B Online Appendix: "Scoring Strategic Agents" by Ian Ball

Nonlinear signaling equilibria are difficult to analyze in general. Here, I give a condition under which no Bayes–Nash equilibrium, pure or mixed, is fully informative.

Proposition 5 (No fully informative equilibrium)
If $\Sigma_{\delta\delta}$ has full rank, then the signaling game has no fully informative Bayes–Nash equilibrium.

Proof. Assume $\Sigma_{\delta\delta}$ is full rank. Suppose for a contradiction that the signaling game has a fully informative equilibrium. I will show that some type of the sender has a profitable deviation.

The first step is to construct the candidate deviating types. The type space $T = \operatorname{supp}(\eta, \delta)$ must contain an ellipse E defined by the equation

$$(\eta - \mu_\eta)^T \Sigma_{\eta\eta}^{-1} (\eta - \mu_\eta) + (\delta - \mu_\delta)^T \Sigma_{\delta\delta}^{-1} (\delta - \mu_\delta) = r^2,$$

for some positive radius r. Choose η^0 such that $(\eta^0 - \mu_\eta)\Sigma_{\eta\eta}^{-1}(\eta^0 - \mu_\eta)$ is strictly between 0 and r^2. Then $(\eta^0, t\mu_\delta)$ intersects E for two positive values of t, which I denote $t_1 < t_2$. Let $\delta^0 = t_1 \mu_\delta$ and set $\kappa = t_2/t_1$ so $\kappa\delta^0 = t_2\mu_\delta$. Next, I construct a sequence of types converging to (η^0, δ^0) as follows. Since $\Sigma_{\delta\delta}$ and $\Sigma_{\eta\eta}$ both have full rank, we can find a strictly positive sequence t^i converging to 0 and a real sequence s^i converging to 0 such that each type

$$(\eta^i, \delta^i) := (\eta^0 + t^i \beta, \delta^0 + s^i \delta^0)$$

lies on the ellipse E. Clearly $(\eta^i, \delta^i) \to (\eta^0, \delta^0)$ as $i \to \infty$.

For all $i \geq 0$, choose a feature vector x^i that type (η^i, δ^i) induces through some equilibrium distortion choice. Since the equilibrium is fully informative, it follows that $y(x^i) = \beta_0 + \beta^T \eta^i$ for each i. Each type (η^i, δ^i) can secure the payoff from mimicking (η^0, δ^0), so the sequence (x^i) for $i \geq 1$ is bounded. After possibly passing to a subsequence, I can assume that this sequence converges to some limit x^*.

Now I obtain the contradiction. To simplify notation, let

$$c(d) = (1/2)\sum_{j=1}^{k} d_j/\delta_j^0.$$

Each type (η^i, δ^i) weakly prefers x^i to x^0, so

$$t^i\|\beta\|^2 \geq \frac{c(x^i - \eta^i) - c(x^0 - \eta^i)}{(1+s^i)^2}.$$

Passing to the limit in i gives

$$c(x^* - \eta^0) \leq c(x^0 - \eta^0). \tag{32}$$

Type $(\eta^0, \kappa\delta^0)$ must be indifferent between x^0 and any feature vector chosen in equilibrium since x^0 yields same decision and cannot be more costly (for otherwise type (η^0, δ^0) would have a profitable deviation). Therefore, type $(\eta^0, \kappa\delta^0)$ weakly prefers x^0 to x^i, so

$$t^i\|\beta\|^2 \leq \frac{c(x^i - \eta^0) - c(x^0 - \eta^0)}{\kappa^2} \leq \frac{c(x^i - \eta^0) - c(x^* - \eta^0)}{\kappa^2}, \tag{33}$$

where the second inequality follows from (32).

Similarly, since each type (η^i, δ^i) prefers x^i to x^j, we have

$$(t^i - t^j)\|\beta\|^2 \geq \frac{c(x^i - \eta^i) - c(x^j - \eta^i)}{(1+s^i)^2}.$$

Passing to the limit as $j \to \infty$ gives

$$t^i\|\beta\|^2 \geq \frac{c(x^i - \eta^i) - c(x^* - \eta^i)}{(1+s^i)^2}. \tag{34}$$

Clear denominators in (33) and (34) and then subtract to get

$$
\begin{aligned}
&((1+s_i)^2 - (1+\kappa)^2)t^i\|\beta\|^2 \\
&\geq [c(x^i - \eta^i) - c(x^* - \eta^i)] - [c(x^i - \eta^0) - c(x^* - \eta^0)] \\
&= [c(x^i - \eta^i) - c(x^i - \eta^0)] + [c(x^* - \eta^0) - c(x^* - \eta^i)] \\
&= [c(x^i - \eta^0 - t^i\beta) - c(x^i - \eta^0)] + [c(x^* - \eta^0) - c(x^* - \eta^0 - t^i\beta)].
\end{aligned}
$$

Divide by t^i and pass to the limit as $i \to \infty$. By the mean value theorem, the terms on the right converge to $-c'(x^* - \eta^0)\beta$ and $c'(x^* - \eta^0)\beta$, so we obtain the contradiction

$$-(\kappa^2 - 1)\|\beta\|^2 \geq 0. \qquad \square$$