

# Uncertainty at the Zero Lower Bound (Online Appendix)

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## A Expected Time Until the Liftoff

In this section, I present the survey-based measures of the expected time until the liftoff to support the claim that the market participants consistently underestimated the duration of the lower bound episode since the federal funds rate hit the lower bound in late 2008. The surveys I examine are (i) the Blue Chip Surveys, (ii) the Survey of Professional Forecasters, and (iii) the Primary Dealers Survey.

The evidence from all three surveys is consistent with the claim that the market participants have consistently underestimated the duration of the lower bound episode. In particular, for the first two years of the lower bound episode, the market participants expected that the federal funds rate to stay at the ELB only for additional few quarters.<sup>1</sup>

### A1 Blue Chip Surveys

The Blue Chip Surveys consists of two monthly surveys, the Blue Chip Economic Indicators Survey and the Blue Chip Financial Forecasts Survey. These two surveys ask their participants (about 50 financial institutions for each survey) their forecast paths of various macroeconomic variables, including the 3-month Treasury Bill rate in the Economic Indicators Survey and the federal funds rate in the Financial Forecasts Survey. The near-term forecast horizon is up until the end of next calendar year and the frequency of the projection is quarterly. Thus, the forecast path of the Treasury rate or the federal funds rate can tell us the expected time until the liftoff when the participants expect the first liftoff to occur within two years.

Twice a year, the surveys ask longer-run projections of certain variables in the special question section (March and October for the Economic Indicators and June and December for the Financial Forecasts). The longer-run forecasts are in annual frequency for next 5 to 6 years. Towards the end of the lower bound episode, the Surveys also asked the participants to provide the expected liftoff date in the special questions section.

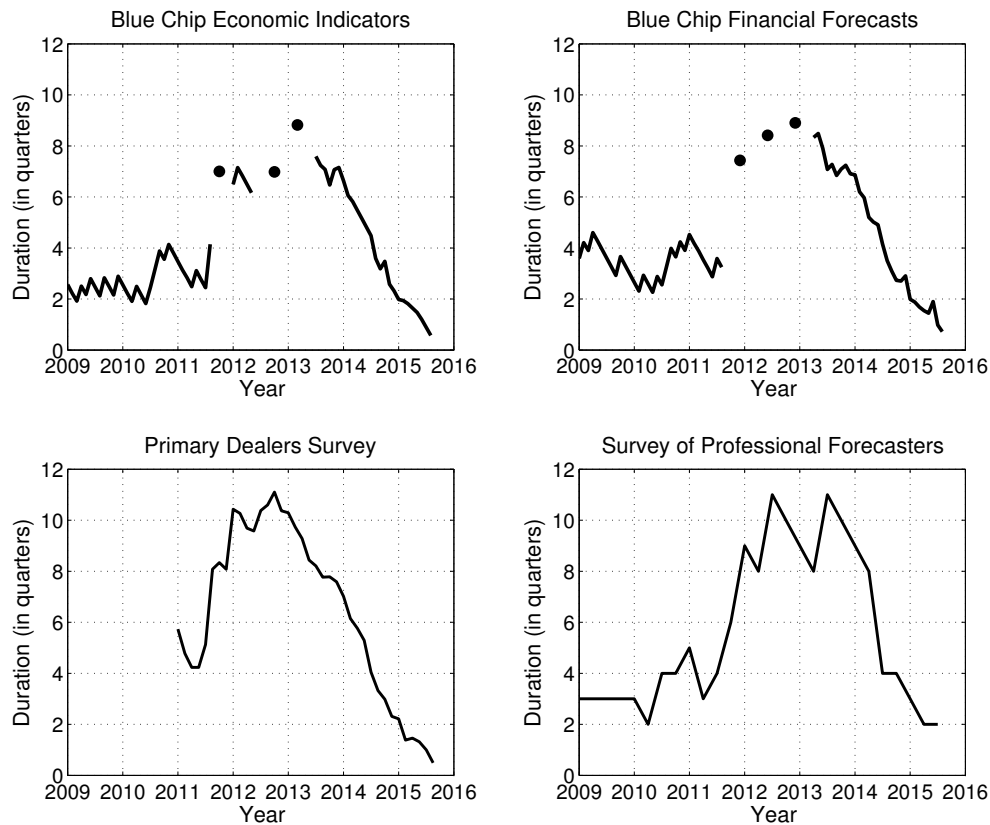
For each survey, I combine these various pieces of information in the following way to construct a series for the expected period until the liftoff. First, I use the average probability distribution over the timing of the liftoff to compute the expected time until the liftoff whenever that information is available. Second, if the probability distribution

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<sup>1</sup>While not shown, the expected duration of the lower bound episode based on the expected policy path implied by the federal funds rate futures is also consistent with this claim.

is not available, then I use the information from the near-term forecasts. The time of liftoff is defined to be the first quarter when the median federal funds rate forecast exceeds 37.5 basis points. Finally, when the policy rate is projected to stay at the ELB until the end of the near-term forecast horizon, I use the information from the long-run projections if the Survey has that information and leave the series blank when the Long-Range section is not available.

Figure A1.1: Expected Time Until Liftoff



Top two panels in figure A1.1 show the evolutions of the expected period until the liftoff based on the Blue Chip Economic Indicators Survey and the Blue Chip Financial Forecasts Survey. According to both panels, the market participants expected the lower bound episode to be transitory in the early stage of the lower bound episode. The market's expectation shifted in the second half of 2011, with the expected duration of staying at the ELB exceeding 2 years. Since late 2012 or early 2013, the market participants started to gradually reduce its expectation for the additional duration of the lower bound episode.

## A2 Primary Dealers Survey

The Primary Dealers Survey (the PD Survey in the remainder of the text), conducted by the Federal Reserve Bank of New York, asks primary dealers about their policy expectations eight times a year. The survey asks its participants their probability distribution over the liftoff timing (quarter or FOMC meeting). I compute the expected time until the liftoff using the average probability distribution over the liftoff timing. The results of the PD Survey are publicly available since January 2011.

The bottom-left panel of figure A1.1 shows the evolution of the expected period until the liftoff based on the PD Survey. Consistent with the measures based on the Blue Chip, the expected duration of the additional period of the lower bound episode increase markedly in the second half of 2011. The expected duration hovers around 10 quarters during 2012, and has declined steadily since then.

## A3 Survey of Professional Forecasters

The Survey of Professional Forecasters (the SPF in the remainder of the text) is a quarterly survey of about 40 individuals in academia, financial industries, and policy institutions, administered by the Federal Reserve Bank of Philadelphia. Like the Blue Chip Surveys, the SPF asks its participants their projections of various macroeconomic variables, including 3-month Treasury rate. For the near-term projection that extends to the end of the next calendar year, the forecasts are available in quarterly frequency. For the longer horizon, the forecast is available in annual frequency.

The bottom-right panel of figure A1.1 shows the evolution of the expected period until the liftoff based on the SPF. Consistent with the Blue Chip Surveys and the Primary Dealers Survey, the SPF shows that the market anticipated the lower bound episode to last for only about one additional year until the second half of 2011. The expected duration averages about 9 quarters in 2012 and 2013. The expected duration started declining in the second half of 2013 and has come down to 2 quarters in February 2015.

## B Analyses of Two-State Shock Models

In this section, I analyze the effects of uncertainty in environments where a two-state discount factor shock is the force that pushes the policy rate to the ZLB. The use of a two-state process is common in the ZLB literature. See, for example, the work of Gauti Eggertsson and Michael Woodford (2003), Ivan Werning (2012), and Anton R. Braun, Lena Mareen Körber and Yuichiro Waki (2013), among many others.

I consider two distinct setups. In the first setup, I study the effect of uncertainty regarding the two-state shock by comparing (i) an economy in which the two-state shock follows a Markov process with the expected duration of a crisis state being  $N$  periods and (ii) an economy with a fixed duration of a crisis state being  $N$  periods. In both economies, the expected discounted sum of the discount rate shock is the same

at time one. Thus, the differences between these two economies at time one reflect the effects of uncertainty regarding the duration of the crisis state.

In the second setup, I add an AR(1) discount rate shock on top of the two-state Markov process and compare (i) an economy where the variance of the AR(1) shock is zero with (ii) an economy where the variance of the AR(1) shock is positive.

Throughout this section, I use a semi-loglinear version of the model to be consistent with the aforementioned papers. I also modify the Taylor-rule to include an intercept that would responds to the discount rate shock. This is a common modification in this two-state environment to guarantee that the policy rate is at the ZLB whenever the shock takes a crisis value.<sup>2</sup>

## B1 First Setup: Fixed Duration versus Stochastic Duration

I use the parameter values of the two-state shock model from Matthew Denes, Gauti Eggertsson and Sophia Gilbukh (2013) (see Table B1.1), except for two parameters—the probability that the shock returns to a steady-state value and the magnitude of the shock. The persistence of the shock is set to 0.75, as opposed to 0.856 in their paper, so that the expected duration of the negative shock is 4 quarters. The magnitude of the shock is chosen so that the initial decline in output is 8 percent.

Table B1.1: Parameterization for the Two-State Shock Models

$\beta$	$\chi_c$	$\chi_n$	$\theta$	$\varphi$	$\phi_\pi$	$\phi_y$	$\mu$
0.9970	0.9760	1.69	13.23	4748.7605	1.50	0	0.75

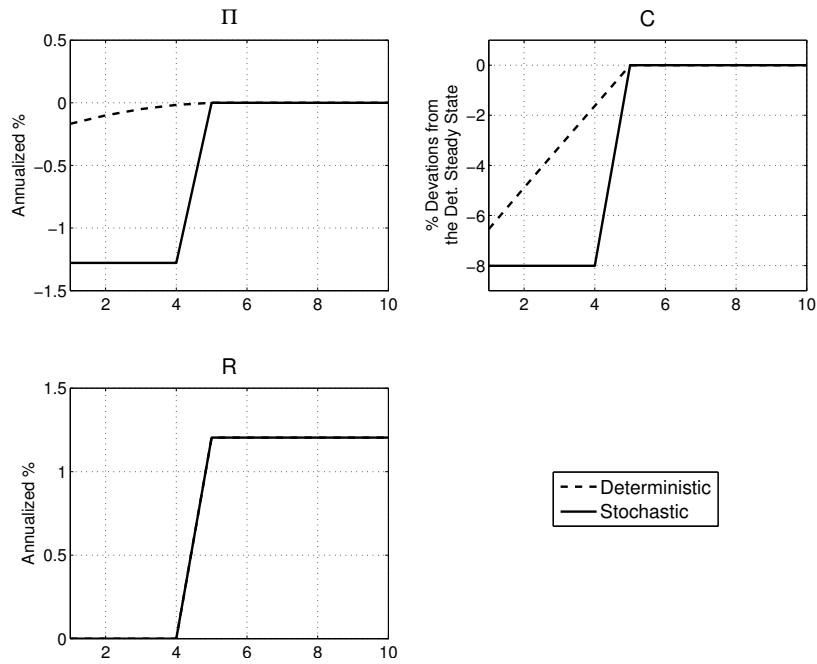
\* $\mu$  is the persistence of the crisis shock. Following the literature, non-crisis state is assumed to be an absorbing state. The size of the crisis shock is chosen to generate 8 percent decline in output in the crisis state.

Figure B1.1 compares the path of the economy with a fixed 4-quarter crisis duration and a realized path of the economy with a stochastic crisis duration when the crisis shock lasts for 4 periods. Under the fixed duration case, shown by the black lines, the declines in inflation and the output gap are smaller as the economy is closer to the period when the shock disappears. On the other hand, under the stochastic duration case, shown by the red lines, the declines are constant whenever the shock is negative and the policy rate is at the ZLB, reflecting the Markov nature of the shock process.

The effects of uncertainty are seen by the difference between these two economies at time one, as the (expected) sum of future discount rates are identical across the two economies at time one. The time-one declines in output and inflation are about 6 percent and 15 basis points under the fixed-duration case, versus 8 and 1.25 percent

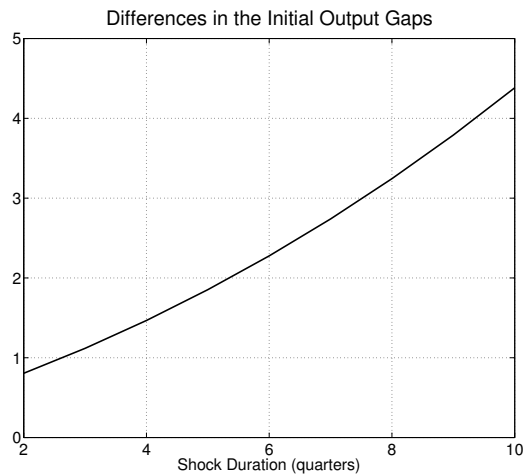
<sup>2</sup>See, for example, Gauti Eggertsson (2011) and Braun, Körber and Waki (2013). In the absence of this modification, the policy rate can be suboptimally above the ZLB even when the shock takes the crisis value.

Figure B1.1: Fixed vs. Stochastic Crisis Duration (4-quarter crisis duration)



under the stochastic-duration case, showing that uncertainty regarding the duration of the crisis reduces output and inflation at the ZLB.

Figure B1.2: Effects of Uncertainty with Different Shock Durations



The importance of uncertainty regarding the duration of the crisis state is robust to alternative durations of the crisis. Figure B1.2 shows how the difference in the

output gap at time one varies with the (expected) duration of the crisis shock. In this experiment, the size of the crisis shock is adjusted so that the initial decline in the stochastic duration economy is 8 percent. According to the figure, the effect of uncertainty is larger the longer the (expected) duration of the shock is. When the duration of the shock is 2 quarters, the effect of uncertainty is about 1 percent. When the duration of the shock is 10 quarters, the effect is more than 4 percent.

## B2 Second Setup: A Composite-Shock Model

Now, we turn to the setup where an AR(1) shock is added on top of a two-state Markov shock.<sup>3</sup> The variance of the AR(1) shock is chosen so that the probability of being at the zero lower bound is 10 percent.

Figure B2.1: IRFs from the Composite-Shock Model

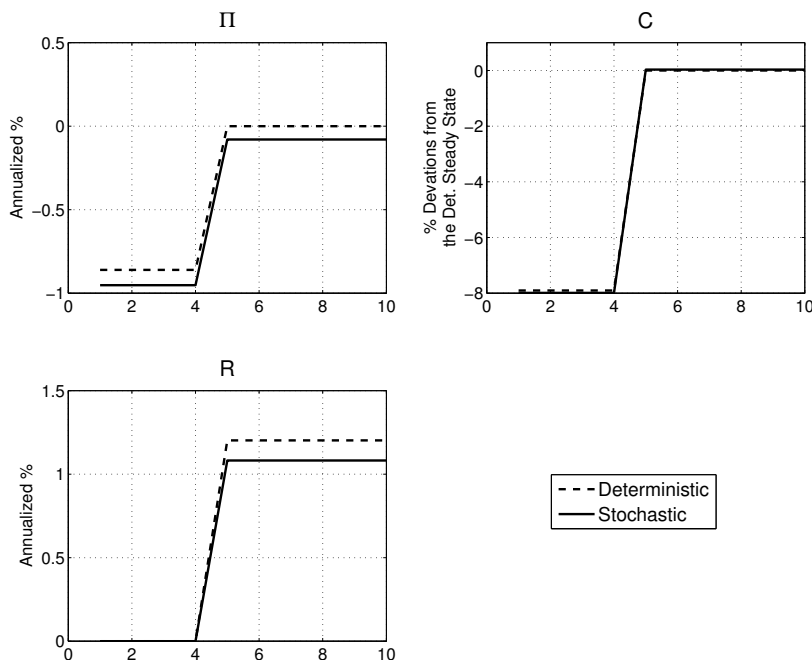
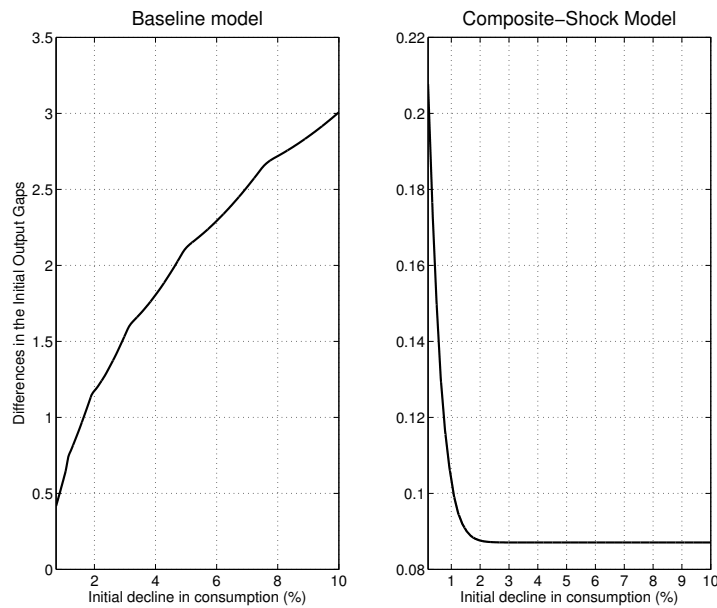


Figure B2.1 shows specific realizations of two models—deterministic and stochastic—in which the crisis shock lasts for 4 quarters and the realization of the innovation to the AR(1) process is zero. The solid black line is for the stochastic economy in which the variance of the shock to the AR(1) discount factor process is positive while the dashed black line is for the deterministic economy in which the variance of the AR(1) process is zero.

<sup>3</sup>Benjamin K. Johannsen (2014) considers a similar setup in which he adds a three-state fiscal shock on top of the two-state Markov shock for investment efficiency in order to analyze the effects of fiscal uncertainty at the ZLB.

Starting with the effect of uncertainty away from the ZLB, inflation and the policy rate is slightly lower, and consumption is slightly higher, with AR(1) uncertainty than without AR(1) uncertainty, consistent with the results from the models of the main text. The effect of uncertainty at the ZLB reflects the effects of uncertainty away from the ZLB; Inflation is lower at the ZLB with uncertainty than without uncertainty because the inflation is low in the future non-crisis state, which lowers inflation expectations. Consumption is lower at the ZLB with uncertainty than without uncertainty because low inflation at the ZLB leads to higher expected real interest rates. The effect of uncertainty at the ZLB arising from AR(1) shock component is small. In the crisis state, the shadow rate is substantially below zero and the policy rate remains zero under most realizations of shocks to the AR(1) process. As a result, uncertainty regarding the process does not affect the private sector's expectations, and thus allocations, in quantitatively important ways.

Figure B2.2: Effects of Uncertainty in the Baseline Model versus the Composite-Shock Model



\*The panels show additional declines in consumption due to uncertainty when the decline in consumption in the crisis state is given by the horizontal axis.

One implication of this reasoning is that, in the two-state shock model, the smaller the declines in consumption and inflation in the crisis state are, the larger the effects of uncertainty are because smaller declines in consumption and inflation mean that the shadow rate is closer to zero. The right panel of Figure B2.2 shows how the effect of uncertainty on consumption varies with the severity of the recession, as measured by the level of consumption in the crisis state. When the recession is less severe, the

shadow rate is less negative. Thus, there is a higher probability that some realization of the AR(1) shock pushes the policy rate above zero.

This is in a sharp contrast to the baseline model studied in the main text where the effects of uncertainty are larger in a severer recession. In the baseline model, even when today’s shadow rate is very low and the policy rate remains at the ZLB under most realizations of shocks tomorrow, the economy gradually recovers and *at some point in the future*, the shadow policy rate is sufficiently close to zero so that uncertainty alters the expectations of relevant prices in quantitatively important ways. Since what matter for the private sector’s decisions today is the expectations at all horizons, the uncertainty on allocations and prices depends on how much uncertainty alters the expectations *cumulatively*. As a result, uncertainty matters more in a deeper recession in the baseline AR(1) model considered in the main text, as shown in the left panel of Figure B2.2.

## C Further Sensitivity Analyses

### C1 Sensitivity to alternative structural parameter values

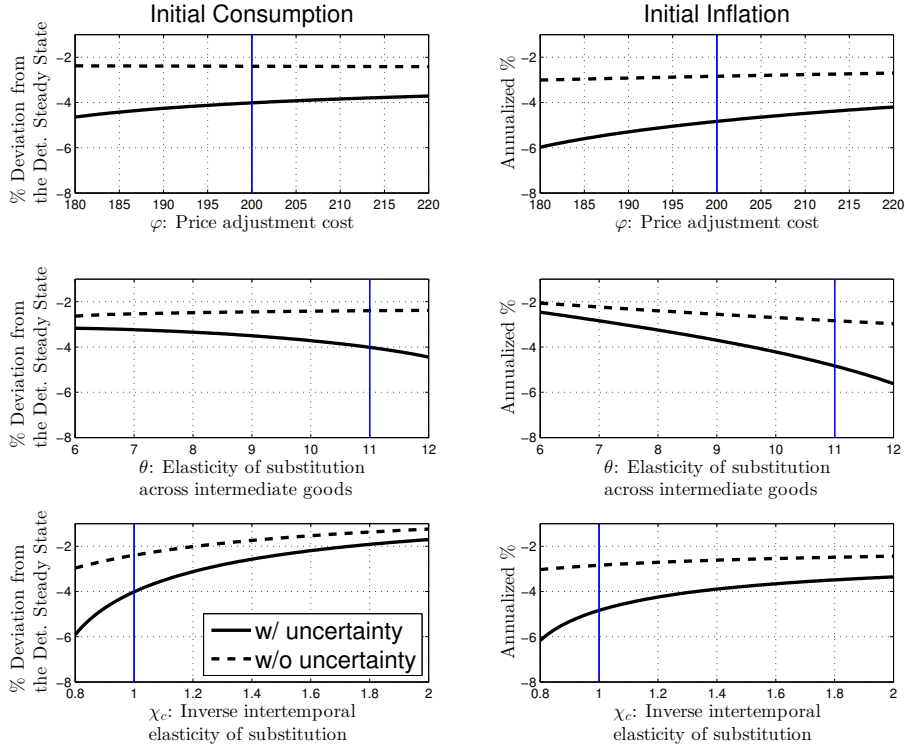
In order to understand how structural parameters affect the magnitude of additional declines due to uncertainty, recall the analysis in Section III. that shows that the presence of uncertainty reduces consumption through its effect on expected future real interest rates, and that the presence of uncertainty leads to a decline in inflation through its effect on expected future real marginal costs. According to the consumption Euler Equation iterated forward (Eq. 22), the same decline in the expected sum of future real interest rates leads to a larger decline in consumption today when  $\chi_C$  is smaller, i.e. when the intertemporal elasticity of substitution (IES) is large. Similarly, according to the Phillips curve iterated forward (Eq. 24), the same decline in the expected discounted sum of future real wages leads to a larger decline in inflation today when the slope of the Phillips curve ( $\kappa \equiv \frac{\theta-1}{\varphi}$ ) is larger. The slope of the Phillips curve is larger when prices are more flexible (i.e., smaller  $\varphi$ ) or when intermediate goods are more substitutable (i.e., larger  $\theta$ ).

Figure C1.1 confirms these predictions. The left panel in the first row of Figure C1.1 shows the declines in consumption when  $\delta_t = 1 + 3\sigma_\delta$ , which is three standard deviations away from the deterministic steady-state level in the stochastic economy, for various values of price adjustment cost parameter ( $\varphi$ ). A larger  $\varphi$  means that prices are more sticky. The solid and dashed black lines correspond to the stochastic and deterministic economies respectively. The right panel in the first row of Figure C1.1 shows the decline in inflation when  $\delta_t = 1 + 3\sigma_\delta$  in a similar manner.

In the deterministic economy, the declines in consumption and inflation at  $\delta_t = 1 + 3\sigma_\delta$  from the steady-state level do not vary much for the range of price flexibility shown. However, in the stochastic economy, the magnitude of the declines depends importantly on price flexibility. Consistent with the aforementioned prediction, the more



Figure C1.1: Sensitivity Analysis:  
Declines in Consumption and Inflation at  $\delta_1 = 1 + 3\sigma_\delta$



Dashed black line: deterministic economy ( $\sigma_\epsilon = 0$ ). Solid black line: stochastic economy ( $\sigma_\epsilon = \frac{0.29}{100}$ ). Blue vertical lines are for the baseline parameter values.

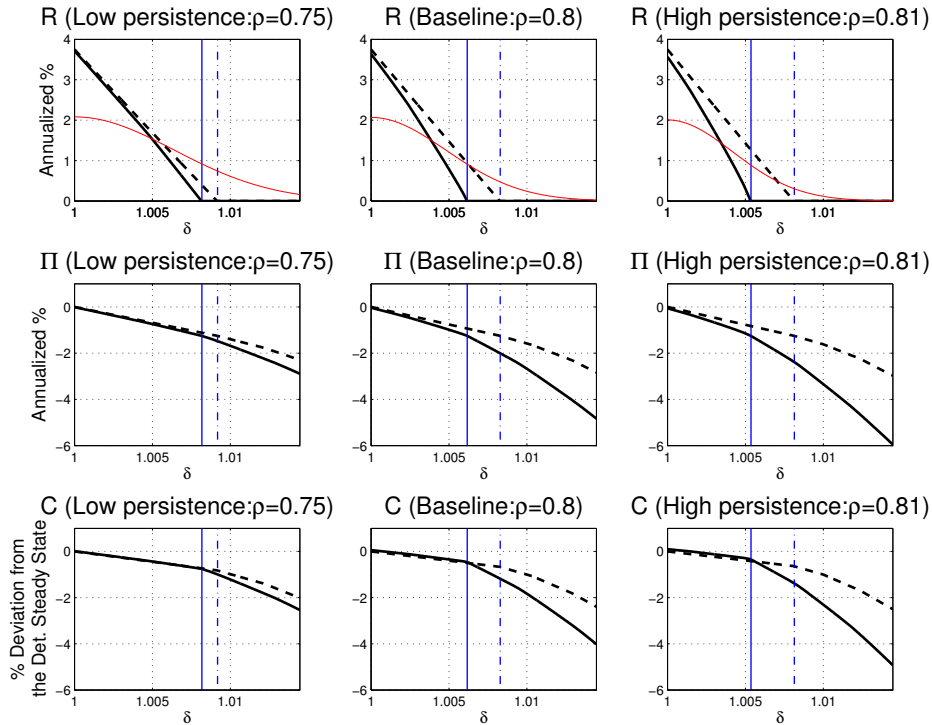
\*For the price adjustment cost,  $\varphi = 180$  implies the slope of the log-linearized Phillips curve that is equivalent to the one in the Calvo model with 79 percent chance of no price adjustment.  $\varphi = 220$  corresponds to the Calvo model with 81 percent chance of no price adjustment.

flexible the price is (i.e. the smaller the price adjustment cost is), the larger the additional declines in consumption and inflation due to uncertainty. While the additional declines in consumption and inflation due to uncertainty are about 4 and 4.8 percent at  $\varphi = 200$ , they are about 4.5 and 6 percent at  $\varphi = 180$ . The second and third rows of Figure C1.1 respectively show how the substitutability of intermediate goods and the inverse intertemporal elasticity of substitution, respectively, affect the quantitative importance of uncertainty on consumption and inflation at the ZLB. Again, consistent with the observations made above, the more substitutable intermediate goods are, or the larger the IES is, the larger the additional declines in consumption and output due to uncertainty.

## C2 Alternative degrees of shock persistence

How do the parameters of the discount factor shock process,  $\sigma_\epsilon$  and  $\rho$ , affect the quantitative significance of uncertainty? It is perhaps obvious that the smaller  $\sigma_\epsilon$  is, the smaller the additional reductions in consumption, output, and inflation are. What is less obvious is how alternative degrees of persistence affect the quantitative significance of uncertainty.

Figure C2.1: Effects of Uncertainty With Alternative Persistence of Shocks



Dashed black lines: deterministic model. Solid black lines: stochastic model. Solid red lines: density function for  $\delta$ .

\*Dashed and solid blue vertical lines indicate the values of  $\delta$  above which the ZLB binds in the deterministic and stochastic economies. For each model, policy functions are shown for the range of  $\delta$  that covers its steady-state level to the level that is 3 standard deviations away from the steady-state. For lower and higher persistence cases, the standard deviations of the shock are chosen so that the unconditional standard deviations of the discount rate is the same as in the baseline.

Figure C2.1 shows policy functions from three economies with alternative degrees of persistence. The second column shows policy functions from the baseline model with  $\rho = 0.8$ , and the first and third columns show the policy function from the model with low ( $\rho = 0.75$ ) and high ( $\rho = 0.81$ ) persistence, respectively. In each figure, solid black and dashed black lines are respectively policy functions for the stochastic and deterministic economies. For the low and high persistence cases, the standard deviation

of the shock is chosen so that the unconditional standard deviations of the discount factor shock are the same as in the baseline case.<sup>4</sup>

These figures show that the more persistent the process is, the more adverse the effects of uncertainty are at the ZLB. At three standard deviations away from the steady-state, the additional declines in consumption due to uncertainty are about 0.5, 1.6, and 2.4 percent when persistence is low, medium, and high, respectively. The additional declines in inflation due to uncertainty are about 0.6 percent, 2 percent, and 3 percent when persistence is low, medium, and high, respectively.

Why are the effects of uncertainty larger when shocks are more persistent? Recall that the key factor that generates large adverse effects of uncertainty at the ZLB is nonlinearity in the policy functions for real interest rates and real marginal costs. If the discount rate shock is not persistent, even if the nominal interest rate is zero today, the economy is expected to be away from the ZLB region in the near future where policy functions for relevant prices are almost linear. On the other hand, if the shock is persistent, the household and firms expect the nominal interest rate to be at the ZLB for a long period where policy functions exhibit nonlinearity. Thus, the adverse effects of uncertainty are larger when the process driving the economy into the ZLB is more persistent.

### C3 A model with price indexation

This section studies the effect of uncertainty at the ZLB when there is a backward-looking element in the firms' price setting decision. Following Peter N. Ireland (2007), I modify the price adjustment cost function to penalize firms for deviating from the lagged aggregate inflation as follows.

$$(C3.1) \quad P_t \frac{\varphi}{2} \left[ \frac{P_{i,t}}{\Pi_{t-1}^\alpha P_{i,t-1}} - 1 \right]^2 Y_t$$

where  $\alpha$  measures the degree of price indexation.  $\alpha = 0$  corresponds to the benchmark case without any indexation considered in the main text. This modification leads to a so-called hybrid Phillips curve in which today's inflation is a function of both expected inflation tomorrow and realized inflation yesterday.

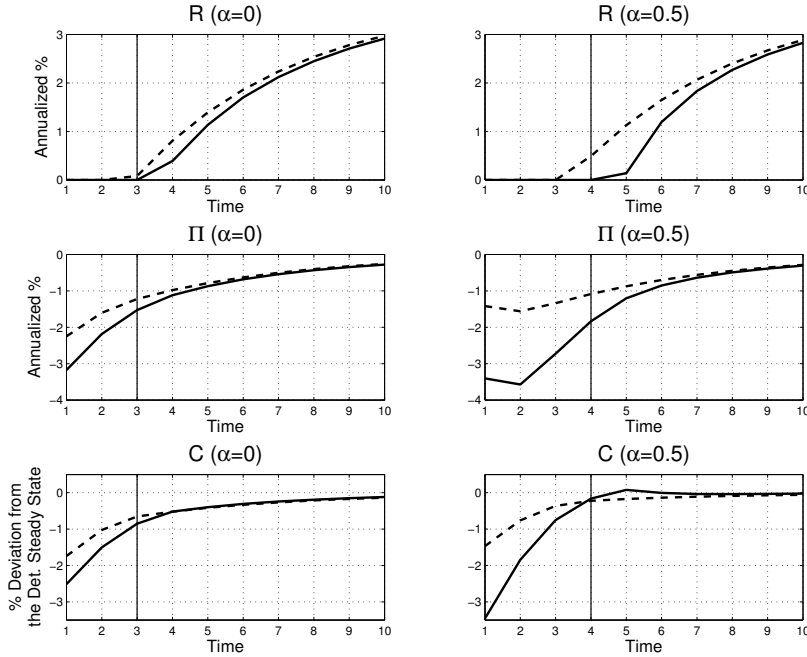
The first and second columns in Figure C3.1 show the impulse response functions from the baseline economy without indexation and an economy with  $\alpha = 0.5$ . In both economies,  $\sigma_\epsilon$  is set to a lower value ( $\frac{0.25}{100}$ ) as the maximum  $\sigma_\epsilon$  consistent with the existence of equilibrium is lower than the original value of  $\frac{0.29}{100}$  when the degree of price indexation is large. The initial  $\delta$  is set to three standard deviations away from the steady state.

These figures show that the additional declines in consumption and inflation due to

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<sup>4</sup>As pointed out by Taisuke Nakata and Sebastian Schmidt (2014) and Alexander W. Richter and Nathaniel A. Throckmorton (2015), a recursive equilibrium does not exist when the shock persistence is sufficiently high. 0.81 is close to the maximum frequency for which the recursive equilibrium exists.

Figure C3.1: IRFs with Alternative Degrees of Price Indexation



Dashed black lines: impulse response functions from the deterministic model ( $\sigma_\epsilon = 0$ ). Solid black lines: modal responses from the stochastic model ( $\sigma_\epsilon = \frac{0.25}{100}$ )

uncertainty are larger in an economy with price indexation. In the model without price indexation, the presence of uncertainty reduces consumption and inflation by about 1 percent at time one. In the model with price indexation, the presence of uncertainty reduces consumption and inflation by about 2 percent at time one. Also, in the model with price indexation, the additional decline in inflation due to uncertainty remains large for a longer period.

To understand why inertia in the price setting behavior magnifies the impact of uncertainty, let us examine the following log-linearized optimality condition of the firm.

$$(C3.2) \quad \hat{\Pi}(\delta_t) - \alpha \hat{\Pi}(\delta_{t-1}) = \kappa \hat{w}(\delta_t) + \beta E_t [\hat{\Pi}(\delta_{t+1}) - \alpha \hat{\Pi}(\delta_t)]$$

To the extent that the discount rate is persistent, we can approximate this equilibrium condition as follows.

$$(C3.3) \quad (1 - \alpha) \hat{\Pi}(\delta_t) \cong \kappa \hat{w}(\delta_t) + \beta E_t [(1 - \alpha) \hat{\Pi}(\delta_{t+1})]$$

Iterating forward and dividing both sides by  $1 - \alpha$ , we obtain

$$(C3.4) \quad \hat{\Pi}(\delta_t) \cong \lim_{s \rightarrow \infty} \frac{E_t \hat{\Pi}(\delta_{t+s})}{1 - \alpha} + \frac{\kappa}{1 - \alpha} E_t \sum_{s=0}^{\infty} \beta^s \hat{w}(\delta_{t+s}).$$

As described in the main text, an increase in uncertainty reduces the expected discounted sum of future real marginal costs,  $E_t \sum_{s=0}^{\infty} \beta^s \hat{w}(\delta_{t+s})$ , due to the concavity of the policy function for the real wage. The coefficient  $\frac{\kappa}{1 - \alpha}$  determines how sensitive today's inflation is to changes in the expected discounted sum of future real marginal costs. The larger  $\alpha$  is, the more sensitive today's inflation is to the change in the expected future real marginal costs. Thus, an increase in uncertainty reduces today's inflation by a larger amount when the degree of indexation,  $\alpha$ , is higher.

#### C4 A model with consumption habits

In this section, I will consider the effect of introducing consumption habits in the household's preference. The period utility function of the household is given by

$$(C4.1) \quad \frac{(C_t - \gamma C_{t-1})^{1 - \chi_c}}{1 - \chi_c} - \frac{N_t^{1 + \chi_n}}{1 + \chi_n}.$$

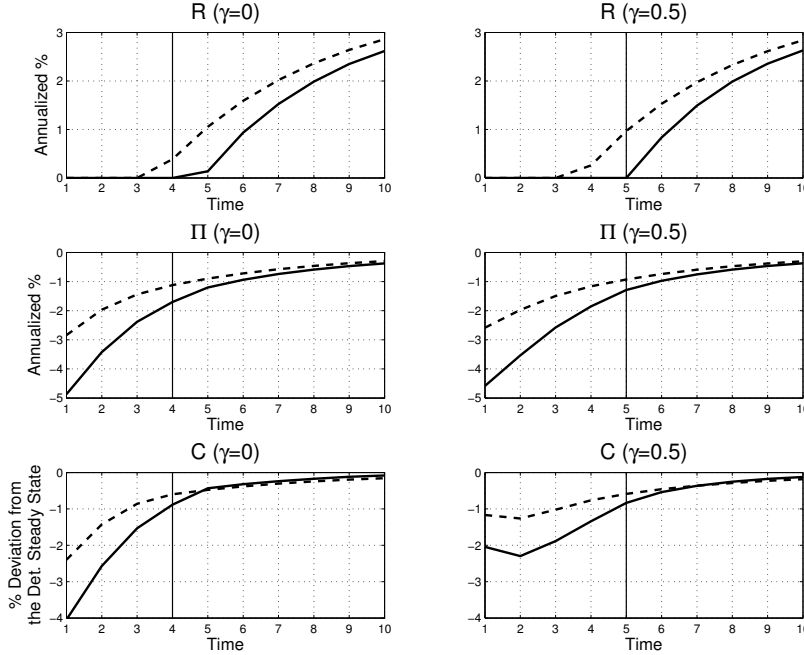
The first and second columns in Figure C4.1 show the impulse response functions for nominal interest rate, inflation, and consumption in the baseline economy without consumption habits and an economy with  $\gamma = 0.5$ . The figure shows that the effects of uncertainty about the same for inflation; At time one, uncertainty reduces inflation by about 2 percentage points in both economies. For consumption, even though the decline in consumption is much smaller in the model with consumption habit, the effects of uncertainty are about the same cumulatively.

#### C5 Calvo Model

In this section, I examine the robustness of the main result on the differential effects of uncertainty at and away from the ZLB to the Calvo pricing setup. The exercise is motivated by recent papers documenting important differences between the Rotemberg and Calvo models at the ZLB (Anton R. Braun and Yuichiro Waki (2010) and Jianjun Miao and Phuong Ngo (2016)). The Calvo parameter is set to 0.8 so that the semi-loglinear version of this Calvo model is identical to the semi-loglinear version of the Rotemberg model of the main text.

The left panel of the figure C5.1 shows the impulse response functions of the deterministic and stochastic economies with the Calvo price-setting. Consistent with the analysis based on the Rotemberg model, the effect of uncertainty is larger at the ZLB than away from the ZLB. The right panel of the figure C5.1 show the effect of uncertainty in the version of the Calvo model without the ZLB constraint. The effects of uncertainty are relatively small, as seen by the fact that solid and dashed black lines

Figure C4.1: IRFs with Alternative Degrees of Consumption Habit



Dashed black lines: impulse response functions from the deterministic model ( $\sigma_\epsilon = 0$ ). Solid black lines: modal responses from the stochastic model ( $\sigma_\epsilon = \frac{0.29}{100}$ ).

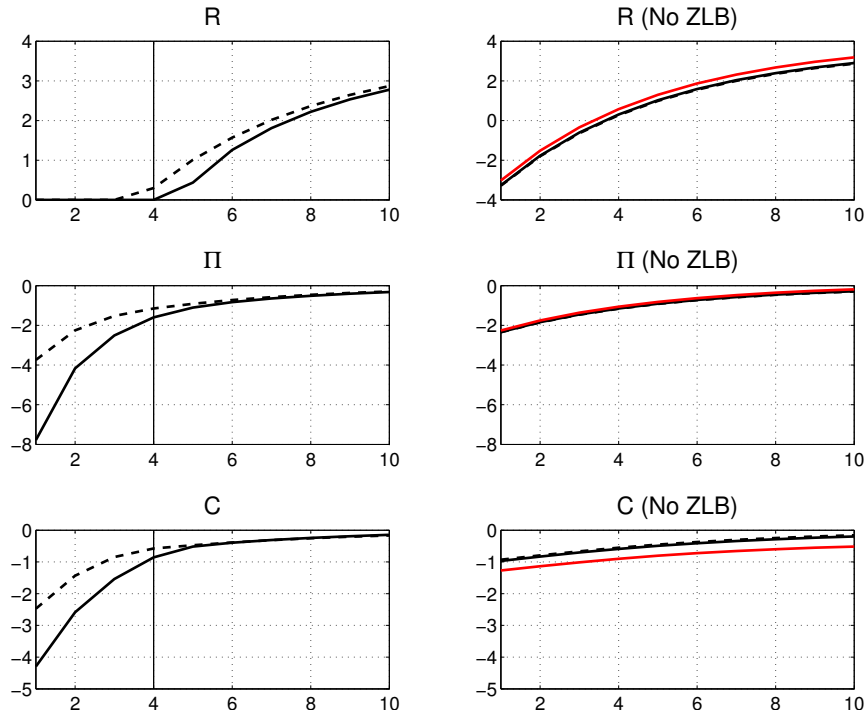
almost overlap. This is consistent with what we saw in the Rotemberg model. To show the effect of uncertainty more clearly, I show the evolution of the policy rate, inflation, and consumption when the standard deviation is three times as large as the baseline standard deviation in the solid red lines. The uncertainty reduces consumption, but increases inflation and the policy rate. The positive effect of uncertainty on inflation is consistent with the findings in Benjamin Born and Johannes Pfeifer (2014) and Jesús Fernández-Villaverde, Pablo Guerrón-Quintana, Keith Kuester and Juan Rubio-Ramírez (2015).

## C6 Alternative Inflation Targets

In the baseline stylized model, I assumed that the inflation target is zero for simplicity. This subsection shows the robustness of the main result to alternative levels of the inflation target.

Figure C6.1 shows the policy functions for the nominal interest rate, inflation and consumption in the model with 2 percent inflation target, with and without uncertainty. The standard deviation of the innovation to the discount rate shock is set so that the frequency of being at the ZLB remains 10 percent. Consistent with the baseline stylized

Figure C5.1: IRFs in the Calvo Model



Dashed black lines: impulse response functions from the deterministic model ( $\sigma_\epsilon = 0$ ). Solid black lines: modal responses from the stochastic model ( $\sigma_\epsilon = \frac{0.29}{100}$ ). Solid red lines: modal responses from the stochastic model with higher uncertainty ( $\sigma_\epsilon = 3 * \frac{0.29}{100}$ )

\*The left column is for the model with the ZLB and the right column is for the model without the ZLB.

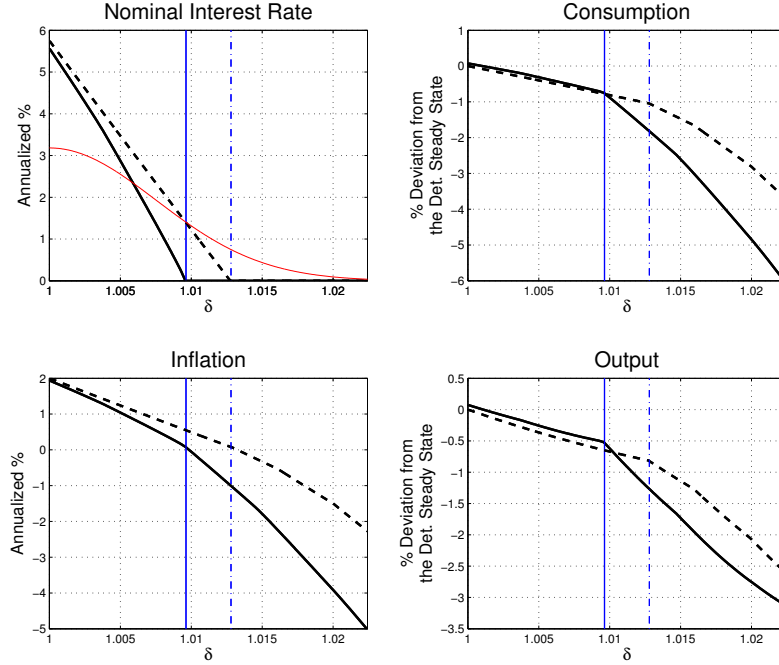
model, the effect of uncertainty is small when the policy rate is away from the ZLB while it is quantitatively important at the ZLB.

## D Solution Method

I will describe the solution method for the stylized model. The method can be extended to other models in a straightforward manner.

The problem is to find a set of policy functions,  $\{C(\cdot), N(\cdot), Y(\cdot), w(\cdot), \Pi(\cdot), R(\cdot)\}$ ,

Figure C6.1: Effects of Uncertainty in the Model with 2% Inflation Target  
(Policy Functions from Deterministic and Stochastic Economies)



Dashed black lines: deterministic model. Solid black lines: stochastic model. \*Dashed and solid blue vertical lines indicate the values of  $\delta$  above which the ZLB binds in the deterministic and stochastic economies. Policy functions are shown for the range of  $\delta$  that covers its steady-state level to the level that is 3 standard deviations away from the steady-state.

that solves the following system of *functional* equations.

$$(D0.1) \quad C(\delta_t)^{-\chi_c} = \beta \delta_t R(\delta_t) E_t C(\delta_{t+1})^{-\chi_c} \Pi(\delta_{t+1})^{-1}$$

$$(D0.2) \quad w(\delta_t) = N(\delta_t)^{\chi_n} C(\delta_t)^{\chi_c} \\ \frac{N(\delta_t)}{C(\delta_t)^{\chi_c}} [\varphi(\Pi(\delta_t) - 1) \Pi(\delta_t) - (1 - \theta) - \theta w(\delta_t)] \dots$$

$$(D0.3) \quad \dots = \beta \delta_t E_t \frac{N(\delta_{t+1})}{C(\delta_{t+1})^{\chi_c}} \varphi(\Pi(\delta_{t+1}) - 1) \Pi(\delta_{t+1})$$

$$(D0.4) \quad Y(\delta_t) = C(\delta_t) + \frac{\varphi}{2} [\Pi(\delta_t) - 1]^2 Y(\delta_t)$$

$$(D0.5) \quad Y(\delta_t) = N(\delta_t)$$

$$(D0.6) \quad R(\delta_t) = \max\left[1, \frac{1}{\beta} \Pi(\delta_t)^\phi\right]$$

Substituting out  $w(\cdot)$  and  $N(\cdot)$  using equations (D0.2) and (D0.5), this system can be



reduced to a system of four functional equations for  $C(\cdot)$ ,  $Y(\cdot)$ ,  $\Pi(\cdot)$ , and  $R(\cdot)$ .

$$(D0.7) \quad C(\delta_t)^{-\chi_c} = \beta \delta_t R(\delta_t) E_t C(\delta_{t+1})^{-\chi_c} \Pi(\delta_{t+1})^{-1} \\ \frac{Y(\delta_t)}{C(\delta_t)^{\chi_c}} [\varphi(\Pi(\delta_t) - 1) \Pi(\delta_t) - (1 - \theta) - \theta Y(\delta_t)^{\chi_n} C(\delta_t)^{\chi_c}] \dots$$

$$(D0.8) \quad \dots = \beta \delta_t E_t \frac{Y(\delta_{t+1})}{C(\delta_{t+1})^{\chi_c}} \varphi(\Pi(\delta_{t+1}) - 1) \Pi(\delta_{t+1})$$

$$(D0.9) \quad Y(\delta_t) = C(\delta_t) + \frac{\varphi}{2} [\Pi(\delta_t) - 1]^2 Y(\delta_t)$$

$$(D0.10) \quad R(\delta_t) = \max[1, \frac{1}{\beta} \Pi(\delta_t)^\phi]$$

Following the idea of Lawrence J. Christiano and Jonas D. M. Fisher (2000) and Christopher Gust, David López-Salido and Matthew Smith (2012), I decompose these policy functions into two parts using an indicator function: One in which the policy rate is allowed to be less than zero, and the other in which the policy rate is assumed to be zero. That is, for any variable  $Z$ ,

$$(D0.11) \quad Z(\cdot) = \mathbf{1}_{\{R(\cdot) \geq 1\}} Z_{unc}(\cdot) + (1 - \mathbf{1}_{\{R(\cdot) \geq 1\}}) Z_{zlb}(\cdot).$$

The problem then becomes finding a set of a *pair* of policy functions,  $\{[C_{unc}(\cdot), C_{zlb}(\cdot)], [Y_{unc}(\cdot), Y_{zlb}(\cdot)], [\Pi_{unc}(\cdot), \Pi_{zlb}(\cdot)], [R_{unc}(\cdot), R_{zlb}(\cdot)]\}$  that solves the system of functional equations above. This method can achieve a given level of accuracy with a considerable less number of grid points relative to the standard approach.<sup>5</sup>

The time-iteration method starts by specifying a guess of the values the policy functions take on a finite number of grid points. Let  $X(\cdot)$  be a vector of policy functions that solves the functional equations above and let  $X^{(0)}$  be the initial guess of such policy functions where the values of the policy functions not on grid points are interpolated or extrapolated.<sup>6</sup> At the  $s$ -th iteration and at each point of the state space, we solve the system of nonlinear equations given by equations (D0.7)-(D0.10) to find today's consumption, output, inflation, and the policy rate, given that  $X^{(s-1)}(\cdot)$  is in place for the next period. In solving the system of nonlinear equations, I use Gaussian quadrature to evaluate the expectation terms in the consumption Euler equation and the Phillips curve, and the value of future variables not on the grid points are evaluated with linear interpolation. The system is solved numerically by using a nonlinear equation solver, `dneqnf`, provided by the IMSL Fortran Numerical Library. For all models, I use 10 grid points for the Gaussian quadrature. If the updated policy functions are sufficiently close to the guessed policy functions, then the algorithm ends. Otherwise, using the updated policy functions just obtained as the guess for the next period's policy

<sup>5</sup>A systematic analysis of the benefits of using the Christiano-Fisher approach is available upon request.

<sup>6</sup>For all models and all variables, I use flat functions at the deterministic steady-state values as the initial guess.

functions, I iterate on this process until the difference between the guessed and updated policy functions is sufficiently small ( $\|vec(X^s(\delta) - X^{s-1}(\delta))\|_\infty < 1\text{E-11}$  is used as the convergence criteria). I used equally spaced 1001 grid points on the interval between  $[1 - 4\sigma_\delta, 1 + 4\sigma_\delta]$ .

For the models with alternative shocks considered Section IV., I have 1001 grid points on each of the interval ranging from  $-4$  to  $4$  unconditional standard deviations away from the steady-state. For the models with alternative policy rules, I have 51 equally spaced grid points on both preference shocks and the lagged state variable (either the lagged actual policy rate, the lagged shadow policy rate, or the lagged price level). For the empirical model, I have 21 grid points on the discount rate shock and 11 grid points on each of the three endogenous state variables. In all models, the same convergence criteria described above was used.

## E Details of the Empirical DSGE Model

This section provides the details of the empirical model considered in the final section of the paper.<sup>7</sup>

### E1 Household markets

#### E11 Labor packer

The labor packer buys labor  $N_{h,t}$  from households at their monopolistic wage  $W_{h,t}$  and resells the packaged labor  $N_t$  to intermediate goods producers at  $W_t$ . The problem can be written as

$$(E1.1) \quad \max_{N_{h,t}, h \in [0,1]} W_t N_t - \int_0^1 W_{h,t} N_{h,t} dh$$

subject to the following CES technology

$$(E1.2) \quad N_t = \left[ \int_0^1 N_{h,t}^{\frac{\theta^w - 1}{\theta^w}} dh \right]^{\frac{\theta^w}{\theta^w - 1}}.$$

The first order condition implies a labor demand schedule

$$(E1.3) \quad N_{h,t} = \left[ \frac{W_{h,t}}{W_t} \right]^{-\theta^w} N_t.^8$$

$\theta^w$  is the wage markup parameter.

<sup>7</sup>Note that the model will be presented allowing for price and wage indexation for the sake of generality, but the indexation parameters will be set to zero in the calibration.

<sup>8</sup>This implies that the labor packer will set the wage of the packaged labor to  $W_t = \left[ \int_0^1 W_{h,t}^{1-\theta^w} dh \right]^{\frac{1}{1-\theta^w}}$ .

## E12 Household

The representative household chooses its consumption level, amount of labor, and bond holdings so as to maximize the expected discounted sum of utility in future periods. As is common in the literature, the household enjoys consumption and dislikes labor. Assuming that period utility is separable, the household problem can be defined by

$$(E1.4) \quad \max_{C_{h,t}, w_{h,t}, B_{h,t}} E_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[ \prod_{s=0}^{t-1} \delta_s \right] \left[ \frac{(C_{h,t} - \zeta C_{t-1}^a)^{1-\chi_c}}{1-\chi_c} - A_t^{1-\chi_c} \frac{N_{h,t}^{1+\chi_n}}{1+\chi_n} \right]$$

subject to the budget constraint

(E1.5)

$$P_t C_{h,t} + R_t^{-1} B_{h,t} \leq W_{h,t} N_{h,t} - W_t \frac{\varphi_w}{2} \left[ \frac{W_{h,t}}{a W_{h,t-1} (\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right]^2 N_t + B_{h,t-1} + P_t \Phi_t - P_t T_t$$

or equivalently

(E1.6)

$$C_{h,t} + \frac{B_{h,t}}{R_t P_t} \leq w_{h,t} N_{h,t} - w_t \frac{\varphi_w}{2} \left[ \frac{w_{h,t}}{a w_{h,t-1} (\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right]^2 N_t + \frac{B_{h,t-1}}{P_t} + \Phi_t - T_t$$

and subject to the labor demand schedule

$$(E1.7) \quad N_{h,t} = \left[ \frac{W_{h,t}}{W_t} \right]^{-\theta^w} N_t.$$

–or equivalently

$$(E1.8) \quad N_{h,t} = \left[ \frac{w_{h,t}}{w_t} \right]^{-\theta^w} N_t.$$

where  $C_{h,t}$  is the household's consumption,  $N_{h,t}$  is the labor supplied by the household,  $P_t$  is the price of the consumption good,  $W_{h,t}$  ( $w_{h,t}$ ) is the nominal (real) wage set by the household,  $W_t$  ( $w_t$ ) is the market nominal (real) wage,  $\Phi_t$  is the profit share (dividends) of the household from the intermediate goods producers,  $B_{h,t}$  is a one-period risk free bond that pays one unit of money at period  $t+1$ ,  $T_t$  are lump-sum taxes or transfers, and  $R_t^{-1}$  is the price of the bond.  $C_{t-1}^a$  represents the aggregate consumption level from the previous period that the household takes as given. The parameter  $0 \leq \zeta < 1$  measures how important these external habits are to the household. Because we are including wage indexation, measured by the parameter  $\iota_w$ , we assume the household takes as given the previous period wage inflation,  $\Pi_{t-1}^w$ , where  $\Pi_t^w = \frac{W_t}{a W_{t-1}} = \frac{w_t P_t}{a w_{t-1} P_{t-1}} = \frac{w_t}{a w_{t-1}} \Pi_t^p$ .

The discount rate at time  $t$  is given by  $\beta\delta_t$  where  $\delta_t$  is the discount factor shock altering the weight of future utility at time  $t+1$  relative to the period utility at time  $t$ .  $\delta_t$  is assumed to follow an AR(1) process

$$(E1.9) \quad (\delta_t - 1) = \rho_\delta(\delta_{t-1} - 1) + \epsilon_t^\delta \quad \forall t \geq 2$$

and  $\delta_1$  is given. The innovation  $\epsilon_t^\delta$  is normally distributed with mean zero and standard deviation  $\sigma_\delta$ . It may therefore be interpreted that an increase in  $\delta_t$  is a preference imposed by the household to increase the relative valuation of the future utility flows, resulting in decreased consumption today (when considered in the absence of changes in the nominal interest rate).

$A_t$  is a non-stationary total factor productivity shock that also augments labor in the utility function in order to accommodate the necessary stationarization of the model later on. See the next section for more details on this process.

## E2 Producers

### E21 Final good producer

The final good producer purchases the intermediate goods  $Y_{f,t}$  at the intermediate price  $P_{f,t}$  and aggregates them using CES technology to produce and sell the final good  $Y_t$  to the household and government at price  $P_t$ . Its problem is then summarized as

$$(E2.1) \quad \max_{Y_{f,t}, f \in [0,1]} P_t Y_t - \int_0^1 P_{f,t} Y_{f,t} di$$

subject to the CES production function

$$(E2.2) \quad Y_t = \left[ \int_0^1 Y_{f,t}^{\frac{\theta^p - 1}{\theta^p}} di \right]^{\frac{\theta^p}{\theta^p - 1}}.$$

$\theta^p$  is the price markup parameter.

### E22 Intermediate goods producers

There is a continuum of intermediate goods producers indexed by  $f \in [0, 1]$ . Intermediate goods producers use labor to produce the imperfectly substitutable intermediate goods according to a linear production function ( $Y_{f,t} = A_t N_{f,t}$ ) and then sell the product to the final good producer. Each firm maximizes its expected discounted sum

of future profits<sup>9</sup> by setting the price of its own good. Any price changes are subject to quadratic adjustment costs.  $\varphi_p$  will represent an obstruction of price adjustment, the firm indexes for prices—measured by  $\iota_p$ —and takes as given previous period inflation  $\Pi_{t-1}^p$ , and  $\bar{\Pi}^p$  represents the monetary authority's inflation target.

$$(E2.3) \quad \max_{P_{f,t}} E_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[ \prod_{s=0}^{t-1} \delta_s \right] \lambda_t \left[ P_{f,t} Y_{f,t} - W_t N_{f,t} - P_t \frac{\varphi_p}{2} \left[ \frac{P_{f,t}}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p} P_{f,t-1}} - 1 \right]^2 Y_t \right]$$

such that

$$(E2.4) \quad Y_{f,t} = \left[ \frac{P_{f,t}}{P_t} \right]^{-\theta^p} Y_t.^{10}$$

$\lambda_t$  is the Lagrange multiplier on the household's budget constraint at time  $t$  and  $\beta^{t-1} \left[ \prod_{s=0}^{t-1} \delta_s \right] \lambda_t$  is the marginal value of an additional profit to the household. The positive time zero price is the same across firms (i.e.  $P_{i,0} = P_0 > 0$ ).

$A_t$  represents total factor productivity which evolves deterministically:

$$(E2.5) \quad \ln(A_t) = \ln(a) + \ln(A_{t-1}).$$

$a$  is the trend growth rate of productivity. This trend growth factor will imply that the model will need to be stationarized. Monetary policy will also have to accommodate this growth factor as well.

### E3 Government policies

It is assumed that the monetary authority determines nominal interest rates according to a truncated notional inertial Taylor rule augmented by a speed limit component.

$$(E3.1) \quad R_t = \max[1, R_t^*]$$

where

$$(E3.2) \quad \frac{R_t^*}{\bar{R}} = \left( \frac{R_{t-1}^*}{\bar{R}} \right)^{\rho_R} \left( \frac{\Pi_t^p}{\bar{\Pi}^p} \right)^{(1-\rho_r)\phi_\pi} \left( \frac{Y_t}{A_t \bar{Y}} \right)^{(1-\rho_r)\phi_y} \exp(\epsilon_t^R)$$

<sup>9</sup>NOTE: Each period, as it is written below, is in *nominal* terms. However, we want each period's profits in *real* terms so the profits in each period must be divided by that period's price level  $P_t$  which we take care of further along in the document.

<sup>10</sup>This expression is derived from the profit maximizing input demand schedule when solving for the final good producer's problem above. Plugging this expression back into the CES production function implies that the final good producer will set the price of the final good  $P_t = \left[ \int_0^1 P_{f,t}^{1-\theta^p} di \right]^{\frac{1}{1-\theta^p}}$ .

where  $\Pi_t^p = \frac{P_t}{P_{t-1}}$  is the inflation rate between periods  $t-1$  and  $t$ ,  $\bar{R} = \frac{\bar{\Pi}^p a^{\chi_c}}{\beta}$  (see the section on stationarization to see why), and  $\epsilon_t^R$  represents white noise monetary policy shocks with mean zero and standard deviation  $\sigma_R$ .  $\bar{Y}$  is the deterministic steady state of the normalized output  $\frac{Y_t}{A_t}$ .

## E4 Market clearing conditions

The market clearing conditions for the final good, labor and government bond are given by

$$(E4.1) \quad Y_t = C_t + \int_0^1 \frac{\varphi_p}{2} \left[ \frac{P_{f,t}}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p} P_{f,t-1}} - 1 \right]^2 Y_t df + \dots$$

$$\dots + \int_0^1 w_t \frac{\varphi_w}{2} \left[ \frac{w_{h,t}}{a w_{h,t-1} (\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right]^2 N_t dh$$

$$(E4.2) \quad N_t = \int_0^1 N_{f,t} di$$

$$(E4.3) \quad C_t^a = C_t = \int_0^1 C_{h,t} dh$$

and

$$(E4.4) \quad B_t = \int_0^1 B_{h,t} dh = 0.$$

## E5 An equilibrium

Given  $P_0$  and stochastic processes for  $\delta_t$ , an equilibrium consists of allocations  $\{C_t, N_t, N_{f,t}, Y_t, Y_{f,t}, G_t\}_{t=1}^\infty$ , prices  $\{W_t, P_t, P_{f,t}\}_{t=1}^\infty$ , and a policy instrument  $\{R_t\}_{t=1}^\infty$  such that

(i) allocations solve the problem of the household given prices and policies

$$(E5.1) \quad \partial C_{h,t} : (C_{h,t} - \zeta C_{t-1}^a)^{-\chi_c} - \lambda_t = 0$$

(E5.2)

$$\begin{aligned}
\partial w_{h,t} : & \theta^w A_t^{1-\chi_c} \frac{N_t^{1+\chi_n}}{w_t} \left( \frac{w_{h,t}}{w_t} \right)^{-\theta^w(1+\chi_n)-1} \\
& + (1-\theta^w) \lambda_t \left( \frac{w_{h,t}}{w_t} \right)^{-\theta^w} N_t \\
& - \lambda_t w_t \varphi_w \left( \frac{w_{h,t}}{aw_{h,t-1}} \frac{\Pi_t^p}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right) N_t \frac{\Pi_t^p}{aw_{h,t-1} (\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} \\
& + \beta \delta_t \mathbf{E}_t \lambda_{t+1} w_{t+1} \varphi_w \left( \frac{w_{h,t+1}}{aw_{h,t}} \frac{\Pi_{t+1}^p}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_t^w)^{\iota_w}} - 1 \right) N_{t+1} \frac{w_{h,t+1}}{aw_{h,t}^2} \frac{\Pi_{t+1}^p}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_t^w)^{\iota_w}} = 0
\end{aligned}$$

(E5.3)

$$\partial B_{h,t} : -\frac{\lambda_t}{R_t P_t} + \beta \delta_t \mathbf{E}_t \frac{\lambda_{t+1}}{P_{t+1}} = 0$$

(ii)  $P_{f,t}$  solves the problem of firm  $i$ 

By making the appropriate substitution (the intermediate goods producer's constraints in place of  $Y_{f,t}$  and subsequently in for  $N_{f,t}$ ) and by dividing each period's profits by that period's price level  $P_t$  so as to put profits in real terms (and thus make profits across periods comparable) we get the following:

(E5.4)

$$\begin{aligned}
\partial P_{f,t} : & \lambda_t \frac{Y_t}{P_t} \left[ \frac{P_t}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p} P_{f,t-1}} \varphi_p \left( \frac{P_{f,t}}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p} P_{f,t-1}} - 1 \right) - (1-\theta^p) \left( \frac{P_{f,t}}{P_t} \right)^{-\theta^p} \right. \\
& \left. - \theta^p \frac{w_t}{A_t} \left( \frac{P_t}{P_{f,t}} \right)^{1+\theta^p} \right] = \beta \delta_t \mathbf{E}_t \frac{\lambda_{t+1} Y_{t+1}}{P_{t+1}} \left[ P_{t+1} \varphi_p \left( \frac{P_{f,t+1}}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p} P_{f,t}} - 1 \right) \frac{P_{f,t+1}}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p} P_{f,t}^2} \right]
\end{aligned}$$

(iii)  $P_{f,t} = P_{j,t} \quad \forall i \neq j$ 

(E5.5)

$$\begin{aligned}
\frac{Y_t}{\lambda_t^{-1}} \left[ \varphi_p \left( \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - 1 \right) \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - (1-\theta^p) - \theta^p \frac{w_t}{A_t} \right] = \dots \\
\dots = \beta \delta_t \mathbf{E}_t \frac{Y_{t+1}}{\lambda_{t+1}^{-1}} \varphi_p \left( \frac{\Pi_{t+1}^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p}} - 1 \right) \frac{\Pi_{t+1}^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p}}
\end{aligned}$$

(iv)  $R_t$  follows a specified rule

and

(v) all markets clear.

Combining all of the results derived from the conditions and exercises in (i)-(v), a symmetric equilibrium can be characterized recursively by  $\{C_t, N_t, Y_t, w_t, \Pi_t^p, R_t\}_{t=1}^\infty$  satisfying the following equilibrium conditions:

$$(E5.6) \quad \lambda_t = \beta \delta_t R_t E_t \lambda_{t+1} (\Pi_{t+1}^p)^{-1}$$

$$(E5.7) \quad \lambda_t = (C_t - \zeta C_{t-1})^{-\chi_c}$$

$$(E5.8) \quad \frac{N_t}{\lambda_t^{-1}} \left[ \varphi_w \left( \frac{\Pi_t^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right) \frac{\Pi_t^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - (1 - \theta^w) - \theta^w \frac{A_t^{1-\chi_c} N_t^{\chi_n}}{\lambda_t w_t} \right] = \dots$$

$$\dots = \beta \delta_t E_t \frac{N_{t+1}}{\lambda_{t+1}^{-1}} \varphi_w \left( \frac{\Pi_{t+1}^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_t^w)^{\iota_w}} - 1 \right) \frac{\Pi_{t+1}^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_t^w)^{\iota_w}} \frac{w_{t+1}}{w_t}$$

$$(E5.9) \quad \Pi_t^w = \frac{w_t}{a w_{t-1}} \Pi_t^p$$

$$(E5.10) \quad \frac{Y_t}{\lambda_t^{-1}} \left[ \varphi_p \left( \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - 1 \right) \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - (1 - \theta^p) - \theta^p \frac{w_t}{A_t} \right] = \dots$$

$$\dots = \beta \delta_t E_t \frac{Y_{t+1}}{\lambda_{t+1}^{-1}} \varphi_p \left( \frac{\Pi_{t+1}^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p}} - 1 \right) \frac{\Pi_{t+1}^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p}}$$

$$(E5.11) \quad Y_t = C_t + \frac{\varphi_p}{2} \left[ \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - 1 \right]^2 Y_t + \frac{\varphi_w}{2} \left[ \frac{\Pi_t^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right]^2 w_t N_t$$

$$(E5.12) \quad Y_t = A_t N_t$$

$$(E5.13) \quad R_t = \max [1, R_t^*]$$

where

$$(E5.14) \quad \frac{R_t^*}{\bar{R}} = \left( \frac{R_{t-1}^*}{\bar{R}} \right)^{\rho_R} \left( \frac{\Pi_t^p}{\bar{\Pi}^p} \right)^{(1-\rho_r)\phi_\pi} \left( \frac{Y_t}{A_t \bar{Y}} \right)^{(1-\rho_r)\phi_y} \exp(\epsilon_t^R)$$



and given the following processes ( $\forall t \geq 2$ ):

$$(E5.15) \quad (\delta_t - 1) = \rho_\delta(\delta_{t-1} - 1) + \epsilon_t^\delta$$

and

$$(E5.16) \quad \ln(A_t) = \ln(a) + \ln(A_{t-1}).$$

## E6 A stationary equilibrium

Let  $\tilde{Y}_t = \frac{Y_t}{A_t}$ ,  $\tilde{C}_t = \frac{C_t}{A_t}$ ,  $\tilde{w}_t = \frac{w_t}{A_t}$ , and  $\tilde{\lambda}_t = \frac{\lambda_t}{A_t^{\chi_c}}$  be the stationary representations of output, consumption, real wage, and marginal utility of consumption respectively. The stationary symmetric equilibrium can now be characterized by the following system of equations.

$$(E6.1) \quad \tilde{\lambda}_t = \frac{\beta}{a^{\chi_c}} \delta_t R_t E_t \tilde{\lambda}_{t+1} (\Pi_{t+1}^p)^{-1}$$

$$(E6.2) \quad \tilde{\lambda}_t = (\tilde{C}_t - \tilde{\zeta} \tilde{C}_{t-1})^{-\chi_c}, \quad \tilde{\zeta} = \frac{\zeta}{a}$$

$$(E6.3) \quad \begin{aligned} \frac{N_t \tilde{w}_t}{\tilde{\lambda}_t^{-1}} \left[ \varphi_w \left( \frac{\Pi_t^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right) \frac{\Pi_t^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - (1 - \theta^w) - \theta^w \frac{N_t^{\chi_n}}{\tilde{\lambda}_t \tilde{w}_t} \right] = \dots \\ \dots = \frac{\beta \varphi_w}{a^{\chi_c - 1}} \delta_t E_t \frac{N_{t+1} \tilde{w}_{t+1}}{\tilde{\lambda}_{t+1}^{-1}} \left( \frac{\Pi_{t+1}^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_t^w)^{\iota_w}} - 1 \right) \frac{\Pi_{t+1}^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_t^w)^{\iota_w}} \end{aligned}$$

$$(E6.4) \quad \Pi_t^w = \frac{\tilde{w}_t}{\tilde{w}_{t-1}} \Pi_t^p$$

$$(E6.5) \quad \begin{aligned} \frac{\tilde{Y}_t}{\tilde{\lambda}_t^{-1}} \left[ \varphi_p \left( \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - 1 \right) \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - (1 - \theta^p) - \theta^p \tilde{w}_t \right] = \dots \\ \dots = \frac{\beta \varphi_p}{a^{\chi_c - 1}} \delta_t E_t \frac{\tilde{Y}_{t+1}}{\tilde{\lambda}_{t+1}^{-1}} \left( \frac{\Pi_{t+1}^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p}} - 1 \right) \frac{\Pi_{t+1}^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_t^p)^{\iota_p}} \end{aligned}$$

$$(E6.6) \quad \tilde{Y}_t = \tilde{C}_t + \frac{\varphi_p}{2} \left[ \frac{\Pi_t^p}{(\bar{\Pi}^p)^{1-\iota_p} (\Pi_{t-1}^p)^{\iota_p}} - 1 \right]^2 \tilde{Y}_t + \frac{\varphi_w}{2} \left[ \frac{\Pi_t^w}{(\bar{\Pi}^w)^{1-\iota_w} (\Pi_{t-1}^w)^{\iota_w}} - 1 \right]^2 \tilde{w}_t N_t$$

$$(E6.7) \quad \tilde{Y}_t = N_t$$

and

$$(E6.8) \quad R_t = \max[1, R_t^*]$$

where

$$(E6.9) \quad \frac{R_t^*}{\bar{R}} = \left( \frac{R_{t-1}^*}{\bar{R}} \right)^{\rho_R} \left( \frac{\Pi_t^p}{\bar{\Pi}^p} \right)^{(1-\rho_r)\phi_\pi} \left( \frac{\tilde{Y}_t}{\bar{Y}} \right)^{(1-\rho_r)\phi_y} \exp(\epsilon_t^R)$$

and given the following processes ( $\forall t \geq 2$ ):

$$(E6.10) \quad (\delta_t - 1) = \rho_\delta(\delta_{t-1} - 1) + \epsilon_t^\delta$$

and

## E7 Stationary deterministic steady-state values

For each variable,  $X_t$ , we denote its corresponding *stationary* deterministic steady-state value as  $\bar{X}$ . The following is a list of analytical expressions for the stationary steady states for each of the variables of the model.

$\bar{\Pi}^p = \bar{\Pi}^p$ , (this parameter is set exogenously by the monetary authority)

$$\begin{aligned} \bar{\Pi}^w &= \bar{\Pi}^p \\ \bar{R} &= \frac{a^{X_c} \bar{\Pi}^p}{\beta} \\ \bar{w} &= \frac{\theta_p - 1}{\theta_p} \\ \bar{C} &= \left( \frac{\bar{w}(\theta_w - 1)}{\theta_w (1 - \tilde{\zeta})^{X_c}} \right)^{\frac{1}{X_c + X_n}} \\ \bar{\lambda} &= \left[ (1 - \tilde{\zeta}) \bar{C} \right]^{-X_c} \\ \bar{N} &= \bar{Y} = \bar{C} \end{aligned}$$

## F Details of Optimal Monetary Policies

In this section, we formulate the problems of the optimizing central bank under discretion and under commitment.

## F1 Optimal discretionary policy

The problem of the discretionary central bank is to choose allocations, prices, and the policy rate today to maximize the welfare—the expected discounted sum of future household utility flows—, taking as given the future value and policy and functions as given:

$$(F1.1) \quad V_t(\delta_t) = \max_{\{d_t\}} \left[ \frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1-\chi_n}}{1-\chi_n} \right] + \beta \delta_t \mathbf{E}_t V_{t+1}(\delta_{t+1})$$

where  $d_t := (C_t, N_t, \Pi_t, R_t)$  and the optimization is subject to the following private-sector equilibrium conditions:

$$(F1.2)$$

$$C_t^{-\chi_c} = \beta \delta_t R_t \mathbf{E}_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1}$$

$$(F1.3)$$

$$\frac{N_t}{C_t^{\chi_c}} [\varphi(\Pi_t - 1) \Pi_t - (1 - \theta) - \theta(1 - \tau) N_t^{\chi_n} C_t^{\chi_c}] = \beta \delta_t \mathbf{E}_t \frac{N_{t+1}}{C_{t+1}^{\chi_c}} \varphi(\Pi_{t+1} - 1) \Pi_{t+1}$$

$$(F1.4)$$

$$N_t = C_t + \frac{\varphi}{2} (\Pi_t - 1)^2 N_t$$

$$(F1.5)$$

$$R_t \geq 1$$

The Markov Perfect Equilibrium is defined to be the set of time-invariant value and policy functions  $\{V(\cdot), C(\cdot), N(\cdot), \Pi(\cdot), R(\cdot)\}$  that solve the Bellman equation above. The following first order conditions, complementary slackness conditions, and constraints constitute the Kuhn-Tucker conditions for this optimization problem:

$$(F1.6) \quad \partial C_t : C_t^{-\chi_c} - \chi_c \phi_{1,t} C_t^{-\chi_c - 1} - \chi_c \phi_{2,t} \frac{N_t}{C_t^{\chi_c + 1}} [\varphi(\Pi_t - 1) \Pi_t - (1 - \theta)] - \phi_{3,t} = 0$$

$$(F1.7) \quad \begin{aligned} \partial N_t : -N_t^{\chi_n} + \phi_{2,t} \left( \frac{1}{C_t^{\chi_c}} (\varphi(\Pi_t - 1) \Pi_t - (1 - \theta)) - \theta(1 - \tau)(\chi_n + 1) N_t^{\chi_n} \right) \\ + \phi_{3,t} \left( 1 - \frac{\varphi}{2} (\Pi_t - 1)^2 \right) = 0 \end{aligned}$$

$$(F1.8) \quad \partial \Pi_t : \phi_{2,t} \left( \frac{N_t}{C_t^{\chi_c}} [\varphi(2\Pi_t - 1)] \right) - \phi_{3,t} \varphi(\Pi_t - 1) N_t = 0$$

$$(F1.9) \quad \partial R_t : -\phi_{1,t} \beta \delta_t \mathbf{E}_t C_{t+1}^{-\chi_c} \Pi_{t+1}^{-1} + \phi_{4,t} = 0 \quad R_t \geq 1, \quad \phi_{4,t} \geq 0, \quad (R_t - 1) \phi_{4,t} = 0$$

## F2 Optimal commitment policy

The problem of the central bank under commitment is to choose a state-contingent sequence of the model's variables at time one in order to maximize the welfare of the household:

$$(F2.1) \quad \max_{\{C_t, N_t, G_t, \Pi_t, R_t\}} \mathbb{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \prod_{s=0}^{t-1} \delta_s \left[ \frac{C_t^{1-\chi_c}}{1-\chi_c} - \frac{N_t^{1-\chi_n}}{1-\chi_n} \right]$$

subject to the private sector equilibrium conditions. First-order necessary conditions at time  $t$  are given by

$$(F2.2) \quad \begin{aligned} \partial C_t : C_t^{-\chi_c} - \chi_c \phi_{1,t} \frac{C_t^{-\chi_c-1}}{R_t} + \chi_c \phi_{1,t-1} \frac{C_t^{-\chi_c-1}}{\Pi_t} - \chi_c \phi_{2,t} \frac{N_t}{C_t^{\chi_c+1}} [\varphi(\Pi_t - 1)\Pi_t - (1 - \theta)] \\ + \chi_c \varphi \phi_{2,t-1} \frac{N_t}{C_t^{\chi_c+1}} (\Pi_t - 1)\Pi_t - \phi_{3,t} = 0 \end{aligned}$$

$$(F2.3) \quad \begin{aligned} \partial N_t : -N_t^{\chi_n} + \phi_{2,t} \left( \frac{1}{C_t^{\chi_c}} (\varphi(\Pi_t - 1)\Pi_t - (1 - \theta)) - \theta(1 - \tau)(\chi_n + 1)N_t^{\chi_n} \right) \\ - \varphi \phi_{2,t-1} \frac{(\Pi_t - 1)\Pi_t}{C_t^{\chi_c}} + \phi_{3,t} \left( 1 - \frac{\varphi}{2} (\Pi_t - 1)^2 \right) = 0 \end{aligned}$$

$$(F2.4) \quad \partial \Pi_t : \phi_{1,t-1} \frac{C_t^{-\chi_c}}{\Pi_t^2} + \phi_{2,t} \left( \frac{N_t}{C_t^{\chi_c}} [\varphi(2\Pi_t - 1)] \right) - \varphi \phi_{2,t-1} \frac{N_t}{C_t^{\chi_c}} (2\Pi_t - 1) - \varphi \phi_{3,t} (\Pi_t - 1)N_t = 0$$

$$(F2.5) \quad \partial R_t : -\phi_{1,t} \frac{C_t^{-\chi_c}}{R_t^2} + \phi_{4,t} = 0$$

$$(F2.6) \quad R_t \geq 1, \quad \phi_{4,t} \geq 0, \quad (R_t - 1)\phi_{4,t} = 0$$

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