - \gamma_1\theta_t + \gamma_2 b_{t-1}$$
$$\mathbb{E}_t\pi_{t+1} = \gamma_0 - (\gamma_2 - 1)b_t + \Big(\gamma_1 + \frac{1}{\beta}\Big)\rho\theta_t$$

where $\gamma_0 = \frac{\frac{\hat{\lambda}}{\hat{\alpha}}\gamma_2}{1-\beta\gamma_2} > 0$, *where* $\gamma_1 = \frac{\gamma_2}{1-\gamma_2\beta\rho}\Big[\rho - \frac{1+\hat{\alpha}}{\hat{\alpha}}\frac{1}{\beta}\Big] < 0$, *and where* $\gamma_2 = \frac{1+\hat{\alpha}(1+\beta)-\sqrt{(1+\hat{\alpha}(1+\beta))^2-4\hat{\alpha}^2\beta}}{2\hat{\alpha}\beta}$ *with* $0 < \gamma_2 < 1$. *Optimal inflation sets* $\pi_t = -b_t + b_{t-1} + \frac{1}{\beta}\theta_t$.

To illustrate the economic forces that govern the dynamic inflation target mechanism, consider the following exercise: We initialize the economy at its risky steady state.[60] Formally, we consider

[59] In both this application and the ones that follow, the proof shows that there are two linear solutions that satisfy the first order conditions of the optimum, and we take the non-explosive solution to remain consistent with the transversality condition.

[60] We define the risky steady state of the economy under a dynamic inflation target as comprising the allocation, prices, and target parameters (τ, ν) that the model converges to if a shock sequence of $\theta_t = 0$ for all t is realized. This is

a particular realization of the stochastic process where $\theta_t = 0$ for sufficiently many periods such that the economy and the mechanism asymptotically converge. It is straightforward to see that the target flexibility converges to $b_t \to b = -\frac{\gamma_0}{1-\gamma_2} = -\frac{1}{1-\gamma_2}\frac{\gamma_2}{1-\beta\gamma_2}\kappa\lambda < 0$ in this limit. In the language of Svensson (1997), the distorted steady state $\lambda > 0$ implies that there is an *average inflationary bias*, which $b < 0$ corrects. Similarly, the target level converges to $\tau_t = \mathbb{E}_t\pi_{t+1} \to \tau = \gamma_0 - (\gamma_2 - 1)b = 0$ in the risky steady state limit. This reflects a common Ramsey intuition: with a distorted steady state, the central bank achieves a better inflation-output trade-off today by promising lower inflation tomorrow, and subsequently achieves a better inflation-output trade-off tomorrow by promising future lower inflation, and so on. This pushes optimal inflation under commitment towards zero in the long run, absent shock innovations. Formally, the allocation rule implies $\pi_t = -b_t + b_{t-1} + \frac{1}{\beta}\theta_t \to b - b = 0$. Our dynamic inflation target implements the long-run Ramsey allocation in the risky steady state of this economy with a target level of $\tau = 0$ and a positive target flexibility $\nu > 0$ that exactly offsets the central bank's time inconsistent incentive to respond to the steady state distortion.[61]

We now initialize the economy at this risky steady state and consider a positive realization of the shock, $\theta_0 > 0$. Intuitively, we consider the economy as having entered a lower bound spell of uncertain duration at date 0. We plot the resulting impulse response functions (IRFs) under the dynamic inflation target mechanism in Figure 4.

Suppose first that the ZLB spell is purely transitory, and hence $\mathbb{E}_0\theta_1 = 0$. We consider a realization of the shock path such that $\theta_t = 0$ for all $t \geq 1$. The red-dashed line in Panel (a) of Figure 4 plots the dynamics of the target flexibility under this path.

The dynamic inflation target becomes more flexible at the lower bound, i.e., b_0 rises since $\gamma_1 < 0$. Intuitively, the transitory lower bound spell increases the value of future inflation and calls for a lower future inflation penalty. Even though the economy escapes from the lower bound at date 1, the added target flexibility is persistent and decays only at the rate $\gamma_2 < 1$. This endogenous persistence in the target response captures the standard intuition that optimal monetary policy in a liquidity trap makes long-lived promises to keep interest rates low even after the economy moves away from the lower bound (Werning, 2011). Intuitively, promising high inflation at date 1 means that unless the central bank also promises high inflation at date 2, the economy experiences a significant output contraction at date 1. The central bank therefore smooths the output contraction by promising to maintain higher inflation for longer.

The associated increase in inflation expectations is also reflected in an upwards adjustment of

distinct from the standard deterministic steady state because agents understand that the environment is stochastic. It is also distinct from the stochastic steady state, which describes the random variables that allocation, prices, and target parameters converge to in distribution as the model is simulated for a sufficiently long period of time under the ergodic stochastic process $\{\theta_t\}$.

[61] Similarly, we have $i_t \to r^*$ and $y_t \to 0$. The allocation in the risky steady state is therefore the same as in the deterministic steady state of this model. This follows from certainty equivalence under a first-order linearization.

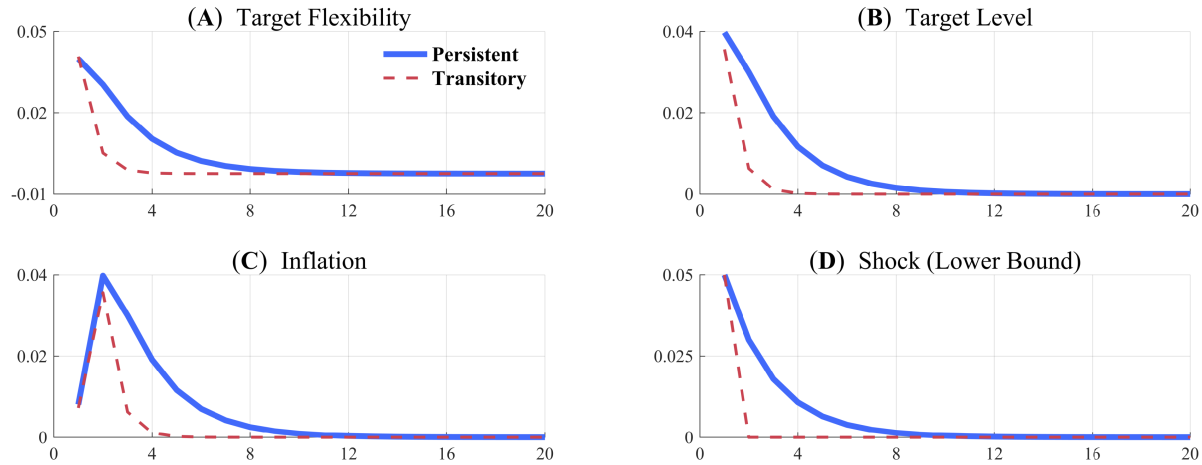

Figure 4. Impulse Responses: Lower Bound Spell

Note. Figure 4 plots the impulse responses of inflation and the dynamic inflation target after a lower bound shock $\theta_0 > 0$. Panels (A) through (D) show target flexibility, target level, inflation, and the shock, respectively. Our illustrative calibration closely follows Galí (2015), except we focus on the limit of a vanishing EIS, $\sigma = 0$. The blue solid line corresponds to a persistent shock ($\rho = 0.6$) and the red dashed line to a transitory shock ($\rho = 0$). In each case, we initialize the economy at the risky steady state and consider a shock at time 0.

the target level—see panel (b) of Figure 4. This reflects the success of the central bank in using the increased target flexibility to raise inflation expectations. It manifests in a higher inflation level in the next period. Coinciding with the gradual decay in target flexibility, the target level and realized inflation also both remain above zero even after the shock has phased out. A persistent shock, $\rho > 0$, leads to qualitatively similar but more persistent dynamics.

B.2.1 Proof of Proposition 24

Using reduced form preferences, our two key equations are

$$\nu_{t-1} = -\pi_t - \hat{\alpha}\left(\pi_t - \beta \mathbb{E}_t \pi_{t+1}\right) + \hat{\lambda}$$

$$\nu_t = -\hat{\alpha}\left(\pi_t - \beta \mathbb{E}_t \pi_{t+1}\right) + \hat{\lambda} - \frac{1}{\beta}\theta_t$$

Summing the two equations, we get $\nu_t = \nu_{t-1} + \pi_t - \frac{1}{\beta}\theta_t$. Guessing and verifying a linear solution $\nu_t = \gamma_0 + \gamma_1\theta_t + \gamma_2\nu_{t-1}$ and using our key equation, we get

$$\pi_t = \nu_t - \nu_{t-1} + \frac{1}{\beta}\theta_t$$

Leading one period and taking expectations,

$$\mathbb{E}_t \pi_{t+1} = \gamma_0 + (\gamma_2 - 1)\nu_t + \left(\gamma_1 + \frac{1}{\beta}\right)\rho\theta_t$$

Now, substituting back in to the equation for ν_t and rearranging,

$$\left(1 + \hat{\alpha} - \hat{\alpha}\beta(\gamma_2 - 1)\right)\nu_t = \hat{\alpha}\beta\gamma_0 + \hat{\lambda} + \left[\hat{\alpha}\beta\left(\gamma_1 + \frac{1}{\beta}\right)\rho - \frac{1+\hat{\alpha}}{\beta}\right]\theta_t + \hat{\alpha}\nu_{t-1}$$

Now, we solve by coefficient matching. Coefficient matching on γ_2, we have

$$0 = \hat{\alpha}\beta\gamma_2^2 - \left(1 + \hat{\alpha} + \hat{\alpha}\beta\right)\gamma_2 + \hat{\alpha}$$

and so the non-explosive root is

$$\gamma_2 = \frac{1 + \hat{\alpha} + \hat{\alpha}\beta - \sqrt{\left(1 + \hat{\alpha} + \hat{\alpha}\beta\right)^2 - 4\hat{\alpha}^2\beta}}{2\hat{\alpha}\beta}$$

Now, we can coefficient match on the constant, $\gamma_0 = \frac{\hat{\alpha}}{1+\hat{\alpha}-\hat{\alpha}\beta(\gamma_2-1)}\frac{\hat{\alpha}\beta\gamma_0+\hat{\lambda}}{\hat{\alpha}}$, giving

$$\gamma_0 = \frac{\gamma_2}{1 - \beta\gamma_2}\frac{\hat{\lambda}}{\hat{\alpha}}$$

Finally, coefficient mathcing on γ_1,

$$\gamma_1 = \frac{\hat{\alpha}}{1 + \hat{\alpha} - \hat{\alpha}\beta(\gamma_2 - 1)} \frac{\left[\hat{\alpha}\beta\left(\gamma_1 + \frac{1}{\beta}\right)\rho - \frac{1+\hat{\alpha}}{\beta}\right]}{\hat{\alpha}}$$

$$\gamma_1 = \frac{\gamma_2}{1 - \gamma_2\beta\rho}\left[\rho - \frac{1+\hat{\alpha}}{\hat{\alpha}}\frac{1}{\beta}\right]$$

B.3 r^* Revisited and the Commitment Curve

We revisit the application to persistent changes in the natural interest rate r_t^* (Section 3.1) but allow for $\sigma > 0$. The realized nominal interest rate is

$$i_t = \mathbb{E}_t \pi_{t+1} + \theta_t + \sigma\left[\mathbb{E}_t y_{t+1} - y_t\right] - \epsilon_t.$$

Intuitively, an expected rise in the output gap means household consumption is expected to rise, raising the nominal interest rate and pushing the central bank away from the ELB. Similar to Section

3.1, we can write $i_t = i_t^* - \epsilon_t$ and write the welfare losses $v(i_t^*)$ from the ELB. In this case with $\sigma > 0$, we have a change in the definition of i_t^* to

$$i_t^* = -\sigma \pi_t + (1 + (1+\beta)\sigma)\mathbb{E}_t \pi_{t+1} - \beta\sigma \mathbb{E}_t \pi_{t+2} + \theta_t,$$

which reflects internalizing the NKPC to substitute out the output gap. Intuitively, higher inflation today, π_t, increases output today and so reduces the required nominal rate. Higher inflation π_{t+1} both directly increases the nominal rate and indirectly increases it by stimulating output y_{t+1}. Conversely, higher inflation π_{t+1} depresses output y_{t+1} and so reduces the nominal rate.

We now characterize the shape of the commitment curve in this setting. Recall that the reduced-form objective is given by $U_t = -\frac{1}{2}\pi_t^2 - \frac{1}{2}\hat{\alpha}(\pi - \beta\mathbb{E}_t\pi_{t+1})^2 + v(i_t^*)$. We can now write

$$\nu_{t,t+1} = \nu_{t,t+1}^y + \nu_{t,t+1}^i,$$

where $\nu_{t,t+1}^y = -\frac{1}{2}\hat{\alpha}(\pi_t - \beta\mathbb{E}_t\pi_{t+1})$ is the usual output gap component, and where $\nu_{t,t+1}^i = -(v_1 - \beta v_2 i_t^*)(1 + (1+\beta)\sigma) < 0$ is the component that comes from the effective lower bound. From here, we can show that

$$\nu_{t,t+2} = -\beta^* \nu_{t,t+1}^i,$$

where $\beta^* = \frac{\sigma}{1+\sigma(1+\beta)} < 1$ is increasing in σ.

Intuitively, in this case the commitment curve can be decomposed into two components. The first component is the output gap commitment curve, where we have $\nu_{t,t+1}^y > 0$ and $\nu_{t,t+k}^y = 0$ for all $k > 1$. This corresponds to the standard one period commitment to stabilize the output gap. The second component is the *effective lower bound commitment curve*, where $\nu_{t,t+1}^i < 0$ and $\nu_{t,t+2}^i = -\beta^* \nu_{t,t+1}^i > 0$. The effective lower bound commitment curve switches signs precisely because of the different effects of inflation at different horizons.

B.4 Costly Transfers: Main Applications Revisited

It is instructive to revisit how costly transfers (Section 5.2) affects the optimal allocation rule in our main applications. In this Appendix, we revisit our applications on declining r_t^* (Section 3.1), the flattening Phillips curve (Section 3.2), cost-push shocks (Appendix B.1), and lower bound spells (Appendix B.2).

We show that costly transfers calls for *less* aggressive unconventional policies (e.g., forward guidance) when the economy experiences a lower bound spell, while it calls for *more* aggressive policies (e.g., raising the inflation target) in response to a decline in r^*. We document competing effects in the case of flattening Phillips curve that can call more more or less aggressive policies.

Declining r^*. In the case of changes in the natural rate $\theta_t = r_t^*$ (Section 3.1), reduced-form preferences satisfy $\frac{\partial U_t}{\partial \pi_t \partial \theta_t} = 0$ and $\frac{\partial U_t}{\partial \mathbb{E}_t \pi_{t+1} \partial \theta_t} = -c_1$ for a constant $c_1 > 0$. Intuitively, high θ_t corresponds to being further from the effective lower bound, which reduces the value of raising inflation expectations to get away from the ELB. The allocation rule under the optimal mechanism is given by

$$\frac{\partial U_t}{\partial \pi_t} = \nu_{t-1} - K\Gamma_{t-1}c_1,$$

where again the RHS is λ_{t-1}. The rule thus parallels the rule under lower bound spells, but in the opposite direction. This is because higher inflation expectations now *reduce* past information rents to the central bank, rather than raising them, by pushing the economy away from the ELB. This leads the planner to prefer a *more* aggressive policy for promoting future inflation.

These results highlight a surprising contrast between the two lower bound applications: costly transfers calls for less aggressive unconventional policies in a lower bound spell, but for more aggressive policies in response to changing a natural rate. Intuitively once the economy is already in a lower bound spell, boosting inflation expectations raises central bank information rents by disproportionately benefiting central banks in worse conditions. By contrast if the economy has not yet hit the lower bound, boosting inflation expectations reduces central bank information rents by pushing all central banks away from the lower bound, reducing the value to the central bank of private information about r^*.

Flattening Phillips curve. In the case of a flattening Phillips curve (Section 3.2), reduced-form preferences satisfy $\frac{\partial U_t}{\partial \pi_t \partial \theta_t} = \frac{1}{\kappa}$ and $\frac{\partial U_t}{\partial \mathbb{E}_t \pi_{t+1} \partial \theta_t} = -\frac{\beta}{\kappa}$. This reflects that a flattening Phillips curve (higher θ_t) increases the value of stimulating current output through current inflation, but also increases the cost of higher inflation expectations that depress output. The optimal allocation rule is given by

$$\frac{\partial U_t}{\partial \pi_t} = \nu_{t-1} + \frac{K}{\kappa}\Delta\Gamma_t,$$

where again the RHS is λ_{t-1} and where $\Delta\Gamma_t \equiv \Gamma_t - \Gamma_{t-1}$. There are two competing effects from costly transfers. On the one hand, high θ_t means that the central bank's value of stimulating output rises, promoting higher current inflation. This increases information rents to the central bank and calls for lower inflation. On the other hand, high inflation also increases past inflation expectations, which reduces information rents to past central banks and calls for higher inflation (similarly to the r^* application). The relative magnitude of the two effects is determined by $\Delta\Gamma_t$, that is the change in the persistent portion of the information rent earned by the central bank between the two dates. From Proposition 17, we can write

$$\Delta\Gamma_t = \Gamma_{t-1}\left(h(\theta_t|\theta_{t-1})\mathbb{E}_t\left[\Lambda(s_t|\theta_{t-1})\middle| s_t \geq \theta_t\right] - 1\right).$$

where recall that $h^{-1}(\theta_t|\theta_{t-1}) = \frac{1-F(\theta_t|\theta_{t-1})}{f(\theta_t|\theta_{t-1})}$ is the inverse hazard rate and $\Lambda(s_t|\theta_{t-1}) = \frac{\partial f(s_t|\theta_{t-1})/\partial\theta_{t-1}}{f(\theta_t|\theta_{t-1})}$ is the derivative of the likelihood ratio. We know that the expected likelihood ratio derivative is zero at $\theta_t = \underline{\theta}$ while we know that the inverse hazard rate is zero at $\theta_t = \overline{\theta}$. Thus local to the two extremes of the shock distribution, we have $\Delta\Gamma_t < 0$ and hence the optimal mechanism promotes *higher* inflation. Interestingly, this suggests a tendency in this environment for the backward looking information rent to dominate the contemporaneous information rent, and hence generate a tendency to promote higher inflation to generate lower past information rents (at the expense of promoting higher current information rents). In the interior, two common assumptions are a nonincreasing inverse hazard rate and a monotone (increasing) likelihood ratio (higher past types signal high future types). These have competing effects on the response to a flattening Phillips curve. Intuitively, a lower inverse hazard rate reduces current virtual surplus whereas a higher likelihood ratio increases virtual surplus.

Cost-push shocks. With costly transfers, note that we have $\frac{\partial U_t}{\partial\pi_t\partial\theta_t} = \frac{1}{2}\alpha$ and $\frac{\partial U_t}{\partial\mathbb{E}_t\pi_{t+1}\partial\theta_t} = -\frac{1}{2}\alpha\beta$. The impacts are analogous to a flattening Phillips curve, and means we can write

$$\frac{\partial U_t}{\partial\pi_t} = \nu_{t-1} + \frac{1}{2}\frac{K}{\alpha}\Delta\Gamma_t$$

Thus relative to the Ramsey solution, the optimal mechanism adjusts the allocation trading off two effects on information rents. On the one hand, higher expected inflation reduces *past* information rents by increasing costs of inflation for central banks that experience large past cost push shocks. On the other hand, higher contemporaneous inflation increases *current* information rents by reducing costs of large contemporanous cost push shocks. The optimal allocation rule trades off these two effects. As once again $\Delta\Gamma_t < 0$ local to the boundaries of the shock distribution, particularly large or particularly small cost push shocks at date t lead past information rents to dominate, and calls for a *more* aggressive inflation response today in order to reduce historical information rents. Interestingly, this amplifies the response of inflation to a large cost push shock, pushing the allocation rule closer to the policy under discretion.

Lower bound spells. In the case of lower bound spells (Section B.2), reduced-form preferences satisfy $\frac{\partial U_t}{\partial\pi_t\partial\theta_t} = 0$ and $\frac{\partial U_t}{\partial\mathbb{E}_t\pi_{t+1}\partial\theta_t} = c_0$ for a constant $c_0 > 0$. This reflects that high $\theta_t > 0$ corresponds to a binding lower bound and thus makes it valuable to promise more *future* inflation. However, because θ_t reflects a benefit of increasing the nominal rate and increasing inflation π_t does not directly increase the nominal rate, changes in the allocation rule π_t does not generate an information rent for the central bank at date t. This leads to an allocation rule given by

$$\frac{\partial U_t}{\partial\pi_t} = \nu_{t-1} + K\Gamma_{t-1}c_0,$$

where the RHS is λ_{t-1}.

Suppose that lower bound spells are persistent and higher current types signal higher future types (monotone likelihood). Then, $\Gamma_{t-1} > 0$, so that the optimal mechanism prescribes a marginal value of contemporaneous inflation that is *higher* under costly transfers, all else equal. Intuitively, higher inflation expectations increase *past* information rents through by pushing the economy away from the lower bound. This leads the planner to prefer a less aggressive policy for promoting future inflation.

B.5 Revisiting Rogoff's Inflation-Conservative Central Banker

We ask whether dynamic inflation targets can be implemented by inflation-conservative central bankers in the spirit of Rogoff (1985). In particular, our inflation-conservative central banker places a greater penalty on inflation than the government. After appropriate intertemporal rearrangement of terms, we represent this by assuming central bank preferences equal to

$$V_t = U_t - c(\pi_t - \mathbb{E}_{t-1}[\pi_t|\tilde{\theta}_{t-1}]),$$

where as before U_t denotes the preferences of society and the government, and where c is the constant linear cost to the conservative central banker of inflation exceeding firm inflation expectations.[62] We obtain the following result.

Proposition 25. *With an inflation-conservative central banker, the full-information Ramsey allocation can then be implemented by a dynamic inflation target with* $b_{t-1} = -\nu_{t-1} + c$*.*

Proposition 25 demonstrates that the appointment of an inflation-conservative central banker does not obviate the fundamental need for a dynamic inflation target. Intuitively, the inflation-conservative central banker applies a constant penalty to inflation, given by c. In the presence of persistent shocks, the target flexibility $-\nu_t$ of the dynamic inflation target changes over time. While an inflation-conservative central bank raises target flexibility on average, in the sense that $b_{t-1} = -\nu_{t-1} + c > -\nu_{t-1}$, the total implied inflation penalty $b_{t-1} - c$ is $-\nu_{t-1}$ just as before. The inflation target mechanism that implements the full-information Ramsey allocation is still time-varying and responds to persistent shocks.

In the language of Svensson (1997), however, appointing an inflation-conservative central banker can resolve *average* inflationary bias when c is set equal to the average value of ν_t in the stochastic steady state. When this average penalty is large (e.g., in the presence of a distorted

[62] This is a special case of preference disagreement in Appendix C.2.

steady state) but time variation in ν_t is small, approximating the dynamic inflation target with an inflation-conservative central bank may result in relatively small welfare losses.

Proposition 25 suggests that an alternative implementation of the dynamic inflation target might be to appoint new central bank chairs with appropriate inflation preferences in response to changes in ν_t. The inflation conservativeness of the central bank would then be time-varying and correspond to $c_t = \nu_{t-1}$. If in response to a shock at date $t-1$ the dynamic inflation target requires $\nu_{t-1} > \nu_{t-2}$, then a more dovish central banker at date $t-1$ should be replaced by a more hawkish central banker at t. Just as the dynamic inflation target must be updated one period in advance, the appointment of a new central banker would also be announced one period in advance.[63]

B.5.1 Proof of Proposition 25

The proof follows the same steps as in Proposition 3. The envelope condition is the same, given that the additional term $-c(\pi_t - \mathbb{E}_{t-1}[\pi_t|\tilde{\theta}_t])$ in V_t depends on reported types and not true types. From here, the value function at date t under our proposed mechanism given by

$$\begin{aligned}\mathcal{W}_t(\theta^t) &= b_{t-1}(\pi_t - \mathbb{E}_{t-1}\pi_t) + V_t + \beta\mathbb{E}_t\Big[\mathcal{W}_t(\theta^{t+1})|\theta_t\Big] \\ &= (-c + b_{t-1})(\pi_t - \mathbb{E}_{t-1}\pi_t) + U_t + \beta\mathbb{E}_t\Big[\mathcal{W}_t(\theta^{t+1})|\theta_t\Big] \\ &= -\nu_{t-1}(\pi_t - \mathbb{E}_{t-1}\pi_t) + U_t + \beta\mathbb{E}_t\Big[\mathcal{W}_t(\theta^{t+1})|\theta_t\Big]\end{aligned}$$

which is the same value function as in the proof of Proposition 3 when evaluated at the constrained efficient allocation. Thus the result follows using the same proof as for Proposition 3.

C Further Extensions

C.1 Welfare Gains from a Dynamic Inflation Target

We characterize the potential welfare gains under a dynamic inflation target. Suppose that the central bank adopts a permanent, static target (ν^*, τ^*) instead of the dynamic inflation target of Proposition 3.[64] The following proposition describes the first-order welfare gains from moving

[63] Importantly, just as a fixed central bank under the optimal mechanism was tasked with updating its own target, in an implementation with time varying conservativeness a central banker would be tasked with appointing her own replacement one period in advance (or at the least, would be responsible for naming her successor). However, this institutional arrangement is not typical (if used at all) in practice. For example, in the U.S. the president is tasked with appointing members of the Board of Governors, who must then be confirmed by the Senate.

[64] To simplify analysis, we will characterize welfare under a static target with full information, even though the dynamic inflation target implements the Ramsey allocation under incomplete information. This streamlines analysis because under a static target absent full information, the central bank's reporting constraints would be nontrivial due to

from the static target to a dynamic inflation target.

Proposition 26. *To first order, the welfare gains in allocative efficiency from moving from a static target* (ν^*, τ^*) *to the dynamic inflation target* (ν_{t-1}, τ_{t-1}) *of Proposition 3 are*

$$\mathbb{E} \sum_{t=1}^{\infty} \beta^t \Bigg[\underbrace{\nu^*_{t-1} - \nu^*}_{\text{Cost of Excess Inflation}} \Bigg] \Bigg[\underbrace{\mathbb{E}_{t-1}\pi^*_t - \tau_{t-1}}_{\text{Amount of Excess Inflation}} \Bigg].$$

The first order welfare gains available from moving to a dynamic inflation target depend on two forces. The first, $\nu^*_{t-1} - \nu^*$, is the intertemporal variation in the time consistency problem under the static target (where ν^*_{t-1} is the time consistency wedge evaluated at the allocation obtained under the static target). When $\nu^*_{t-1} > \nu^*$, the time consistency problem is more severe than the slope imposed ν^*, and hence inflation is too high relative to the efficient trade-off. In other words, the first term reflects the cost of excess inflation. The second term, $\mathbb{E}_{t-1}\pi^*_t - \tau_{t-1}$, is the difference between inflation expectations under the static target and inflation expectations under the dynamic target. High welfare gains are therefore available when a large excess time consistency problem, $\nu^*_{t-1} - \nu^*$, coincides with substantial excess inflation, $\mathbb{E}_{t-1}\pi^*_t - \tau_{t-1}$, relative to the constrained efficient inflation level. The dynamic inflation target thus allows welfare gains not only by allowing for greater inflation when the static target would be too severe, but also by allowing for lower inflation when the static target would be too flexible.

C.1.1 Proof of Proposition 26

To first order, the welfare gains of an inflation perturbation from the static target is

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\partial U_t}{\partial \pi_t} d\pi_t + \frac{\partial U_t}{\partial \mathbb{E}_t \pi_{t+1}} d\pi_{t+1} \right].$$

From here, the first order condition of the central bank is $\nu^* = \frac{\partial U_t}{\partial \pi_t}$, while by definition $\frac{\partial U_t}{\partial \mathbb{E}_t \pi_{t+1}} = -\beta \nu^*_t$. We have $\frac{\partial U_0}{\partial \pi_0} = 0$, so that we have

$$\mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \left[\nu^* - \nu^*_{t-1} \right] d\pi_t.$$

Finally, we have $\mathbb{E}_{t-1} d\pi_t = \tau_{t-1} - \mathbb{E}_{t-1}\pi^*_t$, giving the result.

information effects.

C.2 Preference Differences

We extend the costly transfers model (Section 5.2) to allow for preference disagreement. Formally, the central bank has utility U_t but the government has utility $V_t(\pi_t, \mathbb{E}_t[\pi_{t+1}|\tilde{\theta}_t], \theta_t)$. Social preferences of the government are now

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(V_t(\pi_t, \mathbb{E}_t[\pi_{t+1}|\tilde{\theta}_t], \theta_t) - \kappa T_t\right)\right]. \tag{31}$$

As before there is a central bank participation constraint. Define $K = \frac{\kappa}{1+\kappa}$ as before, and define *weighted reduced form preferences* to be

$$Z_t = (1-K)V_t + KU_t.$$

Weighted reduced form preferences average the preferences of the government and central bank. A higher weight is assigned to central bank preferences the more costly transfers are, that is K rises in κ. The optimal mechanism can be described as follows.

Proposition 27. *The solution to an optimal allocation rule of the relaxed problem is given by the first-order conditions*

$$\frac{\partial Z_t}{\partial \pi_t} - K\Gamma_t \frac{\partial U_t}{\partial \theta_t \partial \pi_t} = \lambda^*_{t-1}$$

where $\lambda^*_{t-1} = -\frac{1}{\beta}\frac{\partial Z_{t-1}}{\partial \mathbb{E}_{t-1}\pi_t} + K\Gamma_{t-1}\frac{1}{\beta}\frac{\partial^2 U_{t-1}}{\partial \theta_{t-1}\partial \mathbb{E}_{t-1}\pi_t}$ *and* Γ_t *is defined as in Proposition 17.*

The optimal allocation rule of Proposition 27 is similar to that of Proposition 17, but with one important difference: the weighted preference Z_t replaces the planner's utility. Intuitively, the government places value on the lifetime utility to the central bank because promising higher lifetime value allows the government to extract more surplus in the form of transfers. Counterveiling this force is information rents, which are analogous to before and only depend on central bank preferences U_t. Intuitively, these terms only depend on central bank preferences as information rents accrue based on central bank preferences. Otherwise, the intuitions of Section 5.2 carry over.

It is helpful to illustrate two dichotomous cases. If $K = 0$ and transfers are costless, we have $Z_t = V_t$ and hence the optimal allocation is the *government*'s Ramsey allocation. This follows intuitively: the government has no cost to designing a scheme that incentives the central bank to choose the government's preferred allocation. At the other extreme, if $K = 1$ then $Z_t = U_t$, that is to first order the planner only values transfers. Interestingly, the optimal allocation collapses to that of Proposition 17. Intuitively when the principal only cares about transfers, the principal on the one hand wants to make utility as high as possible to the agent in order to relax the central bank's participation constraint and extract larger transfers ex ante. On the other hand, the principal also internalizes that higher agent utility increasess agent information rents. This leads to the same allocation rule as in the case where principal and agent preferences are aligned except for transfers.

At intermediate values of K, the optimal allocation rule trades off the two extremes. On the one hand, the planner wishes to push the allocation closer to her Ramsey allocation, which increases her direct utility from allocations. At the same time, the planner wishes to push the allocation closer to the central bank's Ramsey allocation in order to relax the participation constraint and extract greater transfers. This leads to a balancing act determined by K, which encodes a relative weight the principal assigns to the different motivations.

As in Corollary 18, following $\theta_t \in \{\underline{\theta}, \overline{\theta}\}$ the optimal allocation reverts to the Ramsey allocation associated with weighted reduced-form preferences Z_t. If $K = 1$, then this allocation coincides with that of the dynamic inflation target.

C.2.1 Proof of Proposition 27

Observe that the integral envelope condition (27) still holds and implies Lemma 20 characterizes the central bank's value function, given central bank preferences have not changed. Thus the transfer rule is still given by $T_t = \mathcal{W}_t - U_t - \beta \mathbb{E}_t[\mathcal{W}_{t+1} | \theta_t]$. Thus we still have

$$-\mathbb{E} \sum_{t=0}^{\infty} T_t = \mathbb{E} \sum_{t=0}^{\infty} \beta^t U_t - \mathcal{W}_0$$

where $\mathcal{W}_0 = \mathbb{E}_0 \left[\sum_{s=0}^{\infty} \beta^s B_0^s(\theta^s) \middle| \theta_0 \right]$. Given the change in preferences, the government's objective function is now

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t V_t - \kappa T_t \right]$$

thus substituting in the transfer rule and definition of $\mathcal{W}_0$, the government's objective function is

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t \left[V_t + \kappa U_t - B_0^t \right] \right]$$

Finally dividing through by $1 + \kappa$ and defining $K = \frac{\kappa}{1+\kappa}$ $(1 - K = \frac{1}{1+\kappa})$, we obtain

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t \left[(1 - K) V_t + K U_t - K B_0^t \right] \right]$$

Thus we simply define $Z_t = (1 - K) V_t + K U_t$ and the derivation proceeds exactly the same as before with Z_t replacing U_t as the government's effective utility function. This recovers the first order condition given and completes the proof.

C.3 Inaction

Periods of policy inaction may arise between policy meetings or at the zero lower bound. Does the optimality of our dynamic inflation target mechanism extend to such periods of inaction? We extend our baseline model to allow for an inaction state. In this inaction state, the central bank is forced to set its policy variable π_t to an exogenously specified level. It can still influence current utility by communicating its type and the future policies it will set ("forward guidance"). In practice, central bankers use speeches and other forms of communication to convey information between policy meetings or when the economy is at the zero lower bound.

In this extension, the Ramsey allocation involves the central bank adjusting its next-period target in order to manage inflation expectations $\mathbb{E}_t\pi_{t+1}$ even while its policy variable π_t is exogenously fixed. We show that the dynamic inflation target remains locally incentive compatible. That is, the central bank's incentives to report truthfully are not affected by the inaction constraint. Intuitively, the dynamic inflation target mechanism already implements the full extent to which the Ramsey planner would like to use forward guidance. And because the preferences of Ramsey government and central bank over future inflation policy are aligned, this forward guidance is incentive compatible as in the baseline model. Even in the inaction region, therefore, target adjustments under our mechanism implement the forward guidance that the Ramsey planner would like to use.

At the beginning of each period t, a publicly observed and i.i.d. action/inaction state $I_t \in \{0, 1\}$ is realized. With probability p, the "action state" $I_t = 0$ is realized and the central bank is able to choose an inflation level π_t. With probability $1 - p$, the "inaction state" $I_t = 1$ is realized and the central bank must set inflation equal to an exogenous constant, $\pi_t = \pi^I$.

Reduced-form preferences are given by $U_t(\pi_t, \pi_t^e, \theta_t)$ as in the baseline model, where in this extension inflation expectations are

$$\pi_t^e = \mathbb{E}_t\left[\pi_{t+1}\middle|\tilde{\theta}_t\right] = p\mathbb{E}_t\left[\pi_{t+1}\middle|I_{t+1} = 0, \tilde{\theta}_t\right] + (1-p)\pi^I.$$

Parallel to the proof of Proposition 1, the Ramsey allocation π_t in the action state ($I_t = 0$) is given by $\frac{\partial U_t}{\partial \pi_t} = \nu_{t-1}$, where $\nu_{t-1} = -\frac{1}{\beta}\frac{\partial U_{t-1}}{\partial \pi_t^e}$ if $t > 0$ and $\nu_{-1} = 0$. Inflation, inflation expectations, and transfers are now functions of the shock history (θ^t, I^t), that is we have $\pi_t(\theta^t, I^t)$, $\pi_t^e(\theta^t, I^t)$, and $T_t(\theta^t, I^t)$.

Parallel to Section 1.3, we have

$$\begin{aligned}\mathcal{W}_t(\theta^{t-1}, \tilde{\theta}_t, I^t|\theta_t) = U_t\left(\pi_t(\theta^{t-1}, \tilde{\theta}_t, I^t), \pi_t^e(\theta^{t-1}, \tilde{\theta}_t, I^t), \theta_t\right) + T_t(\theta^{t-1}, \tilde{\theta}_t, I^t)\\ + \beta\mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1}, I^{t+1}|\theta_{t+1})\middle|\theta_t\right],\end{aligned}$$

and therefore obtain local incentive compatibility

$$\frac{\partial \mathcal{W}_t(\theta^t, I^t)}{\partial \theta_t} = \frac{\partial U_t\left(\pi_t(\theta^t, I^t), \pi_t^e(\theta^t, I^t), \theta_t\right)}{\partial \theta_t} + \beta \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t+1}, I^{t+1}) \frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)} \middle| \theta_t\right]. \tag{32}$$

Finally, we define our dynamic inflation target in this environment as before by

$$T_t(\theta^t, I^t) = b_{t-1}(\pi_t - \pi_{t-1}^e).$$

It is worth noting that the dynamic inflation target is maintained in both the action and inaction state (i.e., the central bank receives transfers even in the inaction state). This implies the central bank is rewarded/punished based on its inflation policy in the inaction state, even though it has no control over inflation in this state. However, transfers in the inaction state end up washing out, since the target level π_t^e includes the contribution of inflation in the inaction state to expectations.

We now obtain the counterpart of our main result.

Proposition 28. *In the action/inaction model, a dynamic inflation target implements the full-information Ramsey allocation in a locally incentive compatible mechanism, with target flexibility $b_{t-1} = -\nu_{t-1}$. The target (b_{t-1}, τ_{t-1}) is a sufficient statistic at date t for the history θ^{t-1} of past types.*

Why does the dynamic inflation target remain relevant even with inaction? One might have expected the central bank to have a motivation to lie in the inaction state in order to give itself more favorable inflation expectations, since its contemporaneous inflation policy is fixed and would not be affected by a misreport. In fact, the central bank *is* motivated to lie to alter inflation expectations favorably—a force that, crucially, is also present in our baseline model. Misreporting in this manner also changes the inflation target for the next period, however. The combined effect of a marginal change in reported type on current expectations and the next period target level is

$$\underbrace{\frac{\partial U_t}{\partial \pi_t^e} \frac{\partial \pi_t^e}{\partial \tilde{\theta}_t}}_{\text{Effect via Current Expectations}} + \underbrace{\beta \frac{\partial T_{t+1}^e}{\partial \pi_t^e} \frac{\partial \pi_t^e}{\partial \tilde{\theta}_t}}_{\text{Effect via Future Target Level}} = \left[\frac{\partial U_t}{\partial \pi_t^e} - \beta b_t\right] \frac{\partial \pi_t^e}{\partial \tilde{\theta}_t} = \left[-\beta \nu_t + \beta \nu_t\right] \frac{\partial \pi_t^e}{\partial \tilde{\theta}_t} = 0$$

Thus, just as in our baseline model, the benefit of lying to obtain more favorable inflation expectations is offset by the fact that such a lie alters the future target, affecting future penalties. Intuitively, our dynamic inflation target mechanism already provides the central bank with incentives to implement forward guidance to the full extent the Ramsey planner would like to use it.

C.4 Proof of Proposition 28

The derivation of the Envelope condition for local incentive compatibility parallels that of the baseline model, since $f(\theta_{t+1}|\theta_t)$ does not depend on the action/inaction state. We therefore proceed as usual by showing the value function generated by our mechanism satisfies this envelope condition.

Verifying the envelope condition. We now verify the value function under our mechanism satisfies the envelope condition. The value function associated with the mechanism is

$$\mathcal{W}_t(\theta^t, I^t) = -\nu_{t-1}(\theta^{t-1}, I^{t-1})\Big(\pi_t(\theta^t, I^t) - \pi^e_{t-1}(\theta^{t-1}, I^{t-1})\Big) + U_t\left(\pi_t(\theta^t, I^t), \pi^e_t(\theta^t, I^t), \theta_t\right)$$
$$+ \beta\mathbb{E}_t\Big[\mathcal{W}_{t+1}(\theta^{t+1}, I^{t+1})\Big|\theta_t\Big]$$

From here, recall that ν_{t-1} and $\mathbb{E}_{t-1}[\pi_t|\theta_{t-1}]$ are only functions of θ^{t-1}. Therefore, $\frac{\partial \nu_{t-1}}{\partial \theta_t} = \frac{\partial \mathbb{E}_{t-1}[\pi_t|\theta_{t-1}]}{\partial \theta_t} = 0$. Thus differentiating the value function in θ_t, we have

$$\frac{\partial \mathcal{W}_t(\theta^t, I^t)}{\partial \theta_t} = \frac{\partial U_t}{\partial \theta_t} + \beta\mathbb{E}_t\Big[\mathcal{W}_{t+1}\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\Big|\theta_t\Big].$$
$$+ \Big(-\nu_{t-1} + \frac{\partial U_t}{\partial \pi_t}\Big)\frac{\partial \pi_t}{\partial \theta_t} + \frac{\partial U_t}{\partial \pi^e_t}\frac{d\pi^e_t}{d\theta_t} + \beta\mathbb{E}_t\Big[\frac{\partial \mathcal{W}_{t+1}(\theta^{t+1}, I^{t+1})}{\partial \theta_t}\Big|\theta_t\Big]$$

Writing out continuation value function $\mathcal{W}_{t+1}$ in sequence notation, we have

$$\mathcal{W}_{t+1} = -\nu_t\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big)$$
$$-\mathbb{E}_{t+1}\Big[\sum_{s=1}^{\infty}\beta^s\nu_{t+s}\Big(\pi_{t+1+s} - \mathbb{E}_{t+s}[\pi_{t+1+s}|\theta_{t+s}]\Big)\Big|\theta_{t+1}\Big]$$
$$+\mathbb{E}_{t+1}\Big[\sum_{s=0}^{\infty}\beta^s U_{t+1+s}\left(\pi_{t+1+s}, \mathbb{E}_{t+1+s}\left[\pi_{t+2+s}|\theta_{t+1+s}\right], \theta_{t+1+s}\right)\Big|\theta_{t+1}\Big]$$

As in the proof of our main result, since ν_{t+s} is only a function of (θ^{t+s}, I^{t+s}) and so is a constant from the date $t+s+1$ perspective we have

$$\mathbb{E}_{t+1}\Big[\nu_{t+s}\pi_{t+1+s}|\theta_{t+1}\Big] = \mathbb{E}_{t+1}\Big[\mathbb{E}_{t+s}\Big[\nu_{t+s}\pi_{t+1+s}\Big|\theta_{t+s}\Big]|\theta_{t+1}\Big] = \mathbb{E}_{t+1}\Big[\nu_{t+s}\mathbb{E}_{t+s}\Big[\pi_{t+1+s}\Big|\theta_{t+s}\Big]|\theta_{t+1}\Big]$$

and therefore the second line above is equal to zero. Note we did not use anything about whether we are in the action or inaction state in this argument. Therefore, we can write

$$\mathcal{W}_{t+1} = -\nu_t\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) + \mathbb{E}_{t+1}\left[\sum_{s=0}^{\infty}\beta^s U_{t+1+s}\left(\pi_{t+1+s}, \mathbb{E}_{t+1+s}\left[\pi_{t+2+s}|\theta_{t+1+s}\right], \theta_{t+1+s}\right)\Big|\theta_{t+1}\right]$$

Observe that this is an *augmented Lagrangian* at date $t+1$: it is the date $t+1$ lifetime value (second line), plus an augmented penalty on date $t+1$ inflation. The Ramsey solution is a critical point of the augmented Lagrangian, which leads to a simple derivative. Formally, we know that the impact of a change in report θ_t on continuation value through changes in inflation policy at date $t+2+s$, $s \geq 0$, is

$$\left[\frac{dU_{t+1+s}}{\partial \mathbb{E}_{t+1+s}\pi_{t+2+s}} + \beta\frac{\partial U_{t+2+s}}{\partial \pi_{t+2+s}}\right]\frac{d\pi_{t+2+s}}{d\theta_t} = 0.$$

If $I_{t+2+s} = 0$ and the central bank is in the action state, then $\frac{dU_{t+1+s}}{\partial \mathbb{E}_{t+1+s}\pi_{t+2+s}} + \beta\frac{\partial U_{t+2+s}}{\partial \pi_{t+2+s}} = 0$ and hence the above equality holds. If instead $I_{t+2+s} = 1$ and the central bank is in the inaction state, then $\frac{d\pi_{t+2+s}}{d\theta_t} = 0$ and again the above is equal to zero. Thus the above equality holds.

Using this result, we therefore have

$$\begin{aligned}
\frac{\partial \mathcal{W}_{t+1}}{\partial \theta_t} &= -\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) - \nu_t\left(\frac{\partial \pi_{t+1}}{\partial \theta_t} - \frac{d\mathbb{E}_t[\pi_{t+1}|\theta_t]}{d\theta_t}\right) + \frac{\partial U_{t+1}}{\partial \pi_{t+1}}\frac{\partial \pi_{t+1}}{\partial \theta_t} \\
&= -\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) + \nu_t\frac{d\mathbb{E}_t[\pi_{t+1}|\theta_t]}{d\theta_t}
\end{aligned}$$

where the second line follows since $\nu_t = \frac{\partial U_{t+1}}{\partial \pi_{t+1}}$ in the action state (Ramsey) and $\frac{\partial \pi_{t+1}}{\partial \theta_t} = 0$ in the inaction state.

Now substituting back into the expression for $\frac{\partial \mathcal{W}_t}{\partial \theta_t}$, we have

$$\begin{aligned}
\frac{\partial \mathcal{W}_t(\theta^t, I^t)}{\partial \theta_t} =& \frac{\partial U_t}{\partial \theta_t} + \beta\mathbb{E}_t\left[\mathcal{W}_{t+1}\frac{\partial f(\theta_{t+1}|\theta_t)/\partial\theta_t}{f(\theta_{t+1}|\theta_t)}\Big|\theta_t\right]. \\
&+ \left(-\nu_{t-1} + \frac{\partial U_t}{\partial \pi_t}\right)\frac{\partial \pi_t}{\partial \theta_t} + \frac{\partial U_t}{\partial \pi_t^e}\frac{d\mathbb{E}_t[\pi_{t+1}|\theta_t]}{d\theta_t} \\
&+ \beta\mathbb{E}_t\left[-\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big) + \nu_t\frac{d\mathbb{E}_t[\pi_{t+1}|\theta_t]}{d\theta_t}\Big|\theta_t\right]
\end{aligned}$$

The first term on the third line is zero, since

$$\mathbb{E}_t\left[-\frac{\partial \nu_t}{\partial \theta_t}\Big(\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big)\Big|\theta_t\right] = -\frac{\partial \nu_t}{\partial \theta_t}\mathbb{E}_t\left[\pi_{t+1} - \mathbb{E}_t[\pi_{t+1}|\theta_t]\Big|\theta_t\right] = 0.$$

From here, we can rearrange terms to get

$$\begin{aligned}\frac{\partial \mathcal{W}_t(\theta^t, I^t)}{\partial \theta_t} =& \frac{\partial U_t}{\partial \theta_t} + \beta \mathbb{E}_t \left[\mathcal{W}_{t+1}(\theta^{t+1}) \frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)} \middle| \theta_t \right] \\ &+ \left[-\nu_{t-1} + \frac{\partial U_t}{\partial \pi_t} \right] \frac{\partial \pi_t}{\partial \theta_t} + \left[\frac{\partial U_t}{\partial \mathbb{E}_t \left[\pi_{t+1}|\theta_t\right]} + \beta \nu_t \right] \frac{d\mathbb{E}_t \left[\pi_{t+1}|\theta_t\right]}{d\theta_t}\end{aligned}$$

The first term on the second line is zero, since either $-\nu_{t-1} + \frac{\partial U_t}{\partial \pi_t} = 0$ (Ramsey in the action state) or else $\frac{\partial \pi_t}{\partial \theta_t} = 0$ (inaction state). The second term on the second line is zero from the definition $\frac{\partial U_t}{\partial \mathbb{E}_t[\pi_{t+1}|\theta_t]} + \beta \nu_t = 0$. Thus, the entire second line is zero, and we are left with

$$\frac{\partial \mathcal{W}_t(\theta^t, I^t)}{\partial \theta_t} = \frac{\partial U_t}{\partial \theta_t} + \beta \mathbb{E}_t \left[\mathcal{W}_{t+1} \frac{\partial f(\theta_{t+1}|\theta_t)/\partial \theta_t}{f(\theta_{t+1}|\theta_t)} \middle| \theta_t \right]$$

which is the required envelope condition. This concludes the proof.

D Sufficient Statistics for the K-horizon dynamic inflation target

In this appendix, we show how to use two $K \times 1$ vectors as sufficient statistics for the history of shocks under the K-horizon dynamic inflation target. We only need to carry two $K \times 1$ vectors, $\boldsymbol{V}_{t-1} = \{\boldsymbol{V}_{t-1,t}, \ldots, \boldsymbol{V}_{t-1,t-1+K}\}$ and $\boldsymbol{T}_{t-1} = \{\boldsymbol{T}_{t-1,t}, \ldots, \boldsymbol{T}_{t-1,t-1+K}\}$.

We define $\boldsymbol{V}_{t-1,t-1+k}$ as cumulative promises inherited at the beginning of date t (end of date $t-1$) for date $t-1+k$. Thus, $\boldsymbol{V}_{t-1,t} = \bar{\nu}_{t-1}$ corresponds to target flexibility at date t and summarizes all commitments made over the past K periods. By contrast, $\boldsymbol{V}_{t-1,t-1+k}$ for $k > 1$ reflects the cumulative *partial commitments* the central bank has made so far for dates beyond t. We refer to these as partial commitments precisely because they can still be updated at date t. We can track the evolution of partial commitments using the recursion

$$\boldsymbol{V}_{t,t+k} = \boldsymbol{V}_{t-1,t+k} + \nu_{t,t+k}$$

where $\boldsymbol{V}_{t-1,t+K} \equiv 0$ and $\nu_{t,t+k}$ reflects the new promise made at date t for target flexibility k periods ahead. To illustrate, note that $\boldsymbol{V}_{t,t+1} = \boldsymbol{V}_{t-1,t+1} + \nu_{t,t+1} = \bar{\nu}_t$: target flexibility for period $t+1$ results from adding a new partial commitment made in period t, $\nu_{t,t+1}$, to our measure of cumulative promises made in the past, $\boldsymbol{V}_{t-1,t+1}$. Vector $\boldsymbol{V}_{t-1}$ thus summarizes all relevant information for updating target flexiblity at date t to $\boldsymbol{V}_t$.

To update the target level τ_t, the central bank must compute a weighted average of historical inflation forecasts. The evolution of this weighted average of forecasts satisfies the recursion

$$\tau_t = \frac{\nu_{t,t+1}}{\bar{\nu}_t}\mathbb{E}_t[\pi_{t+1}|\tilde{\theta}_t] + \sum_{k=1}^{K-1}\frac{\nu_{t-k,t+1}}{\bar{\nu}_t}\mathbb{E}_{t-k}[\pi_{t+1}|\tilde{\theta}_{t-k}]$$

$$= \frac{\nu_{t,t+1}}{\nu_{t,t+1}+\boldsymbol{V}_{t-1,t+1}}\mathbb{E}_t[\pi_{t+1}|\tilde{\theta}_t] + \frac{\boldsymbol{V}_{t-1,t+1}}{\nu_{t,t+1}+\boldsymbol{V}_{t-1,t+1}}\underbrace{\sum_{k=1}^{K-1}\frac{\nu_{t-k,t+1}}{\boldsymbol{V}_{t-1,t+1}}\mathbb{E}_{t-k}[\pi_{t+1}|\tilde{\theta}_{t-k}]}_{\equiv \boldsymbol{T}_{t-1,t+1}},$$

where the first line expresses τ_t as an average of current and historical inflation forecasts with weights directly taken from Proposition 12. We introduce $\boldsymbol{T}_{t-1}$ to track the evolution of average forecasts and summarize the information needed by the central bank to update its target level. Its first element reflects the current target level, $\boldsymbol{T}_{t-1,t} = \tau_{t-1}$, which is taken as given at date t. For $k > 1$, $\boldsymbol{T}_{t-1,t-1+k}$ summarizes the cumulative weighted average of historical forecasts for inflation in period $t-1+k$. Its evolution satisfies the recursion

$$\boldsymbol{T}_{t,t+k} = \frac{\boldsymbol{V}_{t-1,t+k}}{\boldsymbol{V}_{t-1,t+k}+\nu_{t,t+k}}\boldsymbol{T}_{t-1,t+k} + \frac{\nu_{t,t+k}}{\boldsymbol{V}_{t-1,t+k}+\nu_{t,t+k}}\mathbb{E}_t[\pi_{t+k}|\tilde{\theta}_t].$$

To implement the K-horizon dynamic inflation target, the central bank must therefore keep track of $(\boldsymbol{V}_{t-1}, \boldsymbol{T}_{t-1})$. Intuitively, these two vectors encode a notion of forward guidance in the form of partial commitments for what the central bank will do for the next K periods. At date t, the central bank takes as given its target for the current date, $\tau_{t-1} = \boldsymbol{T}_{t-1,t}$ and $b_{t-1} = \boldsymbol{V}_{t-1,t}$, and lacks any ability to update this target. The central bank has partial ability to update its target for periods $t+k$, for $1 \leq k < K$, taking as given its prior commitments that are encoded in $\boldsymbol{V}_{t-1,t+k}$ and $\boldsymbol{T}_{t-1,t+k}$. Finally, the central bank has no prior commitment over inflation at date $t+K$, and so makes its first partial commitment for this period at date t. This provides a generalized notion of the iterated one-period commitments of the baseline model: The central bank here makes iterated K-period *partial* commitments.

E Global Incentive Compatibility

E.1 K-Horizon Dynamic Inflation Target

As in Section 2.3, let us define the *augmented Lagrangian* as

$$\mathcal{L}_t(\vartheta^t|\theta_t) = -\,\mathbb{E}_t\left[\sum_{k=0}^{K-1}\beta^k \boldsymbol{V}_{t-1,t+k}\pi_{t+k}(\vartheta_t^{t+k})\middle|\theta_t\right]$$
$$+\,\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s U_{t+s}(\pi_{t+s}(\vartheta^{t+s}), \mathbb{E}_{t+s}[\pi_{t+s+1}(\vartheta_t^{t+s+1})|\theta_{t+s}], \ldots, \mathbb{E}_{t+s}[\pi_{t+s+K}(\vartheta_t^{t+s+K})|\theta_{t+s}], \theta_{t+s})\middle|\theta_t\right]$$

where $\boldsymbol{V}_{t-1}$ is defined in Appendix D. We can then obtain a characterization of global incentive compatibility that mirrors that of Lemma 4.

Lemma 29. *The dynamic inflation target is globally incentive compatible if*

$$\begin{aligned}\mathcal{L}_t(\theta^t|\theta_t) - \mathcal{L}_t(\vartheta^t|\theta_t) \geq & U_t(\pi_t(\vartheta^t), \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\tilde{\theta}_t], \ldots, \mathbb{E}_t[\pi_{t+K}(\vartheta_t^{t+K})|\tilde{\theta}_t], \theta_t) \\ & - U_t(\pi_t(\vartheta^t), \mathbb{E}_t[\pi_{t+1}(\vartheta_t^{t+1})|\theta_t], \ldots, \mathbb{E}_t[\pi_{t+K}(\vartheta_t^{t+K})|\theta_t], \theta_t) \\ & + \sum_{k=1}^{K} \beta^k \nu_{t,t+k}(\vartheta_t^t) \left(\mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\tilde{\theta}_t] - \mathbb{E}_t[\pi_{t+k}(\vartheta_t^{t+k})|\theta_t] \right)\end{aligned}$$

Proof. The proof parallels the proof of Lemma 4. Recall from the proof of Proposition 12 that global IC is $\mathcal{W}_t(\theta^t|\theta_t) \geq \mathcal{W}_t(\theta^{t-1}, \tilde{\theta}_t|\theta_t)$ for all $t, \theta^t, \tilde{\theta}_t$, where

$$\begin{aligned}\mathcal{W}_t(\theta^{t-1}, \tilde{\theta}_t|\theta_t) = & U_t\left(\pi_t(\theta^{t-1}, \tilde{\theta}_t), \pi_t^e(\theta^{t-1}, \tilde{\theta}_t), \ldots, \pi_{t,t+k}^e(\theta^{t-1}, \tilde{\theta}_t), \theta_t\right) + T_t(\theta^{t-1}, \tilde{\theta}_t) \\ & + \beta \mathbb{E}_t\left[\mathcal{W}_{t+1}(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1}|\theta_{t+1})\middle|\theta_t\right].\end{aligned}$$

Recall further that

$$\mathcal{W}_{t+1}(\theta^{t+1}) = -\mathbb{E}_{t+1}\sum_{s=0}^{K-1}\beta^s\left[\sum_{s<k\leq K}\nu_{t+1+s-k,t+1+s}\left(\pi_{t+1+s} - \mathbb{E}_{t+1+s-k}[\pi_{t+1+s}|\theta_{t+1+s-k}]\right)\right] + \mathbb{E}_{t+1}\sum_{s=0}^{\infty}\beta^s U_{t+1+s}$$

The result follows immediately from the definitions of $\boldsymbol{V}_{t-1,t+k}$ and from noting that $\mathbb{E}_{t+1+s-k}[\pi_{t+1+s}|\theta_{t+1+s-k}]$ does not depend on $(\theta_t, \tilde{\theta}_t)$ except at $k = s+1$. ■

E.2 Global IC in Quasilinear Models

We conclude by characterizing global incentive compatibility when preferences are quasilinear in inflation expectations,

$$U_t(\pi_t, \pi_t^e, \theta_t) = u(\pi_t, \theta_t) - g(\theta_t)\beta\pi_t^e. \tag{33}$$

This case gives rise to an economically insightful sufficient condition and also nests the flattening Phillips curve application of Section 3.2.[65]

This case is tractable because the Ramsey allocation is time-invariant and does not depend on the density f. In particular, the Ramsey allocation $\pi_t(\theta^t) \equiv \pi(\theta_{t-1}, \theta_t)$ is given implicitly as $\frac{\partial u(\pi(\theta_{t-1},\theta_t),\theta_t)}{\partial \pi_t} = g(\theta_{t-1})$. This allows us to characterize a stronger-than-needed sufficient condition

[65] The results of this section extend readily to the case where u_t and g_t are time-dependent. Policies and value gains are then explicitly indexed by time, and the sufficient condition of Proposition 30 holds for each date t.

for global incentive compatibility by showing that Lemma 4 holds history-by-history, rather than in expectation. In doing so, we show that global incentive compatibility can be guaranteed by a bound on a likelihood ratio.[66]

Proposition 30. *With quasilinear reduced-form preferences (33), a sufficient condition for global incentive compatibility is*

$$\Big(g(\tilde{\theta}_t) - g(\theta_t)\Big)\pi(\tilde{\theta}_t, \theta_{t+1})\Bigg(\overbrace{\frac{f(\theta_{t+1}|\tilde{\theta}_t)}{f(\theta_{t+1}|\theta_t)}}^{\text{Likelihood Ratio}} - 1\Bigg) \leq \Delta(\tilde{\theta}_t, \theta_{t+1}|\theta_t) \tag{34}$$

for all $\theta_t, \tilde{\theta}_t, \theta_{t+1}$, *where*

$$0 \leq \Delta(\tilde{\theta}_t, \theta_{t+1}|\theta_t) \equiv u(\pi(\theta_t, \theta_{t+1}), \theta_{t+1}) - g(\theta_t)\pi(\theta_t, \theta_{t+1}) - \Big[u(\pi(\tilde{\theta}_t, \theta_{t+1}), \theta_{t+1}) - g(\theta_t)\pi(\tilde{\theta}_t, \theta_{t+1})\Big]$$

is the utility gain from the date $t+1$ *inflation policy from truthful reporting* θ_t *as opposed to misreporting* $\tilde{\theta}_t$.

Proof. The result follows readily from Lemma 29 combined with the fact the Ramsey allocation as $\pi(\theta_{t-1}, \theta_t)$. Equation 34 follows by forcing Lemma 29 to hold history-by-history and by discarding gains in value of the augmented Lagrangian that come from the date t inflation (that is, only looking at $t+1$).[67] ■

Proposition 30 highlights that sufficient conditions for global incentive compatibility come as a bound on deviations of the likelihood ratio $\frac{f(\theta_{t+1}|\tilde{\theta}_t)}{f(\theta_{t+1}|\theta_t)}$ from one, where the likelihood ratio measures the likelihood of θ_{t+1} under a misreported type $\tilde{\theta}_t$ as opposed to the truthful type θ_t.[68] Intuitively, equation (34) tells us that violations of global incentive compatibility occur when the central bank can substantially alter firm and government beliefs by misreporting, in excess of the loss from distorting the Ramsey allocation.

There are two special cases of the quasilinear model in which global incentive compatibility is guaranteed. Both conditions also inform the characterization of Proposition 30.

The first special case is that of iid shocks, where the likelihood ratio is one and hence Proposition 30 necessarily holds. Thus it is only when shocks are persistent, and hence the likelihood ratio may deviate from one, that global incentive compatibility may be violated.

[66] Equation 34 is stronger than necessary for two reasons. First, equation 34 is specified history by history rather than in expectation. Second, equation 34 ignores losses in value that arise because a misreport at date t also distorts the date t allocation.

[67] This is valid sufficient condition because the quasilinear form means the Ramsey policy is not just a critical point of the augmented Lagrangian at date t, but also maximizes the augmented Lagrangian.

[68] Observe that Proposition 30 generally provides two bounds on the same likelihood ratio. The first bound comes from true type θ_t misreporting as $\tilde{\theta}_t$, while the second comes from true type $\tilde{\theta}_t$ misreporting as θ_t.

The second case in which global incentive compatibility is guaranteed arises when the quasilinear weight $g(\theta)$ is not a function of θ, that is $g(\theta) = g_0$. Economically, global incentive compatibility is guaranteed in this case because the flexibility of the dynamic inflation target is constant over time and equal to g_0. As a result, the benefits and costs of manipulating firm and government beliefs are not only locally offsetting, but also globally offsetting. Hence, global incentive compatibility may be violated in Proposition 30 because the global benefit of manipulating firm beliefs always depends on the true quasilinear weight $g(\theta)$, whereas the benefit of manipulating government beliefs depends on the reported weight $g(\theta)$. This highlights the offsetting effects of manipulating firm and government beliefs achieved by the dynamic inflation target.