Online Appendix
for "Heterogeneous Noise and Stable Miscoordination"

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B Online Appendix

B.1 Standard Representation of Coordination Games

Symmetric Coordination Games We first show that the one-parameter standard representation (left panel of Table 1 in the main text) w.l.o.g. captures any two-action symmetric coordination game (in line with Harsanyi and Selten's (1988) Axiom 2 of best-response invariance).

A two-action symmetric coordination game is characterized by two strict equilibria on the main diagonal, as represented by the right panel of Table 1. Sampling dynamics (defined in Eq. 2.1 and 2.2) depend solely on the differences between the payoffs a player can achieve by choosing different actions. (This property also holds for best-response and logit dynamics, implying that the sets of Nash equilibria, quantal response equilibria, and evolutionarily stable strategies depend only on these payoff differences.) These differences remain unchanged when a constant is subtracted from all of a player's payoffs while holding the opponent's action fixed (e.g., subtracting u_{21} from all of Player 1's first-column payoffs). Additionally, the sampling dynamics (and the other dynamics and solution concepts mentioned above) are invariant to dividing a player's payoffs by a positive constant (which preserves vN–M utility). The right panel of Table 1 is reduced

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to the left panel by the following steps (which do not affect the sampling dynamics):

1. Subtracting u_{21} from Player 1's payoffs in her first column.

2. Subtracting u_{12} from Player 1's payoffs in her second column.

3. Dividing Player 1's payoffs by $u_{22} - u_{12}$

General Coordination Games Next, we show that two-parameter standard representation (Table 2 in the main text, with $u_1 \geq 1$) can capture any two-action game that admits two strict Nash equilibria. By relabeling Player 1's actions, we can assume w.l.o.g. that these two pure equilibria are (a_1, a_2) and (b_1, b_2) . If the two pure equilibria are instead (a_1, b_2) and (b_1, a_2), we switch the Player 1's action labels: $a_1 \leftrightarrow b_1$). This implies that the left panel of Table 3 shows a parametric representation of all two-action coordination games.

As mentioned above, sampling dynamics (as defined in Eq. 3.1) depend only on the differences between the payoffs a player can get by playing different actions. These differences are invariant to (1) subtracting a constant from all the payoffs of a player while fixing the opponent's action, and (2) dividing player's payoffs by a positive constant (which preserves the vN–M utility). The left panel of Table 3 is reduced to the right panel by the following steps (none of which affect the sampling dynamics):

1. Three changes to Player 1's payoffs: (I) subtracting u_{21} from Player 1's payoffs in her first column, (II) subtracting u_{12} from Player 1's payoffs in her second column, and (III) dividing Player 1's payoffs by $u_{22}-u_{12}$; and

2. Three changes to Player 2's payoffs: (I) subtracting v_{12} from Player 1's payoffs in her first row, (II) subtracting v_{21} from Player 1's payoffs in her first row, and (III) dividing Player 1's payoffs by $v_{22} - v_{21}$.

Table 3: Normalization of General Two-Action Coordination Games

Original Representation

	a_2	b_2
a_1	u_{11}, v_{11}	u_{12}, v_{12}
b_1	u_{21}, v_{21}	u_{22}, v_{22}

$\Rightarrow$

Standard Representation

	a_2	b_2
a_1	$u_1 = \frac{u_{11}-u_{21}}{u_{22}-u_{12}}$, $u_2 = \frac{v_{11}-v_{12}}{v_{22}-v_{21}}$	0, 0
b_1	0, 0	1, 1

$u_{11} > u_{21}, u_{22} > u_{12},\ v_{11} > v_{12}, v_{22} > v_{21}$

Table 4: Normalization of Hawk–Dove Games ($g, l \in (0,1)$)

Original Representation

	h_2	d_2
h_1	0, 0	$1+g$, $1-l$
d_1	$1-l$, $1+g$	1, 1

$\Rightarrow$

Standard Representation

	$a_2 = h_2$	$b_2 = d_2$
$a_1 = d_1$	$\frac{1-l}{g}$, $\frac{g}{1-l}$	0, 0
$b_1 = h_1$	0, 0	1, 1

Observe that the assumption that $u_1 = \frac{u_{11}-u_{21}}{u_{22}-u_{12}} \geq 1$ in the standard representation of Table 2 is w.l.o.g.. If $\frac{u_{11}-u_{21}}{u_{22}-u_{12}} < 1$, then we can multiply all of Player 1's payoffs by $\frac{u_{22}-u_{12}}{u_{11}-u_{21}}$ and all of Player 2's payoffs by $\frac{v_{22}-v_{21}}{v_{11}-v_{12}}$, relabel the actions $a_i \leftrightarrow b_i$ for both players, and obtain a standard representation in which $u_1 \geq 1$ as in Table 2.

Example 1. Consider a *hawk–dove* (aka chicken) game, which can be interpreted as bargaining over the price of an asset (e.g., a house) between a buyer and a seller. Each player can either insist on a more favorable price ("hawk") or agree to a less favorable price in order to close the deal ("dove"). The left panel of Table 4 shows the payoffs of a hawk–dove game. Two doves agree on an equally favorable price (which gives both players a relatively high payoff normalized to one). A hawk obtains a favorable price when matched with a dove (which increases the payoff to the hawk by $g \in (0,1)$, while reducing the dove's payoff by $l \in (0,1)$), but faces a high probability of bargaining failure when matched with another hawk (which yields a low payoff of zero to both hawks).

Observe that a hawk–dove game can be transformed to our standard representation of a coordination game (the right panel of Table 4) as follows: (1) relabel the actions of Player 1 such that $a_1 = d_1$ and $b_1 = h_1$ (while keeping the actions of Player 2 as $a_2 = h_2$ and $b_2 = d_2$), (2) subtract a payoff of 1 from Player 1's payoffs in her second column and from Player 2's payoffs in her first column, and (3) divide all the payoffs of Player 1 by g, and all of the payoffs of Player 2 by $1-l$. Observe that the induced standard representation is antisymmetric, i.e., $u_1 = \frac{1-l}{g} = \frac{1}{u_2}$.

B.2 Rephrasing Our Results in Terms of q-Dominance

In this appendix, we rephrase our results in terms of q-dominance (Morris et al. (1995) of the equilibria, rather than payoffs (u_1, u_2). This serves two purposes: (1) to facilitate comparison with the literature, which often presents results in terms of q-dominance (e.g., Oyama et al., 2015), and (2) to provide a clearer interpretation of the results that is invariant to payoff normalization. Specifically, normalizing a game to fit the standard two-parameter representation (see Appendix B.1), alters the payoffs, yet preserves the q-dominance of the equilibria.

Fix $q \in [0, 1]$. We say that action a_i (resp., b_i) is *q-dominant* (Morris et al., 1995) for player i if it is a strict best response to any opponent's mixed action that assigns a mass of at least q to the counterpart action a_j (resp., b_j). Notice that both actions are 1-dominant (due to being part of a strict equilibrium). Additionally, it can be noted that as q decreases, the q-dominance condition becomes more stringent. In other words, if an action is q-dominant, it also satisfies r-dominance for any r between q and 1. Lastly, it can be observed that action a_i (resp., b_i) is q-dominant iff $q > \frac{1}{1+u_i}$ (resp., $q > \frac{u_i}{1+u_i}$).

Observe that q-dominance depends only on the differences between the payoffs a player can get by playing the different actions. This implies that q-dominance is invariant to the payoff transformations detailed in Appendix B.1. Thus, an action is q-dominant in the standard representation (left panel of Table 3) iff it is q-dominant in the original representation (right panel of Table 3). By contrast, payoff dominance is not invariant to these two transformations. Specifically, adding a constant to the two payoffs of Player 1 (Player 2) in the same column (row) might change a Pareto-dominated equilibrium into a Pareto-dominant equilibrium.

The following definition will be useful for our characterization.

Definition 1. Action profile $\mathbf{a}$ (resp., $\mathbf{b}$) is tightly $\mathbf{q}$-dominant if each action a_i (resp., b_i) is r_i-dominant iff $r_i > q_i$.

Observe that:

1. $\mathbf{a}$ is tightly (q_1, q_2)-dominant iff $\mathbf{b}$ is tightly $(1 - q_1, 1 - q_2)$-dominant.

2. If the payoffs of the coordination game (in its standard representation) are (u_1, u_2), then equilibrium $\mathbf{a}$ is tightly $(\frac{1}{1+u_1}, \frac{1}{1+u_2})$-dominant.

3. if equilibrium $\mathbf{a}$ is tightly (q_1, q_2)-dominant, then the standard representation of the payoff matrix is $u_i = \frac{1-q_i}{q_i}$.

We redefine an environment as a pair $(\mathbf{q}, \boldsymbol{\theta})$, where $\mathbf{q}$ represents the level of tight dominance of equilibrium $\mathbf{a}$ and $\boldsymbol{\theta}$) is the sample-size distribution profile. W.l.o.g. we assume $q_1 \geq \frac{1}{2}$. The above observations imply the following rephrasing of our results.

Theorem $\hat{2}$ (Rephrasing of Theorem 2)**.** *If for each population* i*,* $1 < \max(\text{supp}(\theta_i)) < \frac{1}{q_i}$*. Then, there exists a proportion* α_i *such that significantly increasing the sample sizes of* α_i *of the agents in each population* i *induces an environment with an asymptotically stable interior state.*

Theorem $\hat{3}$ (Rephrasing of Theorem 3)**.** *For any sample size distribution profile, if* q_1 *is sufficiently small and* q_2 *is sufficiently large, then significantly increasing the sample size of half of the agents in each population induces an asymptotically stable interior state with a miscoordination probability of at least 50%.*

Theorem $\hat{4}$ (Rephrasing of Theorem 4)**.**

1. ***Global convergence to miscoordination****: Assume that*

$$\theta_1(1) \cdot \mathbb{E}_{<\frac{1}{1-q_2}}(\theta_2) > 1 \quad \textit{and} \quad \theta_2(1) \cdot \mathbb{E}_{<\frac{1}{q_1}}(\theta_1) > 1.$$

If $\boldsymbol{p}(0) \notin \{(0,0),(1,1)\}$*, then* $\lim_{t\to\infty} \mathbf{p}(t) \notin \{(0,0),(1,1)\}$*.*

2. ***Local convergence to coordination****: Assume that*

$$\theta_1(1) \cdot \mathbb{E}_{<\frac{1}{1-q_2}}(\theta_2) < 1 \quad \textit{or} \quad \theta_2(1) \cdot \mathbb{E}_{<\frac{1}{q_1}}(\theta_1) < 1.$$

Then at least one of the pure equilibria is asymptotically stable.

Proposition $\hat{4}$**.** *Assume that pure equilibrium* $\boldsymbol{a}$ *is tightly* (q_1, q_2)*-dominant. Then*[1]

1. $\mathbb{E}_{<\frac{1}{1-q_1}}(\theta_1) \cdot \mathbb{E}_{<\frac{1}{1-q_2}}(\theta_2) > 1 \Rightarrow (a_1, a_2)$ *is unstable;*
2. $\mathbb{E}_{<\frac{1}{1-q_1}}(\theta_1) \cdot \mathbb{E}_{<\frac{1}{1-q_2}}(\theta_2) < 1 \Rightarrow (a_1, a_2)$ *is asymptotically stable;*
3. $\mathbb{E}_{\le\frac{1}{q_1}}(\theta_1) \cdot \mathbb{E}_{\le\frac{1}{q_2}}(\theta_2) > 1 \Rightarrow (b_1, b_2)$ *is unstable; and*
4. $\mathbb{E}_{\le\frac{1}{q_1}}(\theta_1) \cdot \mathbb{E}_{\le\frac{1}{q_2}}(\theta_2) < 1 \Rightarrow (b_1, b_2)$ *is asymptotically stable.*

B.3 Coordination Games with More Than Two Actions

B.3.1 Extended model

We redefine the underlying game as a two-player coordination game with $M \geq 2$ actions. The action sets $A_i = \left(a_i^1, ..., a_i^M\right)$ are finite, with positive payoffs on the main diagonal

[1] The slight difference in using strict inequalities for $\mathbf{a}$ and weak inequalities for $\mathbf{b}$ is due to our a_i-favorable tie-breaking rule.

Table 5: Payoff Matrix for Coordination Games with $M \geq 2$ Actions

	a_2^1	...	a_2^m	...	a_2^M
a_1^1	u_1^1, u_2^1	0,0	0,0	0,0	0,0
...	0,0	...	0,0	0,0	0,0
a_1^m	0,0	0,0	u_1^m, u_2^m	0,0	0,0
...	0,0	0,0	0,0	...	0,0
a_1^M	0,0	0,0	0,0	0,0	u_1^M, u_2^M

and zero off-diagonal payoffs: (1) $u_i^m \equiv u_i(a_1^m, a_2^m) > 0$ for each m, and (2) $u_i(a_1^m, a_2^n) = 0$ for each $m \neq n$. The payoff matrix is shown in Table 5. To insure results hold for all tie-breaking rules, we take the following mild genericity assumption: there do not exist two pure equilibria that yield the same payoff profile (i.e., if $u_i^m = u_i^{m\prime}$ then $u_j^m \neq u_j^{m\prime}$).

This class of coordination games is significant as it captures common situations in which players must collectively agree on the terms of a joint venture that could yield positive benefits for both parties involved. Each action profile on the main diagonal represents potential mutually agreed-upon terms, while off-diagonal action profiles represent disagreement, resulting in the failure of the joint venture and, consequently, a low payoff (normalized to zero) for both players. These games are called pure coordination games or contracting games (see, e.g., Young, 1998; Hwang and Newton, 2017).

We redefine an environment as a tuple $(G, \boldsymbol{\theta})$, where G is a two-player coordination game with $M \geq 2$ actions, and $\boldsymbol{\theta}$ is the profile of sample size distributions. A state of population i is a distribution $p_i \in \Delta(A_i)$ over the actions of player i. As in the baseline model, each new agent with sample size k samples k random actions of her opponent and plays the action that maximizes her payoff against the sample. The results of this section hold under any tie-breaking rule. All other aspects of the model remain unchanged.

A pure equilibrium $\boldsymbol{a}^m = (a_1^m, a_2^m)$ is *Pareto efficient* if for each n, $u_1^m < u_1^n$ implies that $u_2^m > u_2^n$. Let $\bar{u}_i$ denote the highest feasible payoff of player i, i.e., $\bar{u}_i = \max_{m \leq M}(u_i^m)$. Let $\bar{m}_i$ be the index of an action that induces payoff $\bar{u}_i$, i.e., $u_i^{\bar{m}_i} = \bar{u}_i$.

B.3.2 Generalized Result

In what follows, we generalize Theorem 4 to coordination games with $M \geq 2$ actions. Specifically, we show that similar to the case of two actions, the stability of each pure equilibrium depends on whether the product of the share of agents with sample size one and the truncated expectation of the sample size is larger or smaller than one. Formally,[2]

Proposition 1 (Generalization of Theorem 4)**.**

1. *Assume that for each pure equilibrium* $\boldsymbol{a}^m$*,*

$$\theta_1(1)\cdot\mathbb{E}_{<\frac{\bar{u}_2}{u_2^m}+1}(\theta_2)>1 \text{ or } \theta_2(1)\cdot\mathbb{E}_{<\frac{\bar{u}_1}{u_1^m}+1}(\theta_1)>1.$$

   Then all pure equilibria are unstable.

2. *Assume that there exists a Pareto-efficient pure equilibrium* $\boldsymbol{a}^m$ *that satisfies,*

$$\theta_1(1)\cdot\mathbb{E}_{\leq\frac{\bar{u}_2}{u_2^m}+1}(\theta_2)<1 \text{ and } \theta_2(1)\cdot\mathbb{E}_{\leq\frac{\bar{u}_1}{u_1^m}+1}(\theta_1)<1.$$

   Then equilibrium $\boldsymbol{a}^m$ *is asymptotically stable.*

Sketch of Proof. See Appendix B.3.3 for a formal proof.

1. Assume that $\theta_1(1)\cdot\mathbb{E}_{<\frac{\bar{u}_2}{u_2^m}+1}(\theta_2)>1$ (resp., $\theta_2(1)\cdot\mathbb{E}_{<\frac{\bar{u}_1}{u_1^m}+1}(\theta_1)>1$). Observe that in any initial state in which in each population i almost all agents play a_i^m, while a few agents play $a_i^{\bar{m}_2}$ (resp., $a_i^{\bar{m}_1}$), the product of the shares of agents who play $a_i^{\bar{m}_2}$ (resp., $a_i^{\bar{m}_1}$) in each population i would increase by analogous arguments to those in Proposition 4. This implies that equilibrium $\boldsymbol{a}^m$ is unstable. If condition (1) holds for all pure equilibria, then all of those equilibria are unstable.

2. Consider any initial state in which almost all agents play a_i^m. The fact that $\boldsymbol{a}^m$ is Pareto efficient implies that $u_i^{m'}<u_i^m$ for some $i\in\{1,2\}$. By analogous arguments to those in Proposition 4, the small share of agents who play $a_i^{m'}$ would decrease if $\theta_i(1)\,\mathbb{E}_{\leq\frac{u_j^{m'}}{u_j^m}+1}(\theta_j)<1$. This inequality holds because $\mathbb{E}_{\leq\frac{u_j^{m'}}{u_j^m}+1}(\theta_j)\leq\mathbb{E}_{\leq\frac{\bar{u}_j}{u_j^m}+1}(\theta_2)$ and $\theta_i(1)\,\mathbb{E}_{\leq\frac{\bar{u}_j}{u_j^m}+1}(\theta_2)<1$. This implies that $\boldsymbol{a}^m$ is asymptotically stable. □

Proposition 1 immediately implies that if all agents have the same sample size size, then all pure states are asymptotically stable.

Corollary 1 (Adaptation of Theorem 1')**.** *Assume that* $\theta_i\equiv k_i>1$ *for each* $i\in\{1,2\}$*. Then, all the Pareto-efficient pure equilibria are asymptotically stable.*

[2] The fact that the truncated expectation has strict inequality in part (1) and weak inequality in part (2) allows these conditions to be valid under any tie-breaking rule.

Thus, heterogeneity in sample size is important for the stability of miscoordination also in this extended setup. Finally, observe that it is straightforward to adapt Theorem 3 to the setup with $M \geq 2$ actions, by assuming that (1) the payoffs of two of the pure equilibria satisfy the conditions of Theorem 3, and (2) these two equilibria Pareto dominate the remaining pure equilibria. Formally,

Corollary 2 (Generalization of Theorem 3)**.** *If $\frac{u_1^1}{u_1^2}$ and $\frac{u_2^1}{u_2^2}$ are not too close to one and the u_i^m-s are sufficiently small for each $m > 2$, then replacing some of the agents in each population by agents with sufficiently large sample sizes, can induce an asymptotically stable interior state $\hat{\boldsymbol{p}}$.*

The proof (which is essentially the same as the proof of Theorem 3) is omitted for brevity.

B.3.3 Formal Proof of Proposition 1

Part 1. We have to show that pure equilibrium $\boldsymbol{a}^m$ is unstable if either $\theta_1(1) \cdot \mathbb{E}_{<\frac{\bar{u}_2}{u_2^m}+1}(\theta_2) > 1$ or $\theta_2(1) \cdot \mathbb{E}_{<\frac{\bar{u}_1}{u_1^m}+1}(\theta_2) > 1$. Assume that $\theta_1(1) \cdot \mathbb{E}_{<\frac{\bar{u}_2}{u_2^m}+1}(\theta_2) > 1$ (resp., $\theta_2(1) \cdot \mathbb{E}_{<\frac{\bar{u}_1}{u_1^m}+1}(\theta_2) > 1$). Consider a slightly perturbed state near $\boldsymbol{a}^m$, where in each population i a small share $\epsilon_i << 1$ of the agents play action $a_i^{\bar{m}_2}$ (resp., $a_i^{\bar{m}_1}$), while all the other agents play action a_i^m. Observe that: (1) a new agent in population 1 (resp., 2) with sample size 1 who observes the rare action $a_2^{\bar{m}_2}$ (resp., $a_1^{\bar{m}_1}$) plays action $a_1^{\bar{m}_2}$ (resp., $a_2^{\bar{m}_1}$), and (2) a new agent in population 2 (resp., 1) with sample size k who observes the rare action $a_1^{\bar{m}_2}$ (resp., $a_2^{\bar{m}_1}$) once in her sample plays $a_2^{\bar{m}_2}$ (resp., $a_1^{\bar{m}_1}$) if $(k-1)\,u_2^m < \bar{u}_2 \Leftrightarrow k < \frac{\bar{u}_2}{u_2^m} + 1$ (resp., $k < \frac{\bar{u}_1}{u_1^m} + 1$). This gives the following lower bound for the change in the share of agents who play the rare action (neglecting terms that are of order $O(\epsilon_i^2)$):

$$\dot{\epsilon}_1 \geq \theta_1(1)\,\epsilon_2 - \epsilon_1 \text{ and } \dot{\epsilon}_2 \geq \mathbb{E}_{<\frac{\bar{u}_2}{u_2^m}+1}(\theta_2)\,\epsilon_1 - \epsilon_2$$

$$\left(\text{resp., } \dot{\epsilon}_1 \geq \mathbb{E}_{<\frac{\bar{u}_1}{u_1^m}+1}(\theta_1)\,\epsilon_1 - \epsilon_2 \text{ and } \dot{\epsilon}_2 \geq \theta_2(1)\,\epsilon_1 - \epsilon_2\right).$$

Observe that the Jacobian of the above system of equations is given by $J = \begin{pmatrix} -1 & a_1 \\ a_2 & -1 \end{pmatrix}$ with $a_1 \geq \theta_1(1)$ and $a_2 \geq \mathbb{E}_{<\frac{\bar{u}_2}{u_2^m}+1}(\theta_2)$ (resp., $a_2 \geq \theta_2(1)$ and $a_1 \geq \mathbb{E}_{<\frac{\bar{u}_1}{u_1^m}+1}(\theta_1)$). The larger eigenvalue is given by $-1+\sqrt{a_1 a_2}$, which is larger than $-1+\sqrt{\theta_1(1)\,\mathbb{E}_{<\frac{\bar{u}_2}{u_2^m}+1}(\theta_2)} >$

$-1+1=0$ (resp., $-1+\sqrt{\theta_2(1)\,\mathbb{E}_{<\frac{\bar{u}_1}{u_1^m}+1}(\theta_2)} > -1+1=0$). The fact that this eigenvalue is positive implies that the state is unstable (see, e.g., Perko, 2013, Section 2.9).

Part 2: Any perturbed state near $\boldsymbol{a}^m$ can be represented as a vector $\left(\epsilon_1^{m'},\epsilon_2^{m'}\right)_{m'\neq m}$ with $2\times(M-1)$ components, where $\epsilon_i^{m'}<<1$ represents the share of agents in population i who play action $a_i^{m'}$ (where the remaining share $1-\sum_{m'\neq m}\epsilon_i^{m'}$ of agents in population i play action a_i^m). Observe that a new agent in population i with sample size k who observes the rare action $a_j^{m'}$ once in her sample (and all other observed actions are a_j^m) plays $a_j^{m'}$ only if $(k-1)\,u_i^m \leq u_i^{m'} \Leftrightarrow k \leq \frac{u_i^{m'}}{u_i^m}+1$. This give the following upper bound for the share of agents who play the rare action (neglecting terms that are of order $O\left(\epsilon_i^{m'}\right)^2$):

$$\dot{\epsilon}_i^{m'} \leq \mathbb{E}_{\leq\frac{u_i^{m'}}{u_i^m}+1}(\theta_i)\,\epsilon_j^{m'}-\epsilon_i^{m'}.$$

Observe that the Jacobian of the above system of $(2\times(M-1))$ equations is given by

$$J=\begin{pmatrix} -1 & a_1^1 & & 0 & 0 & & 0 & 0\\ a_2^1 & -1 & & 0 & 0 & & 0 & 0\\ & & & \vdots & & & & \\ 0 & 0 & \dots & -1 & a_1^{m'} & \dots & 0 & 0\\ 0 & 0 & & a_2^{m'} & -1 & & 0 & 0\\ & & & \vdots & & & & \\ 0 & 0 & & 0 & 0 & & -1 & a_1^M\\ 0 & 0 & & 0 & 0 & & a_2^M & -1 \end{pmatrix},$$

where $a_1^{m'} \leq \mathbb{E}_{\leq\frac{u_i^{m'}}{u_i^m}+1}(\theta_i)$. The Jacobian has the following $2\times(M-1)$ eigenvalues: $\left(-1\pm a_1^{m'}\cdot a_2^{m'}\right)_{m'\neq m}$. The fact that $\boldsymbol{a}^m$ is Pareto efficient (and the mild assumption that there do not exist two pure equilibria that yield the same payoff profile) implies that $u_i^{m'}<u_i^m$ for $i\in\{1.2\}$. This implies that $\mathbb{E}_{\leq\frac{u_i^{m'}}{u_i^m}+1}=\theta_i(1)$. Observe that $\mathbb{E}_{\leq\frac{u_j^{m'}}{u_j^m}+1}(\theta_j)\leq \mathbb{E}_{<\frac{\bar{u}_j}{u_j^m}+1}(\theta_j)$. This, in turn, implies that the eigenvalue $-1\pm a_1^{m'}\cdot a_2^{m'}$ is bounded by

$$-1\pm a_i^{m'}\cdot a_j^{m'} \leq -1+\mathbb{E}_{\leq\frac{u_i^{m'}}{u_i^m}+1}(\theta_i)\,\mathbb{E}_{\leq\frac{u_j^{m'}}{u_j^m}+1}(\theta_j)\leq -1+\theta_i(1)\,\mathbb{E}_{\leq\frac{\bar{u}_j}{u_j^m}+1}(\theta_2)<-1+1=0,$$

which implies that all the eigenvalues are positive. Thus, $\boldsymbol{a}^m$ is asymptotically stable.

B.4 Coordination Games with More Than Two Players

In this appendix, we extend our analysis to minimum effort coordination games (Van Huyck et al., 1990) with $N > 2$ players. We analyze one-population dynamics of these games, as this aligns with the typical experimental implementation, where the game is symmetric, and players cannot condition their behavior on specific roles. For simplicity, we assume each agent chooses between two effort levels, $A = \{L, H\}$ (the results are similar with more effort levels). An agent selecting low effort L receives a payoff of 1. An agent selecting high effort (H) earns a payoff of $2 - c$ if all opponents also choose H, and $1 - c$ otherwise, where $c \in (0, 1)$ represents the cost of high effort. The game has two strict equilibria: the *safe equilibrium* $\boldsymbol{L}$ and the *efficient equilibrium* $\boldsymbol{H}$. Let $p \in [0, 1]$ denote the share of agents playing action a in the popualtion.

When generalizing the sampling dynamics to games with more than two players, different assumptions can be made about what each agent observes. We focus on two alternative assumptions on what each agent observes in each element of her sample:

1. the minimum effort level in a random round of the N-player game. This observation structure fits the best the typical feedback in experimental implementations of the minimum effort games (see, e.g., Van Huyck et al., 1990; Avoyan and Ramos, 2023).

2. The action of a randomly chosen opponent.

B.4.1 Observation of Minimum Efforts

Our first result characterizes the asymptotic stability of pure equilibria. The safe equilibrium is always asymptotically stable, while the efficient equilibrium becomes unstable iff the truncated expectation of the sample size is sufficiently large.

Proposition 2. *State L is asymptotically stable for all c-s. State H is:*

1. *asymptotically stable if* $\mathbb{E}_{\leq \frac{1}{1-c}}(\theta) < \frac{1}{N}$, *and*

2. *unstable if* $\mathbb{E}_{< \frac{1}{1-c}}(\theta) > \frac{1}{N}$.

Proof. Consider a perturbed state $(1 - \epsilon)$ near L, where $\epsilon << 1$ represents the small share of agents playing H. A new agent with sample size k will only play H if they observe H as the minimum effort in their sample, but the probability of this is negligible: $O(k \cdot \epsilon^N) < \epsilon$. Thus, L is asymptotically stable.

Now consider a perturbed state $\epsilon << 1$ near H. A new agent with sample size k plays H (resp. L) when they observe a rare minimum effort level L once in their sample if $(k-1)(1-c) > c \Leftrightarrow k > \frac{1}{1-c}$ (resp. $k < \frac{1}{1-c}$). The probability of observing L is approximately $N \cdot k \cdot \epsilon$. Following similar reasoning to Proposition 4, we conclude:

1. converging to everyone playing H, so H is asymptotically stable if $N \cdot \mathbb{E}_{\leq \frac{1}{1-c}}(\theta) < 1$.

2. the share of agents playing L increases, making H unstable if $N \cdot \mathbb{E}_{< \frac{1}{1-c}}(\theta) > 1$. □

Comparative statics for the stability of the efficient equilibrium align with experimental findings: the set of distributions for which the efficient equilibrium is stable decreases with both the cost of effort c and the number of players N. (Numeric analysis also suggests that similar trends apply to the size of the efficient equilibrium's basin of attraction.)

Remark 1. In typical experiments of this game, there are 7 (rather than 2) levels of effort, and a player's payoff is equal to the minimal effort level chosen by any of the players minus c times her own effort. Simple adaptations to the proof of Proposition 2 show that the same condition for the asymptotic stability of the efficient equilibrium holds for any non-safe equilibrium. That is, if $\mathbb{E}_{< \frac{1}{1-c}}(\theta) > \frac{1}{N-1}$, then only the safe equilibrium is asymptotically stable, while if $\mathbb{E}_{\leq \frac{1}{1-c}}(\theta) < \frac{1}{N-1}$, then all pure equilibria are asymptotically stable.

Next we show that heterogeneity induces stable miscoordination also with $N > 2$ players. Specifically, we generalize Theorem 3, and show that if k is not too large, one can always add players with sufficiently large sample sizes, and obtain an environment with stable miscoordination.

Proposition 3 (Adaptation of Theorem 3)**.** *For any $k < \frac{1}{1-c}$, there exists a minimum-effort environment with some agents having sample size k and the others with sufficiently large samples, in which an asymptotically stable interior state exists.*

Proof. Let $p^{NE} \in (0,1)$ be the symmetric interior Nash equilibrium of the minimum effort coordination game. Fix a sufficiently small $\epsilon > 0$. Let $w^{\theta}(p)$ be the probability of a new agent playing action L when the new agent's sample size is distributed according to θ, and when the share of agents playing L is p. An agent with sample size k would play action L if her sample includes at least one observation of L. The probability of this is

$$w^k(p) = 1 - (1-p)^{k(N-1)} = k(N-1)p - \begin{pmatrix} k(N-1) \\ 2 \end{pmatrix} p^2 + O\left(p^3\right).$$

Fix a sufficiently small $\epsilon > 0$. Let $\hat{p} \in \left(0, p^{NE} - \epsilon\right)$ be sufficiently small such that the term $O(p^3) < \epsilon$ is negligible for any $p < \hat{p}$. Let $\alpha \in (0,1)$ be such that: (1) $\alpha k(N-1) > 1$ and (2) $\alpha\left(k(N-1) - \begin{pmatrix} k(N-1) \\ 2 \end{pmatrix} p\right) < 1 - 2\epsilon$. This implies that $w^k(p) > \frac{p}{\alpha}$ in a right neighborhood of zero, and $w^k(p) < \frac{p}{\alpha}$ in a left neighborhood of $\hat{p}$. Observe that $w^{\bar{k}}(p) < \epsilon$ for a sufficiently large $\bar{k}$. This implies that $w^{k\alpha\bar{k}}(p) > p$ in a right neighborhood of zero, and $w^{k\alpha\bar{k}}(p) < p$ in a left neighborhood of $\hat{p}$. This, in turn, implies that there exists a symmetric stationary state $\tilde{p} \in (0, \hat{p})$ that satisfies (1) $w^{k\alpha\bar{k}}(\tilde{p}) = \tilde{p}$, (2) $w^{k\alpha\bar{k}}(p) > p$ in a left neighborhood of $\tilde{p}$, and (3) $w^{k\alpha\bar{k}}(p) < p$ in a right neighborhood of $\tilde{p}$. Due to Part (3) of Fact 1 $\tilde{p}$ is asymptotically stable. □

B.4.2 Observation of Actions

Next, we characterize the asymptotic stability of pure equilibria when new agents observe actions instead of minimum efforts.

Proposition 4. *1. The safe equilibrium L is:*

(a) asymptotically stable if $\mathbb{E}_{\leq\left(\sqrt[(N-1)]{\frac{1}{1-c}}\right)}(\theta) < 1$, *and*

(b) unstable if $\mathbb{E}_{<\left(\sqrt[(N-1)]{\frac{1}{1-c}}\right)}(\theta) > 1$.

2. The efficient equilibrium H is:

(a) asymptotically stable if $\mathbb{E}_{\leq\left(\frac{1}{1-\sqrt[(N-1)]{c}}\right)}(\theta) < 1$, *and*

(b) unstable if $\mathbb{E}_{<\left(\frac{1}{1-\sqrt[(N-1)]{c}}\right)}(\theta) > 1$.

Proof. 1. Consider a perturbed state $(1-\epsilon)$ near L, where $\epsilon << 1$ represents the small share of agents playing H. A new agent with sample size k who observes the rare action H once in their sample estimates the probability that the minimum effort of $N-1$ random opponents is H as $\frac{1}{k^{N-1}}$. This means that the agent will play H (resp., L) if $\left(k^{(N-1)} - 1\right)(1-c) > c \Leftrightarrow k^{(N-1)} > \frac{1}{1-c}$ (resp. $k^{(N-1)} < \frac{1}{1-c}$).

The probability of observing L is approximately $k \cdot \epsilon$. Following similar reasoning to Proposition 4, we obtain conditions (a) and (b).

2. Consider a perturbed state $\epsilon << 1$ near H. A new agent with sample size k who observes the rare action L once estimates the probability that the minimum effort of $N-1$ random opponents is L as $1-\left(\frac{k-1}{k}\right)^{N-1}$. This means that the agent will play H (resp., L) if $\left(1-\left(\frac{k-1}{k}\right)^{N-1}\right)c > (1-c)\left(\frac{k-1}{k}\right)^{N-1} \Leftrightarrow k > \frac{1}{1-\sqrt[(N-1)]{c}}$ (resp. $k < \frac{1}{1-\sqrt[(N-1)]{c}}$). The probability of observing L is approximately $k \cdot \epsilon$. Following similar reasoning to Proposition 4, we obtain conditions (a) and (b). □

Finally, we show that heterogeneity induces stable miscoordination also with $N > 2$ players. Specifically, we generalize Theorem 3, and show that if k is not too large, one can add players with sufficiently large sample sizes, and obtain an environment with stable miscoordination. The proof, which is analogous to Proposition 5, is omitted for brevity.

Proposition 5 (Adaptation of Theorem 3)**.** *For any* $1 < k < \frac{1}{1-\sqrt[(N-1)]{c}}$*, there exists a minimum-effort environment with some agents having sample size* k *and the others with sufficiently large samples, in which an asymptotically stable interior state exists.*

B.5 Logit Dynamics

In this appendix, we show that our result of heterogeneity inducing stable miscoordination applies to logit dynamics. Specifically, we find that (1) standard logit dynamics with uniform noise levels require implausibly high noise to achieve stable miscoordination, while (2) a variant with heterogeneous noise levels achieves stable miscoordination with much lower noise. This suggests our insight into the role of heterogeneity in noise may be relevant across different dynamics, not just sampling dynamics.

Standard (Homogeneous) Logit Dynamics Logit dynamics (introduced in Fudenberg and Levine, 1995; see Sandholm, 2010, Section 6.2.3 for a textbook exposition, and see Nax and Newton, 2022 for a recent application) are characterized by a single parameter η_i that describes the *noise level* for each population i. If player i plays action a_i, she gets a payoff of $p_j \cdot u_j$. If she plays action b_i she gets a payoff of $(1-p_j) \cdot 1$. Logit dynamics assume that the probability of revising agents playing action a_i is proportional to $e^{\frac{\text{Payoff of } a_i}{\eta}}$. Specifically, logit dynamics are given by

$$w_i\left(p_j\right) \equiv w_i\left(\mathbf{p}\right) = \frac{e^{\frac{p_j \cdot u_j}{\eta_i}}}{e^{\frac{1-p_j}{\eta_i}} + e^{\frac{p_j \cdot u_j}{\eta_i}}}. \tag{B.1}$$

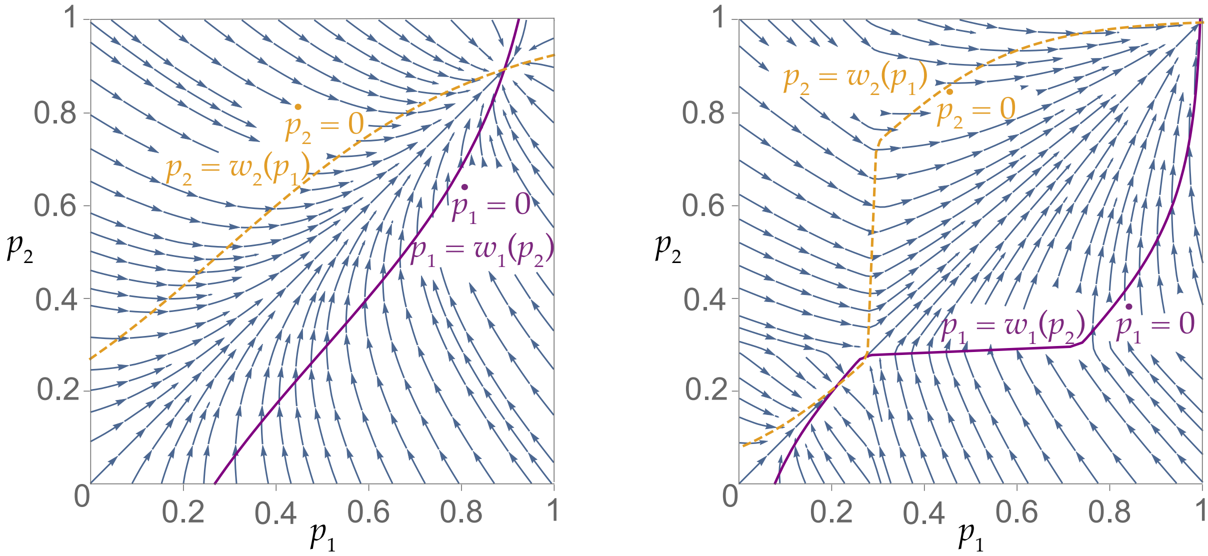
Trivially, logit dynamics can induce substantial miscoordination by having high values of noise. The interesting question is whether stable miscoordination can be supported by a low level of noise. Our numerical analysis suggests that the answer is negative. In what follows, we demonstrate that this is indeed the case. For example, when we revisit the two examples of Figure 3 ($u_1 = u_2 = 2.5$ and $u_1 = \frac{1}{u_2} = 5$), then the minimal level of noise that is required to sustain an asymptotically stable interior state in which each action is played with a probability of at least 10% is $\eta = 1$ (see the left panel of Figure 4 for an illustration of the case of $u_1 = u_2 = 2.5$). Such a high level of noise implies that 27% of the revising agents make the obvious mistake of playing a_i when facing a population in which almost everyone plays b_j; by contrast, this obvious mistake is never made under action-sampling dynamics. Moreover, the average expected payoff obtained by revising agents who follow logit dynamics against an opponent population in which the share of agents playing action a_i is distributed uniformly is 85% (resp., 71%) of the maximal payoff that can be obtained by payoff-maximizing agents in the first (resp., second) environment with $u_1 = u_2 = 2.5$ (resp., $u_1 = \frac{1}{u_2} = 5$). By contrast, this average expected payoff is 98% (resp., 95%) of the maximal payoff under the sampling dynamics. Thus, stable cooperation can be supported by standard (homogeneous) logit dynamics only when the agents have high levels of noise.

Heterogeneous Logit Dynamics Next, consider a variant of logit dynamics in which there is heterogeneity in the level of noise for agents in each population. Specifically, in a population in which there are n groups, the size of the l-th group is μ_i^l, and its members have a noise level of η_i^l, the heterogeneous logit dynamics are given by

$$w_i\left(p_j\right) \equiv w_i\left(\mathbf{p}\right) = \sum_l \mu_i^l \cdot \frac{e^{\frac{p_j \cdot u_j}{\eta_i^l}}}{e^{\frac{1-p_j}{\eta_i^l}} + e^{\frac{p_j \cdot u_j}{\eta_i^l}}}. \tag{B.2}$$

The numerical calculations demonstrate that heterogeneous noise levels can induce asymptotically stable miscoordination with relatively low levels of noise. Specifically, in both of the above examples ($u_1 = u_2 = 2.5$, which is illustrated in the right panel of Figure 4, and $u_1 = \frac{1}{u_2} = 5$), populations in which 55% of the agents have a moderate level of noise (i.e.,

Figure 4: Supporting Stable Coordination with Heterogeneous Logit Dynamics



The figure revisits the symmetric game presented in the left panel of Figure 3 in which $u_1 = u_2 = 2.5$. The left panel shows the phase plot of the minimal homogeneous level of noise, $\eta_i = 1$, that sustains an asymptotically stable state in which each action is played with a probability of at least 10%. The right panel shows the phase plot of a heterogeneous variant of logit dynamics in which 55% of the the agents in each population have a moderate level of noise $\eta_i = 0.55$ and 45% have a small level of noise $\eta_i = 0.01$.

$\eta = 0.55$) and 45% have a small level of noise (i.e., $\eta = 0.01$) induce asymptotically stable states with miscoordination ($(0.21, 0.21)$ in the right panel of Figure 4). Given these heterogeneous levels of noise, only 8% of the agents make the mistake of playing action a_i when facing a population in which everyone plays a_j, and the average expected payoff obtained by playing against opponent populations in which the share of agents playing action a_i is distributed uniformly is 96% (resp., 89%) of the maximal payoff that can be obtained by payoff-maximizing revising agents in the environment with $u_1 = u_2 = 2.5$ (resp., $u_1 = \frac{1}{u_2} = 5$).

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