Bargaining with Arrival of New Traders*

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Abstract

We study a general model of dynamic bargaining between a seller and a privately informed buyer, with arrival of exogenous events. Events can represent arrival of competing buyers (or sellers) or release of information. We characterize a unique limit of stationary equilibria of these games as the time between offers goes to zero. We show that the possibility of arrivals leads to equilibrium dynamics that violate the Coase conjecture. Even in the limit, there is considerable delay of trade in equilibrium, the seller slowly screens out buyers with higher valuations. The limit equilibria are very tractable, allowing us to establish many comparative statics and utilize the model to answer many applied questions. For example, we show that in some applications when buyer valuations fall, average transaction prices drop and the time on the market gets longer. If the arrival rate is high enough, then the division of surplus and equilibrium dynamics are driven more by the relative chances of a competing trader arriving on either side of the market than on the relative discount factors. Finally, even when multiple buyers can arrive, the expected time to trade is a non-monotonic function of the arrival rate.

1 Introduction

In bargaining theory (and in practice) outside options always play an important role. In many economic situations the outside option is to wait for new developments. Maybe another agent will show up and offer better terms of trade, maybe new information will arrive reducing the information asymmetry, etc. Traders compare these potential benefits to costs of delaying trade and the risk that over time the opportunity to trade might disappear (or that unfavorable information will arrive etc.). In this paper we study a general bargaining model that captures outside options of this nature and characterizes their impact on the dynamics of bargaining.

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We start with an abstract, general bargaining game. Our first main finding is that the introduction of this kind of outside options makes the Coase conjecture result disappear. That is, even in stationary equilibria, such outside options lead to inefficient delay. Furthermore, we show that as bargaining frictions disappear (i.e. as the length of periods shrinks to zero allowing the seller to change asking price frequently) the equilibrium becomes incredibly tractable. This allows us to obtain a very clear understanding of the equilibrium dynamics. In particular, the seller's inability to commit to prices decreases his payoffs to his expected outside option, showing that some of the Coasian forces/features are still present.

In the second part of the paper we show how the model can be use in different, more applied problems. For example, we show environments in which a weaker market (characterized by a weaker distribution of buyer's value) implies a lower average transaction prices and a longer time to trade.

In particular, we analyze a model of a thin market in which a seller bargains over a price of a non-divisible asset with a privately informed buyer. As bargaining evolves over time, an external event can happen that influences the competitive positions of the players: a second buyer can arrive, a second seller can arrive or public information may get released. During bargaining the seller (the uninformed player) makes the offers and we characterize limits of stationary equilibria as bargaining frictions disappear.

For example, suppose you have put your house on the market. So far only one buyer has expressed interest. He informs you that your original price is too high and asks you to reduce it. What do you consider before responding? Out of many factors that you may take into account two important ones are: 1) How likely it is that other serious buyers will show up in the short run. 2) How likely it is that if you wait to reduce the price, the current buyer will find another house and "disappear". In fact, these risks in many situations are likely to be more important in evaluating the relative costs and benefits of delay than the standard discounting costs that play a crucial role in many bargaining models. As we show in our applications that intuition gets confirmed by our model.

New traders arriving over time is a common feature of many markets (housing, labor, financial markets to name a few). A key characteristic of such markets is that trade/bargaining over price takes time and the bargaining dynamics are heavily influenced by the market conditions. For example, the asking price of a house takes time to drop, and how long it takes may depend on whether it is a "sellers' market" or a "buyers' market." In this paper we try to shed some light on how external conditions affect the dynamics of bargaining.

The benchmark to compare our results to is that in any stationary equilibrium of our bargaining game but without the arrivals, as bargaining frictions disappear trade is immediate (or the two parties don't trade at all) – a result shown by Fudenberg, Levine and Tirole (1985) and Gul, Sonnenschein and Wilson (1986) (henceforth FLT and GSW, respectively). We show that in that same environment, allowing for external events leads to slow trade in negotiations.

An important ingredient of our model is that the seller's expected payoff upon arrival of the external event is an increasing function of the buyer's type. That assumption is not satisfied if the external event represents an opportunity that another buyer could arrive and pay a fixed price, say $100,000. However, if the second buyer also has a privately known valuation, then we would expect the seller to make the two buyers compete, for example by running an auction for the asset. The revenues from the auction would then depend on the valuation of the first buyer, and the expected payoffs would satisfy our key assumption. This observation together with our first main result implies that delay is a very natural consequence of bargaining in the context of a thin market, where the possibility of arrival of new buyers (and subsequent competition between them) leads to delay in reaching a compromise.

The positive relationship between the current buyer type and the endogenous expected payoff upon arrival is key to the delay. In equilibrium, as time progresses without the buyer accepting a price the seller becomes more and more pessimistic about the value of the buyer.[1] That progressive pessimism makes the seller want to offer lower and lower prices over time (and we assume the seller cannot commit not to reduce prices in the future). The buyer expects the prices to drop soon, which reduces his reservation price. For any reservation prices ("demand function") in a game without arrivals the seller wants to run down the demand function as quickly as possible. But since the buyer anticipates it, in equilibrium the reservation prices collapse.

In contrast, in a game with arrivals, if the reservation prices are too low, it is not optimal for the seller to run down the demand curve: he is better off waiting for the arrival. But (assuming immediate trade is efficient) waiting cannot be an equilibrium either because then the buyer's reservation prices would increase and the seller would be better off trading with them immediately (which cannot be an equilibrium either, as we already argued). Therefore, in equilibrium there has to be a balance: the seller cannot have incentives to either speed up the bargaining or to slow it down. As the seller runs smoothly down the demand curve, he becomes more and more pessimistic about the value of the buyer and, importantly, about his outside option. That reduces the incentives to wait for the arrival and makes it optimal to reduce the price a small amount and so on...

Through most of the paper we assume that only one event can arrive. A general analysis of markets with many opportunities to trade is complicated, and one way to interpret our general model is that the reduced-form payoffs upon arrival of the event are not really terminal payoffs in the game but rather represent expected continuation payoffs from possibly continued bargaining.

However, one may be interested in analyzing explicitly a multiple-arrival game, at the very least to verify which of our assumptions for the reduced-form payoffs hold. Therefore, in the last part of the paper, we analyze a game with infinitely many potential buyers arriving according to some Poisson rate. Although some of the original assumptions turn out to be violated and the dynamics are a little bit different, we show that the main intuitions and results are robust: the seller still slowly screens down the demand and the inability to commit still forces his expected payoffs down to the outside

[1] This is known as the Skimming Property and it happens because higher types face higher costs of delaying the trade and hence trade sooner in equilibrium.

option and allows us to pin down the equilibrium prices.

There are many other papers that show equilibrium delay in bargaining. For example, delay occurs in a model with two sided private information about fundamentals and overlap in values (e.g. Cramton 1984, Chatterjee and Samuelson 1987 or Cho 1990), with irrational players (Abreu and Gul 2000), with higher order beliefs (Feinberg and Skrzypacz 2005) with disagreement about continuation play (Yildiz 2004), with correlation between seller's cost and buyer's value (Evans 1989, Vincent 1989 and Deneckere and Liang 2006) or with the possibility that players can commit to delay (Admati and Perry 1987). The novelty is that even in the simplest FLT/GSW framework adding only the possibility of arrival of the second buyer leads to delay. Such arrivals are a natural possibility of real-life transactions and hence can be a common reason for delay.

The main intuition why there is delay in equilibrium is closely related to the bargaining models with interdependent values, as presented by Evans (1989), Vincent (1989) and Deneckere and Liang (2006). In these models the seller does not know the cost of supplying the asset and the buyer has private information that determines both the value and the cost. If the lowest possible value is below the average cost, delay must occur. The reasoning is that otherwise the individual rationality constraint of at least one of the agents would be violated: to satisfy buyer's IR with no delay implies prices not higher than the lowest buyer's value, but then the seller would lose money on average.

In our model the seller knows his physical cost of delivering the asset, but the buyer has information about the (endogenous) opportunity cost: by trading today the seller forgoes the option to trade after an event arrives. As long as the post-arrival seller profits depend on the buyer type, the delay is necessary. That link leads to the surprising observation that even if the seller's cost and values of potential buyers are all independent, the equilibria look very similar in our paper and in the papers with interdependent types.

The main difference between our model and the previous work on bargaining with interdependent values is that the interdependence is created by market conditions and hence we can obtain interesting insights about thin markets. On the methodological side, we focus on the continuous time limit, which greatly simplifies the analysis and allows us to provide many additional predictions.

There are also other bargaining papers that allow for arrival of new traders (in particular buyers) without obtaining equilibrium delay. This difference in results is caused by different assumptions about post-arrival competition, mainly that the post-arrival profits do not depend on the current buyer type. For example, Inderst (2003) only allows the seller to choose whether to keep the original buyer or switch to the new one but if he does switch, then the value of the original value is irrelevant for his continuation value. As a result, in his model the Coase conjecture continues to hold.[2]

The paper is organized as follows: Section 2 presents the general model. Section 3 characterizes the equilibrium of the game in the continuous time limit. Section 4 presents applications of the general

[2] The same happens in Alberto Trejos and Randall Wright (1995) where the newly arrived traders simply displace the old ones.

model. Section 5 discusses an extension to allow multiple arrivals of buyers and Section 6 concludes. Most proofs are in the Appendix.

2 The Model

We start with a general bargaining game with arrival of a new event. In Section 4 we analyze in detail several applications and in particular a model where the event stands for the arrival of a new trader.

2.1 General Bargaining

There is a seller and a buyer. The seller has an indivisible good (or asset) to sell. The buyer has a privately known type $v \in [0,1]$ that represents his value of the asset. v is distributed according to a *c.d.f.* $F(v)$ which is an atomless distribution with full support.

Time is discrete and periods have length Δ. The timing within periods is as follows. In the beginning of the period an event arrives with probability $\Delta\lambda$ that ends the game (λ represents a Poisson arrival rate; for now, we treat the event as a reduced form of some continuation play). If the event does not arrive, the seller makes a price offer p. Then the buyer decides whether to accept this price or to reject it. If he accepts, the game ends. If he rejects, the game moves to the next period.

A strategy of the seller is a mapping from the histories of rejected prices to current period price offers. A strategy of the buyer is a mapping from the history of rejected prices to an acceptance strategy (which specifies the set of prices that the buyer accepts in the current period).

The payoffs are as follows. If the game ends with the buyer accepting price p at time t, then the seller's payoff is $e^{-rt}p$ and the buyer's payoff is $e^{-rt}(v-p)$, where r is a common discount rate.[3] If the game ends with the event arriving at time t, then the payoffs are:

$$e^{-rt}W(v) \text{ for the buyer,}$$
$$e^{-rt}\Pi(v) \text{ for the seller.}$$

Finally, define $V_A(k) = \int_0^k \Pi(v)\frac{f(v)}{F(k)}dv = E[\Pi(v)|v \leq k]$ as the seller's expected payoff conditional on the arrival of the event and buyer type being distributed according to a truncated $F(v)$ over $v \in [0,k]$.

To justify the reduced-form payoffs consider the following examples. Let the arrival represent a second buyer arriving and suppose the seller runs an English auction upon arrival. If the buyers' valuations are i.i.d. then $\Pi(v) = \int_0^1 \min\{x,v\}\,dF(x)$ and $W(v) = \int_0^v F(x)\,dx$. If their values are perfectly correlated, then $\Pi(v) = v$ and $W(v) = 0$.[4]

[3] We focus on the case $\Delta \to 0$, i.e. no bargaining frictions, so it is more convenient to count time in absolute terms rather than in periods. Period n corresponds to real time $t = n\Delta$.

[4] Alternatively, the same payoffs can be derived by supposing that the buyer has a temporary private informa-

We provide additional examples later.

We assume:

Assumption 1

i) $\frac{e^{-\Delta r}\Delta\lambda}{(1-e^{-\Delta r}(1-\Delta\lambda))}(\Pi(v)+W(v))<v$.

ii) $W(v)$ is continuous and increasing, with $v-W(v)$ strictly increasing.

iii) $\Pi(v)$ is continuous, strictly increasing and differentiable.

iv) $\Pi(0)=W(0)=0$.

These assumptions are not too restrictive and are satisfied in many environments (including the examples above).[5]

Condition (i) is assumed so that from the point of view of the two parties involved in the negotiation delay is inefficient and if it were not for the information frictions there would be no delay in equilibrium. If it was violated delay would be a natural consequence of waiting for the total surplus to grow.[6] (ii) Simply states that higher types are more eager to trade immediately. This guarantees that the *skimming property* holds. The properties of $\Pi(v)$ in (iii), in particular $\Pi'(v)>0$, play an important role in the equilibrium dynamics - they are necessary for slow screening over types in equilibrium. We discuss this in more detail in Section 3. (iv) is assumed to simplify the analysis since it saves us from solving for a fix point problem to find the relevant lowest type that trades. In Section 5 we analyze an environment in which parts (i) and (*iii*) and (iv) of Assumption 1 are violated.

2.2 Stationary Equilibrium

As usual (in dynamic bargaining games), in any equilibrium the buyer types remaining after any history are a truncated sample of the original distribution (even if the seller deviates from the equilibrium prices). This is due to the *skimming property* which states that in any sequential equilibrium after any history of offered prices p^{t-1} and for any current offer p_t, there exists a cutoff valuation $\kappa(p_t,p^{t-1})$ such that buyers with valuations exceeding $\kappa(p_t,p^{t-1})$ accept the offer p_t and buyers with valuations less than $\kappa(p_t,p^{t-1})$ reject it. Best responses satisfy the skimming property because it is more costly for the high types to delay trade than it is for the low types (it can be easily shown using the assumption that $v-W(v)$ is strictly increasing, see FLT Lemma 1 for an analogous proof).

The current cutoff k hence describes the payoff-relevant state of the game and is a natural state variable on which the seller can condition his strategy. If in equilibrium the seller conditions his offers only on the cutoff k and the buyer has an acceptance policy that is independent of the history of the

tion/reputation and the event stands for the type of the buyer becoming public.

[5] For comparison, as we discussed in the Introduction, Inderst's (2003) model violates (iii) because the outside option of the seller is not increasing in the current buyer's valuation in his environment.

[6] A sufficient condition is $\Pi(v)+W(v)\le v$.

game then we call this equilibrium stationary. The classic papers in dynamic bargaining (FLT, GSW, Ausubel and Deneckere 1989, henceforth AD) have shown existence of stationary equilibria and that these equilibria all satisfy the Coase conjecture: as $\Delta \to 0$ the expected time to trade converges to zero and the profit of the seller converges to zero (and prices converge to seller's cost). As shown by AD, there can also exist non-stationary equilibria that exhibit delay and a positive seller's payoff even as $\Delta \to 0$. Since one of our goals is to show that the delay is a consequence of the arrival of external events alone, we limit our analysis to stationary equilibria.

Formally, a stationary equilibrium is characterized by two functions (κ, P) :

1. A buyer's acceptance rule $\kappa(p)$ that specifies the lowest type that accepts offer p.

2. A seller's pricing rule $P(k)$ that specifies the price he offers given truncated beliefs $F(v)$ over $v \in [0, k]$.

A pure stationary equilibrium characterized by (κ, P) is a profile of strategies such the seller offers $P(1)$ in the first period and then in any future period, if $p_{\min}$ is the smallest offered price in the past, he offers $P(\kappa(p_{\min}))$; the buyer follows the acceptance strategy $\kappa(p)$ on and off the equilibrium path. In other words, if the seller ever deviates, the equilibrium strategies call for a return to the equilibrium path as if the seller made the offer $p_{\min}$ in the last period. A general stationary equilibrium allows additionally for mixing by the seller over some prices. However, as shown by AD (Proposition 4.3) in any stationary equilibrium the seller's pricing rule is pure along the equilibrium path except for possibly the first price, $P(1)$.[7] We will refer to (κ, P) as strategies with the understanding that these functions induce proper equilibrium strategies.

Let $k_+ = \kappa(P(k))$ denote next period cutoff given current cutoff k and the strategies (κ, P). Let $V(k)$ be the expected continuation payoff of the seller given a cutoff k and the strategy pair (κ, P). We can express $V(k)$ recursively as:

$$V(k) = \Delta\lambda V_A(k) + (1 - \Delta\lambda)\left[\left(\frac{F(k) - F(k_+)}{F(k)}\right) P(k) + \frac{F(k_+)}{F(k)} e^{-\Delta r} V(k_+)\right] \tag{1}$$

The seller's strategy is a best response to the buyer's strategy $\kappa(p)$ if:

$$P(k) \in \arg\max_p \left[\left(\frac{F(k) - F(\kappa(p))}{F(k)}\right) p + \frac{F(\kappa(p))}{F(k)} e^{-\Delta r} V(\kappa(p))\right] \tag{2}$$

This best response problem captures the seller's lack of commitment: in every period he chooses price to maximize his payoff (instead of committing to a whole sequence of prices at time 0).

Note that the pair (κ, P) determines the future sequence of prices starting at any history described

[7] Additionally, there can be randomization off the equilibrium path. If the seller deviates to a price p' such that $k' = \kappa(p')$ and yet $p' \neq \max\{p | \kappa(p) = k'\}$ (which can happen only if $\kappa(p)$ is constant over a range and is never a seller best response since he can increase the price without changing the probability of trade) then the seller randomizes between prices p_1 and p_2 to rationalize the acceptance of p' by type k'. The prices p_1 and p_2 are the maximum and minimum elements of the seller maximization problem given the cutoff k'.

by k : the current equilibrium price is $P(k)$, the next period price is $P(\kappa(P(k))$ and so on. A necessary condition for the buyer's strategy $\kappa(p)$ to be a best response is that given this expected path of prices the cutoffs satisfy:

$$\underbrace{k_+ - P(k)}_{\text{trade now}} = e^{-\Delta r}(\Delta\lambda \underbrace{W(k_+)}_{\text{arrival}} + (1-\Delta\lambda)\underbrace{(k_+ - P(k_+))}_{\text{trade tomorrow}} \tag{3}$$

The interpretation is that the new cutoff type k_+ has to be indifferent between accepting $P(k)$ today or trading next period at $P(k_+)$ (while facing the risk of arrival and getting $W(k_+)$ instead). These local conditions for optimality of the buyer's strategy are sufficient because the skimming property holds in equilibrium.[8]

Definition 1 *Functions* (κ, P) *describe a stationary sequential equilibrium if given* $V(k)$ *defined in* (1) *they satisfy* (2) *and* (3).

Remark: To be precise, to fully specify the equilibrium strategies, the functions (κ, P) may need to be augmented by an appropriate mixed strategies off-the equilibrium path, as discussed above (yet, the equilibrium-path behavior is completely described by (κ, P).
AD call stationary equilibria weak-Markov (and strong-Markov when $\kappa(p)$ is strictly increasing, which implies that there is no randomization off-equilibrium). The existence of these equilibria is proven in FLT and in AD for the game without arrival of events; and in Deneckere and Liang (2006) in a setup with interdependent values. These proofs can be extended to the present setup. Since we are in the no-gap case these equilibria may not be unique. [9]

The equilibrium strategies (κ, P) implicitly define a decreasing step function $K(t)$ which specifies the highest remaining type in equilibrium as a function of time (with $K(0) = 1$) and a decreasing step function $T(v)$ (with $T(1) = 0$) which specifies the time at which each type v trades conditional on no arrival.

2.3 (Continuous-time) Limit-equilibrium

The equilibrium strategies in discrete time are known to be in general analytically intractable (other than in special cases, for example for uniform distribution of values, see Stokey (1981)). In the Appendix we analyze the stationary equilibria in discrete time and show that they all (even if they are not unique or not pure) converge to the same equilibrium path as $\Delta \to 0$.[10] In contrast, this continuous-time limit

[8] If higher types trade sooner in equilibrium, the buyer's direct-revelation problem of optimizing when to trade by choosing which type to mimic is supermodular in v and the type he mimics.

[9] Finally, in case the equilibria are not unique, there also exist equilibria in which the seller randomizes in the first period over a set of prices that correspond to a set of equilibria without initial randomization.

[10] The general intuition behind the proof is related to the uniform convergence of equilibria shown in AD.

turns out to be relatively easy to characterize so that in the rest of the paper we focus on this limit which we call *the limit-equilibrium.*

For the rest of this section we explicitly note the dependence of the strategies (and values) in the discrete-time game on Δ. For example we write $P(k,\Delta)$ for the seller's strategy. The functions without reference to Δ correspond to the limit-equilibrium.

We say that the three functions $V(k)$, $K(t)$ and $P(k)$ are a limit-equilibrium if:

Definition 2 *The strictly increasing and continuous functions $V(k)$ and $P(k)$ and a strictly decreasing, continuous function $K(t)$ such that $K(0)=1$ are called a limit-equilibrium if for all t such that $K(t)=k$ with $k\in[0,1]$:*
(i) $rV(k)=\lambda(V_A(k)-V(k))+(P(k)-V(k))\frac{f(k)}{F(k)}\left(-\dot{K}\right)+V'(k)\dot{K}$
(ii) $\dot{K}$ *solves*

$$\max_{\dot{K}\in(-\infty,0]}(P(k)-V(k))\frac{f(k)}{F(k)}\left(-\dot{K}\right)+V'(k)\dot{K}$$

(iii) $(r+\lambda)(k-P(k))=\lambda W(k)-P'(k)\dot{K}$
where $\dot{K}$ represents the right derivative of $K(t)$ and $P'(k)$ the left derivative of $P(k)$.

What is the connection between the limit equilibrium and the stationary equilibria of our bargaining game? First, the functions $P(k)$ and $K(t)$ specify behavior in the continuous time: at any moment of time t_0, if $p_{\min}$ is infimum price offered up to time t_0, the seller continuation strategy described as the time-path of prices is given by $p(t+t_0)=P(K(t+t_{\min}))$, where $t_{\min}$ is defined by $P(K(t_{\min}))=p_{\min}$ (while the buyer's behavior is simply the reservation price strategy $P(k)$). That is analogous to the stationary equilibrium behavior in discrete time.

Second, the limit-equilibrium conditions (i)-(iii) are appropriate continuous-time limits of the conditions in the definition of the stationary equilibrium. To see this, note first that given the buyer's acceptance strategy, the seller's best response problem given current cutoff k can be cast as choosing the next cutoff type k_+ rather than a price:

$$V(k,\Delta)=\max_{\frac{k_+-k}{\Delta}}\Delta\lambda V_A(k)+(1-\Delta\lambda)\left[\left(\frac{F(k)-F(k_+)}{F(k)}\right)P(k_+,\Delta)+\frac{F(k_+)}{F(k)}e^{-\Delta r}V(k_+,\Delta)\right]$$

Subtracting $e^{-\Delta r}V(k_+,\Delta)$ from both sides, dividing by Δ and taking the limit $\Delta\to 0$, shows that the seller's best response problem can be thought of as finding the optimal rate $\dot{K}$ at which he goes through types:

$$rV(k)=\max_{\dot{K}\in(-\infty,0]}\lambda(V_A(k)-V(k))+(P(k)-V(k))\frac{f(k)}{F(k)}\left(-\dot{K}\right)+V'(k)\dot{K} \tag{4}$$

That corresponds to conditions (i) and (ii) in the definition of the limit-equilibrium. In taking the limit we used the fact that $(\kappa(P(k,\Delta),\Delta)-k)=k_{+}-k\rightarrow 0$ as $\Delta\rightarrow 0$, and that $P\left(k,\Delta\right)-P\left(k_{+},\Delta\right)$ is $O\left(\Delta\right)$. We prove in the Appendix (See Lemmas 2 and 3) that these two properties hold in any stationary equilibrium. Intuitively, these properties mean that there are no atoms of trade in equilibrium (that the probability of trade in any period goes to zero as the length of the periods goes to zero) and that the asking prices do not drop too quickly in time. These properties hold since otherwise the prices would drop very quickly in time, making it optimal for some types above k_{+} to delay trade.

A more direct interpretation of (4) is that it is the Hamilton-Jacobi-Bellman equation for optimal dynamic control problem of maximizing expected payoffs by choosing $K(t)$ given the reservation prices $P(k)$. One may be concerned that in (4) we restrict the seller to a continuous $K(t)$. The seller does not benefit from jumping down the types if:

$$V\left(k\right)\geq\frac{F\left(k\right)-F(\tilde{k})}{F\left(k\right)}P(\tilde{k})+\frac{F(\tilde{k})}{F\left(k\right)}V(\tilde{k})\text{ for all }\tilde{k}\in[0,k]$$

which is automatically satisfied for any strictly increasing $P(k)$. The intuition is that by jumping down the "demand function" the seller earns only the rectangle under $P(k)$ (from k to $\tilde{k}$ with height $P(\tilde{k})$) while by continuously quickly running down the demand he can earn the whole area below the demand.[11]

For condition (iii) of the limit-equilibrium, recall the buyer optimality condition (3) in discrete time is:

$$k_{+}-P\left(k,\Delta\right)\ =e^{-\Delta r}(\Delta\lambda W\left(k_{+}\right)+\left(1-\Delta\lambda\right)\left(k_{+}-P\left(k_{+},\Delta\right)\right)) \tag{5}$$

where $k_{+}=\kappa\left(P\left(k,\Delta\right),\Delta\right)$. Subtracting $e^{-\Delta r}\left(1-\lambda\Delta\right)\left(k_{+}-P\left(k,\Delta\right)\right)$ from both sides, dividing by Δ and taking $\Delta\rightarrow 0$ (using Lemmas 2 and 3 as in the previous limit) we obtain condition (iii) (see Lemma 7 in the Appendix).

Finally, although the limit-equilibrium conditions are intuitive in their own right, the lemmas mentioned above are also used in the Appendix to show that the limit-equilibrium $P(k)$ and $K(t)$ and $V(k)$ can be formally derived as limits of stationary equilibria:

Theorem 1 *Take any sequence of games indexed by the period lengths that asymptotically decrease to* 0 *and any selection of stationary equilibria of these games* $\{\kappa\left(p,\Delta\right),\ P\left(k,\Delta\right)\}$ *and the corresponding sequences* $\{V\left(k,\Delta\right),\ K\left(t,\Delta\right)\}$.
As $\Delta\rightarrow 0$, *these equilibria converge to the limit-equilibrium (which is unique). That is, as* $\Delta\rightarrow 0$, $V\left(k,\Delta\right)\rightarrow V\left(k\right)$, $P\left(k,\Delta\right)\rightarrow P\left(k\right)$ *and* $K\left(v,\Delta\right)\rightarrow K\left(t\right)$ *(all convergences are point-wise).*

The main intuition behind the proof comes from taking the continuous-time limits of the optimality conditions that have to be satisfied in any equilibrium, and obtaining the limit conditions as we

[11] This intuition mimics the intuition in the durable monopolist problem.

discussed above. The proof of this result is in the Appendix, but we recommend the Reader to study Section 3 first since the proof relates to some of the expressions derived there.

As we show below, focusing the analysis on the limit-equilibria is very convenient in terms of analytical tractability. This theorem guarantees that learning about the limit-equilibrium we also learn about stationary equilibria of the game for small Δ since they all converge to the unique limit-equilibrium.

3 Characterization of the Limit-Equilibrium

We now characterize the limit-equilibrium $V(k)$, $P(k)$ and $K(t)$. At some points of the analysis it is convenient to use the continuous and strictly decreasing function $T(v) = K^{-1}(v)$ which specifies the (equilibrium path) time at which a buyer of type v trades.

Following the interpretation of conditions (i)-(iii) as continuous-time limits of the optimality conditions of the discrete-time game, we refer to equation (4) as the seller's best response problem and to condition (iii) in Definition 2 as the buyer's best response problem.

Seller's problem. Recall from Definition 2 that the seller's problem is:

$$rV(k) = \max_{\dot{K}\in(-\infty,0]} \lambda (V_A(k) - V(k)) + (P(k) - V(k)) \frac{f(k)}{F(k)} \left(-\dot{K}\right) + V'(k)\dot{K} \tag{6}$$

This condition has a direct interpretation. The left-hand side is the expected equilibrium payoff expressed in flow terms. The right hand side represents the possible sources of the flow: upon arrival of the event (which happens with a probability flow λ) the game ends with the seller earning in expectation $V_A(k)$ (and since the game ends he forgoes $V(k)$). With a flow probability $\frac{f(k)}{F(k)}\left(-\dot{K}\right)$ the buyer accepts current offer, $P(k)$, which also ends the game. Finally, if the game does not end immediately, the continuation payoff drops, as the seller becomes more pessimistic about v, as captured by $V'(k)\dot{K}$.

Note that (6) is linear in $\dot{K}$, so that a continuous and strictly decreasing $T(v)$ can be consistent with equilibrium only if the seller is indifferent over all possible $\dot{K}$. This linearity is the source of Coasian dynamics when $\lambda = 0$. In that case, for any strictly increasing $P(k)$ the seller wants to run down the demand function as fast as possible. Therefore the equilibrium $P(k)$ in the limit becomes flat at 0. The outside option in our model provides a counterbalance for the seller's temptation to run down the demand curve, leading to a strictly downward-sloping $P(k)$. The optimality of an interior $\dot{K}$

requires that the coefficients on $\dot{K}$ add up to 0:

$$\begin{aligned}(P(k)-V(k))\frac{f(k)}{F(k)} &= V'(k)\\ &\Updownarrow\\ P(k) &= \frac{\partial}{\partial k}\left[V(k)F(k)\right]/f(k)\end{aligned}$$

If that holds, then we can calculate $V(k)$ by simply evaluating the right hand side of (6) using $\dot{K}=0$:

$$V(k)=\frac{\lambda}{\lambda+r}V_A(k) \tag{7}$$

That implies the equilibrium prices:

$$P(k)=\frac{\lambda}{\lambda+r}\Pi(k) \tag{8}$$

These two equations pin down the unique candidates for the limit-equilibrium $P(k)$ and $V(k)$.

Note that, interestingly, $V(k)$ has the property that at any point in the game (for any k) the expected payoff of the seller is equal to his payoff from waiting for the arrival of the event. Hence, although the Coase conjecture does not hold in terms of the price dropping immediately to zero, the Coasian dynamics force down the seller's profit down to his outside option. In discrete time $\frac{\lambda}{\lambda+r}V_A(k)$ is the lower bound on the payoffs ($V(k,\Delta)\leq\frac{\lambda}{\lambda+r}V_A(k)$) since the seller has the option not to trade until the arrival.[12] In the continuous-time limit the seller completely loses commitment power (in discrete time he commits not to reduce price for Δ units of time) and we have shown that $V(k,\Delta)\rightarrow\frac{\lambda}{\lambda+r}V_A(k)$, so that the outside option becomes also the upper bound on payoffs. In discrete time the limited commitment allows the seller to earn a little bit more that the outside option (which makes the analysis of the equilibrium much more difficult), but that extra amount converges to zero as commitment power disappears.

Finally, for each type k, $P(k)$ is exactly the expected present value the seller would have earned from this type if he waited for the arrival - a kind of no-ex-post regret property - upon the price being accepted the seller does not regret not slowing down the trade.

Buyer's problem. We now turn to the buyer's best response problem. Recall condition (iii) of the limit-equilibrium:

$$(r+\lambda)(k-P(k))=\lambda W(k)-P'(k)\dot{K} \tag{9}$$

It also has a direct interpretation: the LHS is the cost of delaying trade (due to discounting and possibility of arrival) and the RHS is the benefit of waiting consisting of the reduction of price and the payoffs from a possible arrival. The benefits and costs are evaluated at the current cutoff type.

Using $P(k)$ we found above, the buyer's indifference condition allows us to find $K(t)$. Substituting

[12] If τ is the random Poisson arrival time, then $\frac{\lambda}{\lambda+r}=E\left[e^{-r\tau}\right]$ is the expected discount factor of the arrival time.

(8) in (9) yields:

$$-\dot{K} = (\lambda + r) \frac{(r + \lambda) K(t) - \lambda (\Pi (K(t)) + W(K(t)))}{\lambda \Pi' (K(t))} \tag{10}$$

which together with the boundary condition $K(0) = 1$ pins down $K(t)$. By assumption, $\frac{\lambda(\Pi(v)+W(v))}{\lambda+r} < v$ and $\Pi'(v) > 0$, so the numerator and denominator are strictly positive for all $K(t) > 0$. Therefore, this differential equation (with the boundary condition) uniquely defines a strictly decreasing and continuous $K(t)$.

Additionally, we can calculate the buyer's expected payoff. Denote by $B(v)$ the expected payoff of buyer with value v (at the beginning of the game). Looking at the direct-revelation representation of the buyer's strategy, he plays a best response if and only if:

$$B(v) = \max_{v'} e^{-(r+\lambda)T(v')} \left(v - P(v')\right) + \int_0^{T(v')} \lambda W(v) e^{-(\lambda+r)s} ds \tag{11}$$

and $v' = v$ is a solution to this problem. In words, the buyer can mimic another type v' to trade at a different price and time, $P(v')$ and $T(v')$. The first part on the RHS reflects the surplus from trading before the arrival of an event and the second part stands for the possibility that the arrival happens before $T(v')$.[13]

Instead of using $\{T(v), P(v)\}$ to calculate $B(v)$ directly, we can apply the envelope theorem:

$$B'(v) = e^{-(r+\lambda)T(v)} + \frac{\lambda}{\lambda + r} \left(1 - e^{-(r+\lambda)T(v)}\right) W'(v)$$

and use the boundary condition $B(0) = 0$ to pin down $B(v)$.

Summarizing this Section:

Theorem 2 *There exists a unique limit-equilibrium* $\{V(k), P(k), K(t)\}$. *It is characterized by* (7), (8), (10) *and the boundary condition* $K(0) = 1$.

3.1 Properties of the Limit-Equilibrium.

We now present some general properties of the limit-equilibrium.

In the previous section we characterized $T(v) = K^{-1}(v)$, the time at which type v trades conditional on no arrival. If we interpret that the arrival of the event ends the game with an immediate trade (which is true in the applications we present in Section 4 and not true in the application with multiple arrivals in Section 5), we can further define the expected time at which type v trades, $\tau(v)$.

[13] The RHS of (11) is supermodular in v and v' if $T(v')$ is decreasing. Hence, as mentioned above, skimming property guarantees that the local optimality condition (9) - a F.O.C. of (11) – is sufficient.

It takes into account the possibility that arrival takes place before $T(v)$:

$$\tau(v) = \underbrace{\left(\int_0^{T(v)} \lambda e^{-\lambda s} ds\right)}_{\text{Pr arrival before } T(v)} \times \underbrace{\left(\int_0^{T(v)} s \frac{\lambda e^{-\lambda s}}{1 - e^{-\lambda T(v)}} ds\right)}_{E[\text{arrival time } | < T(v)]}$$
$$+ \underbrace{\left(1 - \int_0^{T(v)} \lambda e^{-\lambda s} ds\right)}_{\text{Pr no arrival before } T(v)} \underbrace{T(v)}_{\substack{\text{time to trade conditional} \\ \text{on no arrival}}}$$

Finally, we can define the (unconditional) expected time to trade as $\int_0^1 \tau(v)\, dF(v)$.

Proposition 1
(i) (Delay): For all $0 < \lambda < \infty$ *the expected time to trade is strictly positive.*
(ii) (Coase conjecture): as $\lambda \to 0$, *the expected time to trade and transaction prices converge to* 0 *for all types (i.e.* $T(v) \to 0$ *and* $P(k) \to 0$*).*

Part (i) shows that when the bargaining is subject to external influences, delay is to be expected, which is our first main result.

It follows directly from our characterization, but the intuition is as follows: suppose that there is no delay in equilibrium. Then the transaction prices for all types have to be close to zero, implying seller's payoff close to zero, in particular, less than $\frac{\lambda}{\lambda+r} V_A(k) > 0$. But that leads to a contradiction since the seller can guarantee himself that by just waiting for the arrival of the event. Moreover, the bargaining cannot stop at any type $k > 0$ with the buyer and seller waiting for the arrival of the event, since then for all types $v \leq k$ the reserve price would be $P(v) = v - \frac{\lambda}{\lambda+r} W(v)$. But then, by Assumption 1 that $v > \frac{\lambda}{\lambda+r}(W(v) + \Pi(v))$, the seller would be strictly better off screening through the types quickly than waiting for the arrival.

Part (ii) shows that our limit-equilibrium converges to the equilibria in GSW and FLT: as we take the probability of arrivals to zero (convergence of the model) trade takes place immediately and the buyer captures all the surplus (convergence of equilibrium).

Arrival of new traders or outside options is necessary for delay but another important ingredient for slow equilibrium screening is that the seller's outside value depends on the buyer's type. In particular, we can establish the following general comparative statics:

Proposition 2
(i) Consider two environments, one with $\Pi_1(v)$ *and the other with* $\Pi_2(v)$ *and either* $W_1(v) = W_2(v)$ *or* $\Pi_1(v) + W_1(v) = \Pi_2(v) + W_2(v)$. *Then if* $\Pi_1'(v) \geq \Pi_2'(v)\ \forall v$, *the expected time to trade is shorter in the environment with* $\Pi_2(v)$.

(ii) In the limit as $\Pi'(v) \to 0\ \forall v$*, expected time to trade converges to zero and the buyer asymptotically captures all the surplus.*

The second part of this Proposition shows that the Coase conjecture holds in the limit as $\Pi'(v) \to 0$ $\forall v$. Given our assumption $\Pi(0) = 0$, $\Pi'(v) \to 0\ \forall v$ implies that $\Pi(v) \to 0\ \forall v$. To separate slope versus level effects consider the case where $\Pi'(v) = 0$ *but* $\Pi(v) = c > 0\ \forall v$ (that is, the arrival stands for somebody coming to offer the seller price c). In this case, in equilibrium the seller offers price $p = \frac{\lambda}{\lambda+r}c$ and either trade happens immediately or there is no trade until arrival. For there to be trade with delay it is necessary that $\Pi'(v) > 0$. Intuitively, the seller makes a first offer $p = \frac{\lambda}{\lambda+r}\Pi(1)$. Since this offer is accepted by the highest types, the seller's outside option decreases a bit and next period he is willing to make lower offers. In this way he slowly skims through all buyer types.

But why does it happen slowly? Why don't we get almost immediately to $p = 0$, like in the Coase conjecture? The reason is that if the seller ran 'the clock' too fast then some buyer types would have an incentive to wait for a lower price - their reservation prices would decrease. But then the seller would prefer to stop trading, since he would get a higher expected payoff from just waiting for an arrival than from trading at these low prices. On the other hand, the seller cannot run too slowly through the demand either, since then the reservation prices would be so high, that the seller would prefer to collect the whole area below the demand *before* the arrival. Therefore the speed at which price decreases has to be such that the reservation prices of the buyer keep the balance between the incentive to speed up and slow down the trade.

Following this logic, if $\Pi_1'(v) \geq \Pi_2'(v)\ \forall v$, then under Π_1 the seller's outside option drops faster as his belief of the current buyer cutoff type falls. This makes him offer lower prices at $k' = v - \varepsilon$, that is prices as a function of k decrease at a faster rate for the steeper $\Pi(v)$. Hence, if the seller ran the clock (with respect to $K(t)$) at the same speed, prices would drop faster in time under Π_1. But then the buyers would have an incentive to wait for lower prices, leading to a contradiction that the k changes through time. To keep the current cutoff types willing to trade at the current prices the seller has to go through the types slower under Π_1, as claimed in the first part of the last proposition.

This result allows us to compare our dynamics to existing literature. For example, in Inderst (2003) (and other papers that have the new buyers replace the existing buyer), $\Pi'(v) = 0$ and there is no delay.[14] Taking the limit $\Pi'(v) \to 0$ in our model leads to the same limiting outcome. It is essential for there to be delay that the outside value of the seller depends on the buyer's type. The more sensitive the outside value of the seller to the buyer's private information, the greater the delay/inefficiency. As we explained in the Introduction, the correlation of the seller's outside option with the buyer value endogenously creates a bargaining environment with interdependent values, as studied by Evans (1989), Vincent (1989) and Deneckere and Liang (2006), and hence the main economic intuition behind the delay is similar to that in those papers.

[14] Inderst (2003) assumes that upon the second buyer arriving, the seller can only choose to keep on bargaining with the current buyer or switch to bilateral bargaining with the new one, which implies $\Pi'(v) = 0$ in his model.

The next proposition characterizes how the time on the market and the ex-ante expected payoffs depend on the distribution of values:

Proposition 3 *Suppose* $\Pi(v)$ *and* $W(v)$ *are independent of the distribution of values,* $F(v)$ *.*
(i) The limit-equilibrium $P(k)$ *and* $K(t)$ *are independent of the distribution of values,* $F(v)$*.*
(ii)(Weak markets and time on the market) Consider two distributions of buyer's values F *and* H *such that* F *first order stochastically dominates* H*. The expected time to trade is longer if the distribution of values is* H *(and average prices are lower).*
(iii) (Dispersion of values and efficiency of trade) Consider two distributions of values F *and* H *such that* F *second order stochastically dominates* H*. Then the ex-ante expected sum of payoffs is higher under distribution of values* H*.*

To illustrate this surprising result (that the equilibrium $P(k)$ and $K(t)$ are independent of the distribution of values) consider the following example. Suppose that the event represents an arrival of one more buyer who has the same valuation as the original buyer. Upon arrival the seller runs an English auction.[15] As a result, $\Pi(v) = v$ and $W(v) = 0$ independently of the distribution. In that case, the proposition states that the equilibrium path of prices $P(k)$ and the times at each type trades, $T(v)$ are independent of $F(v)$![16]

The intuition behind this result is as follows. First, since the inability to commit and the associated requirement of time-consistency drives the expected payoff of the seller down to his outside option for any k ($V(k) = \frac{\lambda}{\lambda+r} V_A(k)$) it has to be that prices satisfy $P(k) = \frac{\lambda}{\lambda+r}\Pi(k)$. They depend only on the current cutoff and not the whole distribution (unless $\Pi(v)$ does). Second, $K(t)$ is pinned down by the indifference condition of the buyers. Since the current marginal buyer's incentives do not depend on the distribution (unless $W(v)$ does) the limit-equilibrium is independent of $F(v)$. Clearly low valuation buyers would like the seller not to spend time sorting through high types. The problem is that they have no credible way in which to signal to the seller that they have a low value.

4 Applications

We now turn to four examples to present how the general model can be adapted to different applications and used to derive additional predictions.

[15] Alternatively, the event can represent an arrival of information that reveals the value of the buyer.

[16] If instead the second buyer's value were independent but distributed identically to the first buyer's value then: $W(v) = F(v)(v - E[V_2|V_2 \leq v])$ and $\Pi(v) = F(v)E[V_2|V_2 \leq v] + (1 - F(v))v$. And therefore the equilibrium depends on $F(v)$.

4.1 Arrival of New Traders with Common Value

Suppose that the event represents two possibilities: either a second seller with an identical good arrives or a second buyer with identical valuation arrives (we call it *the common value* case).[17] The arrival rates are λ_s and λ_b respectively with $\lambda = \lambda_b + \lambda_s$. Upon arrival there is Bertrand competition on the long side of the market (for example, the agent on the short side of the market runs an English auction). As a result, the expected payoffs conditional on arrival in this case are:

$$W(v) = \frac{\lambda_s}{\lambda} v \ , \ \Pi(v) = \frac{\lambda_b}{\lambda} v \ .$$

and clearly they satisfy Assumption 1. Using the equilibrium conditions (8) and (10) we can calculate the limit-equilibrium $P(k)$ and $T(v) = K^{-1}(v)$ in a closed form:

$$P(v) = \frac{\lambda_b}{\lambda + r} v \ , \qquad T(v) = -\frac{\lambda_b}{r(\lambda + r)} \ln v \ . \tag{12}$$

The corresponding value functions are:

$$\begin{aligned} V(k) &= \frac{\lambda_b}{\lambda + r} \int_0^k v \frac{f(v)}{F(k)} dv \ , \\ B(v) &= \frac{\lambda_s v + r v^{\frac{\lambda_b + r}{r}}}{\lambda + r} . \end{aligned} \tag{13}$$

Using the general characterization of the limit-equilibrium we can observe that in the limit-equilibrium

1) Market Tightness: Keeping $\lambda = \lambda_b + \lambda_s$ constant (the sum of arrival rates of the second buyer and seller), a decrease in the ratio $\frac{\lambda_b}{\lambda_s}$, implies a shorter equilibrium time on the market, a lower seller's expected payoff and a higher buyer's payoff. In the limit, as $\frac{\lambda_b}{\lambda_s} \to 0$ we get immediate trade with the buyer capturing all the surplus.

2) Market Thickness: Keeping the ratio $\frac{\lambda_b}{\lambda_s}$ fixed, delay is non-monotonic in the sum $\lambda = \lambda_b + \lambda_s$. It converges to zero as $\lambda \to \infty$ and also as $\lambda \to 0$, while it is greater than zero for intermediate values.

The first result shows that trade is more efficient when it is a buyers' market. This is because the higher the likelihood of arrival of the second seller, the more impatient the current seller gets, which makes him offer lower prices. In the limit, if only new sellers can arrive then trade takes place immediately and the buyers capture all the surplus as in FLT or GSW.

The second result shows that the efficiency is not monotonic in the liquidity of the market. In the limit as we approach perfect competition ($\lambda_b + \lambda_s \to \infty$) trade takes place immediately. Trade is also

[17] It can be also described as private values with perfect correlation, since the buyers know their valuations and are not concerned with the winner's curse.

immediate when there is a bilateral monopoly with no possibility of arrival. But when we have a thin market there is some delay in trade.

Since $\Pi(v)$ and $W(v)$ are independent of the distribution $F(v)$, Proposition 3 applies and the equilibrium $P(k)$ and $K(t)$ are independent of the distribution. Does it mean that the distribution of values has no impact on the expected trade dynamics? No. In fact, as a corollary to Proposition 3 we get additional two observations:

3) Weak markets and time on the market: Consider two distributions of buyer's values F and H such that F first order stochastically dominates H. The expected time to trade is longer if the distribution of values is H (and average prices are lower).

4) Dispersion of values and efficiency of trade: Consider two distributions of values F and H such that F second order stochastically dominates H. Then the ex-ante expected sum of payoffs is higher under distribution of values H but the expected time to trade is lower under F.

These results can be derived directly from the expressions (12) and (13) by noting that $T(v)$ is decreasing and convex, the $V(1)$ depends only on the average v and $B(v)$ is convex in v.[18]

These results point to an interesting finding that trade takes longer in markets with weaker distributions of valuations. This could help explain some of the cyclical patterns in real estate markets and in labor markets.

4.1.1 Impatience: Arrivals vs. Discounting

In Rubinstein's (1982) bargaining model the relative discounting of the buyer with respect to the seller is critical in determining the price at which the object is traded. In our model beyond discounting there is an additional source of impatience: the probability of having the arrival of a competing trader on your side of the market. In this Section we study how arrivals compare with discounting in determining the properties of the equilibrium.

Let r_s be the interest rate faced by the seller and r_b the one faced by the buyer. The time at which each type trades and the prices are given by:

$$P(v) = \frac{\lambda_b}{\lambda_s + \lambda_b + r_s} v \quad ; \quad T(v) = -\frac{\lambda_b}{r_b(\lambda_s + r_s) + r_s \lambda_b} \ln v$$

We can see that, in contrast to Rubinstein's model, the seller's discount rate is the only one determining the path of prices. A higher buyer discount rate has no impact on prices. Also note that the seller's two sources of impatience have identical effects for determining prices. Hence having more fear of competition through higher likelihood of arrivals of sellers or a higher discount rate are identical sources of seller's impatience. This is not true on the buyer's side. The buyer discount has a direct effect in

[18]The only new claim is that the expected time to trade is longer when the distribution of values is more dispersed. It follows from $t(v)$ being convex.

the time to trade. This is because the more he discounts the future the faster the seller can lower the prices without violating the buyer's indifference condition. The fear of arrivals of competing buyer's has two effects. A direct effect like the discount factor and an indirect effect via its effect on prices.

Furthermore, if r_s and r_b are much smaller than λ, then the prices paid by different types depend mostly on λ_s and λ_b and very little on r_s and r_b and $\frac{r_s}{r_b}$ affects the equilibrium only via delay:

$$P(v) \approx \frac{\lambda_b}{\lambda_b + \lambda_s} v \quad ; \quad T(v)\, r_b \approx -\frac{\lambda_b}{\lambda_s + \frac{r_s}{r_b}\lambda_b} \ln v$$

Hence in thick markets what matters in terms of bargaining power is driven a lot by the relative arrival rates and much less by the relative rates of time discount.

4.2 Arrival of a New Buyer with Private Values

We now turn to the case were only an additional buyer can arrive and his value $v \sim F(v)$ is independent of the current buyer's value. Also, upon arrival we assume the seller runs a second price auction to allocate the good. In this environment we have the values upon an arrival for the current buyer and seller are given by:

$$\begin{aligned} W(v) &= F(v)\, v - \int_0^v x f(x)\, dx \\ \Pi(v) &= \int_0^v x f(x)\, dx + (1 - F(v))\, v \end{aligned}$$

Given these values, we can use the equations for prices (8) and for timing of trades (10) to characterize the equilibrium in this environment:[19]

$$P(v) = \frac{\lambda}{\lambda + r} \left(\int_0^v x f(x)\, dx + (1 - F(v))\, v \right),$$

$$-\dot{K} = (\lambda + r) \frac{r}{\lambda} \frac{K(t)}{1 - F(K(t))}.$$

The expression for $K(t)$ is quite involved but its inverse:

$$T(v) = \frac{\lambda}{(\lambda + r)\, r} \int_v^1 \frac{1 - F(x)}{x} dx$$

is easier to work with. Therefore, if the distribution changes in a way that $\frac{1-F(k)}{k}$ decreases for all k (which means the distribution is weaker) then every type trades faster. Nonetheless, it is still difficult to rank expected time on the market since there are more weak types. The expected time to

[19] To simplify the equation for $\dot{K}$ We use that $\Pi(v)+W(v) = v$ and that $\Pi'(v) = vf(v)+(1 - F(v))-vf(v) = 1-F(v)$.

trade conditional on no arrival is:

$$E[T(v)] = \frac{\lambda}{(\lambda + r)r} \int_0^1 \frac{(1 - F(v))F(v)}{v} dv \tag{14}$$

In order to carry out comparative statics with respect to the distribution $F(v)$,consider the class of distributions $F(v) = v^a$. In this case the expected time to trade conditional on no arrival simplifies to:

$$E[T(v)] = \frac{\lambda}{2a(\lambda + r)r}.$$

That implies that as we move to a weaker distribution the effect of having more weak types dominates the effect that each type trades faster. Hence, at least within this family of distributions, time to trade is faster for stronger distributions.

Finally, for distributions symmetric around $\frac{1}{2}$ a mean-preserving spread in the distributions leads to a longer average time on the market, as can been seen from a direct inspection of (14).

4.2.1 Auction Format and Time Consistency.

So far we have assumed that the seller runs an English auction with no reserve price upon arrival of the second buyer. McAfee and Vincent (1997) have shown that without commitment (and with symmetric bidders) revenue equivalence holds between first price and second price auctions. Furthermore, when the time between auctions goes to zero, the seller's expected revenues converge to those of a second price auction with no reserve.

However, in our setup since the first and the second buyer are not symmetric at the beginning of the auction[20], different auction formats will yield different expected revenues. In particular, with i.i.d. ex-ante distributions of the two buyers, the first buyer is going to have a weaker (truncated) distribution. In that case Maskin and Riley (1999) have shown that if the distributions of the two bidders are common knowledge (a strong assumption requiring the second buyer to see the full protocol of the bargaining before his arrival), then the first-price auction is going to yield a higher revenue than the second price auction.

In general, optimal auctions usually treat weaker bidders more favorably (to increase competition faced by the stronger bidders). One could expect that treating the first bidder more favorably in the auction would make him more stubborn during the bargaining phase and hence hurt the seller. Can that lead to time-inconsistency of the optimal auction choice of the seller (i.e. that he would like to choose one format ex-ante and another one ex-post)?

Using the analysis above, we can show that in fact no such time-inconsistency would arise. The intuition is that since the lack of commitment to prices drives the seller's payoff down to his outside

[20] The asymmetry arises endogenously even with i.i.d. valuations in our setup because the seller updates his beliefs about v during the bargaining phase with the first buyer.

option, maximizing ex-post revenues, maximizes ex-ante payoffs as well. [21]

4.3 Taste Diversity and Time on the Market.

So far we have assumed that the two buyers have the same valuation. In many markets, however, it is natural to think that there are different groups of potential buyers of the asset, and that even though valuations within a group can be very similar, they would differ across groups quite a bit. For example, families with school age children could be one group with similar valuations for a given house. The group of retirees, on the other hand, could value the same house differently. The first group would put more weight on the quality of the school district while the latter care more about the quality of the walking paths. Similarly, if a firm is being sold, there are different groups of potential buyers such as competing firms and private equity funds that have different motives for purchasing the target.

To illustrate the effects of diverse taste groups of potential buyers on the bargaining dynamics, we parameterize the problem as follows. Assume there are n different groups of buyers. All members of a given group share the same valuation but valuations across groups are *i.i.d.* according to $F(v)$. Now, when the second buyer arrives, with probability $\gamma = \frac{1}{n}$ he belongs to the same group (and has the same valuation) as the current buyer (and this is common knowledge). Otherwise, with probability $(1-\gamma)$, he belongs to a different group and his value is independent of the first buyer value. Therefore, a larger γ stands for a less diverse market place. In either case an English auction is used to allocate the good. For simplicity assume $\lambda_s = 0$.

In this case the expected payoffs conditional on arrival are:

$$\begin{aligned} W(v_1) &= (1-\gamma) F(v_1) (v_1 - E[v_2 | v_2 \leq v_1]) \\ \Pi(v_1) &= \gamma v_1 + (1-\gamma) (F(v_1) E[v_2 | v_2 \leq v_1] + (1 - F(v_1)) v_1) \end{aligned}$$

Applying the general analysis above, we can establish the following comparative statics with respect to the taste diversity:

The limit-equilibrium has the following comparative statics with respect to an increase in the number of groups, $n \uparrow (\downarrow \gamma)$:
(i) The expected time to trade decreases.
(ii) The payoff of the seller falls.
(iii) For any t the price offered is lower.

Part (i) follows from noting that $\frac{\partial \Pi'(v_1)}{\partial \gamma} = F(v_1) > 0$ and using the result from Proposition 2. (ii) and (iii) follow from noting that $\Pi(v_1)$ is decreasing in n (since the second term of $\Pi(v_1)$ is smaller than v_1) and using equations (7) and (8) which respectively characterize the seller's value and prices.

[21] Modeling the impact of different auction formats is somewhat delicate because in general optimal bids depend on the beliefs the new buyer has about the value of the first buyer, and these in turn depend on how much of prior bargaining he can observe. The analysis is tractable only if we assume that he observes the whole history.

This result suggests that sellers would benefit more from specializing in a narrow market, intensively targeting a given group of potential buyers rather than casting a very wide net. Although we do not model it here, this benefit of specialization must be balanced against the potential drop in the contact frequency, λ_b.

4.4 Temporary Reputation/Arrival of Information

Another model we can apply our general setup to is a situation of temporary reputation of the buyer: he starts out privately knowing his type, but over time information can arrive making his type public. Additionally, assume that upon arrival of the information the players split the surplus according to some expected shares $\alpha, (1-\alpha)$ that represent the relative post-arrival bargaining strengths Then the expected post-arrival payoffs are:

$$\begin{aligned} W(v) &= v(1-\alpha), \\ \Pi(v) &= v\alpha . \end{aligned}$$

We can apply our previous results to obtain full characterization of the equilibrium:

$$P(v) = \frac{\lambda}{\lambda + r}\alpha v , \qquad T(v) = -\frac{\alpha\lambda}{r(\lambda + r)} \ln v$$

Finally, using Proposition (2), or directly from the equations above, we can show that as the expected bargaining power of the seller decreases (as measured by α):
i) The expected time to trade decreases.
ii) The surplus for the seller falls.
iii) For any t the price offered is lower.

5 Multiple Arrivals

In many markets the seller can wait for more than one additional buyer. That leads us to a natural extension of the model to multiple arrivals. Unfortunately, a general model in which the seller can bargain with multiple buyers at the same time and have more and more of them arrive is not tractable. To gain some intuition (and to demonstrate that some of the economics we described above are robust), we instead analyze a simpler, more stylized model.

In particular, we assume that there is a constant arrival rate of new buyers, λ, and the buyers have independent private values all drawn from the same distribution $F(v)$ with support $[0,1]$ (there are potentially infinitely many buyers that can arrive). Throughout this section we assume that $F(v)$ satisfies the downward-sloping marginal revenue condition, that is we assume that $v - \frac{1-F(v)}{f(v)}$ is strictly increasing.[22]

When a new buyer arrives we assume that the seller makes a last take-it-or-leave-it offer to the old buyer. If it is accepted, the game is over. If it is rejected, the new buyer replaces the old buyer and the bargaining starts from scratch until the next arrival. These assumptions are a combination of the setups in Fudenberg, Levine and Tirole (1987) (who allow for take-it-or-leave-it offers with replacement of buyers upon rejection, but have an infinite supply of buyers standing by available for immediate replacement) and in Inderst (2003) who has Poisson arrivals but does not allow for final offers. Nonetheless, the resulting equilibrium dynamics are very different from the ones in either of those papers.

We now sketch a characterization of a stationary equilibrium of this model. Note that after the old buyer rejects an offer, the game starts afresh. Stationarity is crucial for tractability, since it allows us to keep track of only one state variable, k, the cutoff of the currently bargaining buyer.

Denote by $V(k)$ the value of the seller (i.e. his expected equilibrium payoff) when he is bargaining with one buyer with a cutoff belief k. Let $V^* = V(1)$. This is the seller's expected value at the beginning of the game and also his expected continuation payoff after the old buyer rejects his final take it or leave it offer. For now, we will take V^* as given.

Let $V_A(k)$ be the expected payoff of the seller upon arrival (before the current buyer responds to the take it or leave it offer). To find it, note that upon arrival the seller will chose the final offer $p_A(k)$ to maximize:

$$p_A(k) = \arg\max_p \left(\frac{F(k) - F(p)}{F(k)} \right) p + \frac{F(p)}{F(k)} V^* \tag{15}$$

and the expected value upon arrival will satisfy:

$$F(k) V_A(k) = \max_p (F(k) - F(p)) p + F(p) V^* \tag{16}$$

From the envelope condition we have:

$$\frac{\partial}{\partial k} (F(k) V_A(k)) = f(k) p_A(k) \tag{17}$$

To pin down the equilibrium we will also use a technical lemma:

Lemma 1 *For any V^* and $k \geq V^*$ there is a unique and strictly increasing $p_A(k)$ that solves* (15).

The proof, which can be found in the Appendix, makes use of our regularity assumption on F.

Therefore, given a V^*, the above equations determine uniquely $V_A(k)$, and $p_A(k)$. Now, take $V_A(k)$ and $p_A(k)$ as given. How does the equilibrium in the one-on-one bargaining phase look like?

[22] Myerson (1981) calls this condition increasing virtual valuation, or the regular case.

As long as the seller gradually (i.e. without atoms) screens down the demand function (which he will do if $P(k)$ is strictly decreasing) then the seller's problem is as in the base model of Section 3:

$$rV(k) = \max_{\dot{K}\in(-\infty,0]} \lambda\left(V_A(k) - V(k)\right) + \left(P(k) - V(k)\right)\frac{f(k)}{F(k)}\left(-\dot{K}\right) + V'(k)\dot{K}$$

as before, in equilibrium we need the coefficients on $\dot{K}$ to add up to zero, which gives:

$$\begin{aligned} f(k)P(k) &= \frac{\partial}{\partial k}\left[V(k)F(k)\right] \\ V(k) &= \frac{\lambda}{\lambda+r}V_A(k) \end{aligned}$$

Therefore, using (17) we pin down the equilibrium prices (for the range that the seller smoothly screens down the demand):[23]

$$P(k) = \frac{\lambda}{\lambda+r}p_A(k)$$

Lemma (1) implies that $P(k)$ is strictly increasing for $k > V^*$. This guarantees no atoms over that range.

The buyer's local IC constraint for types $k > V^*$ still is:

$$(r+\lambda)(k - P(k)) = \lambda W(k) - P'(k)\dot{K} \tag{18}$$

Where the payoff upon arrival of the current cutoff type is:

$$W(k) = k - p_A(k).$$

because the current cutoff type trades for sure upon arrival (since $p_A(k) < k$ for $k > V^*$). Note as well that $k - W(k) = p_A(k)$ is strictly increasing (which implies that the local IC (18) is still sufficient).

Substituting the equilibrium $P(k)$ into the buyer's indifference condition we get:

$$-\dot{K} = \frac{\lambda+r}{\lambda}\frac{rK(t)}{p'_A(K(t))}$$

For example, if values are distributed uniformly we get simple expressions:

$$\begin{aligned} p_A(k) &= \frac{k+V^*}{2} \\ -\frac{\dot{K}}{K(t)} &= \frac{\lambda+r}{\lambda}2r \Longrightarrow T(v) = -\frac{\lambda}{(\lambda+r)2r}\ln(v) \end{aligned}$$

[23] Note that we get this very simple expression for equilibrium prices even though we no longer use $V_A(k) = E\left[\Pi(v)\,|v \le k\right]$.

The next step is to pin down V^*. Note that:

$$V_A(1) = (1 - F(p_A(1)))\, p_A(1) + F(p_A(1))\, V^*$$

and, as we argued, in equilibrium:

$$V(1) = V^* = \frac{\lambda}{\lambda + r} V_A(1)$$

Combining, we get an expression for $V_A(1)$:

$$V_A(1) = \max_p (1 - F(p))\, p + F(p) \frac{\lambda}{\lambda + r} V_A(1) \tag{19}$$

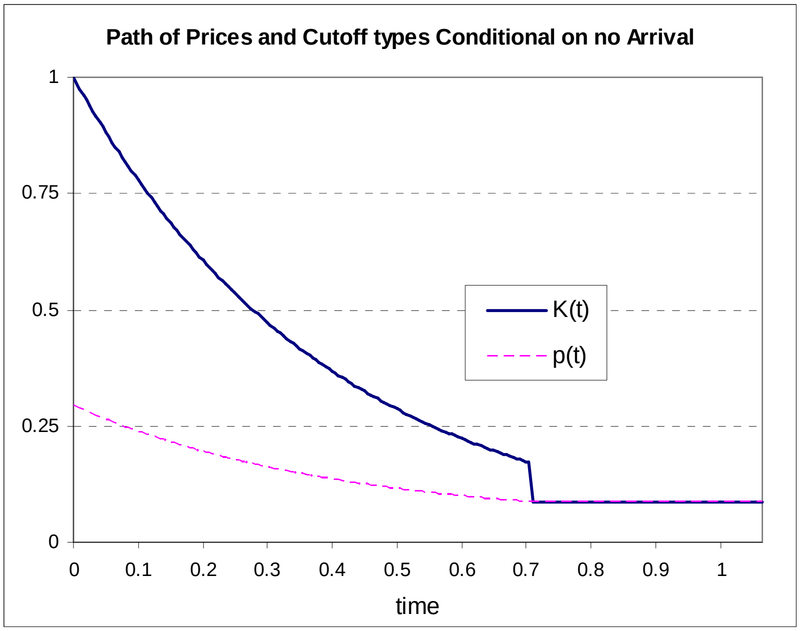
which can be shown to have a unique solution, which implies a unique V^* .

The only part left is to figure out whether the seller is going to smoothly screen down the demand function through all the types or if he is going to stop at some type. Note that the seller would never trade with types below $\frac{\lambda}{\lambda+r}V^*$, since this is the expected payoff he gets by rejecting the current buyer even before the new buyer arrives and restarting the game empty-handed.[24]

That leads to the following equilibrium dynamics: The seller runs smoothly down the demand function up to type $k^* = V^*$ (and the equilibrium $P(k)$, $K(t)$ and V^* are defined above, with the boundary condition $K(0) = 1$). But once he reaches k^*, since $p_A(k) = V^*$ for all $k \in \left[\frac{\lambda}{\lambda+r}V^*, k^*\right]$, the seller reaches the price $P(k^*) = \frac{\lambda}{\lambda+r}V^*$ and never decreases the price again. As a result, that price is immediately accepted by all types $v \in [P(k^*), k^*]$. In other words, the equilibrium reservation price of all types in this range is the same, $P(v) = \frac{\lambda}{\lambda+r}V^*$ (and all types below $\frac{\lambda}{\lambda+r}V^*$ have reservation prices equal to their types, $P(v) = v$, but the seller never trades with them).

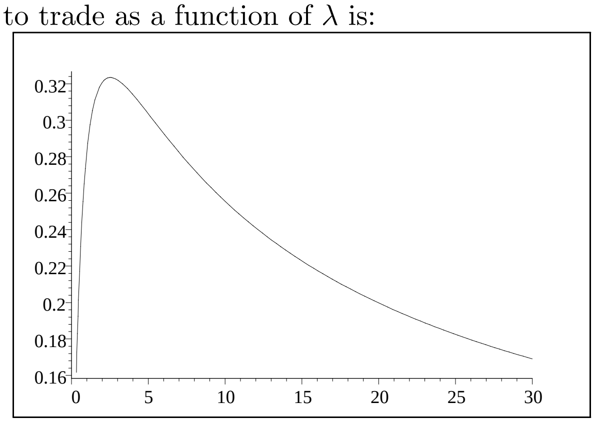
Hence, the violation of these two parts of Assumption 1 introduces a flat part in the reservation price function, $P(v)$ and a corresponding atom at the end of bargaining with the current buyer (the atom is consistent with the equilibrium since after it the seller does not drop the prices any more). Below we plot the path of offered prices and types trading in equilibrium conditional on no arrival for the case $\lambda = r = 1$. This parameterization implies $V^* = 0.172$. Prices fall until they reach $P(k^*) = \frac{\lambda}{\lambda+r}V^* = \frac{0.172}{2} = 0.086$ and then they remain flat at this level. This induces an atom of trade at time $t = 0.705$ when types between 0.172 and 0.086 accept the price $P(k^*) = 0.086$. Prices will not be reduced further since the seller would rather wait to start over than sell at lower prices.

[24] Technically, it violates part (iv) of Assumption 1 and also the spirit of part (i): it is not efficient for all types to trade immediately and the payoff of the seller conditional on the buyer having low value is much higher than this value. These assumptions now hold only over ranges $[\frac{\lambda}{\lambda+r}V^*, 1]$ and the endogenous lower cutoff can be treated as the relevant lowest buyer type that has value higher than the seller's cost.



Finally, we comment on how the time to trade changes with the frequency of arrivals, λ. When $\lambda > 0$ the expected time to trade is strictly positive. But what happens as $\lambda \to 0$ or $\lambda \to \infty$? We argue that in these cases the time to trade converges to zero.

First, in case $\lambda \to 0$, $P(k) \to 0$ for all k and hence the buyer finds it optimal not to delay trade. This is the classic Coase conjecture result. In the second case, $\lambda \to \infty$, we have that $V(1) \to 1$. That high expected payoff is achievable only if the expected time to trade converges to 0, since the transaction prices are bounded uniformly by 1. In the example with $v\tilde{}U[0,1]$, normalizing $r = 1$, we get that the expected time to trade as a function of λ is:



Expected time to trade as a function of λ

In this example, for $\lambda > 3r$, the expected time to trade is decreasing in λ.

6 Conclusions

When bargaining takes place in the context of a thin market, in which other traders might show up, trade will no longer take place immediately with the informed party capturing all the rents. Although many other explanations have been proposed for the observed delay in bargaining, we believe this to be a very natural one. It shows that delay is to be expected outside the extreme cases of perfect competition or bilateral monopolies.

Nonetheless, the Coasian dynamics are still useful in thinking about such markets, because the lack of commitment drives the seller's value down to his outside option of waiting for an arrival. This is what connects the characteristics of the market to the bargaining dynamics. For example, a higher ratio of buyers in the market leads to higher prices and longer times to trade. This, in turn, could affect the decision of traders to enter the market in the first place. The present model does not allow us to capture this general equilibrium effect since the arrival rates are exogenous in the model. Modeling endogenous entry of agents into the market is necessary to further our understanding of such markets.

7 Appendix

The main goal of this appendix is to prove:

Theorem 1: *Take any sequence of games indexed by the period lengths that asymptotically decrease to* 0 *and any selection of stationary equilibria of these games* $\{\kappa(p,\Delta),\ P(k,\Delta)\}$ *and the corresponding sequence* $\{V(k,\Delta),\ K(t,\Delta)\}$.
As $\Delta \to 0$, *these equilibria converge to the limit-equilibrium (which is unique). That is, as* $\Delta \to 0$, $V(k,\Delta) \to V(k)$, $P(k,\Delta) \to P(k)$ *and* $K(v,\Delta) \to K(t)$ *(all convergences are point-wise).*

We start with a series of lemmas that lead up to the proof.. Recall that to keep track of the dependence of the game and equilibrium on Δ we use notation $V(k,\Delta)$ etc. We use $k_+ = \kappa(P(k,\Delta),\Delta)$ to denote next-period cutoff, k_- to denote, the previous period cutoff, and k_{++} to denote the cutoff two periods from now, all given a particular equilibrium and current cutoff k.

Lemma 2 (No Quiet Period) *For all* $\Delta > 0$, *all stationary equilibria must have trade with positive probability in every period.*

Proof. Suppose that there exists an equilibrium in which after a cutoff type k^* is reached, there is a period in which the probability of trade is zero. That implies that next period cutoff type is also k^* and hence (by definition of equilibrium) the price the seller sets in this and all future periods is simply $P(k^*,\Delta)$ and there is no trade till the end of the game.

The seller's expected continuation payoff is simply the expected payoff upon arrival discounted by the expected time of arrival:

$$V\left(k^{*},\Delta\right)=\frac{\Delta\lambda}{1-\left(1-\Delta\lambda\right)e^{-\Delta r}}V_{A}\left(k^{*}\right)$$

Suppose that the seller deviates to a price $p'=\frac{\Delta\lambda}{1-(1-\Delta\lambda)e^{-\Delta r}}V_{A}\left(k^{*}\right)+\varepsilon$. If this price is accepted by some types (i.e. $\kappa\left(p',\Delta\right)<k^{*}$) then for all $\varepsilon>0$ we have a contradiction, since the seller payoff would be greater than $V\left(k^{*},\Delta\right)$ (no matter what the cutoff type $k'=\kappa\left(p',\Delta\right)$ the seller must obtain from the remaining types at least $\frac{\Delta\lambda}{1-(1-\Delta\lambda)e^{-\Delta r}}V_{A}\left(k'\right)$, as in the proposed equilibrium, but obtains a strictly higher payoff from types $\left(k',k^{*}\right)$).

Suppose that this price is rejected for sure for every ε. That implies that in the continuation game $k=k^{*}$ and hence the seller returns to $P\left(k^{*},\Delta\right)$ forever. As a result, the buyer expected continuation payoff is $\frac{\Delta\lambda}{1-(1-\Delta\lambda)e^{-\Delta r}}W\left(k\right)$. But since $\frac{\Delta\lambda}{1-(1-\Delta\lambda)e^{-\Delta r}}\left(W\left(v\right)+\Pi\left(v\right)\right)<v$, there exists an $\varepsilon>0$ such that types close to k^{*} would be strictly better off accepting p', a contradiction. ■

Lemma 3 *For all stationary equilibria as* $\Delta\rightarrow0$, $\frac{P(k,\Delta)-P(k_{+},\Delta)}{\Delta}\rightarrow O\left(const\right)$.

Proof. Recall the buyer's optimality:

$$k_{+}-P\left(k,\Delta\right)\ =e^{-\Delta r}(\Delta\lambda W\left(k_{+}\right)+\left(1-\Delta\lambda\right)\left(k_{+}-P\left(k_{+},\Delta\right)\right))$$

We now re-group the terms:

$$k_{+}\left(1-e^{-\Delta r}\left(1-\Delta\lambda\right)\right)=e^{-\Delta r}\Delta\lambda W\left(k_{+}\right)+\left(P\left(k,\Delta\right)-P\left(k_{+},\Delta\right)\right)+\left(1-e^{-\Delta r}\left(1-\Delta\lambda\right)\right)P\left(k_{+},\Delta\right)$$

divide by Δ :

$$k_{+}\frac{\left(1-e^{-\Delta r}\left(1-\Delta\lambda\right)\right)}{\Delta}=e^{-\Delta r}\lambda W\left(k_{+}\right)+\frac{P\left(k,\Delta\right)-P\left(k_{+},\Delta\right)}{\Delta}+\frac{1-e^{-\Delta r}\left(1-\Delta\lambda\right)}{\Delta}P\left(k_{+},\Delta\right)$$

and take the limit:

$$\lim_{\Delta\rightarrow0}\frac{\left(1-e^{-\Delta r}\left(1-\Delta\lambda\right)\right)}{\Delta}=\lambda+r$$

$$-\frac{P\left(k,\Delta\right)-P\left(k_{+},\Delta\right)}{\Delta}\rightarrow\lambda W\left(k\right)+\left(\lambda+r\right)\left(\lim_{\Delta\rightarrow0}P\left(k,\Delta\right)-k\right)=O\left(const\right)$$

■

Lemma 4 (No Atoms) *For all* $\varepsilon>0$ *there exists a* $\Delta>0$ *such that in all stationary equilibria, the probability of trade in any period (after the initial period) is* $<\varepsilon$. *In other words, as* $\Delta\rightarrow0$, *on the equilibrium path* $k-k_{+}\rightarrow0$ *and* $F\left(k\right)-F\left(k_{+}\right)\rightarrow0$.

Proof. Consider a sequence of equilibria such that for all $\Delta>0$ there is at least a mass of ε of buyers that trade at some time $\tau_{\Delta}>0$ for all Δ, i.e. $\varepsilon\leq F\left(k\right)-F\left(k_{+}\right)$ for some period with current cutoff

$k < 1$. This implies that for all Δ there exists a buyer with value $\bar{k} = \frac{k+k_+}{2}$ that trades at the same time as type k_+ and $\varepsilon_1 > 0$ such that $k - k_+ > \varepsilon_1$.

If type k is the lowest type willing to trade at price $P(k_-, \Delta)$ and type k_+ is the lowest type willing to trade at price $P(k, \Delta)$, then buyer's optimality requires:

$$\begin{aligned} k - P(k_-, \Delta) &= e^{-\Delta r}(\Delta\lambda W(k) + (1-\Delta\lambda)(k - P(k,\Delta))) \\ k_+ - P(k,\Delta) &= e^{-\Delta r}(\Delta\lambda W(k_+) + (1-\Delta\lambda)(k_+ - P(k_+,\Delta))) \end{aligned}$$

Note that

$$\begin{aligned} & v - e^{-\Delta r}(\Delta\lambda W(v) + v(1-\Delta\lambda)) \\ = \ & v\left(1 - e^{-\Delta r}\right) + e^{-\Delta r}\Delta\lambda(v - W(v)) \end{aligned}$$

is strictly increasing in v (because $v - W(v)$ is strictly increasing). Therefore, $\varepsilon_1 > 0$ implies that there exists $\varepsilon_2 > 0$ such that

$$\begin{aligned} \bar{k} - P(k,\Delta) &> e^{-\Delta r}(\Delta\lambda W(\bar{k}) + (1-\Delta\lambda)(\bar{k} - P(k_+,\Delta))) + \varepsilon_2\left(1 - e^{-\Delta r}(1-\Delta\lambda)\right) \\ \bar{k} - P(k_-,\Delta) &< e^{-\Delta r}(\Delta\lambda W(\bar{k}) + (1-\Delta\lambda)(\bar{k} - P(k,\Delta)) - \varepsilon_2\left(1 - e^{-\Delta r}(1-\Delta\lambda)\right) \end{aligned}$$

Rearranging:

$$\begin{aligned} & \frac{e^{-\Delta r}\Delta\lambda}{(1 - e^{-\Delta r}(1-\Delta\lambda))}\left(W(\bar{k}) + (P(k,\Delta) - P(k_+,\Delta))\right) + P(k_+,\Delta) + \varepsilon_2 \\ < \ & \bar{k} < \frac{e^{-\Delta r}\Delta\lambda}{(1 - e^{-\Delta r}(1-\Delta\lambda))}\left(W(\bar{k}) + (P(k_-,\Delta) - P_\Delta(k))\right) + P_\Delta(k) - \varepsilon_2 \end{aligned}$$

Now in the limit as $\Delta \to 0$, using $P(k,\Delta) - P(k_-,\Delta) \to 0$ (proven in Lemma 3) and $\lim_{\Delta\to 0} \frac{\Delta\lambda}{(e^{\Delta r} - (1-\Delta\lambda))} = \frac{\lambda}{r+\lambda}$, we get:

$$\frac{\lambda}{r+\lambda}W(\bar{k}) + \lim_{\Delta\to 0} P(k_+,\Delta) + \varepsilon_2 \le \bar{k} \le \frac{\lambda}{r+\lambda}W(\bar{k}) + \lim_{\Delta\to 0} P(k,\Delta) - \varepsilon_2$$

Which implies there cannot exist such a $\bar{k}$ and therefore there cannot be a mass of buyers trading in any period after the initial period. ■

Lemma 5 *For all stationary equilibria,* $\lim_{\Delta\to 0} V(k,\Delta) = V(k) = \frac{\lambda}{r+\lambda}V_A(k)$.

Proof. First, we can bound the seller's payoff from below by considering a deviation to (completely) slow down the trade. Since the seller can always choose to wait for the arrival of an event, his value must at least be equal to the expected discounted payoff upon arrival. That is, in all stationary equilibria and for all $\Delta > 0$ the seller's value $V(k, \Delta)$ must satisfy:

$$V(k, \Delta) \geq \frac{\Delta\lambda}{1 - (1 - \Delta\lambda) e^{-\Delta r}} V_A(k)$$

As $\Delta \to 0$ the RHS converges to $\frac{\lambda}{r+\lambda} V_A(k)$.

Second, we can bound the seller's payoff from above by considering a deviation to speed up trade. In particular, suppose that the highest remaining type is k and suppose that the seller deviates and instead of asking for $P(k, \Delta)$ he asks for $P(k_+, \Delta)$. For this not to be a profitable deviation, in all stationary equilibria and for all $\Delta > 0$ the seller's payoff must satisfy:

$$P(k, \Delta)[F(k) - F(k_+)] + e^{-\Delta r} U(k_+, \Delta) \geq P(k_+, \Delta)[F(k) - F(k_{++})] + e^{-\Delta r} U(k_{++}, \Delta) \tag{20}$$

where to simplify notation we used $U(k, \Delta) \equiv F(k) V(k, \Delta)$.

By definition of $V(k, \Delta)$ we can write,

$$\begin{aligned} U(k_+, \Delta) &= \Delta\lambda V_A(k_+) F(k_+) + \\ &\quad (1 - \Delta\lambda)\left[P(k_+, \Delta)(F(k_+) - F(k_{++})) + e^{-\Delta r} U(k_{++}, \Delta)\right] \end{aligned}$$

Substituting it to (20) and rearranging terms we get:

$$\begin{aligned} & [P(k, \Delta) - P(k_+, \Delta)][F(k) - F(k_+)] - P(k_+, \Delta)[F(k_+) - F(k_{++})]\left(1 - e^{-\Delta r}(1 - \Delta\lambda)\right) \\ \geq\ & -e^{-\Delta r} \Delta\lambda V_A(k_+) F(k_+) + e^{-\Delta r}\left(1 - (1 - \Delta\lambda) e^{-\Delta r}\right) U(k_{++}, \Delta) \end{aligned}$$

Divide by Δ

$$\begin{aligned} & \frac{P(k, \Delta) - P(k_+, \Delta)}{\Delta}[F(k) - F(k_+)] - P(k_+, \Delta)[F(k_+) - F(k_{++})] \frac{1 - e^{-\Delta r}(1 - \Delta\lambda)}{\Delta} \\ \geq\ & -e^{-\Delta r} \lambda V_A(k_+) F(k_+) + e^{-\Delta r} \frac{1 - e^{-\Delta r}(1 - \Delta\lambda)}{\Delta} U(k_{++}, \Delta) \end{aligned}$$

Now, recall from Lemma 3 that $\frac{P(k,\Delta) - P(k_+,\Delta)}{\Delta} \to O(const)$ and from Lemma 4 that $F(k) - F(k_+) \to 0$. Using again

$$\lim_{\Delta \to 0+} \frac{1 - e^{-\Delta r}(1 - \Delta\lambda)}{\Delta} = r + \lambda$$

we get that in the limit[25]

$$\lambda V_A(k) F(k) \geq \lim_{\Delta \to 0} (r + \lambda) U(k, \Delta)$$

[25] We have used here that $U(k_+, \Delta) \to U(k, \Delta)$, as $\Delta \to 0$. This is true since $U(k, \Delta)$ is continuous and $k \to k_+$.

This implies the lower bound

$$\lim_{\Delta\to 0} V(k,\Delta) \leq \frac{\lambda}{r+\lambda} V_A(k)$$

Combining it with the opposite bound (that we obtained in the first step) yields the result:

$$V(k,\Delta) \underset{\Delta\to 0}{\to} V(k) = \frac{\lambda}{r+\lambda} V_A(k)$$

■

Lemma 6 *For any discrete time stationary equilibria $P(k,\Delta)$ converges to $P(k) = \frac{\lambda}{r+\lambda}\Pi(k)$ as $\Delta \to 0$.*

Proof. First, Lemma 3 and $P(k,\Delta)$ being increasing in k, implies that for every sequence of equilibrium pricing rules:

$$\lim_{\varepsilon\to 0}\left(\lim_{\Delta\to 0} P(k,\Delta) - P(k-\varepsilon,\Delta)\right) = 0$$

in other words, $P(k,\Delta)$ converge to a continuous function.

Hence, if there is a sequence of equilibrium pricing rules $P(k,\Delta)$ converging to something different than $P(k)$, they must differ from $P(k)$ in an open interval. So suppose that there exists a sequence of equilibrium pricing rules $P(k,\Delta)$ such that, as $\Delta \to 0$, $P(k,\Delta) \to \tilde{P}(k) \neq P(k)$ for $k \in (\underline{k}, \overline{k})$.

Consider first the case $\tilde{P}(k) > P(k) = \frac{\lambda}{\lambda+r}\Pi(v)$ for $k \in (\underline{k}, \overline{k})$. Such prices could not be part of an equilibrium because then the expected seller's value would exceed $\frac{\lambda}{r+\lambda} V_A(k)$, contradicting Lemma 5. To see this note that value to the seller at the first cutoff lower than $\overline{k}$, k_0, from following $P(k,\Delta)$ would be:

$$\begin{aligned}\widehat{V}(k_0,\Delta) &= \sum_{n=0}^{N-1} e^{-rn\Delta}\left(\Delta\lambda(1-\Delta\lambda)^n \frac{F(k_n)}{F(k_0)} V_A(k_n) + (1-\Delta\lambda)^{n+1} P(k_n,\Delta)\frac{F(k_n)-F(k_{n+1})}{F(k_0)}\right) \\ &\quad + \frac{F(k_N)}{F(k_0)} e^{-rN\Delta}(1-\Delta\lambda)^N \widehat{V}(k_N,\Delta)\end{aligned}$$

where N is the number of periods for which $k \in (\underline{k}, \overline{k})$ and $\{k_n\}$ is the sequence of equilibrium cutoff types (with k_0 the first cutoff type in this range and k_N the last one).

To bound $\widehat{V}(k_0,\Delta)$ suppose that the seller instead gets prices $\frac{\lambda}{\lambda+r}\Pi(k_n)$ (from the same trades types) and obtains continuation payoff $\frac{\lambda}{\lambda+r} V_A(k_N)$ instead of $\widehat{V}(k_N,\Delta)$. Both are lower bounds, since $P(k_n,\Delta) > \frac{\lambda}{\lambda+r}\Pi(k_n)$ uniformly for all small Δ, and $V(k_N,\Delta)$ converges to $\frac{\lambda}{\lambda+r} V_A(k_N)$ from above.

Call payoffs calculated by that substitution $V_L(k_0,\Delta)$. We get:

$$\lim_{\Delta\to 0}\widehat{V}(k_0,\Delta) > \lim_{\Delta\to 0} V_L(k_0,\Delta) = \frac{\lambda}{\lambda+r} V_A(k_0)$$

The equality follows since conditional on any type k, if the buyer deviated to always reject the offer, then the seller's expected payoff in the limit as $\Delta \to 0$ would be $\frac{\lambda}{\lambda+r}\Pi(k)$. Thanks to the stationarity of the Poisson process, this would be in fact the expected payoff at any moment of time. Moreover, given that the transaction prices are $\frac{\lambda}{\lambda+r}\Pi(k)$ and trade happens only conditional on the event not arriving yet, when the buyer accepts this price, the seller gets the same payoff from that type as he would if the buyer rejected forever.

That establishes that $\tilde{P}(k) > P(k)$ would allow the seller to earn even in the limit strictly more than $V(k)$, a contradiction.

Next, suppose there exists a sequence of equilibrium pricing rules $P(k,\Delta)$ such that, as $\Delta \to 0$, $P(k,\Delta) \to \tilde{P}(k) < P(k)$ for $k \in \left(\underline{k}, \overline{k}\right)$. These pricing rules cannot be part of an equilibrium sequence either, since after an analogous substitution (prices $\frac{\lambda}{\lambda+r}\Pi(k_n)$ and continuation payoff $\frac{\lambda}{\lambda+r}V_A(k_N)$), we would get a strictly higher payoff in the limit, implying that

$$\lim_{\Delta\to 0}\widehat{V}(k_0,\Delta) < \frac{\lambda}{\lambda+r}V_A(k_0)$$

contradicting Lemma 5 again.

Therefore, to satisfy Lemma 5 all equilibrium pricing rules $P(k,\Delta)$ have to converge to $P(k)$. ■

Lemma 7 *In the limit, as $\Delta \to 0$, in any sequential equilibrium there cannot be an atom of trade at $t = 0$.*

Proof. Suppose that in equilibrium there exists some $\bar{k} < 1$ such that all types $v \geq \bar{k}$ trade at $t = 0$. Then the seller payoffs, $\left(1 - F\left(\bar{k}\right)\right)P\left(\bar{k}\right) + F\left(\bar{k}\right)V\left(\bar{k}\right)$, would be strictly less than $\frac{\lambda}{\lambda+r}V_A(1)$, contradicting that he can achieve that payoff by simply asking very high prices. To see this note that:

$$\begin{aligned}
\Pi(\bar{k}) &< \Pi(k) \text{ for all } k > \bar{k} \Longrightarrow \\
\left(1 - F\left(\bar{k}\right)\right)\Pi\left(\bar{k}\right) + F\left(\bar{k}\right)\int_0^{\bar{k}}\frac{\Pi(v)f(v)}{F\left(\bar{k}\right)}dv &< \left(1 - F\left(\bar{k}\right)\right)\int_{\bar{k}}^1\frac{\Pi(v)f(v)}{1-F\left(\bar{k}\right)}dv + F\left(\bar{k}\right)\int_0^{\bar{k}}\frac{\Pi(v)f(v)}{F\left(\bar{k}\right)}dv = V_A(1) \\
&\Downarrow \\
\left(1 - F\left(\bar{k}\right)\right)\frac{\lambda}{\lambda+r}\Pi\left(\bar{k}\right) + F\left(\bar{k}\right)\frac{\lambda}{\lambda+r}V_A\left(\bar{k}\right) &< \frac{\lambda}{\lambda+r}V_A(1) \\
&\Downarrow \\
\left(1 - F\left(\bar{k}\right)\right)P\left(\bar{k}\right) + F\left(\bar{k}\right)V\left(\bar{k}\right) &< \frac{\lambda}{\lambda+r}V_A(1)
\end{aligned}$$

■

Lemma 8 *Consider a sequence of stationary equilibria as $\Delta \to 0$. Let the equilibrium path of cutoff types be defined by $k(0,\Delta) = 1$, $k(t,\Delta) = \kappa(P(k_t,\Delta),\Delta)$ for $t \in \{0, \Delta, 2\Delta...\}$ and for any $t \in (n\Delta, (n+1)\Delta)$ (where $n \in \mathbb{N}$), $k(t,\Delta) = k(n\Delta,\Delta)$. That is, the $k(t,\Delta)$ function is a decreasing step function changing value at times that the seller makes offers.*
Then as $\Delta \to 0$, $\frac{k(t+\Delta,\Delta)-k(t,\Delta)}{\Delta} \to \dot{K}(t)$ and $k(t,\Delta) \to K(t)$.

Proof. Our previous notation k and k_+ corresponds now to $k(t,\Delta)$ and $k(t+\Delta,\Delta)$. Recall the buyer optimality condition in discrete time:

$$k_+ - P(k,\Delta) \ = e^{-\Delta r}(\Delta\lambda W(k_+) + (1-\Delta\lambda)(k_+ - P(k_+,\Delta)))$$

Subtracting $e^{-\Delta r}(1-\lambda\Delta)(k_+ - P(k,\Delta))$ from both sides, dividing by Δ and taking $\Delta \to 0$ we get (using that $P(k,\Delta)$ converges to $P(k)$ and that there are no atoms in the limit) the following limit of the indifference condition:

$$\underbrace{\frac{1-e^{-\Delta r}(1-\lambda\Delta)}{\Delta}}_{\to\lambda+r}(k_+ - P(k,\Delta)) \ = e^{-\Delta r}\lambda W(k_+) + \underbrace{e^{-\Delta r}(1-\Delta\lambda)}_{\to 1}\underbrace{\frac{(P(k,\Delta)-P(k_+,\Delta))}{k-k_+}}_{\to P'(k)}\frac{k-k_+}{\Delta}$$

so that in the limit we get the the optimality condition for the limit-equilibrium:

$$(\lambda+r)(K(t) - P(K(t))) = \lambda W(K(t)) - P'(k)\dot{K}(t)$$

so indeed $\frac{k(t+\Delta,\Delta)-k(t,\Delta)}{\Delta} \to \dot{K}(t)$. Finally, from Lemma 7 we have that $K(0) = 1 = k(0,\Delta)$. Because $k(t,\Delta)$ is bounded and the derivative $\dot{K}(t)$ is bounded, we can use the fundamental theorem of calculus to claim that since derivative of the limit of $k(t,\Delta)$ converges to $\dot{K}(t)$, and $k(0,\Delta) = K(0)$, $k(t,\Delta)$ converges to $K(t)$ for all $t \geq 0$. ■

Proof of Theorem 1. Lemmas 5 and 6 show that in the limit as $\Delta \to 0$ all discrete time equilibria deliver the same value to the seller and the same transaction prices given a current cutoff type. Lemma 8 then shows that how the cutoff types change through time also converges to $K(t)$. ■

Proof of Theorem 2. The fact that equations (8) and (10) together with the boundary condition $K(0) = 1$ characterize an equilibrium is discussed in detail in Section 3. Uniqueness follows from noting that only necessary conditions were used to characterize this equilibrium. ■

Proof of Proposition 1. From equations (8) and (10) we can see that if $\Pi(v)$ and $W(v)$ are independent of $F(v)$ then $P(k)$ and $\dot{K}$ are independent of $F(v)$ and therefore the equilibrium is independent of $F(v)$. ■

Proof of Proposition 2. (1) From equation (*10*) we can see that keeping $\Pi_1(v)+W_1(v) = \Pi_2(v)+W_2(v)$ or simply $W_1(v) = W_2(v)$ leads to $-\dot{K}_2 > -\dot{K}_1$ which implies that buyers with the same valuation will trade faster in the environment with $\Pi_2(v)$. This follows because the boundary conditions are the same and with $\Pi_2(v)$ we go through types faster since $-\dot{K}_2 > -\dot{K}_1$.

(2) As $\Pi'(v) \to 0$ $\Pi(v) \to 0$ this implies the seller's value: $V(k) = \frac{\lambda}{\lambda+r}V_A(k) \to 0$ and prices are also converging to zero $P(k) = \frac{\lambda}{\lambda+r}\Pi(k) \to 0$. Trade on the other hand is taking place faster since $-\dot{K} \to \infty$ therefore there will be no delay in trade and the buyer will capture all the surplus. ■

Proof of Claims in Section 4.3:.

$\Pi(v_1)$ can be re-written as:

$$\Pi(v_1) = \gamma v_1 + (1-\gamma)\left(\int_0^{v_1} x f(x)\, d(x) + (1 - F(v_1))\, v_1\right)$$

Hence,

$$\begin{aligned}\Pi'(v_1) &= \gamma + (1-\gamma)\left(v_1 f(v_1) + (1 - F(v_1)) - f(v_1)\, v_1\right) \\ &= \gamma + (1-\gamma)(1 - F(v_1)) \\ &= 1 - F(v_1) + F(v_1)\,\gamma\end{aligned}$$

Therefore:

$$\frac{\partial \Pi'(v_1)}{\partial \gamma} = F(v_1) > 0$$

Therefore, the larger γ the larger $\Pi'(v)$ $\forall v$ and from Proposition (2) this implies that delay is decreasing in the number of different buyer classes. (ii) and (iii) follow from noting that $\Pi(v_1)$ is decreasing in n since the second term of $\Pi(v_1)$ is smaller than v_1 and using equations (2) and (3) which respectively characterize the seller's value and prices.■

Proof of Lemma 1. $p_A(k)$ is a solution to the F.O.C.:

$$p - \frac{(F(k) - F(p))}{f(p)} = V^*$$

Now, the LHS is decreasing in k.[26] We claim that it is increasing in p if the marginal revenue is downward sloping. The derivative of the LHS with respect to p is:

$$1 - \frac{-f^2(p) - (F(k) - F(p))\, f'(p)}{f^2(p)} = 2 + \frac{(F(k) - F(p))\, f'(p)}{f^2(p)}$$

which if $f'(p) > 0$ is positive for all k and if $f'(p)$ is < 0 it is the smallest for $k = 1$, but then this expression is positive by assumption.

Hence the LHS of the F.O.C. is increasing in p for all k and decreasing in k, which implies that $p_A(k)$ is strictly increasing. ■

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