Online Supplement to "Coarse Competitive Equilibrium and Extreme Prices"

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1 Consumer Behavior in a Dynamic Economy

1.1 Dynamic Decision Problems

The state $i \in N = \{1, \ldots, n\}$ evolves according to an arbitrary stochastic process. A t-period history is a vector $(i_1, \ldots, i_t)$; we call H^t the set of all t-period histories; $H = \bigcup_{t \geq 1} N^t$ the set of all histories, and write λ_h for the probability of history $h \in H$. A function $d \in \mathbb{R}_+^H$ is a (dynamic) consumption plan and $\mathcal{D}$ is the set of all consumption plans. The definition of a coarse consumption plan mirrors the corresponding definition for the static economy:

Definition O.1. *The consumption plan $d \in \mathcal{D}$ is coarse if $|\{d_h \,|\, h \in H\}| \leq k$.*

Let $\mathcal{D}_k$ be the set coarse consumption plans. Hence, each household is restricted to at most k different levels of consumption throughout its entire lifetime. The household's utility from the consumption plan d is

$$V(d) = \sum_{t \geq 1} \sum_{h \in H^t} \beta^{t-1} \lambda_h u(d_h)$$

where $\beta \in (0, 1)$ is the discount factor and u is a CRRA utility index.

A function $q \in \mathbb{R}_+^H$ is a (dynamic) *price* and the consumer's budget is $B_k^*(q, w) = \{d \in \mathcal{D}_k : \sum_{h \in H} q_h d_h \leq w\}$. The coarse consumption plan $d \in B_k^*(p, w)$ is *optimal* if $V(d) \geq V(d')$ for all $d' \in B_k^*(q, w)$. For the price q, define the following normalized *pricing kernel* κ^*:

$$\kappa_h^* = \frac{q_h}{\lambda_h (1 - \beta) \beta^{t-1}} \tag{1}$$

for all $h \in H^t$. The normalized kernel κ_h^* is the price in state $h \in H^t$ divided by the probability of state h (λ_h) suitably discounted. We say that the partition $S = (S_1, \ldots, S_m)$ of H is induced by consumption d if [$d_h = d_{h'}$ if and only if $h, h' \in S_l$ for some l]. We let S^d denote the partition that d induces. A consumption plan is *measurable* if $\kappa^*_h = \kappa^*_{h'}$ implies $d_h = d_{h'}$ and *monotone* if $\kappa^*_h > \kappa^*_{h'}$ implies $d_h \leq d'_h$.

Lemma O.1. Every optimal plan is monotone and measurable.

2 Coarse Competitive Equilibrium in a Dynamic Economy

In this section, we extend our equilibrium analysis to a dynamic Lucas-tree economy. We show that there is a one-to-one correspondence between the stationary coarse competitive equilibria of our dynamic economy and the coarse competitive equilibrium of a corresponding static economy. This correspondence enables us to relate the extreme consumption prices analyzed in Theorem 4 to extreme asset prices.

As in the static economy, $N = \{1, \ldots, n\}$ is a finite set of states and $s_i \in [a, b]$ is the aggregate endowment in state i. Given any history $h = (i_1, \ldots, i_t)$, we let $\jmath(h) = i_t$ and let $H^t_i = \{h \in N^t : \jmath(h) = i\}$ be the set of all t-period histories that end in i.

A matrix of transition probabilities, Φ, describes the evolution of the state; Φ_{ij} is the probability that the state at date $t + 1$ is j given that it is i on date t. We assume that Φ has a stationary distribution π; that is,

$$\pi = \pi \cdot \Phi \tag{2}$$

The initial state (the period 1 history) is drawn from the stationary distribution π. Therefore, the probability of history $h = (i_1, \dots, i_t) \in H^t$ is

$$\lambda_h = \pi_{i_1} \cdot \Phi_{i_1 i_2} \cdots \Phi_{i_{t-1} i_t} \tag{3}$$

The sextuple $E^* = (u, \beta, k, \pi, s, \Phi)$ is a dynamic economy. The assumption that the initial state is chosen according to the invariant distribution means that we can ignore transitory effects of the initial condition. As we show below, the economy has a stationary equilibrium allocation and stationary equilibrium prices. Moreover, we can map the dynamic economy to the two-period economy analyzed in the previous section.

An *allocation* is a probability distribution on $\mathcal{D}$. It is coarse if its support is contained in the set of coarse consumption plans. Thus, the set of dynamic allocations is $\Delta(\mathcal{D})$ and the allocation $\nu \in \Delta(\mathcal{D})$ is coarse if $K(\nu) \subset \mathcal{D}_k$. The allocation ν is *feasible* in E^* if

$$\Sigma_h(\nu) := \sum_{d \in K(\nu)} d_h \cdot \nu(d) \leq s_{\jmath(h)}$$

for all $h \in H$. For any $\mathcal{D}' \subset \mathcal{D}$, let $M^*(\mathcal{D}')$ be the set of all feasible allocations ν such that $\nu(\mathcal{D}') = 1$. Hence, $M^*(\mathcal{D}_k)$ is the set of feasible coarse allocations for E^*.

The set of prices is $Q = \{q \in \mathbb{R}^H_+ : \sum_H q_h = 1\}$. The representative household's budget is

$$B^*_k(q) = \left\{ d \in \mathcal{D}_k : \sum_{h \in H} q_h [d_h - s_{\jmath(h)}] \leq 0 \right\} \tag{4}$$

The coarse consumption plan $d \in B^*_k(q)$ is *optimal at prices* q if $V(d) \geq V(d')$ for all $d' \in B^*_k(q)$.

Definition O.2. *The price-allocation pair* (q, ν) *is a* coarse competitive equilibrium *of E^* if (i) ν is feasible for E^* and (ii) $\nu(d) > 0$ implies d is optimal at prices q.*

Fix a dynamic economy $E^* = (u, \beta, k, \pi, s, \Phi)$ and consider the static economy $E = (u, k, \pi, s)$. The two economies share the same utility function and coarseness constraint. In both economies the initial endowment is chosen according to the distribution π. In the dynamic economy, the endowment is a Markov process while in the static economy the endowment stays fixed. Since π is the stationary distribution of the Markov process with transition matrix Φ, we have

$$\sum_{h \in H_i^t} \lambda_h = \pi_i \tag{5}$$

for all $t \geq 1$. Hence, the ex ante probability of state i in period t is π_i for every t. Since $(1-\beta)\sum_t \sum_{h \in H_i^t} \beta^{t-1}\lambda_h = \pi_i$, the dynamic economy can be thought of as a version of the static economy in which each state i is split into many identical states each corresponding to a branch of the event tree that ends with i. We refer to $E = (u, k, \pi, s)$ as the *static economy for* $E^* = (u, \beta, k, \pi, s, \Phi)$.

A consumption plan is stationary if consumption depends only on the current state. The measurability of coarse competitive equilibrium consumption plans in E will yield the stationarity of coarse competitive equilibrium consumption plans in E^*. Formally, the plan d is *stationary* if there exists a consumption plan, c, for the static economy E such that $d_h = c_{\jmath(h)}$ for all $h \in H$. Let $\bar{\mathcal{D}}$ denote the set of stationary plans. We can associate the stationary consumption plans of the dynamic economy E^* with the consumption plans of the corresponding static economy E: let $T_1 : \mathcal{C} \to \bar{\mathcal{D}}$ be the above one-to-one mapping between static consumption plans and stationary dynamic plans. Thus, $d = T_1(c)$ is the dynamic consumption plan in which a household consumes c_i whenever the state i occurs. The set of stationary allocations is $\Delta(\bar{\mathcal{D}})$. Let $T_3 : \Delta(\mathcal{C}) \to \Delta(\bar{\mathcal{D}})$ be the one-to-one mapping between allocations in the static economy and stationary allocations in the dynamic economy defined by $\nu(d) = \mu(T_1^{-1}(d))$ for $d \in \bar{\mathcal{D}}$ and $\nu(d) = 0$ for $d \notin \bar{\mathcal{D}}$.

A price q is stationary if q_h depends only on the current state $\jmath(h)$ and on the discounted probability of history h appropriately normalized. More precisely, q is *stationary* if there is a static price p such that for all $t \geq 1$ and all $h \in N^t$

$$\begin{aligned} q_h &= \lambda_h(1-\beta)\beta^{t-1}\tfrac{p_{\jmath(h)}}{\pi_{\jmath(h)}} \\ &= \lambda_h(1-\beta)\beta^{t-1}\kappa_{\jmath(h)} \end{aligned} \tag{6}$$

Equations (5) and (6) imply

$$\sum_{t=1}^{\infty} \sum_{h \in H_i^t} q_h = p_i \tag{7}$$

for all i and hence $\sum_{h \in H} q_h = \sum_{i=1}^{n} p_i = 1$. Then, for each static price p, there is a corresponding stationary dynamic price and, conversely, each stationary dynamic price can be mapped to a static price.

Let $\bar{Q} \subset \Delta(H)$ be the set of stationary prices. Let $T_2 : \Delta(N) \to \bar{Q}$ be the one-to-one mapping between prices in the static economy and stationary prices in the dynamic economy defined above. To summarize: $T_1 : \mathcal{C} \xrightarrow{1-1} \bar{\mathcal{D}}$ is the mapping that identifies the unique stationary consumption plan associated with each static consumption plan, $T_2 : \Delta(N) \xrightarrow{1-1} \bar{Q}$ defines the unique stationary price associated with each static price and $T_3 : \Delta(\mathcal{C}) \xrightarrow{1-1} \Delta(\bar{\mathcal{D}})$ defines the unique stationary allocation associated with each static allocation.

Equation (5) implies

$$V(T_1(c)) = (1-\beta)\sum_{t\geq 1}\sum_{h\in N^t} u(c_{\jmath(h)})\beta^{t-1}\lambda_h = U(c) \tag{8}$$

and Equation (7) implies that $c \in B_k(p)$ if and only if $T_1(c) \in B_k^*(T_2(p)) \cap \bar{\mathcal{D}}$. Finally, note that $\mu \in \Delta(\mathcal{C})$ is feasible in E if and only if $T_3(\mu) \in \Delta(\bar{\mathcal{D}})$ is feasible in E^*.

Theorem O.1 below relates coarse competitive equilibria of the dynamic economy to the coarse competitive equilibria of the corresponding static economy. An equilibrium allocation of the static economy yields a stationary equilibrium allocation of the corresponding dynamic economy and an equilibrium price of the static economy yields a stationary equilibrium price of the dynamic economy.

Theorem O.1. (i) $(T_2(p), T_3(\mu))$ is a coarse competitive equilibrium of E^* whenever (p,μ) is a coarse competitive equilibrium of E. (ii) If ν is a coarse competitive equilibrium allocation for E^*, then $\nu \in \Delta(\bar{\mathcal{D}})$ and $T_3^{-1}(\nu)$ is a coarse competitive equilibrium allocation for E.

Theorem O.1 leaves open the possibility of a non-stationary coarse competitive equilibrium price (supporting a stationary coarse competitive equilibrium allocation). Theorem O.1 relies on the assumption that the initial state is chosen according to the stationary distribution π. Without this assumption, there might still be an analogue of Theorem O.1 but the mapping between the dynamic and the static economy would be more complicated.

3 Extreme Asset Prices and the Safe Haven Premium

Recall that a static pure endowment economy $E = (u, k, \pi, s)$ has a unique coarse competitive equilibrium price. In this section, we relate this price p to asset prices in the dynamic economy. To simplify the exposition, we restrict ourselves to iid transitions: $\Phi_{ij} = \pi_j$ for all i, j and refer to a dynamic economy with constant transition probabilities as an *iid economy.* In addition, we assume $\rho = 1$ so that $u(z) = \ln z$. These restrictions are made for expositional ease. All results below can be extended to arbitrary Markov transitions provided that all ratios of transition probabilities stay bounded. For $\rho > 1$, the equilibrium price of consumption may be zero; this would allow us to strengthen some of the results below at the cost of more cumbersome notation. For $\rho < 1$, we would need to weaken slightly the result on extremely low asset prices.

Consider an asset $z = (z_1, \ldots, z_n)$ in zero net supply that delivers z_i units of the consumption good next period if state i occurs. Let $r_h(z)$ be the coarse competitive equilibrium price of this asset in terms of current period consumption after history h. Recall that $\jmath(h) \in N$ is the last state of history h. A standard no-arbitrage argument yields

$$r_h(z) = \frac{\sum_{i \in N} q_{hi} z_i}{q_h} \tag{9}$$

where the numerator is the expected value of the return z after history h and the denominator is the price of consumption after history h. Since q is stationary, the price of the asset depends only on $\jmath(h)$ and therefore $r_h(z) = r_{\jmath(h)}(z)$. Formulas (6) and (9) imply

$$r_h(z) = r_{J(h)}(z) = \beta \frac{E[\kappa z]}{\kappa_{J(h)}} \tag{10}$$

where $E[\xi]$ is the expectation of the random variable ξ on the finite probability space (N, π). Since $\kappa_i > 0$ for all $i \in N$, the asset price is well defined. We let $r(z)$ denote the asset price in period t as a function of the realized state in period t. Since the price of the asset depends only on the current state, the random variable $r(z)$ is well defined.

As in Theorem 4, we consider a sequence of economies that converges to a limit economy with a continuous distribution of endowments. Formally, we say that a sequence of iid-economies $\{E^{*n}\} = \{(u, \beta, k, \pi^n, s^n)\}$ is *convergent* if the corresponding sequence of static economies $\{E^n\} = \{(u, k, \pi^n, s^n)\}$ is convergent. Consider a sequence of asset returns $\{z^n\}$ such that $z^n \in \mathbb{R}^n$. We say that $\{z^n\}$ is bounded if there are $0 < \gamma_1 < \gamma_2 < \infty$ such that $z_i^n \in [\gamma_1, \gamma_2]$ for all i and n. In Theorem O.2 and Theorem O.3 below, we consider convergent sequence of iid economies and corresponding sequences of assets. As in Theorem 4, to avoid having to say "there exists a subsequence such that" multiple times, we let $\lim x^n$ denote an arbitrary limit point of any bounded sequence $\{x^n\}$.

Theorem O.2. If $\{z^n\}$ is bounded, then $\lim \text{Prob}\,(r^n(z^n) < \epsilon) > 0$ for all $\epsilon > 0$ and $\lim \text{Prob}\,(r^n(z^n) > K) > 0$ for all K.

The proof of Theorem O.2 reveals that limit equilibrium asset prices are extremely high if the endowment is near its upper bound and extremely low if the endowment is near its lower bound. Theorem O.2 is a corollary of Theorem 4 and can be used to derive another asset pricing implication that relates *risk-free* and *nearly risk-free* asset prices. Let $\jmath_x^n = \min\{j : \sum_{i \geq j} \pi_i^n \geq 1 - x\}$. Then, let $e^n = (1, \ldots, 1)$ be a risk-free asset and let $e^{n\epsilon}$ be the following nearly risk-free asset:

$$e_i^{n\epsilon} = \begin{cases} 1 & \text{if } i \geq \jmath_\epsilon^n \\ 0 & \text{if } i < \jmath_\epsilon^n \end{cases}$$

Hence $e^{n\epsilon}$ yields 1 in all but the $\epsilon-$fraction of states with the lowest endowment. Let $R(z, z')$ be the ratio of the equilibrium price of asset z over the equilibrium price of asset z' after history h. Equation (10) ensures that $R(z, z')$ does not depend on h.

Theorem O.3. There is $\delta > 0$ such that $\lim R^n(e^{n\epsilon}, e^n) \leq 1 - \delta$ for all $\epsilon > 0$.

Theorem O.3 shows that the risk-free premium does not converge to zero as the returns of the nearly risk-free asset converges in distribution to the returns of the risk-free asset. This safe-haven premium comes about because the price of consumption in the lowest endowment state is bounded away from zero and, therefore, the risk-free asset is always more costly than the nearly risk-free asset.

4 Proofs of the Results in this Supplement

4.1 Proofs of Lemma O.1 and Theorem O.1

Proof of Theorem O.1: Let $\tau(h)$ be the length of history h; that is, $\tau(h) = t$ if $h \in H^t$. Let d be non-measurable or non-monotone optimal consumption plan and let κ^* be its pricing kernel defined in equation (1) above. Then, let $S' = (S'_1, \ldots, S'_m)$ for $m \leq k$, be the partition of H that d induces; that is $d_h = d_{h'}$ if and only if $h, h' \in S'_i$ for some i. Since d is non-monotone or non-measurable, assume w.l.o.g.

that $d(S'_1) > d(S'_2)$ and $\kappa^*_{h_1} \geq \kappa^*_{h_2}$ for $h_i \in S'_i$ for $i = 1, 2$, where $d(S'_i)$ is the consumption level for cell S'_i. Let $S^* = \{\{h_1\}, \{h_2\}, S_3, \ldots, S_m\}$, $S^0 = \{H_1, \ldots, H_n\}$ where N is the set of states in the static economy and let $S = \{S_1, \ldots, S_{\hat{n}}\}$ be the coarsest common refinement of S', S^* and S^0; that is, the coarsest partition of H finer than S', S^* and S^0.

Define the static economy $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ as follows: the utility function and k are as in the dynamic economy. The set of states is $\hat{N} = \{1, \ldots, \hat{n}\}$ and $\hat{s}_i = s_{j(h)}$ for some $h \in S_i$; since S is a refinement of S^0, $\hat{s}_i$ does not depend in the choice of h. Let $\hat{\pi}_i = \sum_{S_i} \lambda_h (1 - \beta) \beta^{\tau(h)-1}$ for all $i \in \hat{N}$.

Let $\hat{p}_i = \sum_{h \in S_i} q_h$ and $\hat{c}_i = d_h$ for some $h \in S_i$. Again, since S is a refinement of S', $\hat{c}_i$ does not depend in the choice of h. It is easy to verify that since d is an optimal consumption at prices q in E^*, $\hat{c}$ is an optimal consumption at price $\hat{p}$ in $\hat{E}$. It is also easy to verify that since d is non-measurable or non-monotone, so is $\hat{c}$, contradicting Theorem 1. □

Proof of Theorem O.1: Assume that (p, μ) is a coarse competitive equilibrium of E but $(T_2(p), T_3(\mu))$ is not a coarse competitive equilibrium of E^*. Let $\{d^1, \ldots, d^m\}$ be the support of $\nu = T_3(\mu)$. Clearly, ν is feasible for E^* since μ is feasible for E. So, there must be some i such that d_i is not an optimal. Since (p, μ) is a coarse competitive equilibrium of E, $T_1^{-1}(d^i)$ is budget-feasible in E and therefore, d^i is budget feasible in E^*. Hence, there is a budget feasible d such that $V(d) > V(d^i)$. If d were stationary, we would have $U(T^{-1}(d)) > U(T^{-1}(d^1))$ contradicting the fact that (p, μ) is an equilibrium of E; that is, d^i is optimal among stationary plans. However, if d is not stationary, then it is not measurable. Then, $V(d) > V(d^1)$ contradicts Lemma O.1.

Next, assume that (q, ν) is a coarse competitive equilibrium of E^* and is not stationary. Let $\{d^1, \ldots, d^m\}$ be the support of ν. Let S^i be the partition that d^i induces and let $S^0 = \{H_1, \ldots, H_n\}$. Let $S = \{S_1, \ldots, S_{\hat{n}}\}$ be the coarsest common refinement of $S^0, \ldots, S^m$. Let $\hat{N} = \{1, \ldots, \hat{n}\}$ and define the static economy $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ as in the proof of Lemma O.1: the utility function and k are the same as in E^*; $\hat{s}_i = s_{j(h)}$ for some $h \in S_i$; $\hat{\pi}_i = \sum_{S_i} \lambda_h (1-\beta)\beta^{\tau(h)-1}$ for all $i \in \hat{N}$.

Let $\hat{p}_i = \sum_{h \in S_i} q_h$ and $\hat{c}_i^l = d_h^l$ for some $h \in S_i$. Again, since S is a refinement of S^l and S^0, both $\hat{s}_i$ and $\hat{c}_i^l$ are well defined for all i, l. It is easy to verify that since d is an optimal consumption at prices q in E^*, $\hat{c}$ is an optimal consumption at price $\hat{p}$ in $\hat{E}$. It is also easy to verify that since d is non-stationary and p is stationary, $\hat{c}$ is non-measurable, contradicting Lemma O.1. Verifying that a stationary equilibrium of E^* is a stationary equilibrium of the static economy is straightforward and omitted. □

4.2 Proof of Theorems O.2 and O.3

Proof of Theorem O.2: Since z^n is bounded, $\sum_N p_i^n z_i^n \in [\gamma_1, \gamma_2]$ for all n. Then, Theorem O.2(i) follows from equation (10) and Theorem 4(i) and Theorem O.2(ii) follows from equation (10) and Theorem 4(ii). □

Proof of Theorem O.3: Note that $\lim E(\kappa^n e^{n\epsilon}) = \lim \sum_{i \geq j_\epsilon^n} p_i^n < 1 - \lim p_1^n$. By Theorem 4(i), $\lim p_1^n > 0$. Hence, equation (10) yields

$$\lim R^n(e^{n\epsilon}, e^n) = \frac{\lim E(\kappa^n e^{n\epsilon})}{\lim E(\kappa^n e^n)} = \lim E(\kappa^n e^{n\epsilon}) \leq 1 - \lim p_1^n$$

Then, setting $\delta = \lim p_1^n$ establishes the desired result. □

5 Remaining Proofs from the Paper

5.1 Proof of Lemma 5

Proof of Lemma 5. Let $Z = \mathbb{R}^n_{++}$, $E = (u, k, \pi, z)$ for $z \in Z$ and let $\Phi^z(\mathcal{C}')$ be the set of feasible allocations for E with support in $\mathcal{C}'$. Then, let $W_k(z) = \max_{\mu \in \Phi^z(\mathcal{C}_k)} W(\mu)$ and $Z^s = \{z \in Z : W_k(z) > W_k(s)\}$. Clearly, $W_k(z) > W_k(y)$ whenever $z_i > y_i$ for all $i \in N$ and hence, Z^s is nonempty.

Suppose $|z_i - y_i| < \epsilon$ and let $y_i^+ = \max\{y_i, z_i\}$, $y_i^- = \min\{y_i, z_i\}$ for all i. Then, let $\mu = (\mathbf{a}, \mathbf{c})$ be optimal for (u, k, π, y^+). Then, $(\mathbf{a}, (1 - \frac{\epsilon}{a})\mathbf{c})$ is feasible for (u, k, π, y^-). Since $W_k(\mathbf{a}, (1 - \frac{\epsilon}{a})\mathbf{c})$ is continuous in ϵ at $\epsilon = 0$ and $W_k(y)$ is nondecreasing in each coordinate, for $\epsilon' > 0$, there is $\epsilon > 0$ such that $|W_k(y) - W_k(z)| \le |W_k(y^+) - W_k(y^-)| < \epsilon'$ proving that W_k is continuous and hence Z^y is open.

We note that since W is a concave function of μ, W_k is a concave function of z and hence the set Z^s is convex. To see that, fix $z^1, z^2 \in Z^s$ and choose $\mu^i \in \Phi^{z^i}(\mathcal{C}_k)$ such that $W(\mu^i) = W_k(z^i)$ for $i = 1, 2$. By Lemma 4, such μ^i exist. Clearly, $\gamma\mu^1 + (1-\gamma)\mu^2 \in \Phi^{\hat{z}}(\mathcal{C}_k)$ for $\hat{z} = \gamma z^1 + (1-\gamma)z^2$ and hence $W_k(\gamma z^1 + (1-\gamma)z^2) \ge W(\gamma\mu^1 + (1-\gamma)\mu^2) = \gamma W(\mu^1) + (1-\gamma)W(\mu_2) = \gamma W_k(z^1) + (1-\gamma)W_k(z^2)$.

Since Z^s is nonempty, open, and convex, and $s \notin Z^s$, by the separating hyperplane theorem, there exists $p \in \mathbb{R}^n$ such that $p_i \neq 0$ for some i and $\sum_i p_i \cdot z_i > \sum_i p_i \cdot s_i$ for all $z \in Z^s$. Since W_k is nondecreasing in each coordinate, we must have $p_i \ge 0$ for every $i \in N$ and hence we can normalize p to ensure that $p \in \Delta(N)$.

Let $\mu = (\mathbf{a}, \mathbf{c})$ be a solution to the planner's problem, where $\alpha^l > 0$ for all l. The argument establishing that each c^l must maximize U given budget $B(p)$ is standard and omitted.

Finally, to see that if (p, μ) is a coarse competitive equilibrium, then μ must be a solution to the planner's problem, note that since every agent has the same endowment, μ must be fair. But then, if μ did not solve the planner's problem, the solution to the planner's problem would Pareto-dominate it. However, it is straightforward to show that every coarse competitive equilibrium allocation is Pareto-efficient. □

5.2 Proof of Theorem 3

Lemma 7 establishes monotonicity, Lemma O.2 below proves essential uniqueness.

Lemma O.2. The coarse competitive equilibrium price of a pure endowment economy is unique.

Proof. Let (p, μ) be a coarse competitive equilibrium. First, we show that $c_i > 0$ for all i and c such that $\mu(c) > 0$. Let $A = \{i : c_i = 0\}$ for some c such that $\mu(c) > 0$. If $A \neq \emptyset$, utility maximization implies $\sum_{i \in A} p_i = 1$ and $\sum_{i \in N \setminus A} p_i = 0$; otherwise increasing c_i by ϵ for all $i \in A$ and lowering c_i by $\frac{\epsilon \sum_{i \in A} p_i}{\sum_{i \in N \setminus A} p_i}$ for all $i \notin A$ results in increase of utility for small ϵ. It follows that c costs the same as $2c$ and since $c_i > 0$ for some i and u is strictly increasing, c cannot be optimal if $A \neq \emptyset$.

Since E is a pure endowment economy, assume without loss of generality that $s_i < s_{i+1}$. For any μ, let $I(\mu) = \{i < n : c_i < c_{i+1} \text{ for some } c \in K(\mu)\}$. Since every coarse competitive equilibrium allocation solves the planner's problem and $k > 1$ (i.e., agents can have at least two distinct consumption levels), $I(\mu) \neq \emptyset$. Hence, for any competitive equilibrium allocation μ, let $J(\mu) = \max I(\mu)$. Let (μ^l, p^l) for $l = 1, 2$ be two coarse competitive equilibria.

We claim that $i \notin I(\mu^l)$ implies $i + 1 \notin I(\mu^l)$. To see why this is the case, note that if $i \notin I(\mu^l)$, then $\Sigma_i(\mu^l) = \Sigma_{i+1}(\mu^l)$ and therefore $\Sigma_{i+1}(\mu^l) < s_{i+1}$ and hence by Lemma 6, $p_{i+1} = 0$. Then, if $c_{i+2} > c_{i+1}$ for any $c \in K(\mu^l)$, define $\hat{c}_j = c_j$ for $j \leq i$,

$\hat{c}_j = c_j$ for $j \geq i+2$ and $\hat{c}_{i+1} = c_{i+2}$ and note that $\hat{c}$ is coarse, costs the same as c but yields strictly higher utility, contradicting the fact that μ^l is a coarse competitive equilibrium allocation.

Next, we claim that $J(\mu^1) = J(\mu^2)$. If not, assume without loss of generality that $J(\mu^1) > J(\mu^2)$. Let $\hat{s}_j = s_j$ for all $j \leq J(\mu^2)$ and $\hat{s}_j = s_j + 1$ for all $j \geq J(\mu^2) + 1$. Then, since we established in the preceding paragraph that $p_j = 0$ for all $j > J(\mu^2)$, we conclude that (p^2, μ^2) is a coarse competitive equilibrium for the economy with endowment $\hat{s}$. Therefore, by Theorem 2, $W_k(s) = W_k(\hat{s})$. But, since $i := J(\mu^2) < J(\mu^1)$, the previous claim implies $i \in I(\mu^1)$. Hence, there exist $c \in K(\mu^1)$ such that $c_i < c_{i+1}$. Since c is monotone (by Theorem 1), $\hat{c}$ defined by $\hat{c}_j = c_j$ for all $j \leq i$ and $\hat{c}_j = c_j + 1$ for all $j \geq k+1$ is coarse. Let $\hat{\mu}$ be the allocation derived from μ^1 by replacing c with $\hat{c}$. Note that $\hat{\mu}$ yields strictly higher mean utility than μ^1 and is feasible for the economy with endowment $\hat{s}$, contradicting $W_k(s) = W_k(\hat{s})$.

Note that if $J(\mu^1) = J(\mu^2) = 1$, then $p^1_1 = p^2_1 = 1$ and hence $p^1 = p^2$ as desired. So, henceforth we assume $J(\mu^1) = J(\mu^2) > 1$. By Theorem 1, both μ^1, μ^2 solve the planner's problem. Then, the linearity of W ensures that $\mu^3 := .5\mu^1 + .5\mu^2$ also solves the planner's problem and hence by Theorem 1, there exists some p^3 such that (p^3, μ^3) is a coarse competitive equilibrium. Then, the previous claim ensures that $J := J(\mu^j) > 1$ for $j = 1, 2, 3$.

For any c such that $c_j > 0$ for all j and for any $i = 1, \ldots, n-1$, define

$$MRS_i(c) = \frac{\sum_{\{j \leq i\}} \pi_j u'(c_j)}{\sum_{\{j > i\}} \pi_j u'(c_j)}$$

For $j = 1, 2, 3$ and $i \in N$, let $q^j_i = \sum_{j \leq i} p^j_i$. For any $i \leq J$, pick $c^i \in K(\mu^1)$ such that $c^i_i < c^i_{i+1}$. Construct $\hat{c}$ by changing c^i_j in all states $j \leq i$ by ϵ and in all states $j > i$

by ϵ' in a budget neutral manner. The optimality of c^i ensures that this alternative plan cannot increase utility which means: $q_i^1 = MRS_i(c^i)(1 - q_i^1)$ for all $i \leq J$. But since $K(\mu^1) \subset K(\mu^3)$, the equations above also hold for q^3 proving that $q_j^3 = q_j^1$ for all j and hence $p^1 = p^3$. A symmetric argument ensures that $p^2 = p^3$. $\square$

Lemma O.3. The coarse competitive equilibrium price of any economy is essentially unique.

Proof. Let $\hat{E} = (u, k, \hat{\pi}, \hat{s})$ be any economy and let $E = (u, k, \pi, s)$ be the corresponding pure endowment economy where $\pi_i = \sum_{j:\hat{s}_j = s_i} \hat{\pi}_j$. For any measurable plan $\hat{c}$ for $\hat{E}$, define the plan c for E in an obvious way. Then, define μ by $\mu(c) = \hat{\mu}(\hat{c})$. Given a price $\hat{p}$ for $\hat{E}$, define p for E as follows: $p_i = \sum_{j:\hat{s}_j = s_i} \hat{p}_j$ for all i.

To prove essential uniqueness, suppose there are two coarse competitive equilibria for $\hat{E}$, $(\hat{p}^l, \hat{\mu}^l)$ for $l = 1, 2$ such that $\hat{p}^1$ and $\hat{p}^2$ are not equivalent. Then, since by Theorem 2 both $\hat{\mu}^l$ are measurable, the corresponding μ^l are well-defined allocations for E. It is easy to see that (p^l, μ^l) are coarse competitive equilibria for E. But since $\hat{p}^1$ and $\hat{p}^2$ are not equivalent, $p^1 \neq p^2$, contradicting Lemma O.2. $\square$

5.3 Proof of Lemma 10

Let (p^n, μ^n) be a coarse competitive equilibrium of E^n. Let U_*^n be the equilibrium utility of a k-course agent with wealth 1 and let Y^n be the maximal utility $k-1$-course agent with wealth 1 could obtain facing prices p^n. Let $f : [a, b] \to \mathbb{R}_{++}$ be the density of the limit endowment.

The proof of this lemma relies on definitions and results in section A.1. of the paper. Let $\{\hat{S}^n\} = \{(\hat{S}_1^n, \ldots, \hat{S}_{m_n}^n)\}$ be the sequence of partitions generated by some

$k-1$-course consumption plan $\hat{c}^n$. Hence, $m_n \leq k-1$ for all n. Let $\pi^n(j) = \pi(\hat{S}_j^n)$ and $p^n(j) = p(\hat{S}_j^n)$ for all n, j.

Suppose that

$$\lim[U_*^n - U(\hat{c}^n)] = 0$$

along some subsequence. Pass to that subsequence. Note that $u(a) \leq U(c^n) \leq U_*^n \leq u(b)$ and $0 \leq \pi^n(l) \leq 1, 0 \leq p^n(l) \leq 1$. Therefore, there exists a subsequence such that $(U(\hat{c}^n), (\pi^n(l), p^n(l))_{l=1}^m)$ converges and $m_n = m$ for all n. Pass to that subsequence and let $(Y, (p(l), \pi(l))_{l=1}^{k-1})$ be its limit. Hence, $\lim U_*^n = Y$. The optimality (in the limit) of $\hat{c}^n$ ensures that $\hat{c}^n(l) > 0$ whenever $\pi(l) > 0$.

For $0 < \epsilon < 1$, the sets $\{B_l^n\}$ are an ϵ-fragment if $B_l^n \subset \hat{S}_l^n$ for all n, $\sum_{B_l^n} \pi_j^n = \epsilon^n \pi^n(l)$ and $\lim \epsilon^n = \epsilon$.

Lemma O.4. Let $\{B_l^n\}$ be an ϵ-fragment. If $\lim U_*^n = Y$, then $\lim \sum_{B_l^n} p_j^n = \epsilon p(l)$.

Proof of Lemma O.4. Since $p_j^n \geq 0$, the conclusion of the lemma is immediate if $p(l) = 0$. Therefore, assume $p(l) > 0$ and that the lemma is false. Pass to a subsequence so that $b_1^n := \sum_{B_l^n} p_j^n$ converges. Define $b_2^n := p^n(l) - b_1^n; a_1^n = \pi^n(l)\epsilon^n, a_2^n = \pi^n(l)(1-\epsilon^n)$ and let $b_i = \lim b_i^n, a_i = \lim a_i^n$ for $i = 1, 2$.

If $b_1 \cdot b_2 = 0$ and $b_1^n \cdot b_2^n > 0$ for all n along some subsequence, then w.l.o.g. assume $b_2 = 0$. Construct $c^n \in C(S^n)$, where

$$S^n = (\hat{S}_1^n, \ldots, \hat{S}_{l-1}^n, B_l^n, \hat{S}_l^n \backslash B_l^n, \hat{S}_{l+1}^n, \ldots, \hat{S}_m^n)$$

as follows: $c_i^n = \hat{c}^n(l) - x$ for all $i \in B_l^n$, $c_i^n = \hat{c}^n(l) + \frac{xb_1^n}{b_2^n}$ for all $i \in S_l^n \backslash B_l^n$, $c(j) = \hat{c}^n(j)$ for all $j = 1, \ldots, m$. Since $\hat{c}(l) > 0$, for $x > 0$ sufficiently small, c is feasible and $\lim[U(\hat{c}^n) - U(c^n)] < 0$, a contradiction. If there exists no subsequence along which

$b_1^n \cdot b_2^n > 0$ for all n, then we can find a (subsequence) such that $b_i^n = 0$, say $b_2^n = 0$, for all n. Then, replace $\hat{c}^n(l) - \epsilon$ in the preceding argument with $\hat{c}^n(l)$ and $\hat{c}^n(l) + \frac{xb_1^n}{b_2^n}$ with $\hat{c}^n(l) + 1$ to get the same contradiction, $\lim[U(\hat{c}^n) - U(c^n)] < 0$.

Hence, we assume $b_1 \cdot b_2 > 0$ and $b_1^n \cdot b_2^n > 0$ for all n and note that $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Consider the sequence of k-coarse plans $c^n \in C(S^n)$ such that $U(c_n) \geq W_\sigma(S^n) + \frac{1}{n}$ for S^n defined above. Then, again we have a contradiction since

$$\lim[U(\hat{c}^n) - U(c^n)] = (b_1 + b_2)\psi_\sigma\left(\frac{a_1 + a_2}{b_1 + b_2}\right) - b_1\psi_\sigma\left(\frac{a_1}{b_1}\right) + b_2\psi_\sigma\left(\frac{a_2}{b_2}\right) < 0$$

where the last inequality follows from the strict convexity of ψ_σ. □

Fix a cell $\hat{S}_l$ of the partition $\hat{S}$ such that $\pi(l) > 0$ and $p(l) > 0$. To see why such a cell must exist, note that the proof of Lemma 10 does not rely on the optimality of the partition S. Let $z = \frac{p(l)}{\pi(l)} > 0$ and let $0 < q_* < q_{**} < \pi(l)$. For $j \in N$, define $L^n(j) := \{i \in N : i < j\}$ and $M^n(j) := \{i \in N : i \geq j\}$. For each n, let $\mathcal{J}^n$ be the set of j such that $q_* \leq \pi^n(\hat{S}_l \cap L^n(j)) \leq q_{**}$. Let $j_*^n := \min\{j \in \hat{S}_l : \pi^n(\hat{S}_l \cap L^n(j)) \geq q_*\}$ and let $j_{**}^n = \max\{j \in \hat{S}_l : \pi^n(\hat{S}_l \cap L^n(j)) \leq q_{**}\}$. Since the limit distribution of the aggregate endowment is nonatomic, $\lim_n \pi^n(\hat{S}_l \cap L^n(j_*^n)) = q_* = \pi^n(\hat{S}_l \backslash M^n(j_*^n))$ and $\lim_n \pi^n(\hat{S}_l \cap L^n(j_{**}^n)) = q_{**} = \pi^n(\hat{S}_l \backslash M^n(j_{**}^n))$.

Let $\tilde{c}^n$ be an optimal consumption for the economy E^n and let $\tilde{S}^n = S^{\tilde{c}^n}$ be the partition that $\tilde{c}^n$ induces. Consider the cells of $\tilde{S}^n$ that have a nonempty intersection with $\mathcal{J}^n$ and, of those cells, let G^n be the one with the minimal consumption and let H^n be the one with the maximal consumption. Hence, by Theorems 1 and 2, either $G^n = H^n$ or $i < j$ for all $i \in G^n$, $j \in H^n$.

Lemma O.5. $\lim \frac{p^n(G^n)}{\pi^n(G^n)} = \lim \frac{p^n(H^n)}{\pi^n(H^n)}$.

Proof of Lemma O.5. If $G^n = H^n$ for all n (sufficiently large), there is nothing to prove. So, assume $G^n \neq H^n$ for all n. Let l_*^n be the minimal element of $\hat{S}_l^n$ and l_{**}^n be the maximal element of $\hat{S}_l^n$. Define, $A^n := G^n \backslash L^n(l_*^n)$, $B^n := H^n \backslash M^n(l_{**}^n)$, $C^n := G^n \setminus A^n$ and $D^n := H^n \setminus B^n$. Let $a_1^n = \pi^n(A^n)$, $a_2^n = \pi^n(B^n)$, $a_3^n = \pi^n(C^n)$, $a_4^n = \pi^n(D^n)$, $b_1^n = p^n(A^n)$ $b_2^n = p^n(B^n)$, $b_3^n = p^n(C^n)$ and $b_4^n = p^n(D^n)$. Choose a subsequence such that the limits $a_i := \lim a_i^n$ and $b_i := \lim b_i^n$ exist for $i = 1, \ldots, 4$; w.l.o.g. assume that this subsequence is the sequence itself.

Let S be the partition derived from $\tilde{S}$ by replacing G^n with $G^n \cup B^n$ and H^n with D^n. Similarly, let T be the partition derived from $\tilde{S}$ by replacing G^n with C^n and H^n with $H^n \cup A^n$. Let $c^n \in C(S)$ be a sequence of consumption plans such that $\lim U(c^n) = \lim W_\sigma(S^n)$ and $\bar{c}^n \in C(T^n)$ be a sequence of consumptions such that $\lim U(\bar{c}^n) = \lim W_\sigma(T^n)$. Again, by passing to a subsequence if necessary, we can ensure that the above limits exist.

Note that setting $\psi_\sigma(0) = 0$ extends ψ_σ continuously to $\mathbb{R}_+$ for all σ. The extended ψ_σ is strictly convex since the continuous extension of a strictly convex function on $\mathbb{R}_{++}$ to $\mathbb{R}_+$ is also strictly convex. Henceforth, we will identify ψ_σ with its extended version.

First, assume $a_1 \cdot a_2 > 0$ and hence by Lemma O.4, $b_i = z \cdot a_i > 0$ for $i = 1, 2$. Clearly, $\lim \frac{p^n(G^n)}{\pi^n(G^n)} = \frac{b_1+b_3}{a_1+a_3}$ and $\lim \frac{p^n(H^n)}{\pi^n(H^n)} = \frac{b_2+b_4}{a_2+a_4}$. The monotonicity of the pricing kernel (Theorem 2) ensures that (i) $b_3 = 0$ implies $a_3 = 0$, (ii) $a_4 = 0$ implies $b_4 = 0$ and

$$\frac{b_1 + b_3}{a_1 + a_3} \geq z \geq \frac{b_2 + b_4}{a_2 + a_4}. \tag{11}$$

If the lemma is false, at least one of the inequalities above must be strict. Note that

$$\lim[V_\sigma(\tilde{S}^n) - V_\sigma(T^n)] = (b_1+b_3)\psi_\sigma\left(\frac{a_1+a_3}{b_1+b_3}\right) + (b_2+b_4)\psi_\sigma\left(\frac{a_2+a_4}{b_2+b_4}\right) - \lim b_3^n \cdot \psi_\sigma\left(\frac{a_3^n}{b_3^n}\right) - (b_1+b_2+b_4)\psi_\sigma\left(\frac{a_1+a_2+a_4}{b_1+b_2+b_4}\right)$$

and

$$\lim[V_\sigma(\tilde{S}^n) - V_\sigma(S^n)] = (b_1+b_3)\psi_\sigma\left(\frac{a_1+a_3}{b_1+b_3}\right) + (b_2+b_4)\psi_\sigma\left(\frac{a_2+a_4}{b_2+b_4}\right) - (b_1+b_2+b_3)\psi_\sigma\left(\frac{a_1+a_2+a_3}{b_1+b_2+b_3}\right) - \lim b_4^n \cdot \psi_\sigma\left(\frac{a_4^n}{b_4^n}\right)$$

Suppose that only the first inequality in equation (11) is strict. Then, (i) above yields $b_3 > 0$; otherwise $\frac{b_1+b_3}{a_1+a_3} = z$, a contradiction. Since $\frac{a_2+a_4}{b_2+b_4} = \frac{a_1+a_2+a_4}{b_1+b_2+b_4} = \frac{1}{z}$ and the extended ψ_σ is strictly convex, $\lim[V_\sigma(\tilde{S}^n) - V_\sigma(T^n)] < 0$, contradicting the limit optimality of $\tilde{c}^n$.

Next, assume that only the second inequality is strict, which given (ii) implies $a_4 > 0$. Hence, $\frac{a_1+a_3}{b_1+b_3} = \frac{a_1+a_2+a_3}{b_1+b_2+b_3} = \frac{1}{z}$. Then, if $b_4 > 0$, $\lim[V_\sigma(\tilde{S}^n) - V_\sigma(S^n)] < 0$ follows from the strict convexity of ψ_σ; if $b_4 = 0$, then, since $a_4 > 0$, we must have $\sigma < 1$; otherwise $\lim U_*^n = \infty$, contradicting feasibility. For $\sigma < 1$, $\lim_{b\to 0} b\psi_\sigma(\frac{a}{b}) = 0$ and hence, $\lim[V_\sigma(\tilde{S}^n) - V_\sigma(S^n)] < 0$ follows from the monotonicity of the pricing kernel and the fact that ψ_σ is strictly decreasing. Again, we have a contradiction.

Finally, assume that both of the inequalities above are strict and hence $a_3+b_3 > 0$, $a_4 + b_4 > 0$ and $\frac{a_1+a_3}{b_1+b_3} < \frac{a_2}{b_2} = \frac{a_1}{b_1} < \frac{a_2+a_4}{b_2+b_4}$. Then, (i) and (ii) above imply $b_3 > 0$ and $a_4 > 0$. As noted above, if $b_4 = 0$, $\sigma < 1$ in which case part (4) of Lemma 1–applied now to the extended ψ_σ to allow for the possibility that $a_3 = 0$–yields either $\lim V_\sigma(S^n) > \lim V_\sigma(\tilde{S}^n)$ or $\lim V_\sigma(T^n) > \lim V_\sigma(\tilde{S}^n)$; if $b_4 > 0$, part (3) of Lemma 1–again applied to the extended ψ_σ to allow for the possibility that $a_3 = 0$–yields the same conclusion, contradicting the limit optimality of $\tilde{c}^n$.

Next, assume $a_1 \cdot a_2 = 0$. If $a_1 = a_2 = 0$, then for all n large enough $a_3^n = a_4^n = 0$ and for $\epsilon > 0$ sufficiently small, we can choose ϵ-fragments $B_l^n, \hat{B}_l^n$ such that $i \in B_l^n, j \in A^n, j' \in B^n, i' \in \hat{B}_l^n$ implies $i < j < j' < i'$. Then, Theorem 3 and Lemma O.4 imply $\lim \frac{p^n(G^n)}{\pi^n(G^n)} = \lim \frac{b_1^n}{a_1^n} = z = \lim \frac{b_2^n}{a_2^n} = \lim \frac{p^n(H^n)}{\pi^n(H^n)}$, proving the lemma. Suppose $a_1 = 0$ and $a_2 > 0$. Then, arguing as above, we get $\lim \frac{p^n(G^n)}{\pi^n(G^n)} = \lim \frac{b_1^n}{a_1^n} = z$ and for n large enough $a_3^n = 0$. Hence, if the lemma is false, we must have $\lim \frac{p^n(H^n)}{\pi^n(H^n)} < z$, which implies that $\lim \frac{a_4^n}{b_4^n} > \frac{1}{z}$. Then,

$$\lim[V_\sigma(\tilde{S}^n) - V_\sigma(S^n)] = (b_2 + b_4)\psi_\sigma\left(\frac{a_2 + a_4}{b_2 + b_4}\right) - b_2\psi_\sigma\left(\frac{a_2}{b_2}\right) - b_4 \cdot \psi_\sigma\left(\frac{a_4^n}{b_4^n}\right) < 0$$

since ψ_σ is convex and $\frac{a_2}{b_2} \neq \frac{a_4}{b_4}$. The last display equation contradicts the limit optimality of $\tilde{c}^n$. Replacing S^n with T^n and making other obvious adjustments in the preceding argument yields the same contradiction for the $a_1 > 0$ and $a_2 = 0$ case.

□

Lemma O.6. $\lim_{n\to\infty}[\Sigma_{j_{**}^n}(\mu^n) - \Sigma_{j_*^n}(\mu^n)] = 0$.

Proof of Lemma O.6. Suppose there is a convergent subsequence along which $\lim[\Sigma_{j_{**}^n}(\mu^n) - \Sigma_{j_*^n}(\mu^n)] > 0$. Thus, there is a sequence of optimal consumptions c^n such that either $\lim[c_{j_{**}^n} - c_{j_*^n}]$ does not exist or exists and is not equal to 0. This implies that the sets G^n and H^n defined above are distinct and, by Lemma O.5, $\lim \frac{\pi^n(G^n)}{p^n(G^n)} = \lim \frac{\pi^n(H^n)}{p^n(H^n)}$. Let $S^n = S^{c^n}$ and x^n, y^n be the optimal consumption levels for cells G^n and H^n. Then, consumption in state j_*^n is x^n and consumption in state j_{**}^n is y^n.

With CRRA utility, $c(S_i^n) = \left(\frac{\pi^n(S_i^n)}{p^n(S_i^n)} \cdot \frac{p^n(S_j^n)}{\pi^n(S_j^n)}\right)^\sigma c(S_j^n)$ and therefore, by Lemma

O.5, $\lim x^n = \lim y^n$ whenever $\lim y_n$ exits. If $\lim y_n = \infty$ (along any subsequence), then, by Theorems 1 and 2, limit consumption in every cell of S^n prior to H^n (i.e., cells containing states $i < j$ for $j \in H^n$) must also go to infinity but since $p_l > 0$, the limit price is strictly greater than zero for at least one cell with infinite limit consumption. Since an infinite limit consumption at a positive limit price would violate the consumer's budget constraint, we have a contradiction. Hence, the sequence y^n is bounded and therefore $\lim |y^n - x^n| = 0$, contradicting $\lim[c_{j^n_{**}} - c_{j^n_*}] \neq 0$. □

Since the density of the aggregate endowment f is continuous on $[a, b]$, it is bounded, which implies $\lim_n [s^n_{j^n_{**}} - s^n_{j^n_*}] > 0$. Then, Lemma 6 and Lemma O.6 imply $p^n_{j^n_*} = 0$ for large n, which, in turn, implies $p^n_i = 0$ for all $i \in M^n(j^*) = 0$. Then, Lemma O.4 and $p(l) > 0$ yield a contradiction.