Amplification and Asymmetric Effects Without Collateral Constraint: Online Appendix

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In this online appendix, beside the additional derivations and analysis for the main paper (Appendices E-J), we present their variants (K, L and M) of the models presented in the main paper. The first variants are models with larger shares of land in the production function or with general CES production functions which generate larger amplification effects. The second variant is a continuous time version of the benchmark model. The Markov equilibrium in this model is similar to the Markov equilibria in He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014). Thus, we can use the algorithms in their papers to solve for the Markov equilibrium in our model. The last variant of the models is in discrete time, and we allow the households to produce using an inefficient technology. In this model, the households will start producing when the wealth of the entrepreneurs reaches its lower bound.

E. Proof of Lemma 1

We prove this result by contradiction. Suppose that in a competitive equilibrium with the optimal plan of the entrepreneurs $\{c_t, h_t, b_t\}_{t,s^t}$, there exists t^* and history $s_*^{t^*}$ such that $\omega_{t^*}(s_*^{t^*}) = q_{t^*}(s_*^{t^*})h_{t^*-1}(s_*^{t^*-1}) + b_{t^*-1}(s_*^{t^*-1}) < 0$. Given the formula for the profit maximization of the entrepreneurs and the definition of financial wealth ω_t, the budget constraint (5) can be re-written as

$$c_{t^*}(s_*^{t^*}) + \left(q_{t^*}(s_*^{t^*}) - \pi_{t^*}(s_*^{t^*}) \right) h_{t^*}(s_*^{t^*}) + p_{t^*}(s_*^{t^*})b_{t^*}(s_*^{t^*}) \leq q_{t^*}(s_*^{t^*})\omega_{t^*}(s_*^{t^*})H.$$

Pick a $\lambda > 1$, and consider an alternative trading and consumption plan $\left\{\tilde{c}_t, \tilde{h}_t, \tilde{b}_t\right\}_{t,s^t}$ for the entrepreneurs which is the same as the initial plan for $t < t^*$ but for $t \geq t^*$:

$$\left\{\tilde{c}_t, \tilde{h}_t, \tilde{b}_t\right\}_{t>t^*,s^t|s_*^{t^*}} = \{\lambda c_t, \lambda h_t, \lambda b_t\}_{t>t^*,s^t|s_*^{t^*}}$$

and

$$\begin{aligned}
\tilde{c}_{t^*}(s_*^{t^*}) &= \lambda c_{t^*}(s_*^{t^*}) - (\lambda-1)q_{t^*}(s_*^{t^*})\omega_{t^*}(s_*^{t^*}) > c_{t^*}(s_*^{t^*}) \\
\tilde{h}_{t^*}(s_*^{t^*}) &= \lambda h_{t^*}(s_*^{t^*}) \\
\tilde{b}_{t^*}(s_*^{t^*}) &= \lambda b_{t^*}(s_*^{t^*}).
\end{aligned}$$

It is easy to verify that $\left\{\tilde{c}_t, \tilde{h}_t, \tilde{b}_t\right\}_{t,s^t}$ satisfy all the constraints, including the no-Ponzi schemes condition. Since $c_t \geq 0$ for all $t', s^{t'}$, $\tilde{c}_t \geq c_t$ for all t, s^t and $\tilde{c}_{t^*}(s_*^{t^*}) > c_{t^*}(s^{t^*})$. This contradicts the property that $\{c_t, h_t, b_t\}_{t,s^t}$ is the optimal plan of the entrepreneurs. So by contradiction $\omega_t \geq 0$ for all t, s^t.

When $0 < \sigma_1 \leq 1$, we prove by contradiction that any feasible plans of the entrepreneurs must have positive financial wealth for all t, s^t. Suppose that in a competitive equilibrium, there is a feasible plan $\left\{\tilde{c}_t, \tilde{h}_t, \tilde{b}_t\right\}_{t,s^t}$ of the entrepreneurs with $\tilde{\omega}_{t^*}(s_*^{t^*}) = q_{t^*}(s_*^{t^*})\tilde{h}_{t^*-1}(s_*^{t^*-1}) + \tilde{b}_{t^*-1}(s_*^{t^*-1}) < 0$ for some t^* and history $s_*^{t^*}$. As argued above, the budget constraint (5) can be re-written as

$$\tilde{c}_{t^*}(s_*^{t^*}) + \left(q_{t^*}(s_*^{t^*}) - \pi_{t^*}(s_*^{t^*}) \right) \tilde{h}_{t^*}(s_*^{t^*}) + p_{t^*}(s_*^{t^*})\tilde{b}_{t^*}(s_*^{t^*}) \leq q_{t^*}(s_*^{t^*})\tilde{\omega}_{t^*}(s_*^{t^*})H.$$

Pick a $\lambda > 1$, and consider an alternative trading and consumption plan $\left\{\tilde{\tilde{c}}_t, \tilde{\tilde{h}}_t, \tilde{\tilde{b}}_t\right\}_{t,s^t}$ for the entrepreneurs which is the same as the initial plan for $t < t^*$ but for $t \geq t^*$:

$$\left\{\tilde{\tilde{c}}_t, \tilde{\tilde{h}}_t, \tilde{\tilde{b}}_t\right\}_{t>t^*,s^t|s_*^{t^*}} = \left\{\lambda\tilde{c}_t, \lambda\tilde{h}_t, \lambda\tilde{b}_t\right\}_{t>t^*,s^t|s_*^{t^*}}$$

and

$$
\begin{aligned}
\tilde{\tilde{c}}_{t^*}(s_*^{t^*}) &= \lambda\tilde{c}_{t^*}(s_*^{t^*}) - (\lambda-1)q_{t^*}(s_*^{t^*})\tilde{\omega}_{t^*}(s_*^{t^*}) > (\lambda-1)q_{t^*}(s_*^{t^*})\left(-\tilde{\omega}_{t^*}(s_*^{t^*})\right) \\
\tilde{\tilde{h}}_{t^*}(s_*^{t^*}) &= \lambda\tilde{h}_{t^*}(s_*^{t^*}) \\
\tilde{\tilde{b}}_{t^*}(s_*^{t^*}) &= \lambda\tilde{b}_{t^*}(s_*^{t^*}).
\end{aligned}
$$

Since $\tilde{c}_t \geq 0$ for all $t', s^{t'}$,

$$
\mathbb{E}_0\left[\sum_{t=0}^{\infty}\gamma^t\frac{(\tilde{\tilde{c}}_t)^{1-\sigma_1}-1}{1-\sigma_1}\right] \geq \mathbb{E}_0\left[\sum_{t=0}^{\infty}\gamma^t\frac{(\tilde{c}_t)^{1-\sigma_1}-1}{1-\sigma_1}\right]
$$
$$
+ \gamma^{t^*}\Pr(s_*^{t^*})\left(\frac{\left((\lambda-1)q_{t^*}(s_*^{t^*})\left(-\tilde{\omega}_{t^*}(s_*^{t^*})\right)\right)^{1-\sigma_1}-1}{1-\sigma_1} - \frac{(\tilde{c}_{t^*}(s_*^{t^*}))^{1-\sigma_1}-1}{1-\sigma_1}\right).
$$

When $\sigma_1 < 1$, since $\dfrac{\left(q_{t^*}(s_*^{t^*})\left(-\tilde{\omega}_{t^*}(s_*^{t^*})\right)\right)^{1-\sigma_1}}{1-\sigma_1} > 0$, by choosing λ sufficiently large, this alternative plan $\left\{\tilde{\tilde{c}}_{t'},\tilde{\tilde{h}}_{t'},\tilde{\tilde{b}}_{t'}\right\}_{t',s^{t'}}$ clearly delivers a higher value to the entrepreneurs compared to their equilibrium value while satisfying all the constraints, including the no-Ponzi schemes condition. This contradicts the fact that the competitive equilibrium plan is optimal. Therefore for all feasible plans, $\tilde{\omega}_t \geq 0$ for all t and s^t.

Similarly when $\sigma_1 = 1$, the last term becomes

$$
\gamma^{t^*}\Pr(s_*^{t^*})\left(\log(\lambda-1) + \log\left(q_{t^*}(s_*^{t^*})\right) + \log\left(-\tilde{\omega}_{t^*}(s_*^{t^*})\right) - \log\tilde{c}_{t^*}(s_*^{t^*})\right)
$$

which also delivers a higher value to the entrepreneurs compared to their equilibrium value for sufficiently high λ; this leads to a contradiction.

F. Steady State and Log-Linearization for Collateral Constraint Model

Becker (1980) shows that in a neoclassical growth model with heterogeneous discount factors, long run wealth concentrates on the most patient agents, in this case the households. However, in our model, due to the collateral constraint, the entrepreneurs can only pledge a fraction of their future wealth to borrow. Therefore, despite their lower discount factor, their wealth does not disappear in the long run. In particular, the model admits a long run steady state in the absence of uncertainty. In this subsection, we solve for the steady state in our model.

Suppose that there is no uncertainty, i.e., $A_t(s_t) \equiv A$. In steady state, all variables are constant, so we can omit the subscript t. For the ease of notation,

denote

$$(F.1) \qquad\qquad \gamma_e = m\beta + (1-m)\gamma,$$

as the average discount factor for the entrepreneurs' investment in land as in Iacoviello (2005). The first order condition in b'_t,

$$(F.2) \qquad\qquad b'_t : -p_t c'^{-\sigma_2}_t + \beta \mathbb{E}_t \left[c'^{-\sigma_2}_{t+1} \right] = 0.$$

In the steady state, $c'_t = c'_{t+1} = c'$, implies that $p = \beta$. Because $\gamma < \beta$, the entrepreneur wants to borrow as much as possible up to the collateral constraint. Indeed, the first order condition for b implies that the collateral constraint is strictly binding and the Lagrange multiplier μ on the constraint is strictly positive:

$$(F.3) \qquad\qquad \mu = (\beta - \gamma) \, c^{-\sigma_1} > 0.$$

Given that the collateral constraint is binding, we have $b = -mqh$.

From the first-order condition in h,

$$(F.4) \qquad (\pi_t - q_t)c^{-\sigma_1}_t + \mu_t m \mathbb{E}_t \left[q_{t+1} \right] + \gamma \mathbb{E}_t \left[q_{t+1} c^{-\sigma_1}_{t+1} \right] = 0,$$

in which the marginal profit

$$\pi_t = \upsilon A_t \left(\frac{h_t}{L_t} \right)^{\upsilon - 1}.$$

In the steady state, we have

$$(F.5) \qquad\qquad q = \frac{1}{1 - \gamma_e} \upsilon A h^{\upsilon - 1} L^{1 - \upsilon}.$$

The steady state version of the first order condition in L_t is

$$(F.6) \qquad\qquad w = (1 - \upsilon) A h^{\upsilon} L^{-\upsilon}.$$

From the budget constraint of the entrepreneurs, we obtain

$$c = \frac{(1 - \gamma)(1 - m)\upsilon}{1 - \gamma_e} A h^{\upsilon} L^{1 - \upsilon}.$$

Combining with the market clearing condition in the market for consumption good, we have $c' = A h^{\upsilon} L^{1 - \upsilon} - c$. The market clearing conditions in the housing market and labor market imply, $h' = H - h$ and $L' = L$.

So in the steady-state all the variables can be expressed as functions of two

unknowns, h and L. The first-order conditions on h' and L' of the households provide two equations that help determine the two unknowns:

$$-q \left(c' \right)^{-\sigma_2} + j \left(h' \right)^{-\sigma_h} + \beta q \left(c' \right)^{-\sigma_2} = 0$$

and

$$w \left(c' \right)^{-\sigma_2} = \left(L' \right)^{\eta - 1}.$$

For example, when $\sigma_2 = 1$ and $\sigma_h = 1$ as in Iacoviello (2005), the second equation, combined with the labor choice equation at the steady state (F.6) implies

$$L = \left[\frac{1 - \upsilon}{1 - \frac{(1 - \gamma)(1 - m)}{1 - \gamma_e} \upsilon} \right]^{\frac{1}{\eta}}.$$

From the first equation, h is determined as

$$\frac{h}{H} = \frac{\upsilon \left(1 - \beta \right)}{\upsilon \left(1 - \beta \right) + j[(1 - \gamma_e) - (1 - \gamma)(1 - m)\upsilon]}.$$

Given the steady state levels of h and L, the steady state level of wealth distribution is $\omega = \frac{(1 - m)h}{H}$.

Following Iacoviello (2005), we assume that the collateral constraint always binds around the steady state. Relative to the standard log-linearization technique, we need to solve for the shadow value of the collateral constraint, i.e., the multiplier μ_t, in addition to prices and allocations. Given a variable x_t, let $\widehat{x}_t$ denote the percentage deviation of x_t from its steady state value, i.e., $\widehat{x}_t = \frac{x_t - x}{x}$. Given the exogenous processes for the productivity shock $\hat{A}_t$, we solve for the endogenous variables $\widehat{c}_t$, $\widehat{c}'_t$, $\widehat{h}_t$, $\widehat{h}'_t$, $\widehat{b}'_t$, $\widehat{q}_t$, $\widehat{w}_t$, $\widehat{p}_t$, $\widehat{L}_t$, $\widehat{\mu}_t$ using the method of undetermined coefficients.[1] The following linear system characterizes the dynamics of the economy around the steady state:

$$(\hat{q}_t - \sigma_2 \hat{c}'_t) = -\sigma_h(1 - \beta)\hat{h}'_t + \beta \mathbb{E}_t[(\hat{q}_{t+1} - \sigma_2 \hat{c}'_{t+1})]$$

$$\hat{p}_t = \sigma_2(\hat{c}'_t - \mathbb{E}_t \hat{c}'_{t+1})$$

$$\hat{w}_t - \sigma_2 \hat{c}'_t = (\eta - 1)\hat{L}_t$$

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[1]Given the special 3-state structure of the stochastic shocks assumed in the main paper, we cannot directly use Dynare to solve for the log-linearized version of the model.

$$(1 - \gamma_e)[\hat{A}_t + (\upsilon - 1)\hat{h}_t + (1 - \upsilon)\hat{L}_t - \sigma_1\hat{c}_t] - (\hat{q}_t - \sigma_1\hat{c}_t)$$
$$+ m(\beta - \gamma)(\hat{\mu}_t + \mathbb{E}_t\hat{q}_{t+1}) + \gamma\mathbb{E}_t(\hat{q}_{t+1} - \sigma_1\hat{c}_{t+1})$$
$$= 0$$

$$\beta(\hat{p}_t - \sigma_1\hat{c}_t) = (\beta - \gamma)\hat{\mu}_t - \gamma\sigma_1\mathbb{E}_t(\hat{c}_{t+1})$$
$$\hat{w}_t = \hat{A}_t + \upsilon\hat{h}_t - \upsilon\hat{L}_t$$
$$\hat{b}'_t = \hat{h}_t + \mathbb{E}_t\hat{q}_{t+1}$$
$$c^*\hat{c}_t + c^{*\prime}\hat{c}'_t = Y^*[\hat{A}_t + \upsilon\hat{h}_t + (1 - \upsilon)\hat{L}_t]$$
$$h^*\hat{h}_t + h'^*\hat{h}'_t = 0$$

$$c'\hat{c}'_t + qh'(\hat{h}'_t - \hat{h}'_{t-1}) + \beta b'^*(\hat{p}_t + \hat{b}'_t)$$
$$= b'^*\hat{b}'_{t-1} + wL(\hat{w}_t + \hat{L}_t).$$

G. Complete Characterization for Model 3

With the parameters chosen in our calibration, in particular, $\sigma_1 = \sigma_2 = \sigma_h = 1$ and $\eta = 1$, we can characterize analytically the policy functions and wealth dynamics. The long-run wealth distribution is degenerated to some positive value ω^{*2} (stationary-state) instead of 0 as in the complete market model (Model 0). We use the guess-and-verify method to solve for the functional forms of the policy functions and wealth dynamics. We summarize the main results below:

R1. The wealth dynamics $\omega_{t+1}(\omega_t)$ is strictly increasing, and is independent of both s_t and s_{t+1}. In the long run, ω_t converges to a positive value ω^*.

R2. There exists a threshold value $\hat{\omega}$ such that the collateral constraints (20) bind when $\omega_t \leq \hat{\omega}$ and does not bind when $\omega_t > \hat{\omega}$. In addition, $\omega^* < \hat{\omega}$ and the collateral constraints are therefore always binding in the stationary state.

R3. The land holding of the entrepreneurs, h and labor demand L depend on ω_t only. In particular, they are independent of s_t.

R4. The entrepreneurs consume a constant fraction $1-\gamma$ of their financial wealth in each period:

(G.1) $$c(\omega_t, s_t) = (1 - \gamma)q(\omega_t, s_t)H\omega_t$$

This result is immediate due to the entrepreneurs' log utility.

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[2]With other commonly used parameter values, we find that the ergodic distribution has very small variance if not degenerated. For example, in an exercise with $\sigma_1 = \sigma_2 = \sigma_h = 2$ and $\eta = 1.5$, the mean entrepreneurs' wealth share ω is 0.0206, and the standard deviation is 8.7×10^{-5}.

The policy functions and wealth dynamics depend on in which region ω_t is. We first take $\hat{\omega}$ as given and discuss how its value is determined by the system later.

G.1. The Binding Region $\omega_t \leq \hat{\omega}$

When $\omega_t \leq \hat{\omega}$, we know the collateral constraints are binding. The wealth dynamics satisfies the following second-order sequential equation:

$$\text{(G.2)} \quad \left[\frac{\frac{jm\omega_{t+1}}{(1-m)-\omega_{t+1}} - \upsilon}{\upsilon(1-\gamma)+\gamma - \frac{\omega_{t+1}}{\omega_t}} \right] \frac{\omega_{t+1}}{\omega_t} - \frac{j(1-m)\omega_{t+1}}{(1-m)-\omega_{t+1}} = \beta \left[\frac{\frac{jm\omega_{t+2}}{(1-m)-\omega_{t+2}} - \upsilon}{\upsilon(1-\gamma)+\gamma - \frac{\omega_{t+2}}{\omega_{t+1}}} \right]$$

and the wealth dynamics $\omega_{t+1}(\omega_t)$ is the sequence with $\omega_\tau \to \omega^*$ as $\tau \to \infty$. By setting $\omega_t = \omega_{t+1} = \omega_{t+2}$, we get the expression for ω^* as

$$\omega^* = \frac{\upsilon(1-\beta)(1-m)}{j(1-\upsilon)(1-\gamma_e) + \upsilon(1-\beta)(1+jm)}.$$

In other words, denote the wealth mapping when $\omega_t \leq \hat{\omega}$ as $f(\omega_t)$, and $f^{(2)}(\omega_t) = f(f(\omega_t))$, and $f^{(n)}(\omega_t)$ as iterating $f(\cdot)$ n times forward, we can solve the mapping $f(\omega_t)$ with

$$\left[\frac{\frac{jmf(\omega_t)}{(1-m)-f(\omega_t)} - \upsilon}{\upsilon(1-\gamma)+\gamma - \frac{f(\omega_t)}{\omega_t}} \right] \frac{f(\omega_t)}{\omega_t} - \frac{j(1-m)f(\omega_t)}{(1-m)-f(\omega_t)} = \beta \left[\frac{\frac{jmf^{(2)}(\omega_t)}{(1-m)-f^{(2)}(\omega_t)} - \upsilon}{\upsilon(1-\gamma)+\gamma - \frac{f^{(2)}(\omega_t)}{f(\omega_t)}} \right]$$

for any $\omega_t \leq \hat{\omega}$, with the restriction that $f^{(\infty)}(\omega_t) = \omega^*$.[3]

To obtain equation (G.2), we define[4]

$$\hat{\mu}_t(s_{t+1}) = \frac{\mu_t(s_{t+1})c_{t+1}(s_{t+1})}{\Pr(s_{t+1}|s_t)},$$

where $\mu_t(s_{t+1})$ denote the Lagrangian multiplier on the collateral constraints (20).

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[3]We can solve for $f(\omega_t)$ using (G.2) and a shooting algorithm.
[4]We drop ω_t from the equations below to save some notations, but it should be clear that all the variables are functions of ω_t (or ω_{t+1}).

The FOCs in h and ϕ can be written as

(G.3) $\qquad \dfrac{q_t}{c_t} = \dfrac{\upsilon A_t h_t^{\upsilon-1} L_t^{1-\upsilon}}{c_t} + \displaystyle\sum_{s_{t+1}|s^t} \left[(m\hat{\mu}_t(s_{t+1}) + \gamma) \Pr(s_{t+1}|s_t) \dfrac{q_{t+1}(s_{t+1})}{c_{t+1}(s_{t+1})} \right]$

(G.4)
$\qquad \dfrac{p_t(s_{t+1})}{c_t} = [\hat{\mu}_t(s_{t+1}) + \gamma] \dfrac{\Pr(s_{t+1}|s_t)}{c_{t+1}(s_{t+1})}.$

From (G.4) and the FOC in bond holding of the household, we have

$$p_t(s_{t+1}) = \Pr(s_{t+1}|s_t)\left[\hat{\mu}_t(s_{t+1}) + \gamma\right] \frac{c_t}{c_{t+1}(s_{t+1})}$$

$$= \beta \Pr(s_{t+1}|s_t) \frac{c_t'}{c_{t+1}'(s_{t+1})}.$$

Thus

(G.5) $\qquad\qquad\qquad \dfrac{c_t}{c_t'} = \dfrac{\beta}{\hat{\mu}_t(s_{t+1}) + \gamma} \dfrac{c_{t+1}(s_{t+1})}{c_{t+1}'(s_{t+1})}.$

We conjecture that result R3 above holds and verify it later, i.e., h and labor demand L only depend on ω_t. With the parameters in Table 1, labor supply condition (15) becomes

$$c_t' = wage_t = (1-\upsilon)A(s_t)\left(\frac{h(\omega_t)}{L(\omega_t)}\right)^{\upsilon}.$$

From the market clearing condition $c_t + c_t' = Y_t$ and the output function,

$$c_t = Y_t - c_t' = [L(\omega_t) - (1-\upsilon)]A(s_t)\left(\frac{h(\omega_t)}{L(\omega_t)}\right)^{\upsilon}$$

(G.6) $\qquad\qquad \dfrac{c_t}{c_t'} = \dfrac{L(\omega_t) - (1-\upsilon)}{1 - \upsilon}.$

Combine this result with (G.5), we have $\hat{\mu}_t(s_{t+1}) \equiv \hat{\mu}_t > 0 \ \forall s_{t+1}$. Because the collateral constraint (20) binds for all s_{t+1},

$$\omega_{t+1} = \frac{q_{t+1}h_t - mq_{t+1}h_t}{q_{t+1}H} = (1-m)\frac{h_t}{H}.$$

From (G.1) and (G.3),

$$\frac{1}{(1-\gamma)H\omega_t} = \frac{\upsilon L_t}{[L_t-(1-\upsilon)]h_t} + (m\hat{\mu}_t+\gamma)\frac{1}{(1-\gamma)(1-m)h_t}.$$

From (13) and (G.5),

$$\left[\frac{1}{(1-\gamma)H\omega_t} - \frac{\hat{\mu}_t+\gamma}{(1-\gamma)(1-m)h_t}\right]\frac{L_t-(1-\upsilon)}{1-\upsilon} = \frac{j}{H-h_t}.$$

From the last two equations, we can solve for the two unknowns $\hat{\mu}_t$ and L_t:

(G.7)
$$\hat{\mu}_t = \frac{(1-m)h_t}{H\omega_t} - \frac{j(1-\upsilon)(1-\gamma)(1-m)h_t}{[L_t-(1-\upsilon)](H-h_t)} - \gamma$$

(G.8)
$$L_t = \frac{\gamma(1-\upsilon) - (1-\upsilon)\frac{(1-m)h_t}{H\omega_t} + \frac{jm(1-\upsilon)(1-\gamma)h_t}{(H-h_t)}}{\upsilon(1-\gamma) + \gamma - \frac{(1-m)h_t}{H\omega_t}}.$$

Furthermore, with (G.6), equation (G.5) can be written as

(G.9)
$$\frac{L_t-(1-\upsilon)}{(1-\upsilon)} = \frac{\beta}{\hat{\mu}_t+\gamma}\frac{L_{t+1}-(1-\upsilon)}{(1-\upsilon)}.$$

After replacing $\hat{\mu}_t$, L_t and L_{t+1} using (G.7) and (G.8), and then replacing h_t with $h_t = \frac{H\omega_{t+1}}{1-m}$, we get (G.2).

After getting $f(\omega_t)$, the other variables can be derived by the following equations and $L(\omega_t)$ by (G.8). Our conjure, Results R1-R3 above, therefore is verified when $\omega_t \leq \hat{\omega}$. In addition,

$$h(\omega_t) = \frac{Hf(\omega_t)}{1-m}$$

$$c'(\omega_t, s_t) = (1-\upsilon)A(s_t)\left(\frac{h(\omega_t)}{L(\omega_t)}\right)^{\upsilon}$$

$$c(\omega_t, s_t) = Y(\omega_t, s_t) - c'(\omega_t, s_t)$$

$$= A(s_t)[L(\omega_t) - (1-\upsilon)]\left(\frac{h(\omega_t)}{L(\omega_t)}\right)^{\upsilon},$$

and

$$q(\omega_t, s_t) = \frac{c(\omega_t, s_t)}{(1-\gamma)H\omega_t}.$$

The threshold value $\hat{\omega}$ can be derived by setting $\hat{\mu}_t = 0$ in (G.7) and solving for the corresponding wealth level.

G.2.   The Unbinding Region $\omega_t > \hat{\omega}$

If $\omega_t > \hat{\omega}$, we know the collateral constraints are not binding. When $\omega_t > \hat{\omega}$, the wealth dynamics can be derived by

$$(G.10) \qquad \frac{1}{\omega_{t+1}} - \frac{\gamma}{\omega_{t+2}} - \upsilon(1-\gamma) = \frac{\beta}{\gamma}\left[\frac{1}{\omega_t} - \frac{\gamma}{\omega_{t+1}} - \upsilon(1-\gamma)\right].$$

This relationship holds in Model 0 with complete markets as well. In Model 0 we use guess-and-verify method to show that $\omega_{t+1} = \frac{\gamma\omega_t}{\beta - \upsilon(\beta-\gamma)\omega_t}$ with $\omega_t \to 0$. However, in Model 3 with collateral constraint, the result is not the same for the following reason. Starting at some wealth level $\omega_t > \hat{\omega}$, the wealth dynamics will eventually enter the binding region with $\omega_t \leq \hat{\omega}$ and then the wealth dynamics will switch to equation (G.2). Denote $t + \tau$ as the first period that the wealth dynamics enter the binding region, such that $\omega_{t+\tau} \leq \hat{\omega}$, and $\omega_{t+\tau-1} > \hat{\omega}$. Given $f(\omega)$ as the solution for (G.2), based on (G.9) and (G.14), we have

$$\omega_{t+\tau-1} = \frac{1}{\frac{\frac{\gamma}{\beta}(1-\gamma)(\upsilon+j)(1-\upsilon)}{[L(\omega_{t+\tau})-(1-\upsilon)]} + \frac{\gamma}{\omega_{t+\tau}} + \upsilon(1-\gamma)}$$

by which we can solve $\omega_{t+\tau-1}$. The expression of $f(\omega)$ is embedded in $L(\omega_{t+\tau})$ from (G.14) with $\omega_{t+\tau+1} = f(\omega_{t+\tau})$. Keep iterating backward, we have the full path of wealth dynamics starting from any $\omega_t > \hat{\omega}$. Denote the wealth dynamics when $\omega_t > \hat{\omega}$ as $\omega_{t+1} = g(\omega_t)$.

To derive (G.10), from the FOC of h_t (10) and (G.1), we have

$$(G.11) \qquad \frac{1}{(1-\gamma)H\omega_t} = \frac{\upsilon L_t}{[L_t - (1-\upsilon)]h_t} + \frac{\gamma}{(1-\gamma)H\omega_{t+1}}.$$

From the FOC of ϕ_t and ϕ'_t,

$$(G.12) \qquad \frac{c_t}{c'_t} = \frac{\beta}{\gamma}\frac{c_{t+1}}{c'_{t+1}}.$$

Using this equation, the FOC of h_t and h'_t as well as (G.6), we obtain

$$(G.13) \qquad h_t = \frac{\upsilon L_t}{j(1-\upsilon) + \upsilon L_t}H.$$

Combining (G.11) with (G.13), we arrive at

$$(G.14) \qquad L_t - (1-\upsilon) = \frac{(1-\gamma)(\upsilon+j)(1-\upsilon)}{\frac{1}{\omega_t} - \frac{\gamma}{\omega_{t+1}} - \upsilon(1-\gamma)}.$$

On the other hand, from (G.12) and (G.6), we also have

$$L_t - (1 - v) = \frac{\beta}{\gamma}[L_{t+1} - (1 - v)].$$

Combining the two equations above, we get (G.10).

After finding the expression for wealth dynamics $g(\omega_t)$, we can get the expressions for L_t and h_t from (G.13) and (G.14). So far we have verified that our conjecture, Results R1-R3, holds. The expressions for c_t, c_t' and q_t are given by the same formulae as in the binding region.

G.3. The Degenerated Wealth Level

We get the degenerated wealth level ω^* based on (G.2). In particular, at $\omega_t \equiv \omega^*$, all the endogenous variables $x(\omega_t, s_t)$ can be written as $x(s_t)$, and the holdings of Arrow securities $\phi(\omega_t, s_t, s_{t+1})$ as $\phi(s_t, s_{t+1})$ and their prices as $p(s_t, s_{t+1})$. These variables take the following values:

$$h^* = \frac{v(1-\beta)H}{j(1-v)(1-\gamma_e) + v(1-\beta)(1+jm)}$$

$$L^* = 1 - \frac{jmh^*}{H - h^*}$$

(G.15) $$\frac{q(s_t)}{A(s_t)} \equiv \frac{1}{1-\gamma_e} v(h^*)^{v-1}(L^*)^{1-v}, \quad \forall s_t = 1, 2, \ldots S.$$

The price of Arrow securities is

(G.16) $$p(s_t, s_{t+1}) = \frac{\beta \Pr(s_{t+1}|s_t)c'(s_t)}{c'(s_{t+1})} = \frac{\beta \Pr(s_{t+1}|s_t)A(s_t)}{A(s_{t+1})}.$$

Denote the Lagrangian multipliers as $\mu(s_t, s_{t+1})$, the first-order-condition of $\phi_t(s_t, s_{t+1})$ is

$$-\frac{p(s_t, s_{t+1})}{c(s_t)} + \mu(s_t, s_{t+1}) + \frac{\gamma \Pr(s_{t+1}|s_t)}{c(s_{t+1})} = 0.$$

Thus we have

$$\mu(s_t, s_{t+1}) = \Pr(s_{t+1}|s_t)\left[\frac{\beta A(s_t)}{c(s_t)A(s_{t+1})} - \frac{\gamma}{c(s_{t+1})}\right]$$

$$= \frac{\Pr(s_{t+1}|s_t)(\beta - \gamma)}{A(s_{t+1})(h^*)^v(L^*)^{-v}[L^* - (1-v)]} > 0$$

which suggests imperfect risk sharing due to the financial frictions.

At $\omega_t = \omega^*$, the allocations and prices share the same features with those under complete markets in the long run, Model 0, that allocations are constant over time and across states and land price is proportional to aggregate productivity. However the two equilibria are not the same. For example, using the expression above, we can show that $h^* < h^\infty$, the limit value of the entrepreneurs' land holding in complete market economy (Model 0) in Appendix A.

G.4.   Amplification with Unexpected Shocks

We can think of Kiyotaki and Moore (1997) as a counter part of our complete hedging model in their deterministic setting. Because the transition in Kiyotaki and Moore is deterministic, the number of state-contingent assets - the bond - other than land is equal to the number future states, i.e., one in Kiyotaki and Moore. So why is it that there is amplification in Kiyotaki and Moore but not in ours at the stationary state? The difference comes from whether shocks are expected or unexpected. In Kiyotaki and Moore, the first negative shock is not expected, i.e., an MIT shock (but the subsequent reactions of the economy to the shock are rationally anticipated by the agents in the economy). However, in the complete hedging version of our economy, all shocks are expected and hedged against by the agents in the economy using state-contingent securities, therefore there is no amplification. Following this reasoning, if we introduce an unexpected shock to the complete hedging model, we should also observe amplification as in Kiyotaki and Moore.

The unexpected shocks that we consider are the following. Assume that at time $t = 0$, the economy is at the stationary state $\omega_0 = \omega^*$. The households and the entrepreneurs trade contingent claims for $t = 1$, i.e. the entrepreneurs hold

$$\{\phi_0(s_1)\}_{s_1 \in \{G,N,B\}} \, .$$

We also know that the collateral constraint is binding at $\omega_0 = \omega^*$, therefore

$$\phi_0(s_1) + mq_1(s_1)h_0 = 0$$

for $s_1 \in \{G, N, B\}$. Now at $t = 1$ and $s_1 = B$, we assume that productivity is lower than what is expected by the agents, i.e. $A' = 0.96 < A_1(B) = 0.97$. In this sense the shock is unexpected. We assume that the dynamics of A_t goes back to the rational expectation version after this unexpected shock.

Figure G.1 shows the amplification in land price and output:

$$\frac{q' - q_1(B)}{q_1(B)(A_1(B) - A')} \quad \text{and} \quad \frac{Y' - Y_1(B)}{Y_1(B)(A_1(B) - A')}$$

as functions of the unexpected decrease in productivity: $A_1(B) - A'$ in percentage. As in Kiyotaki and Moore, the response of land price and output to the unexpected decline in productivity is amplified. For example when productivity declines 1% more than expected, i.e. $A' = 0.96$, land price declines by $3 \cdot 1\% = 3\%$ more than when the decrease is expected, i.e., land price decreases by 6% as opposed to 3%. Similarly output declines by $2 \cdot 1\% = 2\%$ more than when the decrease is expected, i.e., output decreases by 5% instead of 3%.

FIGURE G.1. AMPLIFICATION AFTER UNANTICIPATED SHOCKS
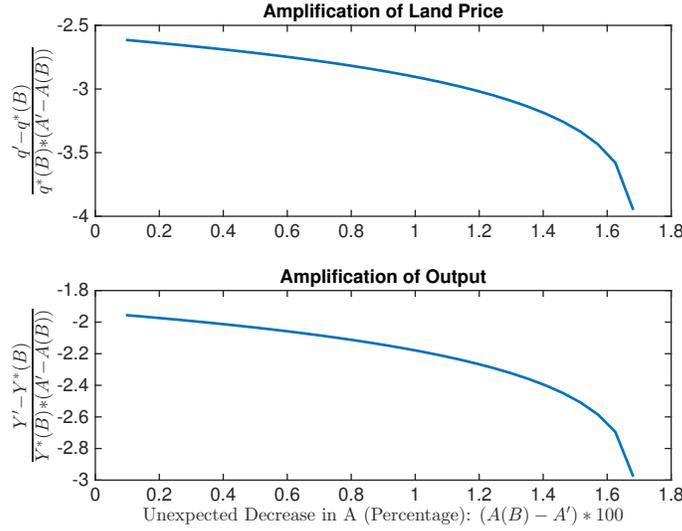
H. Models with Incomplete Markets and Exogenous Borrowing Constraint

In this appendix, we examine an alternative model with exogenous borrowing constraint. The model has the same ingredients as the ones in the benchmark model except for the following exogenous borrowing constraint instead of the collateral constraint:

$$(\text{H.1}) \qquad b_t \geq -\overline{B}.$$

The borrowing constraint $\overline{B}$ is chosen exogenously and we study the behavior of the model when we vary $\overline{B}$. Let μ_t denote the Lagrangian multiplier associated to this borrowing constraint. The first-order conditions with respect to h_t and b_t in the maximization problem of the entrepreneurs are

$$(\text{H.2}) \qquad (\pi_t - q_t)c_t^{-\sigma_1} + \gamma \mathbb{E}_t[q_{t+1}c_{t+1}^{-\sigma_1}] = 0$$

and

(H.3) $$-p_t c_t^{-\sigma_1} + \mu_t + \gamma \mathbb{E}_t \left[c_{t+1}^{-\sigma_1} \right] = 0,$$

and the complementary-slackness condition is satisfied:

(H.4) $$\mu_t \left(b_t + \overline{B} \right) = 0.$$

Other conditions are the same as in the benchmark model.

We first solve for the steady state of this model. From equations (F.2), (H.3), and (H.2), we have

$$p = \beta,$$
$$\mu = (\beta - \gamma) c^{-\sigma_1}$$
$$q = \frac{1}{1 - \gamma} \upsilon A h^{\upsilon - 1} L^{1 - \upsilon}.$$

This expression of land price is different from equation (F.5) in the discount factor γ instead of γ^e. Since $\gamma < \gamma^e$, given the same steady state level of h and L, the land price is lower under the exogenous borrowing constraint than under the endogenous collateral constraint since land loses its value as collateral in the former. Consequently, these two models do not share the same steady state.

We can use the global nonlinear solution method presented in Appendix B to solve for the Markov equilibrium in this economy. Table H.1 is the counterpart of Table 3 in the paper for this model with the exogenous borrowing constraint. In particular, Row 3 of Table H.1 corresponds to Row 4 (Model 2) in Table 3 in the paper, in which there is no upper bound on the borrowing of the entrepreneurs. To make the experiments comparable, In Row 4 of Table H.1, the exogenous limit is equal to the unconditional expectation of the entrepreneur's borrowing in the economy with the endogenous collateral constraint of the benchmark model. When we tighten the exogenous constraint, the amplification and asymmetric effects are actually reduced. At first sight, this result seems counter-intuitive. However, this result is in line with the discussions in Mendoza (2010) and Kocherlakota (2000). The exogenous borrowing constraint reduces the borrowing of the entrepreneurs, and thus reduces the net worth effect in the benchmark incomplete markets model. An important difference here compared to Kocherlakota (2000) is that under uncertainty, it is possible to have infinite exogenous borrowing constraint in the incomplete markets model (the entrepreneurs limit themselves from borrowing too much because of the precautionary saving motive). Infinite exogenous borrowing constraint leads to the maximal net worth effect, thus significant amplification and asymmetric effects.

Table H.1—Average land price and output changes in normal state, 3% shock

|	Land price		Output	
Type of Friction	Expansion	Recession	Expansion	Recession
Complete Markets (Model 0)	3.00%	3.00%	3.00%	3.00%
Collateral Constraint (Model 1a)	3.60%	-4.22%	3.25%	-3.65%
Incomplete markets ($\bar{B} = \infty$, Model 2)	3.46%	-3.81%	3.21%	-3.43%
Incomplete markets ($\bar{B} = 2.73$)	3.47%	-3.52%	3.24%	-3.29%

I. Volatility Paradox

In Brunnermeier and Sannikov (2014), the authors discover an interesting feature called the *volatility paradox*, i.e., with incomplete financial market, lower exogenous risk can lead to higher endogenous risk. Their intuition is that lower exogenous risk encourages the entrepreneurs to borrow more and use higher leverage which result in larger output/asset price declines during crisis. We first show that in Model 1a with collateral constraint, our fully nonlinear solution does not exhibit the volatility paradox because the entrepreneurs' borrowing capacity is constrained. As we decrease the size of the exogenous shocks, the binding probability goes to 1 and the nonlinear solution becomes closer to the log-linear solution, and both converge to the steady state with no endogenous risk. See Table I.1 for the details. The stationary distribution is thus degenerated, as shown in Figure I.1 with different sizes of exogenous shocks.

Table I.1—Probabilities of Binding Collateral Constraint

| Δ	Expansion	Normal	Recession	Overall
0.1%	100%	100%	100%	100%
0.5%	5.3%	90.7%	99.9%	78.7%
1%	0.2%	42.9%	95.5%	44.4%
2%	0%	4.9%	68.9%	14.2%
3%	0%	0.2%	45.1%	7.2%
4%	0%	0%	29.1%	4.6%
5%	0%	0%	19.2%	3.0%

On the contrary, in Model 2 with incomplete market but no collateral constraint, we recover the volatility paradox. The stationary distribution is not degenerated when the size of the exogenous shock goes to 0, as shown in Figure I.2.

Figure I.1. Stationary Distributions with Collateral Constraint, $s_t = N$ (Model 1a)
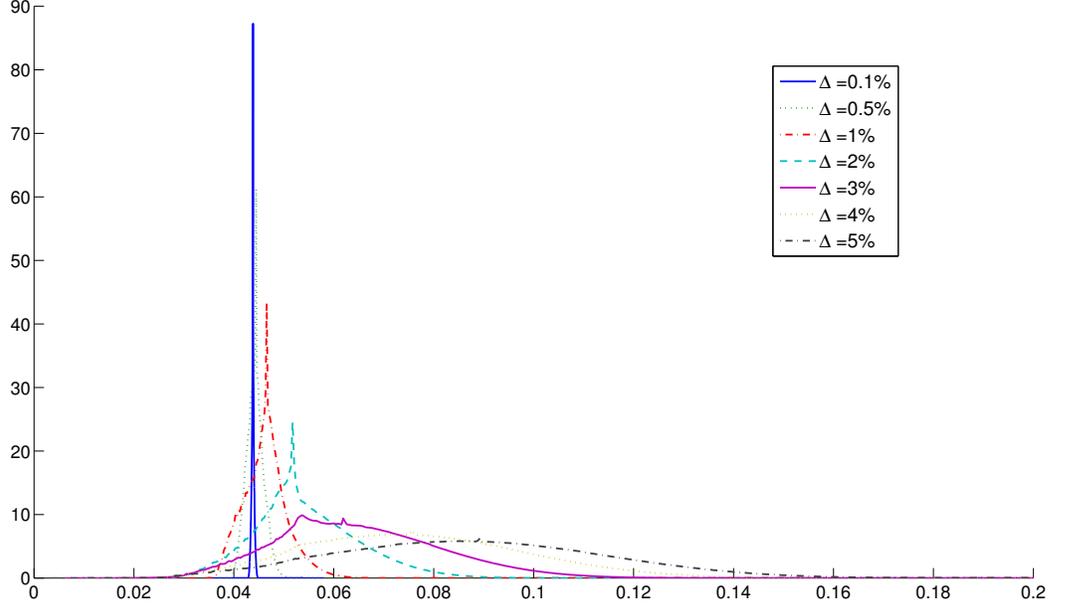
J. Welfare Analysis

We compare the welfares of the entrepreneurs and the households in different economies by computing the consumption equivalence, i.e., the percentage increase (decrease) of the consumptions required to make the agents indifferent between the benchmark economy (Model 1a) and the targeted economy. The way we do this is to assume an unexpected switch from the benchmark (B) economy to the targeted (T) economy by keeping agents' portfolios unchanged, in this case housing h and bond b. We compute the consumption equivalence for each possible value of wealth, and then present the average welfare effects using the ergodic distribution in the benchmark economy. Below are the steps to derive the results:

1) Pick a wealth level ω_t^B and aggregate shock s_t, find the portfolio $h^B(\omega_t^B, s_t)$ and $b^B(\omega_t^B, s_t)$ in the benchmark economy using its policy functions. Then fix (h, b), and use the fixed-point iteration to locate the corresponding wealth level ω_t^T in the targeted economy: $\omega_t^T = \frac{q^T(\omega_t^T, s_t)h + b}{q^T(\omega_t^T, s_t)H}$, where $q^T(\omega_t^T, s_t)$ is the land pricing function in the targeted economy.

2) Using ω_t^T from step 1, we can get the value functions for the entrepreneurs

FIGURE I.2. STATIONARY DISTRIBUTIONS WITHOUT COLLATERAL CONSTRAINT, $s_t = N$ (MODEL 2)
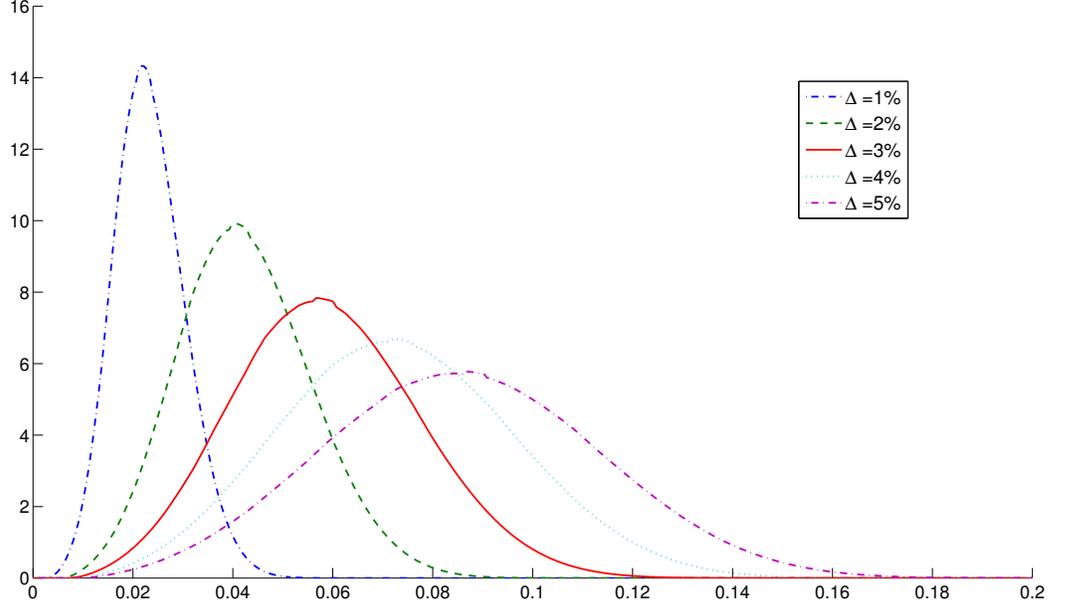
and the households in the targeted economy: $V_e^T(\omega_t^T, s_t)$ and $V_h^T(\omega_t^T, s_t)$. We would like to compute the percentage increase of consumption for the entrepreneurs, $\rho^e(\omega_t^B, s_t)$ and the households $\rho^h(\omega_t^B, s_t)$ from the benchmark economy to the targeted economy to make them indifferent. With log utilities as in the paper, we can get

$$\rho^e(\omega_\tau^B, s_\tau) = \exp\left[(1-\gamma)\left(\hat{V}^e(\omega_t^T, s_t) - V^e(\omega_t^B, s_t)\right)\right] - 1$$
$$\rho^h(\omega_\tau^B, s_\tau) = \exp\left[(1-\beta)\left(\hat{V}^h(\omega_t^T, s_t) - V^h(\omega_t^B, s_t)\right)\right] - 1$$

3) Computing average welfare effects $\bar{\rho}^e$ and $\bar{\rho}^h$ across the ergodic distribution in the benchmark economy.

The results for welfare analysis are listed in Table J.1. In particular, Model 1a generates the highest welfare for the entrepreneurs, followed by the partial hedging model (Model 4), incomplete markets model (Model 2) and the complete hedging model (Model 3), and their welfare in the complete market economy (Model 0) comes as the lowest. The entrepreneurs have lower discount factor so

TABLE J.1—AVERAGE CONSUMPTION INCREASE FROM THE BENCHMARK ECONOMY

Type of Friction	Entrepreneur	Households
Incomplete Markets (Model 2)	-3.97%	0.09%
Partial Hedging (Model 4)	-3.49%	0.16%
Complete Hedging (Model 3)	-6.18%	0.25%
Complete Market (Model 0)	-11.93%	0.42%

they always want to consume early. In the complete market case (Model 0), their consumptions are zero in the long run. Model 1a yields the highest welfare to the entrepreneurs because the collateral constraint prevents them from pledging all their future incomes and thus they are able to maintain the highest average wealth and consumption among all models.

As an example, we look into the 6.18% welfare loss of the entrepreneurs when moving from Model 1a to Model 3. First, we observe that with the parameters in Table 1 which assume log utility for the entrepreneurs, the following relationship holds in all the market structures:

$$(J.1) \qquad\qquad\qquad c_t = (1 - \gamma)q_t H \omega_t$$

which means the entrepreneurs consume a constant fraction $1 - \gamma$ of their wealth.

Based on (J.1), welfare reduction for the entrepreneurs could come from two sources: a reduction in their wealth share ω_t, or/and a reduction in land price. We first compare the entrepreneur's wealth in these two economies. In the left figure in Figure J.1, we plot the stationary distribution in Normal state for Model 1a and Model 3. The vertical line is the degenerated wealth level in Model 3. We can see that the wealth share in Model 3 is significantly lower than the wealth in Model 1a. Actually the average wealth share of Model 1a is 0.0645, which is about 50% more than the degenerated wealth level in Model 3. Entrepreneurs in the complete hedging economy are simply poorer. On the other hand, average land price in Model 1a is 2% lower than in Model 3. Thus if we directly apply equation (J.1) the welfare loss should be around 30%. Why is it only 6.18% as in Table J.1? The reason is by keeping the entrepreneurs' portfolio (h, b) and switching from Model 1a to Model 3, the entrepreneurs get higher wealth distribution ω_t immediately. This is because their debt holding b is negative in the support of the ergodic distribution, and land price is higher in Model 3 than in Model 1a as shown by Figure 9. Thus after this transition the entrepreneurs enjoy higher consumption in the first several periods but as time passes by their wealth and consumption get lower gradually. Lastly, ω_t varies less in Model 3 than in Model 1a and together with the concavity of the entrepreneurs' utility function, this lower variance further reduces the loss from lower ω_t.

The ranking of the welfare comparison for the households is in the opposite order. Welfare of the households in complete markets economy (Model 0) is the

FIGURE J.1. COMPARE MODEL 1A AND MODEL 3, $s_t = N$
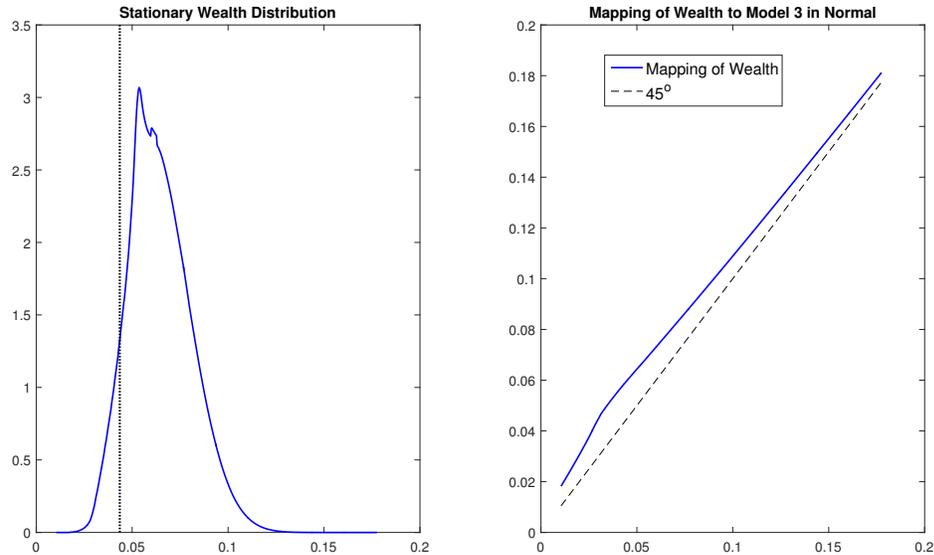
highest followed by the complete hedging economy (Model 3). Since the collateral constraints are binding in Model 3, the allocation of resources is less efficient than in the complete markets economy. Besides, the entrepreneurs survive in Model 3 and share part of the output. The model with incomplete markets and collateral constraints (Model 1a) generates the lowest welfare for the households because of the financial frictions.

We also observe that, moving away from Model 1a, the relative decrease in the entrepreneurs' welfare is significantly larger than the relative increase in the households' welfare. This finding suggests that financial markets reforms aiming at reducing aggregate volatility might not be popular among some agents in the economy.

Now assume that one still wants to reform the financial markets from Model 1a to the other models, when is the best time to implement the reform to get support from the households and face the least resistance from the entrepreneurs? To answer this question, we look at the degree of welfare changes upon reform depending on the state in which they are implemented. Figure J.2 shows the change in the entrepreneurs' welfare if we move from Model 1a to Model 3. The entrepreneurs' welfare decreases the least when ω is high and in expansions. To the extent that the entrepreneurs need to be compensated to support the reform from Model 1a to Model 3, it might be the cheapest to enact such a reform in expansions and when ω is high.

FIGURE J.2. CHANGES IN ENTREPRENEURS' WELFARE FROM MODEL 1A TO MODEL 3
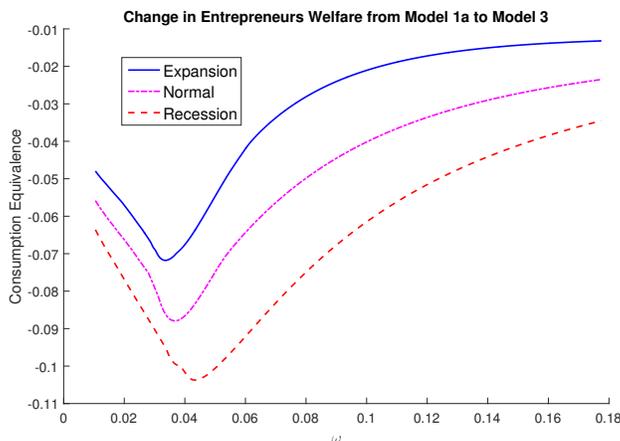
K. Alternative Models with Larger Amplification

In the paper, we show that market incompleteness plays a dominant role in amplifying shocks instead of collateral constraints. We would like to see whether this result would likely carry over to settings with larger amplification. Kocherlakota (2000) suggests two ways to generate larger amplification: (1) by increasing the weight of collateralizable assets (in this case, land) in the production function and (2) by using specifications of the production function in which the elasticity of substitution between land and labor is less than 1. We explore both cases and show that our main result still holds.

K.1. Models with Larger Shares of Land in the Production Function

In the benchmark model, the calibrated share of the land in the production function is $\upsilon = 0.041$. We vary the value of υ to see the relative importance of market incompleteness and collateral constraint in amplifying shocks. Other parameter values are not changed. The results are listed in Table K.1 with two types of financial market structures: incomplete markets structure without borrowing constraint, and incomplete markets with collateral constraint.

First, we can see that as υ gets larger, the amplification effect is stronger. For example, with 3% negative TFP shock and collateral constraint, land price will decrease by -3.95% when $\upsilon = 0.03$, but when $\upsilon = 0.07$, land price decreases by -4.76%. Second, as υ gets larger, market incompleteness by itself explains a larger part in the total amplification effect. With $\upsilon = 0.03$ and a 3% negative TFP shock, output decreases by -3.28% with market incompleteness alone compared to -3.47% in the economy with collateral constraint. As a result, about 60% $\left(\frac{3.28\%-3\%}{3.47\%-3\%}\right)$ of the amplification effect is generated by market incompleteness.

Table K.1—Average land price and output changes in normal state, 3% shock

	Land price		Output	
Type of Friction	Expansion	Recession	Expansion	Recession
Incomplete Markets ($\upsilon = 0.03$)	3.34%	-3.61%	3.11%	-3.28%
Collateral Constraint ($\upsilon = 0.03$)	3.45%	-3.95%	3.14%	-3.47%
Incomplete Markets ($\upsilon = 0.041$)	**3.46%**	**-3.81%**	**3.21%**	**-3.43%**
Collateral Constraint ($\upsilon = 0.041$)	**3.60%**	**-4.22%**	**3.25%**	**-3.65%**
Incomplete Markets ($\upsilon = 0.05$)	3.56%	-3.97%	3.24%	-3.49%
Collateral Constraint ($\upsilon = 0.05$)	3.72%	-4.40%	3.29%	-3.71%
Incomplete Markets ($\upsilon = 0.06$)	3.66%	-4.13%	3.30%	-3.60%
Collateral Constraint ($\upsilon = 0.06$)	3.85%	-4.59%	3.37%	-3.82%
Incomplete Markets ($\upsilon = 0.07$)	3.76%	-4.29%	3.37%	-3.70%
Collateral Constraint ($\upsilon = 0.07$)	3.96%	-4.76%	3.45%	-3.93%

When $\upsilon = 0.07$, this ratio increases to 75% ($\frac{3.70\%-3\%}{3.93\%-3\%}$). The comparison suggests that in situations with larger amplification, market incompleteness quantitatively plays a larger role in amplifying shocks, and collateral constraint plays a smaller role. This result should not be surprising because with larger amplification, the entrepreneurs' precautionary saving motive is stronger and they will borrow less to avoid regions where the collateral constraint is binding. Although in regions with binding collateral constraint, higher υ generates stronger amplification, the probability of the economy to enter those regions are lower. The binding probabilities with different υ are listed in Table K.2. On average collateral constraint has less effect in amplifying shocks as υ gets larger.

Table K.2—Probabilities of Binding Collateral Constraint

υ	Expansion	Normal	Recession	Overall
0.03	0%	3.64%	51.11%	10.52%
0.041	0%	0.2%	45.08%	7.22%
0.05	0%	0%	40.07%	6.29%
0.06	0%	0%	35.72%	5.61%
0.07	0%	0%	32.13%	5.04%

K.2. Models with CES Production Function

In this subsection, instead of using a Cobb-Douglas production, we consider a constant-elasticity-of-substitution (CES) production function with the elasticity between land and labor is set as $\phi = 0.8$, as in equation (K.1). Reducing land input would have a larger effect on output than the Cobb-Douglas case since more

labor would be required to neutralize the effect due to lower substitutability.

$$(K.1) \qquad Y_t = A_t \left(v h_t^{\frac{\phi-1}{\phi}} + (1-v) L_t^{\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}},$$

With CES production function, j - household's weight of housing in utility, and ν - share of housing in the production function, are re-calibrated using the steady-state equilibrium. Again, we use two moments, the average value of residential real estate over total output and the average value of commercial real estate over total output from the Flow of Funds, to calibrate the two parameters. We solve for the steady-state with CES production function and find (j, v) such that the model's moments match exactly the empirical moments. We find $j = 0.071$, and $v = 0.042$. The amplification effects are listed in Table K.3. For convenience, we replicate the results with Cobb-Douglas production function for comparison.

TABLE K.3—AVERAGE LAND PRICE AND OUTPUT CHANGES WITH CES PRODUCTION, 3% SHOCK

	Land price		Output	
Type of Friction	Expansion	Recession	Expansion	Recession
Collateral Constraint, (Cobb-Douglas, Model 1a)	3.60%	−4.22%	3.25%	−3.65%
Incomplete Markets, (Cobb-Douglas, Model 2)	3.46%	-3.81%	3.21%	-3.43%
Collateral Constraint, CES	3.58%	-4.26%	3.30%	-3.80%
Incomplete Markets, CES	3.43%	-3.83%	3.24%	-3.53%

We find that, in general, using a CES production function with the elasticity of substitution less than 1 increases the amplification effects. For example, with collateral constraint, output decreases by -3.80% with a negative 3% TFP shock compared to -3.65% with Cobb-Douglas production function. We find market incompleteness still explains a large part in amplifying shocks. In particular, 66% of the response in land price and also 66% of the response in output to negative shocks are explained by market incompleteness.

L.   Continuous Time Model

In this section, we present a continuous time version of the incomplete markets model, Model 2, in the main paper. The equilibrium in this model can be solved using the methods developed in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013).

L.1. Economic Environment

The environment is exactly the same as in the main paper, except that time is continuous.

The aggregate productivity A_t follows a diffusion process

$$dA_t = \mu^A(A_t)dt + \sigma^A(A_t)dZ_t,$$

where Z_t is a standard Brownian motion.

The dynamics of land price q_t is determined by

(L.1) $$dq_t = \mu_t^q dt + \sigma_t^q dZ_t,$$

where μ_t^q and σ_t^q are endogenously determined.

Given the process of land price q_t, interest rate r_t and wage rate w_t, households maximize a lifetime utility function given by

(L.2) $$\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left\{ \frac{(c_t')^{1-\sigma_2} - 1}{1 - \sigma_2} + j\frac{(h_t')^{1-\sigma_h} - 1}{1 - \sigma_h} - \frac{1}{\eta}(L_t')^\eta \right\} dt \right],$$

where $\mathbb{E}_0[.]$ is the expectation operator, $\rho > 0$ is the discount rate, c_t' is consumption at time t, h_t' is the holding of land. L_t' denotes the hours of work. Households can trade in the market for land as well as a non state-contingent bond market that yield instantaneous rate of return r_t. Let b_t' denote the holding of non state-contingent bond of the households. The households are subject to the following constraint on the dynamics of their net worth n_t':

(L.3)
$$dn_t' = h_t' \left(\mu_t^q dt + \sigma_t^q dZ_t \right) + r_t b_t' dt - c_t' dt + w_t L_t' dt$$
$$n_t' = q_t h_t' + b_t'.$$

Given land in the utility function of households, implicitly $h_t' \geq 0$. The households are also subject to the No-Ponzi scheme conditions

$$\lim{}_{T\to\infty}\mathbb{E}_t \left[e^{-\int_t^T r_s ds} \left(q_T h_T + b_T \right) \right] \geq 0.$$

Entrepreneurs use a Cobb-Douglas constant-returns-to-scale technology that uses land and labor as inputs. They produce consumption good Y_t according to

(L.4) $$Y_t = A_t h_t^v L_t^{1-v},$$

where A_t is the aggregate productivity which depends on the aggregate state s_t, h_t is real estate input, and L_t is labor input.

The entrepreneurs discount the future at the discount rate $\gamma > \rho$. The en-

trepreneurs maximize

(L.5)
$$\mathbb{E}_0 \left[\int_0^\infty e^{-\gamma t} \frac{(c_t)^{1-\sigma_1} - 1}{1 - \sigma_1} dt \right]$$

subject to the following constraint on the dynamic of their net worth, n_t:

$$dn_t = h_t \left(\mu_t^q dt + \sigma_t^q dZ_t \right) + \pi_t h_t + r_t b_t dt - c_t dt$$
(L.6)
$$n_t = q_t h_t + b_t.$$

Output Y_t is produced by combining land and labor using the production function (L.4). Given the production function of the entrepreneurs, we have implicitly $h_t \geq 0$. The entrepreneurs are also subject to the no-Ponzi schemes condition:

$$\lim_{T \to \infty} \mathbb{E}_t \left[e^{- \int_t^T r_s ds} \left(q_T h_T' + b_T' \right) \right] \geq 0.$$

L.2.   Equilibrium

The definition of the sequential competitive equilibrium for this economy is standard and is a continuous version of the competitive equilibrium in the main paper.

DEFINITION 1:   *A competitive equilibrium is sequences of prices* $\{q_t, r_t, w_t\}_{t=0}^\infty$ *and allocations* $\{c_t, h_t, b_t, L_t, c_t', h_t', b_t', L_t'\}$ *such that (i)* q_t *follows the dynamics* (L.1), *(ii) the* $\{c_t', h_t', b_t', L_t'\}$ *maximize* (L.2) *subject to the dynamic net worth constraint* (L.3) *and the no-Ponzi condition and* $\{c_t, h_t, b_t, L_t\}$ *maximize* (L.5) *subject to dynamic net worth constraint* (L.6) *and the no-Ponzi condition, and production technology* (L.4) *given* $\{q_t, r_t, w_t\}$ *and initial asset holdings* $\{h_0, b_0, h_0', b_0'\}$; *(iii) land, bond, labor, and good markets clear:* $h_t + h_t' = H$, $b_t + b_t' = 0$, $L_t = L_t'$, $c_t + c_t' = Y_t$.

Let ω_t denote the *normalized financial wealth* of the entrepreneurs

$$\omega_t = \frac{n_t}{q_t H}$$

and ω_t' denote the normalized financial wealth of the households:

$$\omega_t' = \frac{n_t'}{q_t H}.$$

By the land and bond market clearing conditions, we have $\omega_t' = 1 - \omega_t$ in any competitive equilibrium. Therefore in order to keep track of the normalized financial wealth distribution between the entrepreneurs and the households, (ω_t, ω_t'),

in equilibrium, we only need to keep track of ω_t. To simplify the language, we use the term *wealth distribution* for normalized financial wealth distribution.

Markov equilibrium is also a continuous version of Markov equilibrium in the main paper.

DEFINITION 2: *A Markov equilibrium is a competitive equilibrium in which prices and allocations at time t, depend only on the wealth distribution at time t, ω_t and the exogenous state A_t.*

This Markov equilibrium is the same as the Markov equilibria in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013). We can use the algorithms in their papers to solve for the Markov equilibrium in our paper. However, there are two important differences. First, because of persistent TFP shocks (instead of I.I.D. depreciation shocks to capital stock as in Brunnermeier and Sannikov (2014) or I.I.D. return shocks on dividend as in He and Krishnamurthy (2013)), in our Markov equilibrium, we need to keep track of both wealth distribution and the current exogenous shock. Second, we allow for general utility functions, instead of linear or log utility functions as in Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013).

M. Multiple Production Technologies

In this section, we simplify our model in the main paper in the spirit of Brunnermeier and Sannikov (2014) as well as Cordoba and Ripoll (2004) and Kiyotaki and Moore (1997). We assume that the households do not have a preference for housing but have access to an inefficient production function

$$(M.1) \qquad Y' = \underline{A}\left(h'\right)^{\upsilon'}\left(L'\right)^{1-\upsilon'}$$

with $\min(A_t) < \underline{A} < \max\left(A_t\right)$. Households maximize a lifetime utility function given by

$$(M.2) \qquad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{(c_t')^{1-\sigma_2}-1}{1-\sigma_2} - \frac{1}{\eta}(\widetilde{L}_t)^\eta \right\},$$

where $\widetilde{L}_t$ is the hours of work (instead of L_t' in the benchmark model). The budget constraint of the households is

$$(M.3) \qquad c_t' + q_t(h_t' - h_{t-1}') + p_t b_t' \leq b_{t-1}' + w_t \widetilde{L}_t + Y_t' - w_t L_t'.$$

Housing is no longer in the utility function of households, so we have to impose explicitly

$$h_t' \geq 0.$$

Given their land holding at time t, h_t, the households choose labor demand L'_t to maximize profit

$$\max_{L'_t} \left\{ Y'_t - w_t L'_t \right\}$$

subject to their production technology (M.1) if they produce. The first order condition with respect to L'_t implies

$$w_t = (1 - \upsilon')\underline{A} \left(h'_t \right)^{\upsilon'} (L'_t)^{-\upsilon'},$$

i.e. $L'_t = \left(\frac{(1-\upsilon')A_t}{w_t} \right)^{\frac{1}{\upsilon'}} h'_t$ and profit

$$Y'_t - w_t L'_t = \pi'_t h'_t$$

where $\pi'_t = \upsilon'\underline{A} \left(\frac{(1-\upsilon')\underline{A}}{w_t} \right)^{\frac{1-\upsilon'}{\upsilon'}}$ is profit per unit of land for the households.

DEFINITION 3: *A competitive equilibrium is sequences of prices $\{p_t, q_t, w_t\}_{t,s^t}$ and allocations $\left\{ c_t, h_t, b_t, L_t, c'_t, h'_t, b'_t, L'_t, \widetilde{L}_t \right\}_{t,s^t}$ such that (i) $\left\{ c'_t, h'_t, b'_t, L'_t, \widetilde{L}_t \right\}_{t,s^t}$ maximize (M.2) subject to the budget constraint (M.3), the production technology (M.1), $h'_t \geq 0$ and the No-Ponzi condition, and $\{c_t, h_t, b_t, L_t\}_{t,s^t}$ maximize (L.5) subject to the entrepreneur's budget constraint, production technology (L.4), and a collateral constraint, given $\{p_t, q_t, w_t\}$ and some initial asset holdings*

$$\left\{ h_{-1}, b_{-1}, h'_{-1}, b'_{-1} \right\};$$

(ii) land, bond, labor, and good markets clear: $h_t + h'_t = H$, $b_t + b'_t = 0$, $L_t + L'_t = \widetilde{L}_t$, $c_t + c'_t = Y_t$.

In the steady state, the entrepreneurs own the whole supply of land. Outside the steady state, we can use the definition of Markov equilibrium and the associated solution method as in the benchmark model in the main paper. The main difference between the solution of this model and the benchmark model is that at the natural borrowing limit for the entrepreneurs, i.e. $\omega_t = 0$, the households start producing using their inefficient production function. This puts a higher lower bound on the total output as well as land price compared to when the households cannot produce as in the main paper.

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