Online Appendix for 'The Choice Channel of Financial Innovation'

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Online Appendix B: Omitted Extensions

In this appendix, we present various extensions of the analysis in the main text. The proofs are relegated to Online Appendix C.

1. *Background risks and the precautionary channel*

In the main text, we focused on the cases in which investors effectively do not face any background risks so that they do not have precautionary savings concerns. We next illustrate the effect of financial innovation in an environment with precautionary savings. We isolate an alternative precautionary channel of financial innovation and contrast it with our choice channel. We also obtain a testable implication of the precautionary channel with respect to market participation, which we empirically analyze in Section IV.

Isolating the precautionary channel requires putting additional structure on the model. To this end, we replace Assumption 1 with the following set of assumptions.

Assumption 1^{P}.
(i) Investors share the same beliefs, $E^i[\cdot] = E[\cdot]$ for some common $E[\cdot]$.
(ii) There exists a stochastic discount factor, $(M(\mathbf{z}))_{\mathbf{z}\in Z}$, that prices each asset, that is,

$$P_j = E[M(\mathbf{z})\varphi_j(\mathbf{z})] \text{ for each } j \in \mathbf{J},$$

(iii) Investors' background risks are orthogonal to the stochastic discount factor,

$$cov[L(\mathbf{z}) M(\mathbf{z})] = 0 \text{ for each } i.$$

Here, the first assumption rules out speculation (and the income and the substitution effects that it generates). While speculation plays a central role for the choice channel, it is typically assumed away in traditional analyses that rely on rational (and thus, common) expectations. The second assumption is a no arbitrage condition. The third assumption holds if the background risks are idiosyncratic (and thus, uncorrelated with aggregate risk)—the typical case analyzed in the literature. Finally, we also set the elasticity of substitution to one, to ensure that the choice channel is completely shut down.

PROPOSITION 5 (Precautionary Channel): *Suppose Assumption 1^P holds and $\varepsilon = 1$. Suppose also that investors initially only have access to the risk-free asset, that is, $J^{old} = \{f\}$, and that financial innovation completes the market, that is, $|\mathbf{J}| = |Z|$. Then, financial innovation reduces the investor's asset holdings (and thus, savings), $A_0^{i,new} \leq A_0^{i,old}$.*

The result establishes conditions under which financial innovation induces the investor to save less—consistent with much of the precautionary savings literature (see the references in the introduction). Intuitively, when markets are incomplete, the investor saves for precautionary reasons. This is because she faces some background risks, and the constant elasticity preferences satisfy the prudence condition. Financial innovation enables the investor to hedge her risks. By doing so, it alleviates the precautionary demand for saving, thereby reducing savings. Depending on the stochastic discount factor, financial innovation can also increase the investors' risk-adjusted return (even if there are no belief disagreements)

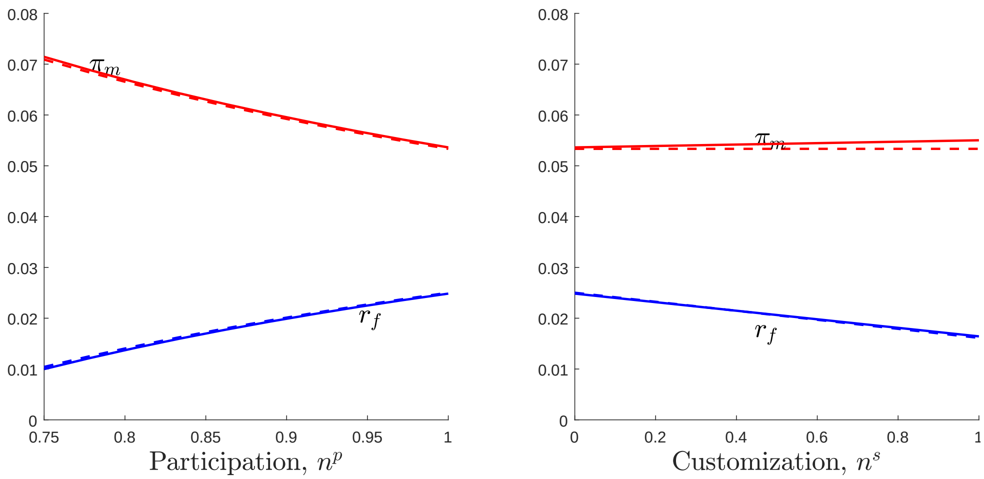

FIGURE B.1. COMPARISON OF EQUILIBRIUM INTEREST RATES AND RISK PREMIA UNDER THE EXACT SOLUTION TO THE PORTFOLIO PROBLEM (SOLID LINES) AND THE SECOND-ORDER APPROXIMATION (DASHED LINES).

by enabling her to participate in aggregate risk sharing. The assumption, $\varepsilon = 1$, ensures that the substitution and the income effects exactly cancel, so that the precautionary channel is the only influence on saving.[21] The result also focuses on a specific type of innovation that takes the investor from the risk-free asset to a fully complete market. This is a technical requirement that enables us to obtain a theoretical result. The economic insights should apply more broadly as long as financial innovation provides new opportunities to hedge background risks.

2. Exact solutions to the portfolio problem and equilibrium

Section III assumes that the agents' asset demand is given by an approximate solution to their portfolio problem. In this section, we solve for an alternative equilibrium, under the assumption that agents solve an exact version of that problem. In this exact version, portfolio weights and asset demand functions do not admit closed-form solutions, so we resort to a numeric approach.

We first define an exact equilibrium as follows.

DEFINITION 2 (Exact Equilibrium): *Under Assumptions 1^G and 2, an exact equilibrium,* $\left\{ \left(\omega^i_{J^i}, A^i_0 \right)_i, P_j \right\}$, *is a collection such that the investors' beliefs for asset returns are given by (7), their portfolio weights are obtained from the solution to (4) using the definition* $\omega^i_j = \frac{P_j x^i_j}{\sum_{\tilde{j} \in J^i} P_j x^i_j}$, *their asset holdings (A^i_0) are given by the solution to problem (5) taking as given the certainty equivalent implicit in* $V_1(A_0) = R^i_{ce} A_0$, *and the asset markets clear [cf. Eq. (12)].*

The environment we study is otherwise identical to Section III.C. Figure B.1 compares the exact and approximate equilibrium solutions for varying degrees of participation (n^p) and customization (n^s). The exact equilibrium risk premium and risk-free rate are displayed in solid lines, while the approximate solutions from Section III.C are displayed in dashed lines. The approximation error is small. The responses of the riskless rate and the risk premium to the comparative statics remain largely unchanged: participation leads to a decreasing risk premium and increasing interest rate, while customization leads

[21]If there is no aggregate risk, as in Aiyagari (1994), then the first part of Assumption 1^P is sufficient to shut down the choice channel. In this special case, the result can be generalized beyond $\varepsilon^i = 1$.

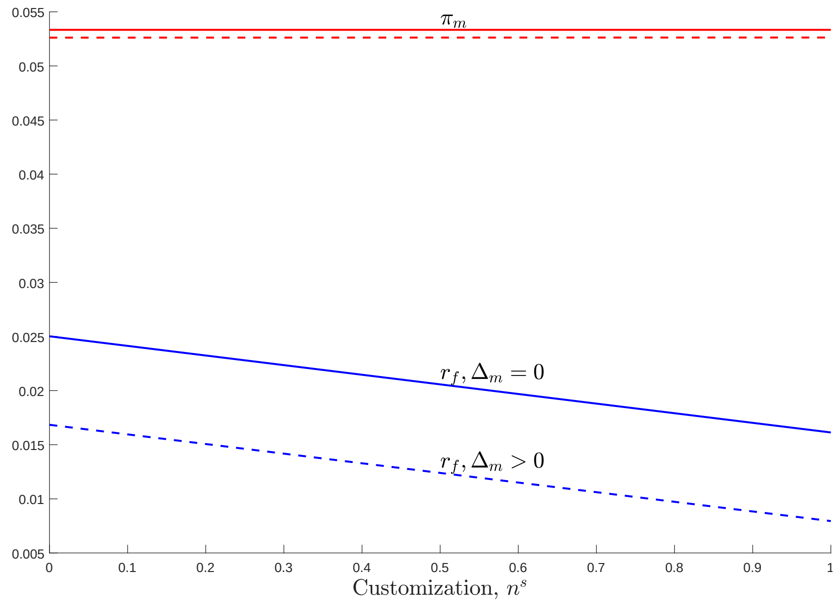

FIGURE B.2. EFFECT OF CUSTOMIZATION WHEN THERE IS DISAGREEMENT ON THE MARKET PORTFOLIO.

to a decreasing interest rate. In the exact solution, increased customization leads to a minor increase in risk premia, originating from third and higher order terms in preferences.

3. Robustness of the effect of customization on the interest rate

Proposition 4 in Section III.B characterizes the effect of increasing portfolio customization on asset returns in a relatively stylized setting with strong assumptions. In this section, we show that the effect on the interest rate is robust to relaxing many of these assumptions. We first illustrate (numerically) that allowing for disagreement on the market portfolio does not overturn the effect on the interest rate. We then show that the result is also robust to allowing for short-selling constraints. Finally, we introduce investment into the model and show that the effect on the interest rate holds also in this setting.

CUSTOMIZATION WITH DISAGREEMENT ON THE MARKET PORTFOLIO. — Recall that we rule out disagreement on the market portfolio (see the third part of Assumption 4). Absent this assumption, the effect of customization is largely unchanged in our numerical simulations, even though we are unable to prove an analytical result. To see this, consider the numerical example we analyze in Section III.C with the only difference being that investors also disagree about the market portfolio. Specifically, an investor who is optimistic (resp. pessimistic) about the speculative asset is also optimistic (resp. pessimistic) about the market portfolio, $\mu_m^{\Delta_s} = \mu_m + \Delta_m$ (resp. $\mu_m^{-\Delta_s} = \mu_m - \Delta_m$) for some $\Delta_m > 0$. We calibrate the level of disagreement on the market portfolio by assuming, $\left|\Delta_m/\sqrt{\Lambda_m}\right| = \left|\Delta_s/\sqrt{\Lambda_s}\right|$, so that the extent of disagreement on the market portfolio (normalized by risk) is the same as the extent of disagreement on the speculative asset. As in Sections III.B and III.C, we assume everyone has access to the market portfolio, $n^n = 0$, and numerically investigate the effect of increasing the fraction of investors that have access to the speculative asset in addition to the market portfolio, $n^s \in [0, 1]$.

Figure B.2 illustrates the results of increased customization in this case. Compared to the earlier case with $\Delta_m = 0$, the risk-free rate is uniformly lower. The risk premium is also slightly lower, but the difference is not discernible. More importantly, increased customization reduces the risk-free rate and does not have a discernible effect on the risk premium, as in Proposition 4, even though the third part of Assumption 4 is violated.

When investors disagree on the market portfolio, they take speculative positions on the market portfolio as well as the speculative asset. This generates an additional increase in their certainty-equivalent returns and reduces the risk-free rate further. However, speculation on the market portfolio also breaks the symmetry between optimists' and pessimists' returns in Eq. (19). Since the asset m is in positive supply, all investors are its natural buyers. Even if optimists did not adjust their positions (relative to the average investor), their perceived return would be higher simply because they are already holding the market portfolio. Therefore, in equilibrium, optimists have a greater (perceived) certainty-equivalent return—and hold more assets—relative to pessimists. This asymmetry implies that belief disagreements can potentially also affect relative asset prices and risk premia, which makes a theoretical characterization difficult. However, for empirically relevant parameters, these asymmetric effects are very small, as illustrated by Figure B.2, and the effect of greater customization remains qualitatively unchanged.

CUSTOMIZATION WITH SHORT-SELLING CONSTRAINTS. — In our model, we assume the investors can short sell the risky assets without constraints. When short-selling constraints bind on some assets, there are additional complications but the effect of customization on the interest rate remains qualitatively unchanged.[22] To see this, suppose the investors cannot short sell a fraction of the nonmarket assets , $\tilde{\mathbf{J}} \subset \{1, .., K-1\}$. Formally, the portfolio problem (8) features an additional constraint,

$$\omega_j^i \geq 0 \text{ for each } j \in \tilde{\mathbf{J}}. \tag{B.1}$$

We continue to make all of the other assumptions in Lemma 2 (including no disagreement on the market portfolio). We also assume $n^{i_A} > 0$ for each $i_A \in I_A$, that is, there is a positive mass of investors of each access type (even before customization improves the market access).

The following lemma characterizes the equilibrium with short-selling constraints. To state the result, we define the notation $\Lambda_{J^{i_A}} = \begin{bmatrix} \Lambda_{\tilde{J}^{i_A}} & \tilde{\Lambda}_{\tilde{J}^{i_A}} \\ \left(\tilde{\Lambda}_{\tilde{J}^{i_A}}\right)' & \Lambda_{J^{i_A}\setminus\tilde{\mathbf{J}}} \end{bmatrix}$ for any investor with market access i_A, where $\tilde{J}^{i_A} = J^{i_A} \cap \tilde{\mathbf{J}}$. We also let $\tilde{\Lambda}_{j,\tilde{J}^{i_A}}$ denote the j-th row of $\tilde{\Lambda}_{\tilde{J}^{i_A}}$.

LEMMA 3: *Consider the setting in Lemma 2 with the short-selling constraints in* (*B*.1). *Then, in equilibrium, the risk premia satisfy*

$$\pi_j = \begin{cases} \frac{\Lambda_{jm}}{\Lambda_m}\pi_m & \text{for } j \notin \mathbf{J}\setminus\tilde{\mathbf{J}} \\ \frac{\Lambda_{jm}}{\Lambda_m}\pi_m - \max_{(\mathbf{i}_A,\mathbf{i_B})}\Delta_j^{(\mathbf{i}_A,\mathbf{i_B})} & \text{for } j \in \tilde{\mathbf{J}} \end{cases}, \tag{B.2}$$

where $\pi_m = \gamma\Lambda_m$ and $\Delta_j^{(\mathbf{i}_A,\mathbf{i_B})} \equiv \mathbf{F}'_j\mathbf{i_B} - \tilde{\Lambda}_{j,\tilde{J}^{i_A}}\Lambda^{-1}_{J^{i_A}\setminus\tilde{\mathbf{J}}}\left(\mathbf{F}_{J^{i_A}\setminus\tilde{\mathbf{J}}}\right)'\mathbf{i_B}$. The risk-free rate is the unique solution to Eq. (18), where $r_{ce}^{(\bar{i}_A,\mathbf{i}_B)}$ satisfies

$$r_{ce}^{(\bar{i}_A,\mathbf{i}_B)} = r_f + \frac{\pi_m^2}{2\gamma\Lambda_m} + \frac{1}{2\gamma}\left(\mathbf{F}'_{J^{i_A}\setminus\tilde{\mathbf{J}}}(\mathbf{i}_B)\right)'\Lambda^{-1}_{J^{i_A}\setminus\tilde{\mathbf{J}}}\left(\mathbf{F}'_{J^{i_A}\setminus\tilde{\mathbf{J}}}(\mathbf{i}_B)\right). \tag{B.3}$$

[22] We should note that our modeling strategy makes short selling seem more relevant than it would be in practice. For tractability, we assume there is a single asset m in positive net supply, and nonmarket assets $j \neq m$ are in zero net supply. Thus, an investor who would like to reduce her exposure to a nonmarket asset is required to short sell. In practice, most nonmarket assets (such as stocks or bonds) would be in positive net supply. An investor who is pessimistic about these assets could simply not include them in her portfolio. The short-selling constraints would start to bind only if the investor is substantially pessimistic.

Here, $\Delta_j^{(\mathbf{i}_A,\mathbf{i}_\mathbf{B})}$ captures an investor's excess valuation of the asset relative to the average investor. The first part says that the asset is now priced by the investor that has the highest valuation. Hence, short-selling constraints change the characterization of relative asset prices. However, they leave the characterization of the risk-free rate largely unchanged. In particular, the second part says that the risk-free rate is determined by Eq. (18) as before. The difference is that the investors' certainty-equivalent returns are determined as if the assets on which the short-selling constraints bind are not available for trade (see $(B.3)$). This is intuitively because short-selling constraints dampen speculation.

Lemma 3 leads to the following generalization of Proposition 4.

PROPOSITION 6 (Customization with Short Selling Constraints): *Consider the setting in Lemma 2 with the short-selling constraints in* $(B.1)$. *Consider financial innovation that increases the scope of customization for some market participants,* $\tilde{n}^{i_A^1} = n^{i_A^1} + \Delta n$ *and* $\tilde{n}^{i_A^0} = n^{i_A^0} - \Delta n$ *where* $i_A^1 > i_A^0$ *and* $\Delta n > 0$. *This change reduces the risk free rate* r_f, *and leaves unchanged the average risk premia,* $\{\pi_j\}_{j\in\mathbf{J}}$.

The result follows by observing that increasing the scope of customization does not affect the characterization of the risk premia in $(B.2)$. This is because the maximum excess valuation, $\max_{(\mathbf{i}_A,\mathbf{i}_\mathbf{B})}\Delta_j^{(\mathbf{i}_A,\mathbf{i}_\mathbf{B})}$, remains unchanged before and after the innovation. In contrast, greater customization does affect the characterization of the risk-free rate by enabling more speculation, which reduces the risk-free interest rate as in Proposition 4.

CUSTOMIZATION WITH INVESTMENT. — In the main text, we examined the asset pricing implications of financial innovations in an environment with a fixed supply of the market portfolio, m, given by $\eta_m > 0$. We next show that the effect of customization on the interest rate continues to hold if there is investment and η_m is determined endogenously.

Suppose that the output of the economy at date $t = 1$ is produced via a constant returns to scale neoclassical production function $Y_1 = \Phi(\mathbf{z})\, G(K, L)$. Here, we assume that Φ is a Hicks-neutral productivity shock that satisfies,

$$\log \Phi(\mathbf{z}) = \mathbf{F}_m^{'}\mathbf{z}.$$

Investors in this economy are also workers and supply one unit of labor inelastically. Therefore, we modify Assumption $\mathbf{1^G}$ slightly to allow for a positive $t = 1$ endowment by investors. We also maintain the structure of investors' market access and beliefs from Section III.B.

The economy starts with zero units of capital at time 0. Capital is produced at time 0 by a competitive sector of investment goods firms that can convert one unit of consumption good at time 0 into one unit of capital at time 1. Since this is only a two-period model, we assume that the capital depreciates fully after use at time 1. Capital and labor are rented at time 1 by a competitive sector of production firms that have access to the production technology of the economy. Given linearity in the investment good technology and production technology for the final good, both types of firms earn zero profits in equilibrium.

We use a similar equilibrium concept for this economy as our "approximate equilibrium" notion in Definition 1 but add additional market clearing conditions that capture the endogeneity of investment. In equilibrium, the price of a unit of capital at $t = 0$ equals its cost of production (namely unity). In addition, the supply of the market portfolio, η_m, equals the supply of capital K. Thus, the price of the market portfolio also equals the price of capital, $P_m = 1$. Finally, there is market clearing in the labor market.

We let $R_m(\eta_m) = \Phi G_K(\eta_m, 1)$ denote the gross return on the market portfolio in equilibrium given supply η_m . We let $r_m(\eta_m)$ denote the log return. Then, the expected log return is given by,

$$E[r_m(\eta_m)] = E[\log\Phi] + \log G_K(\eta_m, 1),$$

Note also that (log) expected return on the market portfolio is equal to the sum of the risk-free rate and the risk premium,

$$E[r_m(\eta_m)] + \frac{\Lambda_m}{2} = r_f + \pi_m.$$

Here, $\Lambda_m = var(\Phi)$ denotes the variance of the market portfolio as in the main text. Combining the last two equations yields the key equation of the characterization,

$$\log G_K(\eta_m, 1) + \left(\frac{\Lambda_m}{2} + E[\log \Phi]\right) = r_f + \pi_m. \tag{B.4}$$

The terms in parentheses are exogenous variables. Hence, the equation says that the supply of capital, η_m, is decreasing in the return on the market portfolio, $r_f + \pi_m$. As we will see, the risk premium on the market portfolio, π_m, will also be determined by exogenous variables. Hence, the equation implies that a lower interest rate, r_f, increases the equilibrium supply of capital, η_m.

To characterize the rest of the equilibrium, we denote the equilibrium wage rate at time 1 by $w(\eta_m) = \Phi G_L(\eta_m, 1)$. Notice that, given the assumption of a Hicks-neutral productivity shock, Φ, the return on the market portfolio and the wage rate are perfectly positively correlated. Since all investors are assumed to have access to the market portfolio, it follows that agents in this economy do not face uninsurable background risks. In particular, the investor's labor endowment is equivalent to holding $\frac{G_L(\eta_m,1)}{G_K(\eta_m,1)}$ units of the market portfolio. Therefore, investor i's effective wealth at time 0 is given by,

$$\tilde{W}_0^i = Y_0 + \frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)} P_m = Y_0 + \frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)}.$$

The investor's effective asset holding is given by $\tilde{A}_0^i = a^i(r_{ce}^i)\tilde{W}_0^i$, where r_{ce}^i is the investor's (log) certainty-equivalent return as before. It follows that the investor's savings are given by,

$$A_0^i = a^i(r_{ce}^i)\tilde{W}_0^i - \frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)}.$$

The asset market clearing conditions can then be written as,

$$\eta_j P_j = \sum_{\{i|j\in\{f\}\cup J^i\}} n^i \omega_j^i \left[a^i(r_{ce}^i)\tilde{W}_0^i - \frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)}\right]. \tag{B.5}$$

The following result characterizes the equilibrium.

LEMMA 4: *Consider the setting with limited portfolio customization (and full market participation) and endogenous investment. In equilibrium, the aggregate risk premium on each risky asset satisfies* $\pi_j = \frac{\Lambda_{jm}}{\Lambda_m}\pi_m$, *where* $\pi_m = \gamma\Lambda_m$. *The supply of the market portfolio,* η_m, *and the risk-free rate,* r_f, *are jointly determined by Eq.* $(B.4)$ *and*

$$\frac{\eta_m + \frac{G_L(\eta_m,1)}{G_K(\eta_m,1)}}{Y_0 + \frac{G_L(\eta_m,1)}{G_K(\eta_m,1)}} = \sum_{i\in I} n^{i_A} n^{\mathbf{i_B}} a\left(r_{ce}^{(i_A,\mathbf{i_B})}\right), \tag{B.6}$$

where the certainty-equivalent return for an investor with type $(i_A, \mathbf{i_B})$ *is given by Eq. (16) as in the main text.*

Compared to Lemma 2, the only difference is that the supply of the market portfolio, η_m, is endogenous and inversely related to the interest rate according to $(B.4)$. This leads to the following result, which generalizes Proposition 4 to this setting.

PROPOSITION 7: *Consider the equilibrium characterized in Lemma 4. Consider financial innovation that increases the scope of customization for some investors,* $\tilde{n}^{i_A^1} = n^{i_A^1} + \Delta n$ *and* $\tilde{n}^{i_A^0} = n^{i_A^0} - \Delta n$, *where* $i_A^1 > i_A^0$ *and* $\Delta n > 0$. *This change reduces the risk free rate* r_f *and the expected return on risky* *assets $E[r_j]$, $j \in \mathbf{J}$ and leaves unchanged the average risk premia. It also increases aggregate investment and the supply of the market portfolio, η_m.*

As with the case of a fixed supply of the market portfolio, increased customization decreases the returns on all assets in the economy. In this case, the lower required returns (or the lower hurdle rates) also translate into greater investment and increased supply, η_m. It is illustrative to consider how the two responses compare. The proof in Online Appendix C implies that,

$$\frac{\partial r_f}{\partial \Delta n} \Big/ \frac{\partial \eta_m}{\partial \Delta n} = \frac{G_{KK}}{G_K}.$$

Therefore, the relative response depends on properties of the aggregate production function. Specifically, if capital and labor are perfect substitutes in production, then $\frac{G_{KK}}{G_K} = 0$, and only the quantity margin responds. If they are perfect complements, then $\frac{G_{KK}}{G_K} \to \infty$, and there is only a price respond. In between, the relative response should depend on the elasticity of substitution between capital and labor.

Online Appendix C: Omitted Proofs

In this appendix, we first present the proofs for the results in the main text, which are not included in Appendix A.2. We then present the proofs for the results in Online Appendix B.

1. *Proofs of results in the main text*

Proof of Proposition 1. Included in Appendix A.2.

Proof of Proposition 2. The equilibrium is described by the following system of equations.

$$\eta_j P_j = \sum_{\{i \;\mid\; j \in \{f\} \cup J^i\}} n^i \omega_j^i a^i \left(r_{ce}^i\right) \left(Y_0^i + W_0^i\right) \text{ for each } j \in \{f\} \cup \mathbf{J},$$

where returns and risk premia are related to prices according to,

$$E^i[r_j] = (\mathbf{F}_j)' \mu_{\mathbf{z}}^i - \log P_j$$

$$\pi_j^i = E^i[r_j] + \frac{\Lambda_j}{2} - r_f$$

and investors' asset demands and savings are determined by,

$$\omega_{J^i}^i = \frac{1}{\gamma^i} \Lambda_{J^i}^{-1} \pi_{J^i}^i$$

$$r_{ce}^i = r_f + \frac{1}{2\gamma^i} \left(\pi_{J^i}^i\right)' \Lambda_{J^i}^{-1} \pi_{J^i}^i$$

$$a^i\left(r_{ce}^i\right) = \frac{\left(\beta^i\right)^{\varepsilon^i} \exp\left(r_{ce}^i\left(\varepsilon^i - 1\right)\right)}{1 + \left(\beta^i\right)^{\varepsilon^i} \exp\left(r_{ce}^i\left(\varepsilon^i - 1\right)\right)}.$$

Note that a lower price P_j increases the return and the risk premium on asset j, which tends to increase the demand for that asset. We are looking for a vector, $P = \{P_j\}_{j \in \{f\} \cup J}$, that ensures all asset markets are simultaneously in equilibrium.

We work with a truncated economy, where prices satisfy $P_j \leq \alpha$ for each asset $j \in \{f\} \cup J$. We are only interested in sufficiently large α so that the truncation becomes inconsequential. First, let us define extended portfolio weights over assets that agent i cannot trade, so that

$$\hat{\omega}_j^i(P) \equiv \begin{cases} \omega_j^i(P), & \text{whenever } j \in \{f\} \cup J^i \\ 0, & \text{otherwise.} \end{cases}$$

For $P \gg 0$ we have individual excess demand for asset $j \in \{f\} \cup J$ defined as

$$z_j^i(P) \equiv \frac{\hat{\omega_j^i}(P)}{P_j} A_0^i(P) - x_{-1,j}^i \tag{C.1}$$

and we analogously define the excess demand for consumption at date $t = 0$ as $z_0^i(P) \equiv c_0^i(P) - Y_0^i$. Aggregate excess demands are then simply defined as $z_j(P) \equiv \sum_i n^i z_j^i(P)$ and $z_0(P) \equiv \sum_i n^i z_0^i(P)$. Walras' Law, i.e., $z_0(P) + \sum_{j \in J} P_j z_j(P) = 0$ can be trivially verified from individual optimality.

First, we impose a lower bound on prices $\hat{\epsilon} > 0$, which we successively relax later. Define $S_{\hat{\epsilon}} \equiv \left\{P \in \mathbb{R}_{++}^{|J|} | P_j \geq \hat{\epsilon} \text{ and } P_j \leq \alpha, \forall j \in \{f\} \cup J\right\}$ which is compact and convex. We are only interested in $\alpha > \hat{\epsilon}$ as to ensure the non-emptiness of $S_{\hat{\epsilon}}$.

We next define a continuous price updating function. Let each entry, which describes the update to the price of asset $j \in J$, be defined by

$$P_j^{upd}(P, \hat{\epsilon}) \equiv \begin{cases} \hat{\epsilon},\ if\ z_j(P) < \hat{\epsilon} - P_j \\ P_j + z_j(P),\ if\ \hat{\epsilon} - P_j \leq z_j(P) \leq \alpha \\ \alpha,\ if\ z_j(P) > \alpha \end{cases}. \tag{C.2}$$

Then, let the function $P^{upd}(P, \hat{\epsilon}) : S_{\hat{\epsilon}} \to S_{\hat{\epsilon}}$ be defined as $P^{upd}(P, \hat{\epsilon}) = \left\{P_j^{upd}(P, \hat{\epsilon})\right\}_{j \in \{f\} \cup J}$. As excess demand functions are continuous, so is the function $P^{upd}(\cdot, \hat{\epsilon})$, which maps the non-empty, convex, and compact set $S_{\hat{\epsilon}}$ into itself. From Brouwer's Fixed Point Theorem, there exists $P^{\hat{\epsilon}} \in S_{\hat{\epsilon}}$ such that $P_j^{upd}\left(P^{\hat{\epsilon}}, \hat{\epsilon}\right) = P^{\hat{\epsilon}}$.

We now take a sequence $\{\hat{\epsilon}_k\}_{k \in \mathbb{N}}$ such that $\hat{\epsilon}_k \to 0$. Let $\left\{P^{\hat{\epsilon}_k}\right\}_{k \in \mathbb{N}}$ be the associated sequence of fixed points. As each price lies in $[0, \alpha]$ that sequence is bounded and admits a converging subsequence. To save on notation, assume we have selected such subsequence from the start. Define its limit by $P^* = \left(P_1^*, P_2^*, ..., P_{|J|}^*\right)$. Naturally $P^* \in \overline{\cup_k S_{\hat{\epsilon}_k}} = \left\{P \in \mathbb{R}_+^{|J|} | P_j \leq \alpha, \forall \{f\} \cup J\right\}$. We now show that $P^* \in \mathbb{R}_{++}^{|J|}$.

Consider the case with $P_j^* = 0$ for risky assets, which w.l.o.g. we call assets $1, .., m$, while the riskless rate remains bounded away from zero. In this case, the risk premia for assets $1, .., m$ approach $+\infty$, and the risk premia for the remaining assets remain finite. Consider all investors that have access to at least one of the assets $1, .., m$ and call that set $I_{r \to \infty}$. It is easy to check that each of these investors have $r_{ce} \to \infty$, and thus, they save all their wealth.

Now consider the net demand for assets that comes from these investors only, $z_j^{I_{r \to \infty}} \equiv \sum_{i \in I_{r \to \infty}} n^i z_j^i(P)$. We claim that regardless of how the prices for $1, .., m$ approach 0 (or conversely, regardless of the risk premia approach infinity), there exists at least one asset within $1, .., m$ such that the total demand from these investors for that asset becomes unboundedly positive. Since the demand from the other investors is finite, this will provide a contradiction.

Let us rewrite risk premia along the sequence. Take a given agent $i \in I_{r \to \infty}$, then the (individually perceived) risk-premium $\pi_j^{i,k}\left(P^{\hat{\epsilon}_k}\right)$ on any asset $j \in J$ can be appropriately rewritten as $\pi_j^{i,k} = \left\|\pi^{i,k}\right\| \hat{\pi}_j^{i,k}$ where $\left\|\pi^{i,k}\right\| := \sum_j \left|\pi_j^{i,k}\right|$ denotes a norm and

$$\hat{\pi}_j^{i,k} \equiv \frac{\pi_j^{i,k}}{\left\|\pi^{i,k}\right\|}$$

denotes the $j - th$ entry of a normalized risk-premium vector.[23] The vector $\hat{\pi}^{i,k} = \left\{\hat{\pi}_j^{i,k}\right\}_{j \in J}$ belongs to the surface of the unit ball centered at zero.

[23]As prices are converging to zero, there are finitely many elements with $\sum_j \pi_j^{i,k} = 0$. We can move to a subsequence that disregards these.

As that surface is a compact set, $\{\hat{\pi}^{i,k}\}_{k\in\mathbb{N}}$ admits a converging subsequence, which we can index by $k_i \in \mathbb{N}$. That forms another price sequence $\left\{P^{\hat{\epsilon}_{k_i}}\right\}_{k_i\in\mathbb{N}}$, from which we can extract a subsequence to ensure that the analogously defined vector $\hat{\pi}^{i',k_i}$ converges for any second agent $i' \in I_{r\to\infty}$. Given that $I_{r\to\infty}$ is finite, this step can be iteratively repeated until a subsequence, indexed by $\tilde{k} \in \mathbb{N}$, is extracted and ensures that each $\hat{\pi}^{i,\tilde{k}}$ converges. Additionally, for each $i \in I_{r\to\infty}$, $\lim_{\tilde{k}\to\infty} \hat{\pi}^{i,\tilde{k}} = \hat{\pi}$, i.e., the limit of the normalized risk-premia are the same and independent of $i \in I_{r\to\infty}$, since disagreements are bounded, while at least one return goes to infinity.

Take a given agent $i \in I_{r\to\infty}$. Define $\hat{\pi}^{i,\tilde{k}}_{J_i}$ and $\hat{\pi}_{J_i}$ to be respectively the restriction of the normalized risk premia vectors $\hat{\pi}^{i,\tilde{k}}$ and $\hat{\pi}$ to the assets that agent i can trade. Notice that along that subsequence portfolio weights of the form $\omega^i_{J^i}\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) = \frac{1}{\gamma^i}\Lambda^{-1}_{J^i}\hat{\pi}^{i,\tilde{k}}_{J_i}\left\|\pi^{i,\tilde{k}}\right\|$ are optimal from equation (10). Therefore, we take the following limit of an inner product

$$\lim_{\tilde{k}\to\infty}\left\langle \hat{\pi}^{i,\tilde{k}}_{J_i}, \frac{\omega^i_{J^i}\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)}{\left\|\pi^{i,\tilde{k}}\right\|}\right\rangle = \frac{1}{\gamma^i}\hat{\pi}'_{J_i}\Lambda^{-1}_{J^i}\hat{\pi}_{J_i} > 0$$

from the positive-definiteness of $\Lambda^{-1}_{J^i}$ and the fact that $\hat{\pi}_{J_i}$ is not null. It follows that it is possible to find $\delta > 0$ and a sufficiently large element $\overline{k}$ such that

$$\left\langle \hat{\pi}, \frac{\hat{\omega}^i\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)}{\left\|\pi^{i,\tilde{k}}\right\|}\right\rangle > \delta,$$

whenever $i \in I_{r\to\infty}$ and $\tilde{k} > \overline{k}$. Given that $A^i_0\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)$ is bounded from below for sufficiently high $\tilde{k}$ for all $i \in I_{r\to\infty}$, there exists $\delta_1 > 0$

$$\left\langle \hat{\pi}, \sum_{i\in I_{r\to\infty}} n^i A^i_0\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)\frac{\hat{\omega}^i\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)}{\left\|\pi^{i,\tilde{k}}\right\|}\right\rangle > \delta_1, \tag{C.3}$$

for all $\tilde{k} > \overline{k}$. This directly implies that there exists one asset $j \in \{1, ..., m\}$ such that $\sum_{i\in I_{r\to\infty}} n^i A^i_0\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)\hat{\omega}^i\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)$ grows without bounds. It follows that excess demand for that asset is unbounded along the subsequence that is indexed by $\tilde{k}$. From (C.2) this means that $P^{upd}_j\left(P^{\hat{\epsilon}_k}, \hat{\epsilon}_k\right) = \alpha$ infinitely many times as $k \to \infty$, reaching a contradiction with $P^*_j = 0$.

Suppose now, towards a different contradiction, that $r_f \to \infty$. Using arguments similar to the previous ones, it is possible to select a subsequence, indexed by $\tilde{k} \in \mathbb{N}$, in which the risk premium, $\pi^i_j\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)$, perceived by each agent $i \in I$ for each asset $j \in J$ either converges to a finite constant, diverges to $+\infty$ or diverges to $-\infty$. Also, a premium can only diverge for all agents at the same time and in the same direction.

First, we deal with the case in which no premium diverges. In this situation, each asset price converges to zero. Adding equations C.1 over agents and assets, properly multiplied by prices and individual population shares, we get

$$\sum_{i,j} P^{\hat{\epsilon}_{\tilde{k}}}_j n^i z^i_j\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) = \sum_{i,j} n^i\left[\hat{\omega}^i_j\left(P^{\hat{\epsilon}_{\tilde{k}}}\right)A^i_0\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) - P^{\hat{\epsilon}_{\tilde{k}}}_j x^i_{-1,j}\right],$$

which after simplifications leads to

$$\sum_j P_j^{\hat{\epsilon}_{\tilde{k}}} z_j\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) = \sum_i n^i A_0^i\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) - \sum_{i,j} P_j^{\hat{\epsilon}_{\tilde{k}}} n^i x_{-1,j}^i.$$

As $P^{\hat{\epsilon}_{\tilde{k}}} \to 0$, the right hand side converges to $\sum_i n^i Y_0^i > 0$. As a consequence, the excess demand for at least one asset j needs to approach $+\infty$ along a subsequence. Along this subsequence then $P_j^{upd}\left(P, \hat{\epsilon}_{\tilde{k}}\right) = \alpha$ infinitely often, leading to a contradiction of the zero price limit.

For the case in which some premia diverge, we still obtain

$$\lim_{\tilde{k}\to\infty} A_0^i\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) - P_j^{\hat{\epsilon}_{\tilde{k}}} n^i x_{-1,j}^i = Y_0^i > 0$$

and

$$\sum_j P_j^{\hat{\epsilon}_{\tilde{k}}} z_j\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) \to \sum n^i Y_0^i > 0$$

If $P^{\hat{\epsilon}_{\tilde{k}}} \to 0$, we find the same contradiction as before. Therefore, for at least one asset $j \in J$, we need to have $P_j^{\hat{\epsilon}_{\tilde{k}}} \to P_j^* \neq 0$ which implies that $\pi_j^{i,\tilde{k}} \to -\infty$ for each $i \in I$. We can therefore follow all the previous steps leading to C.3, with the exception that $\hat{\pi}$ can now have negative entries. This means that we can find a subsequence and an asset $j^{'} \in J$, such that either $\pi_{j^{'}}^{i,\tilde{k}} \to -\infty$ and $z_{j^{'}}\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) \to -\infty$ or $\pi_{j^{'}}^{i,\tilde{k}} \to +\infty$ and $z_{j^{'}}\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) \to +\infty$. For the latter case, we would reach the same contradiction as before since $z_{j^{'}}\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) \to +\infty$ implies that $P_{j^{'}}^{upd}\left(P^{\hat{\epsilon}_k}, \hat{\epsilon}_k\right) = \alpha$ infinitely many times which contradicts positive infinity limits for both the riskless rate and the risk premium on $j^{'}$. Therefore, we need to rule out the former situation. Given that $P_j^{\hat{\epsilon}_{\tilde{k}}} \to P_j^* > 0$, $\pi_j^{i,\tilde{k}} \to -\infty$ and $\hat{\pi}_{j^{'}} \neq 0$ together imply that $P_{j^{'}}^* > 0$. But from (C.2), $z_{j^{'}}\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) \to -\infty$ implies $P_{j^{'}}^{\hat{\epsilon}_{\tilde{k}}} = \hat{\epsilon}_{\tilde{k}}$ infinitely many times with $\hat{\epsilon}_{\tilde{k}} \to 0$, reaching a contradiction with $P_{j^{'}}^* > 0$.

We have, therefore, ruled out any possibility that $P_j^* = 0$ for some asset $j \in J \cup \{f\}$. We still need to show that for sufficiently high α, market clearing is ensured in all markets at prices P^*. Given that $P_j^* \gg 0$, it is possible to find a sufficiently high $\hat{k}$ and $\delta_2 > 0$, such that

$$P_j^{\hat{\epsilon}_k} > \delta_2 > \hat{\epsilon}_k,$$

for all $k > \hat{k}$. As a consequence, from (C.2), for $k > \hat{k}$, $P_j^{\hat{\epsilon}_k} \geq 0$ and $z_j\left(P^{\hat{\epsilon}_k}\right) \geq 0$.

Additionally, for each $i \in I$, $C_0^i\left(P^{\hat{\epsilon}_k}\right) \in \left[0, Y_0^i + \alpha\sum_j x_{-1}^i, j\right]$ implying that

$$-\alpha\sum_{i,j} n^i x_{-1,j}^i \leq \sum_j P_j^{\hat{\epsilon}_{\tilde{k}}} z_j\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) \leq \sum_i n^i Y_0^i.$$

For $\alpha^2 > \sum_i n^i Y_0^i$, it follows that $z\left(P^{\hat{\epsilon}_{\tilde{k}}}\right) \to z\left(P^*\right) = 0$ ensuring market-clearing in the limit and existence of a Walrasian Equilibrium.

■

To establish the remaining proofs, we state a useful lemma that concerns the asset holding function defined in (11), which we replicate here

$$a\left(r_{ce}\right) = \frac{\beta^{\varepsilon}\exp\left(r_{ce}\left(\varepsilon - 1\right)\right)}{1 + \beta^{\varepsilon}\exp\left(r_{ce}\left(\varepsilon - 1\right)\right)}.$$

LEMMA 5: *Whenever $\varepsilon > 1$, the semi-elasticity $\frac{a'(r_{ce})}{a(r_{ce})}$ is decreasing in r_{ce}.*

PROOF:

From the Euler Equation in logarithmic form

$$\log a\left(r_{ce}\right) - \log\left(1 - a\left(r_{ce}\right)\right) = \varepsilon \log \beta + (\varepsilon - 1)\, r_{ce}$$

thus differentiating with respect to r_{ce} and simplifying

$$\frac{a'\left(r_{ce}\right)}{a\left(r_{ce}\right)} = (\varepsilon - 1)\left(1 - a\left(r_{ce}\right)\right) \tag{C.4}$$

so $\frac{a'(r_{ce})}{a(r_{ce})}$ is decreasing in a and therefore in r_{ce}, whenever $\varepsilon > 1$.

Proof of Lemma 1. This proof is mostly included in Appendix A.2. There, we establish Eqs. (14) – (15). It remains to prove that this system of equations has a unique solution.

Toward that end let us first define the average level of savings out of wealth as $\overline{a}\left(r_f, \pi_m, n^p\right) \equiv n^p a\left(r_{ce}^p\right) + \left(1 - n^p\right) a\left(r_f\right)$, and the relative value of the asset endowment as $v\left(r_f + \pi_m\right) \equiv \frac{\eta_m P_m}{Y_0 + \eta_m P_m}$. Combined they characterize

$$\varphi_1\left(r_f, \pi_m, n^p\right) \equiv \overline{a}\left(r_f, \pi_m, n^p\right) - v\left(r_f + \pi_m\right).$$

Notice that $v'\left(r_f + \pi_m\right) \propto -Y_0 v\left(r_f + \pi_m\right) < 0$. As a consequence, $\frac{\partial \varphi_1\left(r_f, \pi_m, n^p\right)}{\partial r_f} = \frac{\partial \overline{a}}{\partial r_f} - v' > 0$, and $\frac{\partial \varphi_1\left(r_f, \pi_m, n^p\right)}{\partial \pi_m} = \frac{\partial \overline{a}}{\partial \pi_m} - v' > 0$.

Additionally, we define

$$\varphi_2\left(r_f, \pi_m, n^p\right) \equiv n^p\left(1 - \omega_m^p\right) a\left(r_{ce}^p\right) + \left(1 - n^p\right) a\left(r_f\right).$$

An equilibrium then is a solution to $\varphi_1\left(r_f, \pi_m, n^p\right) = \varphi_2\left(r_f, \pi_m, n^p\right) = 0$.

Notice then that, $\frac{\partial \varphi_2}{\partial r_f} = n^p\left(1 - \omega_m^p\right) a'\left(r_{ce}^p\right) + \left(1 - n^p\right) a'\left(r_f\right)$. Additionally, $\varphi_2\left(r_f, \pi_m, n^p\right) = 0 \implies \left(1 - \omega_m^p\right) = -\frac{\left(1 - n^p\right)}{n^p} \frac{a\left(r_f\right)}{a\left(r_{ce}^p\right)}$ and $\frac{\partial \varphi_2}{\partial r_f} = \left(1 - n^p\right) a\left(r_f\right)\left[\frac{a'\left(r_f\right)}{a\left(r_f\right)} - \frac{a'\left(r_{ce}^p\right)}{a\left(r_{ce}^p\right)}\right]$ which is positive whenever $\varepsilon > 1$, given Lemma 5. Last, $\frac{\partial \varphi_2}{\partial \pi_m} = -\frac{\partial \omega_m^p}{\partial \pi_m} n^p a\left(r_{ce}^p\right) + n^p\left(1 - \omega_m^p\right) a'\left(r_{ce}^p\right) \frac{\partial r_{ce}^p}{\partial \pi_m} < 0$ since $\left(1 - \omega_m^p\right) < 0$ whenever $\varphi_2 = 0$.

As a consequence, locus $\varphi_1\left(r_f, \pi_m, n^p\right) = 0$ is downward slopping in $\left(r_f, \pi_m\right)$-space while locus $\varphi_2\left(r_f, \pi_m, n^p\right) = 0$ is upward slopping. Both loci are characterized by continuous functions. We can use $\varphi_1\left(r_f, \pi_m, n^p\right) = 0$, with $\frac{\partial \varphi_1}{\partial \pi_m} \neq 0$, and the Implicit Function Theorem to define a decreasing function $\pi_m^{\varphi_1}(\cdot)$ of the interest rate r_f over the first locus. We then look for a solution to $\varphi_2\left(r_f, \pi_m^{\varphi_1}\left(r_f\right), n^p\right) = 0$, where the left-hand side is a strictly increasing function of r_f. The existence of a solution is guaranteed by Proposition **2** and uniqueness follows from strict monotonicity. ■

Proof of Proposition 3 Let $J \equiv \begin{bmatrix} \frac{\partial \varphi_1}{\partial r_f} & \frac{\partial \varphi_1}{\partial \pi_m} \\ \frac{\partial \varphi_2}{\partial r_f} & \frac{\partial \varphi_2}{\partial \pi_m} \end{bmatrix}$ and $\Delta_J < 0$ denote its determinant. Then,

$$\begin{bmatrix} \frac{dr_f}{dn^p} \\ \frac{d\pi_m}{dn^p} \end{bmatrix} = -\frac{1}{\Delta_J} \begin{bmatrix} \frac{\partial \varphi_2}{\partial \pi_m} & -\frac{\partial \varphi_1}{\partial \pi_m} \\ -\frac{\partial \varphi_2}{\partial r_f} & \frac{\partial \varphi_1}{\partial r_f} \end{bmatrix} \begin{bmatrix} a\left(r_{ce}^p\right) - a\left(r_f\right) \\ -\frac{a\left(r_f\right)}{n^p} \end{bmatrix}.$$

Therefore, $\frac{d\pi_m}{dn^p} < 0$. Also,

$$\begin{aligned}\frac{d\left[r_f+\pi_m\right]}{dn^p} &\propto \left(a\left(r_{ce}^p\right)-a\left(r_f\right)\right)\left(\frac{\partial\varphi_2}{\partial\pi_m}-\frac{\partial\varphi_2}{\partial r_f}\right)+\left(\frac{\partial\varphi_1}{\partial\pi_m}-\frac{\partial\varphi_1}{\partial r_f}\right)\frac{a\left(r_f\right)}{n^p}\\ &= \left(a\left(r_{ce}^p\right)-a\left(r_f\right)\right)\left(\frac{\partial\varphi_2}{\partial\pi_m}-\frac{\partial\varphi_2}{\partial r_f}\right)+\left(\frac{a'\left(r_{ce}^p\right)}{a\left(r_{ce}^p\right)}-\frac{a'\left(r_f\right)}{a\left(r_f\right)}\right)\frac{\left(1-n^p\right)}{n^p}\left(a\left(r_f\right)\right)^2<0\end{aligned}$$

again using Lemma 5. ■

Proof of Lemma 2. Included in Appendix A.2.

Proof of Proposition 4. To show this result, notice first that part (i) of Lemma **2** implies that $r_{ce}^{\left(i_A^1,\mathbf{i}_B\right)} \geq r_{ce}^{\left(i_A^0,\mathbf{i}_B\right)}$. Next, re-write (18) as

$$\sum_{i\in I} n^{i_A} n^{\mathbf{i}_B} a\left(r_{ce}^{\left(i_A,\mathbf{i}_B\right)}\right)-\frac{\eta_m P_m}{Y_0+\eta_m P_m}=0 \tag{C.5}$$

and notice that the left-hand side is increasing in r_f, since $r_{ce}^{\left(i_A,\mathbf{i}_B\right)}$ is increasing in r_f and $a\left(.\right)$ is an increasing function, so the first term is increasing in r_f, and also P_m is decreasing in r_f, so the second term is also increasing in r_f. Finally, since $r_{ce}^{\left(i_A^1,\mathbf{i}_B\right)} \geq r_{ce}^{\left(i_A^0,\mathbf{i}_B\right)}$, it follows that $\sum_{i\in I} n^{i_A} n^{\mathbf{i}_B} a\left(r_{ce}^{\left(i_A,\mathbf{i}_B\right)}\right)$ is increasing in Δn, and so, the left-hand side of $(C.5)$ is increasing in Δn. Hence, r_f is decreasing in Δn.

Showing that $\left\{\pi_j\right\}_{j\in\mathbf{J}}$ remain unchanged follows directly from Lemma **2**, part (ii). Finally, showing that the average expected return on risky assets decreases follows from the behavior of r_f and $\left\{\pi_j\right\}_{j\in\mathbf{J}}$.■

2. Proofs of results in Online Appendix B

Proof of Proposition 5. Let $R_f = 1/P_f$ denote the risk-free return. First consider the case after financial innovation. Since the market is complete, the background risks are effectively tradable . Thus, Assumption 1 holds and the analysis is similar to the proof of Proposition 1. In view of Assumption 1^{P}, the value of the investors' background risks is given by, $E\left[M\left(\mathbf{z}\right)L\left(\mathbf{z}\right)\right]=E\left[L\left(\mathbf{z}\right)\right]/R_f$. Using this observation, and following similar steps as before, we obtain (assuming an interior solution),

$$A_0^{i,new}+\frac{E\left[L\left(\mathbf{z}\right)\right]}{R_f}=\frac{\beta}{1+\beta}\left(W_0+\frac{E\left[L\left(\mathbf{z}\right)\right]}{R_f}\right). \tag{C.6}$$

In view of the assumption, $\varepsilon = 1$, the desired total asset holdings is a constant fraction of the investor's total lifetime wealth. The calculation of the desired total assets as well as the total lifetime wealth also include the implicit background income.

Next consider the case before financial innovation. The investor's problem can be written as,

$$\begin{aligned}&\max_{A_0}\log\left(W_0-A_0\right)+\beta\log V_1,\\ &\text{s.t. } V_1=\left(E\left[\left(A_0R_f+L\left(\mathbf{z}\right)\right)^{1-\gamma}\right]\right)^{1/(1-\gamma)}.\end{aligned}$$

Assuming an interior condition, the optimality condition implies,

$$
\begin{aligned}
\frac{1}{W_0 - A_0^{i,old}} &= \beta R_f \frac{1}{V_1} V_1^{\gamma} E\left[\frac{1}{\left(A_0^{i,old} + L(\mathbf{z})\right)^{\gamma}}\right] \\
&= \beta R_f \frac{1}{E\left[\left(A_0^{i,old} R_f + L(\mathbf{z})\right)^{1-\gamma}\right]} E\left[\frac{1}{\left(A_0^{i,old} + L(\mathbf{z})\right)^{\gamma}}\right] \\
&= \beta R_f \frac{1}{E\left[C_1(\mathbf{z})^{1-\gamma}\right]} E\left[\frac{1}{C_1(\mathbf{z})^{\gamma}}\right]. \qquad \text{(C.7)}
\end{aligned}
$$

Here, the last line substitutes, $C_1(\mathbf{z}) = A_0^{i,old} R_f + L(\mathbf{z})$. Next note that $C_1(\mathbf{z})$ and $C_1(\mathbf{z})^{-\gamma}$ are negatively correlated, and strictly so if $C_1(\mathbf{z})$ is not constant. In particular, we have,

$$
cov\left(C_1(\mathbf{z}), C_1(\mathbf{z})^{-\gamma}\right) = E\left[C_1(\mathbf{z})^{1-\gamma}\right] - E\left[C_1(\mathbf{z})\right] E\left[C_1(\mathbf{z})^{-\gamma}\right] \leq 0,
$$

with strict inequality whenever $C_1(\mathbf{z})$ is not constant. Combining this observation with Eq. $(C.7)$, we obtain,

$$
\frac{1}{W_0 - A_0^{i,old}} \geq \beta R_f \frac{1}{E\left[C_1(\mathbf{z})\right]} = \beta R_f \frac{1}{E\left[A_0^{i,old} R_f + L(\mathbf{z})\right]}.
$$

After rearranging terms, this implies,

$$
A_0^{i,old} + \frac{E\left[L(\mathbf{z})\right]}{R_f} \geq \frac{\beta}{1+\beta}\left(W_0 + \frac{E\left[L(\mathbf{z})\right]}{R_f}\right). \qquad \text{(C.8)}
$$

Comparing Eqs. $(C.6)$ and $(C.8)$ implies $A_0^{i,old} \geq A_0^{i,new}$, with strict inequality if $C_1^{i,old}(\mathbf{z})$ is not constant.■

Proof of Lemma 3. First, we show that given prices characterized by Eqs. $(B.2)$, (18), and $(B.3)$, the average portfolio shares for investors with market access i_A, are independent of the heterogeneity in beliefs or market access and satisfy Eq. $(A.2)$. An investor's perceived risk premium for a risky asset j is

$$
\pi_j^{(i_A,\mathbf{i_B})} = \mathbf{F}_j^{'} \mu_z^i + ((\Lambda_j)/2) - \log P_j - r_f = \pi_j + \mathbf{F}_j^{'} \mathbf{i_B}. \qquad \text{(C.9)}
$$

The first-order conditions for the investor can be written as

$$
\boldsymbol{\pi}_{J^{i_A}}^{(i_A,\mathbf{i_B})} - \gamma \Lambda_{J^{i_A}} \boldsymbol{\omega}_{J^{i_A}}^{(i_A,\mathbf{i_B})} + \boldsymbol{\kappa}_{J^{i_A}}^{(i_A,\mathbf{i_B})} = 0, \qquad \text{(C.10)}
$$

where $\boldsymbol{\kappa}_{J^{i_A}}^{(i_A,\mathbf{i_B})}$ consists of a $|\tilde{J}^{i_A}|$-by-1 vector $\boldsymbol{\kappa}_{\tilde{J}^{i_A}}^{(i_A,\mathbf{i_B})}$ of Lagrange multipliers for the respective short-selling constraints and a $|J^{i_A} \backslash \tilde{\mathbf{J}}|$-by-1 vector of zeros for the assets that do not have short-selling constraints. Substituting for $\pi_j^{(i_A,\mathbf{i_B})}$ from (C.9), we have,

$$
\boldsymbol{\pi}_{J^{i_A}} + \mathbf{F}_{J^{i_A}}^{'} \mathbf{i_B} - \gamma \Lambda_{J^{i_A}} \boldsymbol{\omega}_{J^{i_A}}^{(i_A,\mathbf{i_B})} + \boldsymbol{\kappa}_{J^{i_A}}^{(i_A,\mathbf{i_B})} = 0. \qquad \text{(C.11)}
$$

Next, we show that $\boldsymbol{\omega}_{J^{i_A}}^{(i_A,\mathbf{i_B})} = \left[\mathbf{0}_{|\tilde{J}^{i_A}|}, \left(\boldsymbol{\omega}_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{(i_A,\mathbf{i_B})}\right)'\right]'$ satisfies the FOCs in (C.11), where

$$\boldsymbol{\omega}_{J^{i_A}/\tilde{\mathbf{J}}}^{(i_A,\mathbf{i_B})} = \left(\frac{1}{\gamma}\Lambda_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{-1}\left(\boldsymbol{\pi}_{J^{i_A}\backslash\tilde{\mathbf{J}}} + \left(\mathbf{F}_{J^{i_A}\backslash\tilde{\mathbf{J}}}\right)'\mathbf{i_B}\right)\right), \tag{C.12}$$

To show this, first note that for assets in $J^{i_A}\backslash\tilde{\mathbf{J}}$ we have

$$\boldsymbol{\pi}_{J^{i_A}\backslash\tilde{\mathbf{J}}} + \left(\mathbf{F}_{J^{i_A}\backslash\tilde{\mathbf{J}}}\right)'\mathbf{i_B} - \gamma\Lambda_{J^{i_A}\backslash\tilde{\mathbf{J}}}\boldsymbol{\omega}_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{(i_A,\mathbf{i_B})} = 0,$$

which is satisfied given the definition of $\boldsymbol{\omega}_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{(i_A,\mathbf{i_B})}$. Also, notice that the individual weights in $\boldsymbol{\omega}_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{(i_A,\mathbf{i_B})}$ are equivalent to the optimal individual portfolio weights in an equilibrium in which only $\mathbf{J}\backslash\tilde{\mathbf{J}}$ assets are available, with risk premia given in $(B.2)$, and the individual investor i_A has access to $J^{i_A}\backslash\tilde{\mathbf{J}}$ of those. Therefore, the results from Lemma 2 apply for the average portfolio weights, $\boldsymbol{\omega}_{J^{i_A}/\tilde{\mathbf{J}}}^{i_A}$, across investor with different beliefs, and so, a version of Eq. $(A.2)$ holds for these average weights. This in turn implies that we can simplify (C.12) to

$$\boldsymbol{\omega}_{J^{i_A}/\tilde{\mathbf{J}}}^{(i_A,\mathbf{i_B})} = \boldsymbol{\omega}_{J^{i_A}/\tilde{\mathbf{J}}}^{i_A} + \frac{1}{\gamma}\Lambda_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{-1}\left(\mathbf{F}_{J^{i_A}\backslash\tilde{\mathbf{J}}}\right)'\mathbf{i_B}.$$

For assets $j \in \tilde{J}^{i_A}$, these weights imply that

$$\begin{aligned}
\boldsymbol{\kappa}_j^{(i_A,\mathbf{i_B})} &= -\pi_j - \left[\mathbf{F}_j'\mathbf{i_B} - \left(\gamma\tilde{\Lambda}_{\tilde{J}^{i_A}}\boldsymbol{\omega}_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{(i_A,\mathbf{i_B})}\right)_j\right] \\
&= -\pi_j - \left[\mathbf{F}_j'\mathbf{i_B} - \left(\gamma\tilde{\Lambda}_{\tilde{J}^{i_A}}\boldsymbol{\omega}_{J^{i_A}/\tilde{\mathbf{J}}}^{i_A} + \tilde{\Lambda}_{\tilde{J}^{i_A}}\Lambda_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{-1}\left(\mathbf{F}_{J^{i_A}\backslash\tilde{\mathbf{J}}}\right)'\mathbf{i_B}\right)_j\right] \\
&= -\pi_j + \frac{\Lambda_{jm}}{\Lambda_m}\pi_m - \left[\mathbf{F}_j'\mathbf{i_B} - \tilde{\Lambda}_{j,\tilde{J}^{i_A}}\Lambda_{J^{i_A}\backslash\tilde{\mathbf{J}}}^{-1}\left(\mathbf{F}_{J^{i_A}\backslash\tilde{\mathbf{J}}}\right)'\mathbf{i_B}\right] \geq 0,
\end{aligned}$$

where the last inequality follows given the equilibrium values of π_j in $(B.2)$.

To show the rest of the lemma, we use the observation that the individual portfolio weights on assets without short-selling constraints are equivalent to those in an equilibrium in which only $\mathbf{J}\backslash\tilde{\mathbf{J}}$ assets are available, with risk premia given in $(B.2)$, and the individual investor i_A has access to $J^{i_A}\backslash\tilde{\mathbf{J}}$ of those. Therefore, an application of Lemma 2 implies that the investors' certainty-equivalent returns are given by equation $(B.3)$. Finally, Lemma 2 implies that Eqs. $(B.2)$, (18), and $(B.3)$ uniquely characterize the equilibrium prices on all assets and also all market clearing conditions are satisfied at these prices.■

Proof of Proposition 6. The result on assets $\mathbf{J}\backslash\tilde{\mathbf{J}}$ follows by observing that investors' portfolio weights and certainty-equivalent returns are equivalent to those in an equilibrium in which only $\mathbf{J}\backslash\tilde{\mathbf{J}}$ assets are available, and an individual investor with market access i_A has access to $J^{i_A}\backslash\tilde{\mathbf{J}}$ of those, and applying Proposition 4 to the environment with $\mathbf{J}\backslash\tilde{\mathbf{J}}$ available assets. The result on the remaining assets $\tilde{\mathbf{J}}$ follow by applying Lemma 3 before and after customization, and observing that $\max_{(i_A,\mathbf{i_B})}\Delta_j^{(\mathbf{i}_A,\mathbf{i_B})}$ is the same in both cases.■

Proof of Lemma 4. We proceed along the lines of the proof of Lemma 2. Specifically, we show that there exists an equilibrium in which the risk premia, risk-free rate, and the supply of the market portfolio are uniquely determined by the equations stated in the proposition. First, observe that the endogenous supply of the market portfolio does not affect any investor's portfolio problem directly, but only indirectly through the equilibrium prices. Therefore, given equilibrium prices, average portfolio shares for investors with market access $i_A \in I_A$, $\omega_{J^{i_A}}^{i_A}$, defined in (A.1) still satisfy (A.2) and investors' certainty-equivalent returns are given by (16).

Next, note that the equations stated in the proposition still uniquely characterize the equilibrium returns of all assets and the supply of the market portfolio. To see this, notice that after substituting $\pi_m = \gamma\Lambda_m$, Eq. $(B.4)$ describes a downward sloping relation between r_f and η_m in (r_f, η_m)-space. In addition, condition (B.6) describes an upward sloping relation, since the left-hand side of (B.6) is increasing in η_m given that $Y_0 \geq \eta_m$ in equilibrium, and the right-hand side of (B.6) is increasing in r_f since $a(.)$ is assumed to be an increasing function. Also, there exists a solution to these equations since by (B.6) and an Inada condition for the production function, $\lim_{\eta_m \to 0} r_f(\eta_m) = -\infty$ and also $\lim_{\eta_m \to Y_0} r_f(\eta_m) = \infty$.

Finally, we check that these returns and the supply of the market portfolio satisfy the market clearing conditions (B.5). The market clearing conditions for $j \neq m$ are clearly satisfied since $\omega_j^{i_A} = 0$, for each i_A and $j \neq m$. The market clearing condition for the risk-free asset is also satisfied since $\omega_m = 1$ and $\omega_f = 1 - \omega_m$, so each investor has a zero weight on the risk-free asset. Finally, the market clearing condition for asset m is equivalent to (B.6), so it also holds.■

Proof of Proposition 7. First, as in Proposition 4, we have $r_{ce}^{(i_A^1, \mathbf{i_B})} \geq r_{ce}^{(i_A^0, \mathbf{i_B})}$. Next, we re-write (B.6) as

$$\sum_{i \in I} n^{i_A} n^{\mathbf{i_B}} a\left(r_{ce}^{(i_A, \mathbf{i_B})}\right) - \frac{\eta_m + \frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)}}{Y_0 + \frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)}} = 0.$$

Implicitly differentiating this equation (while keeping r_f constant), we obtain,

$$\text{(C.13)} \qquad \frac{\partial \eta_m}{\partial \Delta n} \propto \frac{\sum_{\mathbf{i_B}} n^{\mathbf{i_B}} \left[a\left(r_{ce}^{(i_A^1, \mathbf{i_B})}\right) - a\left(r_{ce}^{(i_A^0, \mathbf{i_B})}\right)\right]}{Y_0 + \frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)} + \frac{\partial}{\partial \eta_m}\left(\frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)}\right)(Y_0 - \eta_m)} \geq 0.$$

Here, we use the fact that $\frac{\partial}{\partial \eta_m}\left(\frac{G_L(\eta_m, 1)}{G_K(\eta_m, 1)}\right) \geq 0$ and $\eta_m \leq Y_0$ in equilibrium. The proof of Lemma 4 shows that r_f and η_m are jointly determined by Eqs. $(B.4)$ and (B.6), which are respectively downward and upward sloping in (r_f, η_m)-space. Eq. (C.13) implies that an increase in Δn leads to an upward shift in relation (B.6). This in turn implies that the equilibrium value of r_f declines and the value of η_m increases. By part (i) of Lemma 4, the decline in r_f translates into a decline in the expected return on all assets.■

Online Appendix D: Data details and additional analysis

The PSID makes a distinction between family units and household units. Throughout the paper we will use the term *households* to refer to family units.

Active saving definitionWe construct active saving by following the definition used in the PSID to construct active saving between 1984 and 1989. The general idea behind the active savings that the PSID constructs is to sum all net active asset purchases. For example, to obtain net active purchases of stocks, the PSID takes the amount of stocks purchased since 1984 and subtracts the amount of stocks sold since 1984. Similarly, to obtain net purchases of other real estate, they take the value of other real estate purchased since 1984 and subtract the value of other real estate sold since 1984. For the assets for which measures of net active purchases are not available, the PSID takes the change in wealth between the different survey periods and subtracts from it the change in values of assets for which there is available information (such as home value, stocks, farm and business, etc.). Finally, the PSID removes changes of assets and debt due to changes in the household, as well as inheritances, since those are arguably not active savings. We use the same method to construct active savings between subsequent survey waves.

Formally, active saving during that period is generated by summing total wealth in 1989, the 1984 home value (unless the household has moved), the 1989 value of private annuities, equity in other real estate in 1984, equity in a farm or business in 1984, the value of stocks held in 1984, the value of other

Table D.1—Summary Statistics

Variable	Observations	Mean	Std. Dev.	Min	Max
active saving rate	17,118	.037	.341	-1.193	1.142
stock mkt. participation	17,118	.288	.453	0	1
share of assets in stocks	15,393	.113	.237	0	1
income (log)	17,118	10.734	.608	9.10	13.869
wealth (log)	17,118	11.061	1.666	0	17.737
age	17,118	43.083	9.871	25	64

real estate purchased since 1984, the cost of additions and repairs to real estate since 1984, the amount invested in own business or farm since 1984, the amount of stocks purchased since 1984, assets removed by movers out of the household since 1984, and debts added by movers into the household since 1984. Out of this one subtracts total wealth in 1984, the 1989 home value, equity in other real estate in 1989, equity in a farm or business in 1989, the value of stocks held in 1989, the value of annuities and pensions cashed in since 1984, the value of other real estate sold since 1984, the value of farm or business sold since 1984, the value of stocks sold since 1984, debts removed by movers out of the household since 1984, assets added by movers into the household since 1984, and the inheritances received since 1984.[24]

Sample selectionWe make the following sample restrictions. We look at households whose head is between 25 and 65 years old and who have positive net worth. In addition, we remove households with extremely low incomes by first removing zero-income heads and then removing heads with income below the 20th percentile. We remove such low income households because their active savings tend to be extremely volatile. Furthermore, we remove outliers by excluding the top and bottom 10 percent of the saving rate distribution in a given survey year. We opt for a relatively aggressive trimming procedure to alleviate measurement error problems since wealth and saving components are self-reported in the PSID. All of our results about the saving behavior of participants versus non-participants are present with a more conservative trimming choice. Finally, we also drop household-year observations that are not in subsequent survey waves and households that do not reside in the 50 US states plus the District of Columbia.

We are left with an unbalanced panel of household saving and stock market participation variables containing 6,410 unique households between 1984 and 2011 with a total of 17,118 household-year observations. Table D.1 contains summary statistics for our main variables of interest.

In addition for our analysis of return dispersion we remove outliers in the implicit log return by excluding the top and bottom 10 percent of log returns in a given survey year. We also remove households that have owned a farm or business in any survey year. A more conservative trimming procedure leads to an increase in the level of return dispersion without affecting the positive time trend, which is the focus of our analysis.

Additional data analysis

Trends in savings. — Column 1 documents that the widening gap in savings between market participants and nonparticipants is robust to the inclusion of demographics, wealth controls and state fixed effects. While statistical significance is lost in the presence of household fixed effects, due to an increase in standard errors, the magnitude of the coefficient on the evolution of the gap is largely unchanged.

In columns 3 and 4, we study how savings comove with respect to an alternative, continuous measure of participation: the share of the household's assets that is invested in stocks (excluding their primary

[24]In case the household has moved between 1984 and 1989 the PSID constructs separate capital gains for each home owned. For simplicity, we drop households that have moved in between survey waves.

Table D.2—Trends in saving and stock market participation.

	(1)	(2)	(3)	(4)
stock mkt. participation	0.0308**	0.0168		
	(0.0098)	(0.0248)		
year	-0.0027**	-0.0027	-0.0029**	-0.0018
	(0.0005)	(0.0055)	(0.0005)	(0.0057)
stock market participation × year	0.0019*	0.0015		
	(0.0008)	(0.0016)		
share of assets in stocks			0.0430*	0.0399
			(0.0189)	(0.0392)
share of assets in stocks × year			0.0056**	0.0078**
			(0.0015)	(0.0026)
Additional controls	Yes	Yes	Yes	Yes
Household FE	No	Yes	No	Yes
R^2	0.018	0.274	0.021	0.286
Observations	16,610	14,696	14,934	13,069

Note: Standard errors in parenthesis. The standard errors are clustered at the state level. The active saving rate is defined as the annualized amount of active savings between two survey years, divided by the average of the household head's income in the two survey years. Active savings are constructed by extending the definition of active savings in the PSID for the period 1984-1989 to later survey waves. Year denotes the year of the survey relative to 1984 (the first year in the sample). Stock market participation is an indicator variable for whether the household holds shares of publicly traded companies or mutual funds (outside of IRAs post 1999). Share of assets in stocks is defined as the value of stocks (held outside of IRAs post 1999) relative to the value of all household assets, excluding the value of the household's primary residence. Additional controls include log of the household head's income and total household wealth, an educational category for the household head, gender of the household head, log of age for the household head and state of residence. For specifications (2) and (4) the additional control include log of the household head's income and total household wealth, and log age of the household head. All regressions are weighted using the PSID sampling weights. ** denotes significance at 1%, and * denotes significance at 5%.

residence). To focus on the intensive margin, we also restrict the analysis to stock market participants. There is a widening gap in saving rates based on this measure, even after controlling for household wealth, household head income and demographics, and household fixed effects. The share of wealth in stocks can be seen as a proxy of the household's risk tolerance. Through the lens of the choice channel, the saving of more risk tolerant investors increases relative to less risk tolerant investors as financial innovation accumulates.[25]

Trends in return dispersion. — One potential issue that could be causing a spurious positive trend in return dispersion (Figure 3) is the change in the frequency of data availability. For the early years of our data, annualized wealth growth is obtained by averaging across five years, while from 1999 onward, it is obtained by averaging across two years. Away from perfect autocorrelation in annual returns, this mechanically lowers the measured cross-sectional dispersion in the first half of the period relative to the second half.

[25]For the same beliefs, certainty equivalent returns are always higher for more risk tolerant agents. A formalization of this result was present in a previous working paper version and is available upon request.

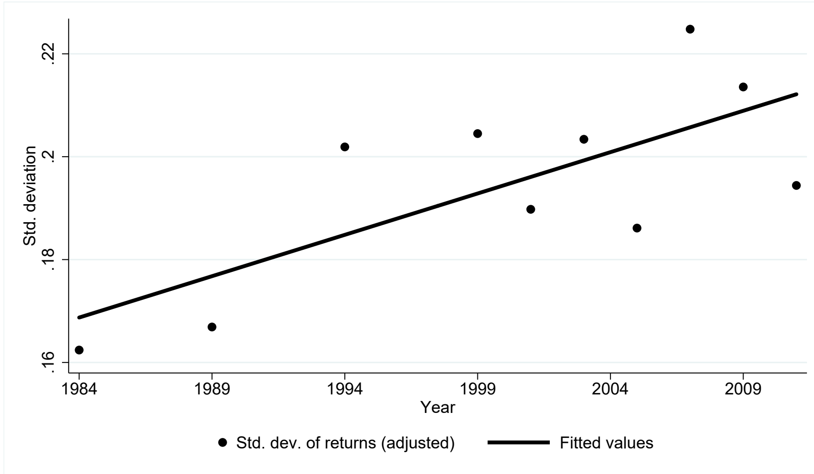

Figure D.1. Trends in dispersion of (implicit) returns on wealth among stock holders (excluding business owners). Adjusted averages.

A conservative adjustment for this problem is to assume independence of household returns across years and multiply the dispersion measures by the square root of five for the first period, and by the square root of two for the second. In this way, we obtain measures of annual return dispersion from the original dispersion of annual averages.[26] Figure D.1 replicates Figure 3 after this adjustment. Similarly, Figure D.2 replicates Figure 7 after this adjustment. The conclusions are unchanged.

[26]This adjustment is conservative because any alternative (constant) yearly return autocorrelation would require multiplication by smaller proportional factors and generate steeper positive trends.

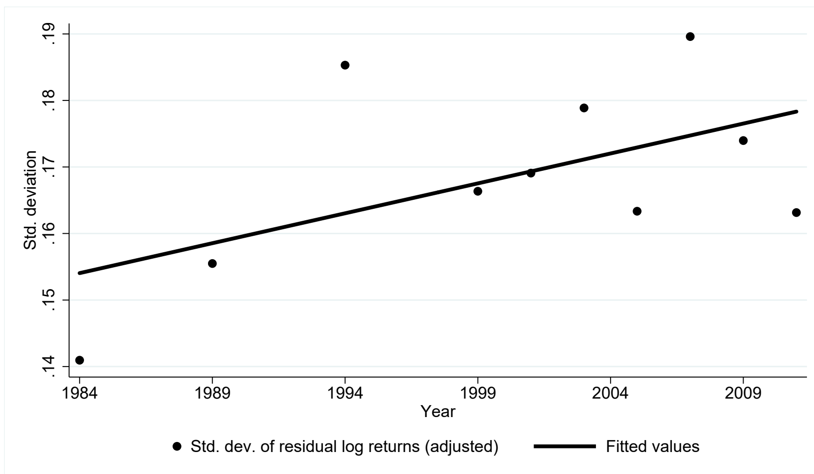

FIGURE D.2. TRENDS IN CONDITIONAL STANDARD DEVIATION OF (IMPLICIT) RETURNS ON WEALTH AMONG STOCK HOLDERS (EXCLUDING BUSINESS OWNERS). ADJUSTED AVERAGES.