Online Appendix for "Management Quality and Firm Hierarchy in Industry Equilibrium"

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I Proof of Proposition 3

Proof. I fixed the number of layers and prove the first part of the proposition first. Equation (14) in the paper implies that the span of control increases at all layers when θ increases and its number of layers is unchanged. Second, as the wage defined in equation (9) of the paper is positively affected the span of control, wages increase at all layers. Third, the FOCs with respect to employment in equation (12) of the paper show that

$$\frac{w_i(q(\theta, T(\theta)), T(\theta))}{w_{i+1}(q(\theta, T(\theta)), T(\theta))} = \frac{1}{2}\frac{m_{i+1}(q(\theta, T(\theta)), T(\theta))}{m_i(q(\theta, T(\theta)), T(\theta))} = \frac{1}{2}x_i(q(\theta, T(\theta)), T(\theta))$$

for $T(\theta) > i \geq 1$. As the span of control increases at all layers, relative wages increase at all layers as well. Fourth, I prove the employment hierarchy that the number of workers is smaller in upper layers.[1] As I consider the employment hierarchy for workers, the minimum value for T is two. Equation (13) in the paper shows

$$\frac{m_{i+1}(q(\theta, T(\theta)), T(\theta))}{m_i(q(\theta, T(\theta)), T(\theta))} = 2\Big[\frac{q(\theta, T(\theta))}{2^{T(\theta)}}\Big]^{\frac{2^{T(\theta)-(i+1)}}{2^{T(\theta)}-1}} \geq \Big[\frac{q(\theta, T(\theta))}{2^{T(\theta)-1}}\Big]^{\frac{2^{T(\theta)-(i+1)}}{2^{T(\theta)}-1}},$$

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[1]This result will be used later.

as $T(\theta) \geq 2$ and $T(\theta) > i \geq 1$. Now, I show the following property of q_{T-1} that is the key step to prove the result of the employment hierarchy:[2]

$$\frac{q_{T-1}}{2^{T-1}} = \left[\frac{2^T - 1}{2^T - 2}\right]^{\frac{(2^{T-1}-1)(2^T-1)}{2^{T-1}}} 2^{\frac{1}{2^{T-1}}-1} > 1.$$

This is because

$$\left[\frac{2^T - 1}{2^T - 2}\right]^{\frac{(2^{T-1}-1)(2^T-1)}{2^{T-1}}} 2^{\frac{1}{2^{T-1}}-1}$$

increases in T for $T \geq 2$ and achieves its minimum value of 1.299 when $T = 2$. In total,

$$\frac{m_{i+1}(q(\theta, T(\theta)), T(\theta))}{m_i(q(\theta, T(\theta)), T(\theta))} \geq \left[\frac{q(\theta, T(\theta))}{2^{T(\theta)-1}}\right]^{\frac{2^{T(\theta)-(i+1)}}{2^{T(\theta)-1}}} > \left[\frac{q_{T(\theta)-1}}{2^{T(\theta)-1}}\right]^{\frac{2^{T(\theta)-(i+1)}}{2^{T(\theta)-1}}} > 1.$$

Therefore, the employment hierarchy holds for workers.

Now, I prove the second part of the proposition. Namely, I consider the case in which a small increase in θ (from $\theta_{T0,2} - \Delta$ to $\theta_{T0,2} + \Delta$) triggers the addition of one layer into the hierarchy. Note that $\theta_{T0,2}$ is the demand threshold where the firm switches from having $T0 + 1$ layers to having $T0 + 2$ layers.

As the change in the span of control is the key to prove this proposition, I prove that the span of control falls at all existing layers first. From equations (13) and (15) of the paper, I have

$$\left.\frac{m_i^*}{m_{i-1}^*}\right|_{T0} = 2\left[\frac{\beta A(\theta_{T0,2} - \Delta)^{\frac{1}{\sigma}}}{4b\psi 2^{\frac{T0}{\sigma}}}\right]^{\frac{\sigma 2^{T0-i}}{\sigma+(2^{T0}-1)}}.$$

and

$$\left.\frac{m_{i+1}^*}{m_i^*}\right|_{T0+1} = 2\left[\frac{\beta A(\theta_{T0,2} + \Delta)^{\frac{1}{\sigma}}}{4b\psi 2^{\frac{T0+1}{\sigma}}}\right]^{\frac{\sigma 2^{T0-i}}{\sigma+(2^{T0+1}-1)}},$$

where Δ is infinitesimally small. Thus, what I have to prove is that

$$Z(\theta_{T0,2}, T0) = \left[\frac{\beta A \theta_{T0,2}^{\frac{1}{\sigma}}}{4b\psi 2^{\frac{T0}{\sigma}}}\right]^{\frac{1}{\sigma+(2^{T0}-1)}}$$

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[2] q_T is defined in Definition 1 of the paper.

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decreases with $T0$ at $\theta_{T0,2}$. Calculation shows that

$$Sign\left[\frac{dZ(\theta_{T0,2}, T0)}{dT0}\right] = Sign\left[\ln 2\left[-2^{T0}\frac{(\ln\frac{\beta A\theta_{T0,2}^{\frac{1}{\sigma}}}{4b\psi} - \ln 2\frac{T0}{\sigma})}{2^{T0} + (\sigma - 1)} - \frac{1}{\sigma}\right]\right].$$

Obviously, if

$$\frac{\beta A\theta_{T0,2}^{\frac{1}{\sigma}}}{4b\psi 2^{\frac{T0}{\sigma}}} \geq 1,$$

then the proof is done. So, I only need to consider the case where

$$\frac{\beta A\theta_{T0,2}^{\frac{1}{\sigma}}}{4b\psi 2^{\frac{T0}{\sigma}}} < 1.$$

For this case, there is a lower bound on the above term due to the result that $\theta_{T0,2} > \theta_{T0,1}$. Thus, I only have to prove that

$$-2^{T0}\frac{(\ln\frac{\beta A\theta_{T0,1}^{\frac{1}{\sigma}}}{4b\psi} - \ln 2\frac{T0}{\sigma})}{2^{T0} + (\sigma - 1)} - \frac{1}{\sigma} < 0.$$

Based on Definition 1 in the paper, $\theta_{T0,1}$ can be rewritten as

$$MR(\theta_{T0,1}, q_{T0}) = A\beta\theta_{T0,1}^{\frac{1}{\sigma}}q_{T0}^{-\frac{1}{\sigma}} = MC(q_{T0}, T0 + 1) = b\psi 2^{2 - \frac{T0+1}{2^{T0+1}-1}}q_{T0}^{\frac{1}{2^{T0+1}-1}}.$$

Thus, I can solve $\theta_{T0,1}$ as

$$\theta_{T0,1} = \frac{(b\psi 2^{2 - \frac{T0+1}{2^{T0+1}-1}})^{\sigma}q_{T0}^{\frac{\sigma + (2^{T0+1}-1)}{2^{T0+1}-1}}}{(A\beta)^{\sigma}}.$$

Consequently, I have

$$2^{T0}\frac{(\ln\frac{\beta A\theta_{T0,1}^{\frac{1}{\sigma}}}{4b\psi} - \ln 2\frac{T0}{\sigma})}{2^{T0} + (\sigma - 1)} = \frac{2^{T0}(\sigma + (2^{T0+1} - 1))}{(\sigma + (2^{T0} - 1))\sigma(2^{T0+1} - 1)}\ln\left(\frac{q_{T0}}{2^{T0}}\right)$$
$$- \frac{2^{T0}}{(\sigma + (2^{T0} - 1))(2^{T0+1} - 1)}\ln 2.$$

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As I have shown that $\frac{q_{T0}}{2^{T0}} > 1$ for $T0 \geq 1$, it must be true that

$$-2^{T0}\frac{(\ln\frac{\beta A\theta_{T0,1}^{\frac{1}{\sigma}}}{4b\psi} - \ln 2\frac{T0}{\sigma})}{2^{T0} + (\sigma - 1)} - \frac{1}{\sigma} < \frac{2^{T0}}{(\sigma + (2^{T0} - 1))(2^{T0+1} - 1)}\ln 2 - \frac{1}{\sigma} < 0$$

for all $T0 \geq 1$. In total, I conclude that

$$-2^{T0}\frac{(\ln\frac{\beta A\theta_{T0,2}^{\frac{1}{\sigma}}}{4b\psi} - \ln 2\frac{T0}{\sigma})}{2^{T0} + (\sigma - 1)} - \frac{1}{\sigma} < 0$$

for all $\theta_{T0,2}$. As Δ is infinitesimally small, It must be true that

$$\frac{m_i^*}{m_{i-1}^*}\bigg|_{T0} > \frac{m_{i+1}^*}{m_i^*}\bigg|_{T0+1}$$

for all i and $T0 \geq 1$. Therefore, the span of control must fall at all existing layers when the firm adds a layer.

Next, as the wage at layer i is

$$w_i(\theta) = b\psi\frac{m_i(\theta, T)}{m_{i-1}(\theta, T)},$$

wages fall at all existing layers when the firm adds a layer.

Third, as the relative wage is proportional to the span of control or

$$\frac{w_{i-1}(\theta)}{w_i(\theta)} = \frac{m_i(\theta, T)}{2m_{i-1}(\theta, T)},$$

relative wages also fall at all existing layers when the firm adds a layer.

Finally, total employment increases discontinuously when the firm adds layer, as output increases discontinuously, and the span of control fall at existing layers. □

II Proof of Proposition 4

Proof. The strategy to prove this proposition is the following. First, I assume that the incentive compatible wage defined in equation (9) of the paper satisfies the constraint indicated in equation (27) of the paper in *every* labor submarket

and prove that there is a unique equilibrium with unemployment in every labor submarket. Second, I show that there is a non-empty set of parameter values within which the incentive compatible wage defined in equation (9) of the paper satisfies the constraint indicated in equation (27) of the paper in *every* labor submarket.

First, I redefine the equilibrium using three conditions. Substituting equation (11) of the paper into equation (21) of the paper leads to the homogeneous sector's employment expressed as

$$(1) \qquad L_h = \frac{(1-\gamma)A^\sigma P^{1-\sigma}}{\gamma p_h}.$$

Substituting the above equation and equation (23) of the paper into equation (25) of the paper yields the following labor market clearing condition:

$$(2) \qquad \frac{WP(\bar{\theta}, A, M) - \psi LD(\bar{\theta}, A, M)}{p_h} + \frac{(1-\gamma)A^\sigma P^{1-\sigma}}{\gamma p_h} = L.$$

Now, the equilibrium of the economy can be solved using three equations (i.e., equations (19), (20) of the paper, and (2)). As a result, I obtain value of three endogenous variables: $\bar{\theta}$, A and p_h.

Value of other equilibrium variables can be solved using θ, A and p_h derived above. First, the ideal price index is

$$(3) \qquad P = \frac{1}{p_h^{\frac{1-\gamma}{\gamma}}}$$

due to equation (4) of the paper. Second, the ideal price index defined in equation (5) of the paper can be re-expressed as

$$(4) \qquad P = \left(\int_{\theta=\bar{\theta}}^{\infty} \theta p(\theta)^{1-\sigma} M \frac{g(\theta)}{1-G(\bar{\theta})} d\theta \right)^{\frac{1}{1-\sigma}} \equiv P_1(\bar{\theta}, A) M^{\frac{1}{1-\sigma}}.$$

This is because prices charged by various firms in the CES sector only depend on A and θ. Thus, the mass of firms M can be derived by using equations (3), (4), and value of $\bar{\theta}$, A and p_h. Third, the aggregate income E can be derived by using equation (11) of the paper and value of A and P. Finally, the allocation of labor can be obtained by using equations (25) of the paper, equation (1), and value of A, P and p_h.

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Now I show why I can use three variables (i.e., $\bar{\theta}$, A and M) to derive both the aggregate wage payment and the number of employed workers in the CES sector. In equation (10) of the paper, only A and θ affect firm's optimal choices given value of exogenous parameters b and ψ. As firms endogenously choose whether or not to stay in the market, wage payment per active firm and employment per active firm are functions of $(A, \bar{\theta})$ only. Therefore, I can use three variables (i.e., A, θ and M) to derive both the aggregate wage payment and the number of employed workers in the CES sector.

Next, the following claim shows the existence and uniqueness of the equilibrium in the CES sector.

Claim 1. *There exists a unique equilibrium for the CES sector characterized by a unique pair of $(\bar{\theta}, A)$.*

Proof. I have two equilibrium conditions: the ZCP condition and the FE condition. I have two endogenous variables to be pinned down: the exit cutoff $\bar{\theta}$ and the adjusted market size A. Let us think about the ZCP condition first. The goal is to establish a negative relationship between $\bar{\theta}$ and A from this condition. Suppose A increases from A_0 to $A_1(> A_0)$ in equation (19) of the paper. If the exit cutoff $\bar{\theta}$ increased from $\bar{\theta}_0$ to $\bar{\theta}_1(\geq \bar{\theta}_0)$, the following contradiction would appear.

$$
\begin{aligned}
0 &= \Pi(\bar{\theta}_1, A_1) \equiv \pi(\bar{\theta}_1, T(\bar{\theta}_1, A_1), A_1) - f \\
&\geq \pi(\bar{\theta}_1, T(\bar{\theta}_0, A_0), A_1) - f \\
&> \pi(\bar{\theta}_0, T(\bar{\theta}_0, A_0), A_0) - f = \Pi(\bar{\theta}_0, A_0) = 0.
\end{aligned}
$$

The first inequality comes from firm's revealed preference on the number of layers, and the second inequality is due to the fact that firm's profit function defined in equation (16) of the paper strictly increases with both θ and A. Therefore, equation (19) of the paper leads to a negative relationship between $\bar{\theta}$ and A. Of course, when $\bar{\theta}$ approaches zero, A determined from equation (19) of the paper approaches infinity. And when $\bar{\theta}$ goes to infinity, A determined from equation (19) of the paper approaches zero.

Second, let me discuss the FE condition. The goal is to show that for all pairs of $(\bar{\theta}, A)$ that satisfy the ZCP condition, there is a positive relationship between these two variables determined by the FE condition. Suppose $\bar{\theta}$ decreases

from $\bar{\theta}_0$ to $\bar{\theta}_1(< \bar{\theta}_0)$ in equation (20) of the paper. If the adjusted market size A increased from A to $A_1(\geq A_0)$, the following result must be true.

$$
\begin{aligned}
f_e &= \int_{\bar{\theta}_1}^{\infty} \Pi(\theta, A_1) g(\theta) d\theta \\
&= \int_{\bar{\theta}_1}^{\bar{\theta}_0} \Pi(\theta, A_1) g(\theta) d\theta + \int_{\bar{\theta}_0}^{\infty} \Pi(\theta, A_1) g(\theta) d\theta \\
&> \int_{\bar{\theta}_0}^{\infty} \Pi(\theta, A_1) g(\theta) d\theta \\
&> \int_{\bar{\theta}_0}^{\infty} \Pi(\theta, A_0) g(\theta) d\theta \\
&= f_e,
\end{aligned}
$$

which is a contradiction. In the above derivation, I have implicitly used the ZCP condition which implies $\Pi(\theta, A_1) \geq 0$ for all $\theta \in [\bar{\theta}_1, \bar{\theta}_0]$. In total, the downward sloping ZCP curve and upward sloping FE curve intersects only once, and the intersection pins down a unique pair of $(\bar{\theta}, A)$ for the product market equilibrium.

Now, I prove the uniqueness. Suppose there were two pairs of $(\bar{\theta}, A)$ (i.e., $(\bar{\theta}_1, A_1)$ and $(\bar{\theta}_2, A_2)$) that satisfy both the ZCP condition and the FE condition. Without loss of generosity, let me assume that $\bar{\theta}_1 > \bar{\theta}_2$. Due to the property of the ZCP condition, it must be true that $A_1 < A_2$ which contradicts the positive relationship between $\bar{\theta}$ and A implied by the FE condition. Therefore, the equilibrium must be unique.

Finally, I prove the existence. For any $A \in (0, \infty)$, there exists a unique $\bar{\theta}(A)$ with $\bar{\theta}'(A) < 0$ determined by the ZCP condition. Furthermore, $\bar{\theta}(A)$ decreases continuously in A, as the firm's profit function with the optimal number of layers increases *continuously* with θ conditional on A. Therefore, among those $(A, \bar{\theta}(A))$ that satisfy the ZCP condition, there must be a pair of $(\bar{\theta}, A)$ that satisfies the FE condition. $\qquad \square$

Third, the following claim shows that there is a unique p_h that clears the labor market in general.

Claim 2. *When $\frac{\sigma-1}{\sigma} \neq \gamma$ and parameter values satisfy certain conditions, there exists a unique wage p_h that clears the labor market given that the product markets are cleared.*

Proof. First, let me decompose the total wage payment of the CES sector and

the number of workers employed in the CES sector into the following two parts:

$$WP(\bar{\theta}, A, M) = WPper(A, \bar{\theta}) * M$$

and

$$LD(\bar{\theta}, A, M) = LDper(A, \bar{\theta}) * M,$$

where "*per*" means per firm. Second, Substituting the above two expressions into equation (2) yields

(5) $$\frac{WCper(A, \bar{\theta}) - \psi LDper(A, \bar{\theta})}{p_h} M + \frac{(1 - \gamma)A^\sigma P^{1-\sigma}}{\gamma p_h} = L.$$

Next, substituting equation (4) into equation (4) of the paper leads to the expression of M in terms of p_h and $P_1(\bar{\theta}, A)$ as follows:

(6) $$M = p_h^{\frac{(1-\gamma)(\sigma-1)}{\gamma}} P_1(\bar{\theta}, A)^{\sigma-1}.$$

Finally, substituting equations (4) and (6) into equation (2) results in the following labor market clearing condition:

(7) $$\left[WCper(A, \bar{\theta}) - \psi LDper(A, \bar{\theta}) \right] P_1(\bar{\theta}, A)^{\sigma-1} + \frac{(1 - \gamma)A^\sigma}{\gamma} = p_h^{1 - \frac{(1-\gamma)(\sigma-1)}{\gamma}} L.$$

There exists a unique p_h that satisfies the above equation, as long as $\frac{(1-\gamma)(\sigma-1)}{\gamma} \neq$ 1.[3] Moreover, equilibrium p_h must satisfy the condition that

$$w_{min} \geq \psi(i) + p_h,$$

where w_{min} is the minimum wage offered in the CES sector. This puts a constraint on parameter values, which I will discuss soon.   □

There are three effects on the labor market when the price of the homogeneous good goes up. First, as p_h is the wage offered in the homogeneous sector, labor demand of firms in the homogeneous sector goes down. Second, as p_h is the outside option for workers entering the CES sector, the number of them must go down in order to make the worker who chooses to enter the CES sector earn

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[3]Note that I have implicitly used the product market equilibrium conditions to derive the above equation.

higher expected payoff.[4] These two negative effects on the labor demand are reflected by p_h that appears in the left hand side of equation (5). Finally, increasing market size due to a bigger p_h makes the aggregate income $E(= A^\sigma P^{1-\sigma})$ and the mass of firms M increase which pushes up the aggregate labor demand in the end. Therefore, whether or not the aggregate labor demand increases with p_h depends on whether or not the third (positive) effect dominates the first two negative effects. However, in either case, the aggregate labor demand is a *monotonic* function of p_h which assures the uniqueness of p_h that clears the labor market.

With Claim 1 and Claim 2 in hand, I only have to show that there is a non-empty set of parameter values within which the incentive compatible wage defined in equation (9) of the paper satisfies the constraint indicated in equation (27) of the paper in *every* labor submarket. In other words, I have to show that the minimum wage offered in the CES sector is weakly bigger than the wage offered in the homogeneous sector plus the disutility of exerting effort, or

$$w_{min} \geq \psi(i) + p_h.$$

First, note that labor endowment L does not affect wages and the minimum wage offered in the CES sector. This is because the solution of $(\bar{\theta}, A)$ in equilibrium does not depend on L, and wages offered by firms in the CES sector only depend on $(\bar{\theta}, A, b)$.[5] Second, equation (7) indicates that p_h approaches zero when L approaches zero and $\frac{\sigma-1}{\sigma} > \gamma$, and p_h approaches zero when L goes to infinity and $\frac{\sigma-1}{\sigma} < \gamma$. Therefore, I conclude that there must exist a small enough L such that

$$w_{min} - \psi > p_h,$$

when $\frac{\sigma-1}{\sigma} > \gamma$. Similarly, there must exist a big enough L such that

$$w_{min} - \psi > p_h,$$

when $\frac{\sigma-1}{\sigma} < \gamma$.

In total, I show that with restrictions on parameter values, there must exist a

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[4]Remember that the labor demand per firm in the CES sector is independent of p_h conditional on $(A, \bar{\theta})$.

[5]Labor endowment L affects the job-acceptance-rates in various labor submarkets and accordingly the *expected* wage of entering the CES sector.

unique equilibrium with unemployment in every labor submarket. The equilibrium is characterized by a unique quadruplet $(\bar{\theta}, M, p_h, E)$. $\square$